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THE GENERALIZED SINGULAR VALUE DECOMPOSITION AND THE GENERAL (A-λB)-PROBLEM

BO KÅGSTRÖM

Institute of Information Processing, University of Umeå, S-90187 Umeå, Sweden

Dedicated to Germund Dahlquist, on the occasion of his 60th birthday.

Abstract.

In the study of the canonical structure of matrix pencils $A-\lambda B$, the column and row nullities of A and B and the possible common nullspaces give information about the Kronecker structure of $A-\lambda B$. It is shown how to extract the significant information concerning these nullspaces from the generalized singular value decomposition (GSVD) of the matrix pair (A, B). These properties are the basis for the RGSVD-algorithm that will be published elsewhere [9]. An algorithm for the numerical computation of the transformation matrices (V, X), used in an equivalence transformation $V^H(A-\lambda B)X$, is presented and discussed. A generalized QZ-decomposition (GQZD) of a matrix pair (A, B) is also formulated, giving a unitary equivalent transformation $V^H(A-\lambda B)Q$. If during the computations X gets too illconditioned we switch to the GQZD, sacrificing a simple diagonal structure of the transformed B-part in order to maintain stability.

1. Introduction.

In the study of the canonical structure of matrix pencils $A-\lambda B$ the column and row nullities of $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{m\times n}$ and their possible common nullspaces play an important role. Parts of the singularities in A and B correspond to zero and infinite eigenvalues of $A-\lambda B$, respectively, and further if, for example, A and B have a common column nullspace there exist minimal column indices in the *Kronecker canonical form* (KCF) of $A-\lambda B$. For arbitrary m and n the KCF of a singular pencil $A-\lambda B$ under strictly equivalent transformations is given by (see [2])

$$P(A-\lambda B)Q = \operatorname{diag}\{J-\lambda I, I-\lambda N, L_{\varepsilon_1}, L_{\varepsilon_2}, \ldots, L_{\varepsilon_p}, L_{\eta_1}^T, L_{\eta_2}^T, \ldots, L_{\eta_q}^T\} \tag{1.1}$$

where

J corresponds to the finite eigenvalues (including zero-eigenvalues) of $A-\lambda B$. N is nilpotent and corresponds to the infinite eigenvalues of $A-\lambda B$.

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Both N and J are direct sums of k by k Jordan blocks N_k and $J_k(\lambda_i)$ respectively.
L_ε and L_η^T are $\varepsilon \times (\varepsilon+1)$ and $(\eta+1)\times\eta$ respectively, and bidiagonal. For example

$$L_3 = \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \end{bmatrix}.$$

The ε_i and η_i are called the *minimal right* (*column*) and *left* (*row*) *indices* of $A-\lambda B$ respectively. Here we assume that $0 \leqq \varepsilon_1 \leqq \varepsilon_2 \leqq \ldots \leqq \varepsilon_p$ and $0 \leqq \eta_1 \leqq \eta_2 \leqq \ldots \leqq \eta_q$.

Notice that the blocks L_ε and/or L_η^T will appear only if $A-\lambda B$ is singular and they expose the singularity of $A-\lambda B$ in the following way. For L_ε there exists a polynomial column vector such that

$$L_\varepsilon \underbrace{[1, \lambda, \ldots, \lambda^\varepsilon]^T}_{\varepsilon+1} = \underbrace{[0, \ldots, 0]^T}_{\varepsilon}$$

while for L_η^T there exists a polynomial row vector such that

$$\underbrace{[1, \lambda, \ldots, \lambda^\eta]}_{\eta+1} L_\eta^T = \underbrace{[0, \ldots, 0]}_{\eta}.$$

For a *regular pencil* $A-\lambda B$ ($A, B \in \mathbb{C}^{n\times n}$, $\det(A-\lambda B) \not\equiv 0$, i.e. a nonzero determinant for all except a finite number of $\lambda \in \mathbb{C}$) the KCF simplifies to the *Weierstrass-Kronecker canonical form* (W-KCF) diag $\{J-\lambda I, I-\lambda N\}$, i.e. the canonical form consists of finite and infinite eigenvalues but no minimal indices. If $A-\lambda B$ is of the form $A-\lambda I$, i.e. a standard eigenvalue problem, the KCF simplifies further to the *Jordan normal form* (JNF) diag $\{J-\lambda I\}$. We illustrate the case with a common column nullspace by considering a $3(=m)$ by $6(=n)$ singular pencil, already in canonical form:

$$A-\lambda B = \left[\begin{array}{cc:cc:cc} -\lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -\lambda & 1 \end{array}\right] = \operatorname{diag}\{J_2(0), L_0, L_0, L_1\}. \tag{1.2}$$

Its KCF consists of one Jordan block of order 2 corresponding to the zero eigenvalue (one $J_2(0)$-block), 2 minimal column indices of degree 0 (two L_0-blocks), and 1 minimal column index of degree 1 (one L_1-block). Notice that the 0 by 1 blocks L_0 only contribute to the column dimension of $A-\lambda B$. Studying A and B separately and pivoting A and B so that all zero-columns of

A are the leading ones we obtain:

(1.3a)
$$A' = \left[\begin{array}{cccc:cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} \} r_1 = 2 \\ \\ \\ \end{matrix}$$
$$\underbrace{\qquad\qquad}_{n_1 = 4}$$

(1.3b)
$$B' = \overbrace{\left[\begin{array}{cccc:cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]}^{\qquad\qquad} \begin{matrix} \} r_1 = 2. \\ \\ \\ \end{matrix}$$

The transpositions made can be summarized in the following way. We retrieve the L_1-block from columns 3, 5 and row 1, and the $J_2(0)$-block from columns 4, 6 and rows 2, 3. Finally the two L_0-blocks form columns 1, 2.

