Discussion 2

1. Review: Arithmetic Progression and Geometric Progression, and their sums

(a). *Arithmetic progression* is a sequence of numbers such that the difference between the consecutive terms is constant.

$$\begin{aligned} a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} \\ a_n = a_1 + (n-1)d \end{aligned}$$

The closed-form formula of its sum (a.k.a. Arithmetic series) is given by [1]:

$$\sum_{i=1}^{n} a_1 + (i-1)d = \frac{(a_1 + a_n)n}{2}$$

(b). *Geometric progression* is a sequence of non-zero numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio.

$$\begin{aligned} &\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{a_n}{a_{n-1}} = r \neq 0 \\ &a_n = a_1 r^{n-1} \end{aligned}$$

The closed-form formula of its sum (a.k.a. Geometric series) is given by [2]:

$$\sum_{i=1}^{n} a_1 r^{i-1} = \frac{a_1(r^n - 1)}{r - 1}$$

2. Solving Simple Recurrence by Substitution

Example. Solve $a_n = 2a_{n-1} - 1$ with initial condition $a_0 = 1$.

$$\begin{aligned} a_n &= 2\underline{a_{n-1}} - 1 && (1) \\ &= 2(\underline{2a_{n-2} - 1}) - 1 && (2) \\ &= 2^2\underline{a_{n-2}} - 2 - 1 && (3) \\ &= && (4) \\ &= && (5) \\ &= \cdots && (6) \\ &= && (7) \\ &= && (8) \\ &= && (9) \end{aligned}$$

[1] n is the number of terms in the series.
[2] n is the number of terms in the series.

3. Review: Polynomials, Exponentials, Logarithms, and their Derivatives [3]

(a). Polynomials

$p(n) = \sum_{i=0}^{d} a_i n^i$ is called a polynomial in n of degree d where $a_d \neq 0$.

A polynomial is asymptotically positive if and only if $a_d > 0$, and we have $p(n) = \Theta(n^d)$.

(b). Exponential functions

a^n is the basic form of an exponential, where a is called base.

$(a^n)^m = a^{mn} = (a^m)^n$.

For all real constants a and b such that $a > 1$,

$$\lim_{x\to\infty} \frac{n^b}{a^n} = 0$$

from which we can conclude that any exponential function with a base strictly greater than 1 grows faster than any polynomial function.

(c). Logarithms

$\lg n = \log_2 n$; $\ln n = \log_e n$ (natural logarithm).

For all real $a > 0$, $b > 0$, $c > 0$, and n, we have the following properties:

i. $a = b^{\log_b a}$

ii. $\log_c(ab) = \log_c a + \log_c b$

iii. $\log_b a^n = n \log_b a$

iv. $\log_b a = \frac{\log_c a}{\log_c b}$

(d). Derivatives

$f(x) = ax^n$, $\dfrac{\mathrm{d}f}{\mathrm{d}x} = anx^{n-1}$.

$f(x) = a^x$, $\dfrac{\mathrm{d}f}{\mathrm{d}x} = a^x \ln a$.

$f(x) = \ln x$, $\dfrac{\mathrm{d}f}{\mathrm{d}x} = \dfrac{1}{x}$, $x > 0$; $f(x) = \log_a x$, $\dfrac{\mathrm{d}f}{\mathrm{d}x} = \dfrac{1}{x \ln a}$, $x > 0$.

4. Asymptotic Notation: O, Ω, Θ

Example: We have $f(n) = \lg n$, $g(n) = \ln n$, what is their relation represented in asymptotic notation? There are two ways.

(a). By definition

To-do: find positive constants c and n_0

[3] For floors and ceilings, modular arithmetic and factorials, read Section 3.2 of the textbook.

(b). By using limits (L'Hopital's rule)

5. Growth relation between frequently used functions

Frequently used functions	Order of Growth	Example
constant	$O(1)$	$f(n) = 10$
logarithm	$O(\log n)$	$f(n) = 4\log(n+2)$
linear	$O(n)$	$f(n) = 2n+1$
linearithmic	$O(n \log n)$	$f(n) = 3n\log 4n + 1$
quadratic	$O(n^2)$	$f(n) = 2n^2 + 3n + 1$
cubic	$O(n^3)$	$f(n) = n^3 + 1$
exponential	$O(a^n), a \geq 2$	$f(n) = 3^n$