From (1.3a-b) we deduce that the column nullity n_1 of A is 4 ($= \dim \mathcal{N}_c(A)$) and that A and B have a common nullspace of dimension 2 ($= n_1 - r_1$). Here r_1 denotes the dimension of $\mathcal{N}_c(A)$ which is not in common to B.

We conclude that the number of L_0-blocks in the KCF is equal to the dimension of the common column nullspace of A and B. Further the KCF of $A - \lambda B$ consists of r_1 structure-blocks $J_k(0)$ or L_k of order $\geqq 1$. More information about these structure blocks can be retrieved by first deflating $A - \lambda B$ at row r_1 and column n_1 and then analyzing the deflated pencil $A^{(1)} - \lambda B^{(1)} = [0\ 0] - \lambda[0\ 1]$ similarly. Notice that if $n_1 = r_1$ then $A - \lambda B$ has no column index of degree 0, but might have column indices of higher degree. In that case the A-part and B-part of at least one deflated pencil $A^{(i-1)} - \lambda B^{(i-1)}$ must have a common column nullspace.

The main purpose of this paper is to show how we can extract the significant information concerning these nullspaces from *the generalized singular value decomposition* (GSVD) ([12], [14-15], [20-21]). In fact the GSVD plays a similar role for the general $(A - \lambda B)$-problem as the standard singular value decomposition (SVD) does for the $(A - \lambda I)$-problem (see e.g. [4] and [6-7]). In [9] we present the *Reiterating Generalized Singular Value Deflation* (RGSVD) *algorithm* for computing the Kronecker structure of an arbitrarily singular pencil. RGSVD is based on a reduction theorem that apart from the Jordan structure of the zero eigenvalue computes the column minimal indices (see the example above). At the same time we obtain pairs of reducing subspaces [19], which is a generalization of deflating subspaces [13] to the singular case. The reduction theorem (proved in [9]) is based on a finite sequence of range-nullspace separations, in terms of generalized singular value decompositions of matrix pairs (A, B). For other approaches see [5] and [18].

Let $A^{(i-1)} - \lambda B^{(i-1)}$ denote the deflated pencil from step $i-1$ which will be studied

next, and let $n_i = \dim \mathcal{N}_c(A^{(i-1)})$ and $n_i - r_i = \dim \mathcal{N}_c(A^{(i-1)}) \cap \mathcal{N}_c(B^{(i-1)})$. In general these column nullities expose the Jordan structure of the zero-eigenvalue and the right (column) indices of a singular $A - \lambda B$ in the following way.

THEOREM 1.1

(1.5a) *The number of L_k-blocks of order $(i-1) \times i$ is equal to $n_i - r_i$*

(1.5b) *while the number of $J_k(0)$-blocks of order i is equal to $r_i - n_{i+1}$.*

For proof see [9], [18] or [22].

The rest of this paper is mainly devoted to the GSVD and its properties which are the basis for the RGSVD-algorithm, (to be published elsewhere) and is organized in the following way. In section 2 we present an algorithmic derivation of the GSVD. A geometric interpretation of the GSVD, that displays the row and column nullspaces of the matrix pair (A, B), is presented in section 3. Further the behaviour in finite precision arithmetic of the GSVD-algorithm from section 2 is discussed in section 4. We derive perturbation bounds to a nearby pencil with prescribed column nullities by deleting small generalized singular values. We also explain why, in our application of the GSVD, we do not come into the numerical difficulties one encounters, when trying to compute the complete GSVD ([12], [14-15]). In section 5 we treat the case $m < n$ (A and $B \in \mathbb{C}^{m \times n}$). A *generalized QZ*[11] *decomposition* (GQZD) of a matrix pair (A, B), that displays the column nullities of A and B similarly to GSVD, is formulated in section 6. In addition we get unitary bases for the desired nullspaces, i.e. X, the matrix of right generalized singular vectors, is replaced by a unitary Q. But possible ill-conditioning in X (e.g. the nonzero singular values of $M = \begin{pmatrix} A \\ B \end{pmatrix}$ vary widely or the rank (nullity) of M is not well determined) is transformed to the upper triangular matrices in the GQZD. Therefore the a priori perturbation bounds in section 4 will be of the same size for th GQZD-algorithm. In section 7 we show how we can make use of either the GSVD or the GQZD in a strictly equivalent transformation of $A - \lambda B$ that extracts some of the Kronecker structure. Finally we make use of the GSVD and display the results after an equivalent transformation $V^H(A - \lambda B)X$ where $A - \lambda B$ is a singular pencil, corresponding to one step of the RGSVD-algorithm.

2. Algorithmic derivation of the GSVD.

The GSVD was introduced in connection with generalized eigenvalue problems by Van Loan [20, 21]. In [12] Paige and Saunders describe how the problem of computing the GSVD of a matrix pair (A, B) can be reduced to that of computing a *Cosine-Sine decomposition* (CSD) of a partitioned orthonormal

matrix (see also [1], [13]). Stewart [14] has recently developed their algorithm further and he proposes a reorthogonalization technique (based on Jacobi's method for the symmetric $(A-\lambda I)$-problem) that gives a stable algorithm for computing the CSD [15]. Sun [16-17] has made a perturbation analysis of the GSVD-problem.

In the following theorem we formulate the GSVD of the m by n matrices A and B when $m \geqq n$. The proof given here is very similar to the one in [12]. We use an SVD and a QR-decomposition instead of a QR-decomposition and an LQ-decomposition respectively. This alternative derivation of the GSVD gives us the strictly equivalent transformations that we use in the RGSVD-algorithm for computing the Kronecker structure of $A-\lambda B$ (see section 7 and [9]), and is therefore presented here.

THEOREM 2.1. *Let $A, B \in \mathbb{C}^{m\times n}$ where $m \geqq n$ and*

$$r = \operatorname{rank}\begin{bmatrix} A \\ \hline B \end{bmatrix} \leqq n. \tag{2.1}$$

Then there exist unitary matrices U, $V \in \mathbb{C}^{m\times m}$ and a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that

$$\left[\begin{array}{c|c} U^H & 0 \\ \hline 0 & V^H \end{array}\right]\begin{bmatrix} A \\ \hline B \end{bmatrix} X = \begin{bmatrix} D_A \\ \hline \underbrace{D_B}_{n} \end{bmatrix}\begin{matrix}\} m \\ \} m\end{matrix} \quad \text{where} \tag{2.2}$$

$$D_A = [\ 0 \ | \ \Sigma_A\ ], \qquad \Sigma_A = \begin{bmatrix} C \\ \hline 0 \end{bmatrix}\begin{matrix}\} r \\ \} m-r\end{matrix} \tag{2.3a}$$

$$D_B = [\underbrace{\ 0\ }_{n-r} \mid \underbrace{\Sigma_B}_{r}\ ], \qquad \Sigma_B = \begin{bmatrix} S \\ \hline 0 \end{bmatrix}\begin{matrix}\} r \\ \} m-r\end{matrix} \quad \text{and} \tag{2.3b}$$

$$C = \operatorname{diag}\{c_1, c_2, \ldots, c_r\}; \quad 0 \leqq c_1 \leqq c_2 \leqq \ldots \leqq c_r \leqq 1 \tag{2.4a}$$

$$S = \operatorname{diag}\{s_1, s_2, \ldots, s_r\}; \quad 1 \geqq s_1 \geqq s_2 \geqq \ldots \geqq s_r \geqq 0 \tag{2.4b}$$

satisfying

$$C^2 + S^2 = I. \tag{2.5}$$

PROOF. Compute the SVD of the $2m$ by n matrix $\begin{bmatrix} A \\ B \end{bmatrix}$:

$$\begin{bmatrix} A \\ \hline B \end{bmatrix} = \begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} \Sigma Z^H \tag{2.6}$$

where $Q_i \in \mathbb{C}^{m\times m}$, $Z \in \mathbb{C}^{n\times n}$ and

$$\Sigma = \operatorname{diag}\{\underbrace{0,\ldots,0}_{n-r}, \sigma_1, \ldots, \sigma_r\} \tag{2.7}$$

and $0 < \sigma_1 \leqq \sigma_2 \leqq \ldots \leqq \sigma_r$. By partitioning $Q_1 = [Q_{11}, Q_{12}]$ and $Q_2 = [Q_{21}, Q_{22}]$ we have

$$\begin{bmatrix} A \\ B \end{bmatrix} Z = \left(\begin{bmatrix} 0_{m\times(n-r)} \\ \hline 0_{m\times(n-r)} \end{bmatrix}, \quad \begin{bmatrix} Q_{12} \\ \hline Q_{22} \end{bmatrix} \cdot \Sigma_1 \right) \text{ where} \tag{2.8}$$

$$Q_{12}^H Q_{12} + Q_{22}^H Q_{22} = I_r \quad \text{and} \tag{2.9}$$

$$\Sigma_1 = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\}, \text{ with } \sigma_r = \sigma_{\max} \begin{bmatrix} A \\ B \end{bmatrix}. \tag{2.10}$$

Since $\|Q_{i2}\|_2 \leqq \left\| \begin{bmatrix} Q_{12} \\ \hline Q_{22} \end{bmatrix} \right\|_2 = 1$ the singular values of Q_{i2} $(i = 1, 2)$ lie between 0 and 1 and we write the SVD of Q_{12} as

$$Q_{12} = U \begin{bmatrix} C \\ 0 \end{bmatrix} W^H \tag{2.11}$$

where $U \in \mathbb{C}^{m\times m}$, $W \in \mathbb{C}^{r\times r}$ are unitary matrices and $C \in \mathbb{C}^{r\times r}$ satisfies 2.4a. From the orthonormality condition 2.9 we get

$$(Q_{22}W)^H Q_{22}W = I_r - C^2, \tag{2.12}$$

i.e. $Q_{22}W$ has orthogonal columns except for some that can be zero. Since C has the diagonal elements in increasing order, possible zero columns of $Q_{22}W$ will be in the last positions. We see that

$$Q_{22}W = V \begin{bmatrix} S \\ 0 \end{bmatrix} \tag{2.13}$$

where $V \in \mathbb{C}^{m\times m}$ is unitary and $S \in \mathbb{C}^{r\times r}$ is a diagonal matrix with decreasing diagonal elements (see 2.4b and 2.5).

V and S in 2.13 can be computed from a normalized QR decomposition of $Q_{22}W$. Combining 2.11 with 2.13 we obtain a CSD of $\begin{bmatrix} Q_{12} \\ \hline Q_{22} \end{bmatrix}$:

$$\left[\begin{array}{c|c} U^H & 0 \\ \hline 0 & V^H \end{array}\right] \begin{bmatrix} Q_{12} \\ \hline Q_{22} \end{bmatrix} W = \begin{bmatrix} C \\ \hline 0 \\ \hline S \\ \hline 0 \end{bmatrix} \equiv \begin{bmatrix} \Sigma_A \\ \hline \Sigma_B \end{bmatrix} \tag{2.14}$$

which in 2.8 give us the GSVD 2.2 where

$$X^{-1} = \left(\begin{array}{c|c} I_{n-r} & 0 \\ \hline 0 & W^H \Sigma_1 \end{array}\right) Z^H. \tag{2.15}$$

This completes the proof. ■

3. A geometric interpretation of the GSVD.

From the introductory example in section 1 we observed that certain column nullities of A and B are of interest and particularly

the column nullspace of A, $\mathcal{N}_c(A)$

the common nullspace of A and B, $\mathcal{N}_c(A) \cap \mathcal{N}_c(B)$.

We introduce the notations

$$n_1 = \dim \mathcal{N}_c(A) \tag{3.1a}$$

$$n_1 - r_1 = \dim \mathcal{N}_c(A) \cap \mathcal{N}_c(B) \tag{3.1b}$$

where r_1 is the dimension of that part of $\mathcal{N}_c(A)$ that is not in common with $\mathcal{N}_c(B)$.

Using the GSVD of A and B (see 2.2–5) we can write

$$A = U \cdot \underbrace{\left[\begin{array}{c|c} 0 & \begin{array}{c} C \\ \hline 0 \end{array} \end{array}\right]}_{\substack{n-r \quad r}} \Big\} m-r \cdot X^{-1} \quad \text{and} \tag{3.2a}$$

$$B = V \cdot \overbrace{\left[\begin{array}{c|c} 0 & \begin{array}{c} S \\ \hline 0 \end{array} \end{array}\right]}^{\substack{\quad r}} \Big\} m-r \cdot X^{-1}. \tag{3.2b}$$

From our assumptions the r_1 first c_i will be zero and therefore the r_1 first s_i will be one (see 2.4–5). From a computational point of view we first obtain $n-r$ from 2.6–7 and then r_1 as the number of zero singular values of Q_{12}. Finally we get $n_1 = (n-r)+r_1$. By multiplying A and B in 3.2a–b with X from the right we see that the columns x_i of X span the different column nullspaces:

$$\mathcal{N}_c(A) = \text{span}\{x_1, \ldots, x_{n_1}\} \tag{3.3a}$$

$$\mathcal{N}_c(A) \cap \mathcal{N}_c(B) = \text{span}\{x_1, \ldots, x_{n_1 - r_1}\} \tag{3.3b}$$

$$\mathcal{N}_c(A) \setminus \mathcal{N}_c(A) \cap \mathcal{N}_c(B) = \text{span}\{x_{n_1 - r_1 + 1}, \ldots, x_{n_1}\}. \tag{3.3c}$$

Analogously the part of $\mathcal{N}_c(B)$ that is not in common to $\mathcal{N}_c(A)$ corresponds to

the number of s_i equal to zero (say p_1) and

$$\mathcal{N}_c(B) = \mathcal{N}_c(A) \cap \mathcal{N}_c(B) + \operatorname{span}\{x_{n-p_1+1}, \ldots, x_n\}. \tag{3.3d}$$

If we sort C and S in the opposite order we instead get a basis of $\mathcal{N}_c(B)$ from the $(n-r)+p_1$ first consecutive columns of X. Notice that the bases above are not orthogonal. Row nullities and bases for the row nullspaces of A and B, $\mathcal{N}_r(A)$ and $\mathcal{N}_r(B)$ respectively, can also be obtained from the GSVD. Since A and $B \in \mathbb{C}^{m\times n}$ we have

$$\dim \mathcal{N}_r(A) = (m-r)+r_1 \tag{3.4a}$$

$$\dim \mathcal{N}_r(B) = (m-r)+p_1 \tag{3.4b}$$

$$\dim \mathcal{N}_r(A) \cap \mathcal{N}_r(B) = m-r \tag{3.4c}$$

where as before r_1 is the number of c_i equal to zero and p_1 is the number of s_i equal to zero. By premultiplying A (3.2a) with U^H and B (3.2b) with V^H we get orthonormal bases for $\mathcal{N}_r(A)$ and $\mathcal{N}_r(B)$ from the columns of U and V:

$$\mathcal{N}_r(A) = \operatorname{span}\{u_1, \ldots, u_{r_1}, u_{m-r+1}, \ldots, u_m\} \tag{3.5a}$$

$$\mathcal{N}_r(B) = \operatorname{span}\{v_{m-(r+p_1)+1}, \ldots, v_m\}. \tag{3.5b}$$

Evidently we cannot get a basis for $\mathcal{N}_r(A) \cap \mathcal{N}_r(B)$ from the GSVD of (A, B). Since $\mathcal{N}_r(B) = \mathcal{N}_c(A^H)$ and $\mathcal{N}_r(B) = \mathcal{N}_c(B^H)$ we can compute the GSVD of (A^H, B^H) and obtain bases of the row nullspaces from 3.3a–d. The values of r_1 and p_1 will still be the same but $n_1 = (m-r)+r_1$. In the following sections r_1 and n_1 denote column nullites and are defined by 3.1a–b.

By considering $A \in \mathbb{C}^{m\times n}$ as a mapping from $\mathbb{C}^n$ to $\mathbb{C}^m$ we can express $\mathbb{C}^n$ and $\mathbb{C}^m$ as direct sums of the range and nullspaces of A and A^H:

$$\mathbb{C}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^H); \quad \mathbb{C}^n = \mathcal{R}(A^H) \oplus \mathcal{N}(A).$$

In our notation $\mathcal{N}_c(A) = \mathcal{N}(A)$ and $\mathcal{N}_r(A) = \mathcal{N}(A^H)$. We find the notation of column and row nullspaces natural for our problem, since these nullities of A and B give us information about the column and row minimal indices of $A-\lambda B$.

4. GSVD in the presence of rounding errors.

Given the pencil $A-\lambda B$ where the rank $\begin{bmatrix} A \\ B \end{bmatrix} = r \leqq n$, find, in the presence of rounding errors, a nearby pencil $C-\lambda D$ which has exactly the geometric

properties of 3.1a–b. In finite arithmetic Σ (2.7), C and S (2.14) will look like

$$\Sigma = \operatorname{diag}\{\Sigma_0, \Sigma_1\} \tag{4.1}$$

where $\Sigma_0 = \operatorname{diag}\{\sigma_1^{(0)}, \ldots, \sigma_{n-r}^{(0)}\}$ are the singular values of $\begin{bmatrix} A \\ B \end{bmatrix}$ interpreted as zeros, Σ_1 (2.10) the nonzero ones, and

$$C = \operatorname{diag}\{C_0, C_1\} \tag{4.2a}$$

$$S = \operatorname{diag}\{S_0, S_1\} + S_{01} \tag{4.2b}$$

where the cosines $C_0 = \operatorname{diag}\{c_1^{(0)}, \ldots, c_{r_1}^{(0)}\}$ are the singular values of Q_{12} interpreted as zeros, C_1 the nonzero ones, and $S_0 = \operatorname{diag}\{s_1^{(0)}, \ldots, s_{r_1}^{(0)}\}$, S_1 the corresponding sines such that the cosine-sine identity 2.5 is exactly satisfied. The r by r matrix S_{01} is upper triangular and contains possible off-diagonal elements from the QR-decomposition of $Q_{22}W$ (see 2.13). The main diagonal of S_{01} accumulates small errors such that $C_0^2 + S_0^2 = I$ and $C_1^2 + S_1^2 = I$ are fulfilled to the rounding level. To summarize we have the following decomposition

$$A = U\left(U^H Q_{11}\Sigma_0, \left[\begin{array}{cc} C_0 & 0 \\ 0 & C_1 \\ \hline 0 & 0 \end{array}\right]\right) X^{-1} \tag{4.3a}$$

$$B = V\left(V^H Q_{21}\Sigma_0, \left[\begin{array}{cc} S_0 & 0 \\ 0 & S_1 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c} S_{01} \\ \hline 0 \end{array}\right]\right) X^{-1} \tag{4.3b}$$

where U, V and X denote the computed transformation matrices. It is now easy to derive a pencil $C - \lambda D$ where the GSVD of (C, D) is given by 2.2. By taking norms of $A - C$ and $B - D$, and applying well-known inequalities for matrix norms ($\|A+B\|_F \leqq \|A\|_F + \|B\|_F$, $\|A \cdot B\|_F \leqq \|A\|_F \|B\|_2$ where $\|\cdot\|_F$ is the Frobenius norm and $\|A\|_2$ is the largest singular value of A) and using that $(a^2+b^2)^{1/2} \leqq a+b$, $a, b \geqq 0$ we can formulate the following theorem.

THEOREM 4.1. *Let A and $B \in \mathbb{C}^{m\times n}$, $m \geqq n$, and rank $\begin{bmatrix} A \\ B \end{bmatrix} = r \leqq n$. Then in the presence of rounding errors there exist nearby matrices C and $D \in \mathbb{C}^{m\times n}$ such that $C - \lambda D$ has exactly the geometric properties of* 3.1*a–b and the GSVD given by* 2.2, *where*

$$\|C - A\|_F \leqq \|\Sigma_0\|_F + \|C_0\|_F \sigma_r \tag{4.4a}$$

$$\|D - B\|_F \leqq \|\Sigma_0\|_F + (\|S_0 - I\|_F + \|S_{01}\|_F)\sigma_r. \tag{4.4b}$$

The diagonal matrices Σ_0, C_0, S_0 and S_{01} are defined by 4.1–2 *and* $\sigma_r = \|\Sigma_1\|_2 = \left\|\begin{bmatrix} A \\ B \end{bmatrix}\right\|_2$ (*see* 2.10).

Since C and D are constructed from 4.3a–b the perturbation inequalities 4.4a–b hold for arbitrary A and B, but we can only expect the perturbations $\Delta A = C - A$ and $\Delta B = D - B$ to be small when the numerical ranks of $\begin{bmatrix} A \\ B \end{bmatrix}$ and $\begin{bmatrix} C \\ D \end{bmatrix}$ are equal. In this case we (normally) have that

$$\|\Delta A\|_F \leqq O(\varepsilon\sigma_r) \tag{4.5a}$$

$$\|\Delta B\|_F \leqq O(\varepsilon\sigma_r) + \|S_{01}\|_F\sigma_r. \tag{4.5b}$$

where ε is the relative machine precision.

Notice that if $A - \lambda B$ is regular, i.e. rank $\begin{bmatrix} A \\ B \end{bmatrix} = n$, then $\Sigma_0 = 0$.

Before ending this section we emphasize that the term $\|S_{01}\|_F\sigma_r$ in the perturbation bound 4.4b is algorithm-dependent and only present when we compute the complete GSVD (U, V, X^{-1}, S and C), i.e. when we execute 2.13. We can eliminate the effects of the S_{01}-block by using Stewart's stabilization technique [14–15]. However, in our application of the GSVD we only need two of the three transformation matrices (V and X, or U and X) in order to make a strictly equivalent transformation of $A - \lambda B$ (see section 7 and [9]), and consequently we never execute the QR-decomposition 2.13.

For example we get both V and X^{-1} from SVD's (we can start to decompose Q_{22} instead) and the only possible unstable step in computing X is the inversion of Σ_1. But this diffulty is inherent in the problem, i.e. rank r (2.1) is not well-determined. We get C from an SVD of Q_{12} (2.11) and S consists of the singular values of Q_{22}. So we can compute either U, C, S, X^{-1} or V, C, S, X^{-1} in a numerically stable way, and X in a controlled way, where (see 2.15)

$$K(X) = \|X\|_2\|X^{-1}\|_2 = \sigma_r/\sigma_1. \tag{4.6}$$

It is quite clear that we cannot guarantee that the GSVD-algorithm gives the closest matrix pencil $C - \lambda D$ with the prescribed nullities n_1 and r_1 (see 3.1a–b). However, the interpretation of n_1 and r_1 in terms of the generalized singular values shows that the GSVD plays a similar role for the $(A - \lambda B)$-problem as the standard SVD does for the $(A - \lambda I)$-problem, where we delete small singular values of A in order to find the closest C with a prescribed nullity.

5. The case $m < n$.

Let $A, B \in \mathbb{C}^{m\times n}$ where $m < n$ and rank $\begin{bmatrix} A \\ B \end{bmatrix} = r \leqq n$. The GSVD of the matrix

pair (A, B) is given by 2.2. However, the structure of D_A (2.3a) and D_B (2.3b) will be different. We have two cases. First, if $m \geqq r$ (but $m < n$, then the structure of D_A and D_B is the same as for the case $m \geqq n$. Second, if $m < r \leqq n$ then (see [12])

$$\text{(5.1a)} \qquad D_A = [\underbrace{0}_{n-r} \mid \underbrace{\Sigma_A}_{r}], \qquad \Sigma_A = [\underbrace{0}_{r-m} \mid \underbrace{C}_{m}]\}m$$

$$\text{(5.1b)} \qquad D_B = [\overbrace{0}^{} \mid \overbrace{\Sigma_B}^{}], \qquad \Sigma_B = [\underbrace{S}_{m} \mid \underbrace{0}_{r-m}]\}m$$

where $C, S \in \mathbb{C}^{m \times m}$ are defined by 2.4–5 with r replaced by m.

As before, let r_1 and p_1 be the number of c_i and s_i equal to zero, respectively. The geometric properties from section 3 can then easily be retrieved:

$$\text{(5.2a)} \qquad n_1 = \dim \mathcal{N}_c(A) = (n-m)+r_1$$

$$\text{(5.2b)} \qquad n-r = \dim \mathcal{N}_c(A) \cap \mathcal{N}_c(B)$$

$$\text{(5.2c)} \qquad q_1 = \dim \mathcal{N}_c(B) = (n-m)+p_1.$$

Using n_1, r_1 and p_1 we decompose Σ_A and Σ_B in a form which clearly enlightens the geometric structure of (A, B):

$$\text{(5.3a)} \qquad \Sigma_A = \left[\begin{array}{c|c|c} \underbrace{0}_{r-m+r_1} & \begin{array}{c} 0 \\ \hline C_* \\ \hline 0 \end{array} & \begin{array}{c} 0 \\ \hline \\ \hline I \end{array} \end{array}\right] \begin{array}{l} \}\, r_1 \\ \}\, (2m-r)-(r_1+p_1) \\ \}\, r-m+p_1 \end{array}$$

$$\qquad\qquad\qquad\qquad r-m+r_1 \quad (2m-r)-(r_1+p_1) \quad r-m+p_1$$

$$\text{(5.3b)} \qquad \Sigma_B = \left[\begin{array}{c|c|c} \overbrace{I}^{} & \overbrace{\begin{array}{c} 0 \\ \hline S_* \\ \hline 0 \end{array}}^{} & \\ \hline 0 & & 0 \end{array}\right] \begin{array}{l} \}\, r-m+r_1 \\ \}\, (2m-r)-(r_1+p_1) \\ \}\, p_1 \end{array}$$

Here C_* and S_* denote the nonzero and nonunity c_i and s_i in 5.1a–b, respectively, i.e. $0 < c_1^* \leqq c_2^* \leqq \ldots < 1$ and $1 > s_1^* \geqq s_2^* \geqq \ldots > 0$.

6. A generalized QZ-decomposition (GQZ).

In this section we formulate a *generalized QZ* (*GQZ*) decomposition of a matrix pair (A, B) in terms of unitary transformations. The GQZ decomposition displays $\mathcal{N}_c(A)$ and $\mathcal{N}_c(A) \cap \mathcal{N}_c(B)$ similar to GSVD. See Moler-Stewart [11] and EISPACK [3] for details about standard QZ decomposition. See also Wilkinson [23].

THEOREM 6.1. *Let $A, B \in \mathbb{C}^{m\times n}$ where $m \geq n$ and rank $M = r \leq n$ (see 2.1). Then there exists unitary $U, V \in \mathbb{C}^{m\times n}$ and $Q \in \mathbb{C}^{n\times n}$ such that*

$$\left[\begin{array}{c|c} U^H & 0 \\ \hline 0 & V^H \end{array}\right]\left[\begin{array}{c} A \\ \hline B \end{array}\right] Q = \underbrace{\left[\begin{array}{c} T_A \\ \hline T_B \end{array}\right]}_{n} \begin{matrix} \}m \\ \}m \end{matrix} \quad \text{where} \tag{6.1}$$

$$T_A = [\underbrace{0}_{n-r} \mid \underbrace{R_A}_{r}], \qquad R_A = \left[\begin{array}{c} R_C \\ \hline 0 \end{array}\right] \begin{matrix} \}r \\ \}m-r \end{matrix} \tag{6.2a}$$

$$T_B = [\overbrace{0}^{n-r} \mid \overbrace{R_B}^{r}], \qquad R_B = \left[\begin{array}{c} R_S \\ \hline 0 \end{array}\right] \begin{matrix} \}r \\ \}m-r \end{matrix} \tag{6.2b}$$

The upper triangular R_C and R_S satisfy

$$\|R_C^H R_C + R_S^H R_S\|_F = \|\Sigma_1^2\|_F \tag{6.3}$$

where Σ_1 is the nonzero singular values of M (see 2.7 and 2.10).

PROOF. The GQZ-decompsition is derived via the GSVD of (A, B). From 2.8 and 2.14–15 we get

$$\left[\begin{array}{c|c} U^H & 0 \\ \hline 0 & V^H \end{array}\right]\left[\begin{array}{c} A \\ \hline B \end{array}\right] Z = \left[\begin{array}{c|c} 0 & C \\ 0 & 0 \\ \hline 0 & S \\ 0 & 0 \end{array}\right] \begin{matrix} \}m \\ \}m \end{matrix} \cdot \left[\begin{array}{c|c} I_{n-r} & 0 \\ \hline 0 & W^H\Sigma_1 \end{array}\right] \}r \tag{6.4}$$

$$\underbrace{}_{n-r} \quad \underbrace{}_{r} \qquad\qquad \underbrace{}_{r}$$

Since Σ_1 is nonsingular we can make an RQ-decomposition of the r by r matrix $W^H\Sigma_1$ giving

$$W^H\Sigma_1 = R_1 Q_1 \tag{6.5}$$

where $R_1 \in \mathbb{C}^{r\times r}$ is upper triangular and $Q_1 \in \mathbb{C}^{r\times r}$ is unitary, which in 6.4 gives $R_C = CR_1$ and $R_S = SR_1$, i.e. T_A and T_B are of the form 6.2a–b. Next we see that

$$R_C^H R_C + R_S^H R_S = R_1^H R_1 = Q_1 \Sigma_1^2 Q_1^H \tag{6.6}$$

and by taking norms we get the identity 6.3. Finally from 6.4–5 we get

$$Q = Z\left(\begin{array}{c|c} I_{n-r} & 0 \\ \hline 0 & Q_1^H \end{array}\right). \tag{6.7}$$

■

If as before we assume that the r_1 first c_i and the p_1 last s_i are zero, then the r_1 first columns of R_C and the p_1 last columns of R_S will be zero columns. So it is also possible to derive the geometric properties desired in section 4 from the GQZ-decomposition. Besides we get unitary bases for $\mathcal{N}_c(A)$, $\mathcal{N}_c(A) \cap \mathcal{N}_c(B)$ and $\mathcal{N}_c(B)$ (Q takes the place of X in 3.2a–c).

By comparing the GQZ-decomposition and the GSVD of a matrix pair (A, B) we see that possible ill-conditioning in X due to Σ_1 (see 2.15 and 4.6) is transformed to R_C (6.2a) and R_S (6.2b). From 6.5 we have $\|\Sigma_1\|_2 = \|R_1\| = \sigma_r$ and when looking for a matrix pair (C, D) with prescribed column nullities according to theorem 4.1 we get upper bounds similar to 4.4a–b. To summarize, the unitary Q (6.7) and the norm-identity 6.3 take the place of X and the cosine-sine identity 2.5, respectively. In [11] Paige and Saunders also mention a decomposition in the style of 6.1.

7. Using GSVD in an equivalent transformation of A—λB.

If the GSVD of a matrix pair $(A, B) \in \mathbb{C}^{m \times n}$ is known we can apply a strictly equivalent transformation to $A - \lambda B$. Assume $m \geqq n$ and make use of V and X from the GSVD of (A, B) as transformation matrices and we get (see 3.2a–b):

$$V^H(A-\lambda B)X = \underbrace{\left[\begin{array}{c|c} 0 & M_1 \\ \hline 0 & A^{(1)} \end{array}\right]}_{n_1} - \lambda \underbrace{\left[\begin{array}{c|c} 0\ I_{r_1} & 0 \\ \hline 0 & B^{(1)} \end{array}\right]}_{n_1} \Big\} r_1 . \tag{7.1}$$

Note that the pencil is left-multiplied by V^H which makes $B^{(1)}$ diagonal:

$$B^{(1)} = \begin{pmatrix} s_{r_1+1} & & 0 \\ & \ddots & \\ 0 & & s_r \end{pmatrix}; \quad 1 > s_{r_1+1} \geqq \ldots \geqq s_r \geqq 0. \tag{7.2}$$

Multiplying the pencil by U^H would have diagonalized A, and so in our case M_1 and $A^{(1)}$ in 7.1 are full matrices. From the geometric interpretation of the GSVD in section 3 and theorem 1.1 in section 1 we can now conclude that $A - \lambda B$ has $n_1 - r_1$ minimal column indices of degree 0 (L_0-blocks), and further there exist r_1 $J_k(0)$-blocks or L_k-blocks of order $\geqq 1$ in its KCF. This is the first step in a reduction theorem of a singular pencil that apart from the Jordan structure of the zero eigenvalue displays the column minimal indices of the Kronecker structure. In the next step we decompose $A^{(1)} - \lambda B^{(1)}$ similarly and we get the column indices of degree 1. For more details the reader is referred to the RGSVD-algorithm (see [9]).

We can also make use of the GQZ-decomposition (see section 6) in a similar equivalent transformation of $A - \lambda B$. We use the unitary V and Q as trans-

formation matrices and get (see 6.1–2):

$$V^H(A-\lambda B)Q = \left(\begin{array}{c|c} \underbrace{0}_{n_1} & M_1 \\ \hline 0 & A^{(1)} \end{array}\right) - \lambda \left(\begin{array}{c|c} \underbrace{0 \quad R_{11}}_{n_1} & N_1 \\ \hline 0 & B^{(1)} \end{array}\right) \Big\} r_1 . \tag{7.3}$$

Here R_{11} is r_1 by r_1 and upper triangular and $B^{(1)}$ has only nonzero elements on and above the diagonal starting at the top left corner. $M_1, A^{(1)}$ and N_1 are full matrices. We obtain the same information about the Kronecker structure but the B-part no longer has the nice and simple structure of V^HBX in 7.1. Of course there is a trade-off between the simplicity of the structure of the transformed pencil and the conditioning of X. However, this choice can be controlled in terms of a tolerance parameter, say tol. If $K(X) >$ tol then we switch to the GQZD, i.e. we compute R_1 and Q_1 from 6.5. On the other hand if, except for the sizes of the blocks (n_1 and r_1), we want the diagonal structure of the B-part, then the GSVD will give a better conditioned X than performing Gaussian eliminations without pivoting on $V^H(A-\lambda B)Q$ in 7.3. In addition we always know the conditioning of X as we proceed in the algorithm (see 4.6).

Finally we display the results after one strictly equivalent transformation of an 8 by 12 singular pencil $A-\lambda B$ according to 7.1. Notice that this is the case $m < n$ (see section 5).

$V^HAX =$

COLUMNS 1 THRU 8

−0.0000	−0.0000	−0.0000	−0.0000	0.0000	−0.0000	−0.0668	−0.1574
0.0000	0.0000	0.0000	0.0000	−0.0000	0.0000	−0.0136	−0.1574
−0.0000	−0.0000	−0.0000	−0.0000	0.0000	−0.0000	0.0336	0.1268
−0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	−0.1293	−0.4060
0.0000	−0.0000	−0.0000	−0.0000	0.0000	−0.0000	−0.2096	0.4951
0.0000	0.0000	0.0000	0.0000	−0.0000	0.0000	0.1426	0.2745
0.0000	0.0000	0.0000	0.0000	−0.0000	0.0000	−0.3806	0.0037
−0.0000	−0.0000	0.0000	0.0000	0.0000	−0.0000	−0.2073	0.0162

(r_1 rows; n_1 columns)

COLUMNS 9 THRU 12

−0.1657	−0.4944	0.6045	−0.4940
−0.6703	0.5165	−0.0763	−0.3407
−0.4991	−0.6171	−0.1623	0.3538
−0.2000	0.0278	−0.0154	0.5692
−0.1907	0.2377	0.3990	0.2245
−0.2049	−0.0905	−0.3650	−0.2054
0.0757	−0.1909	−0.5407	−0.3043
0.0543	0.0561	0.1320	0.0743

$V^HBX =$

COLUMNS 1 THRU 8

0.	0.	1.0000	0.	0.	0.	0.	0.
0.	0.	0.	1.0000	0.	0.	0.	0.
0.	0.	0.	0.	1.0000	0.	0.	0.
0.	0.	0.	0.	0.	1.0000	0.	0.
0.	0.	0.	0.	0.	0.	0.8517	0.
0.	0.	0.	0.	0.	0.	0.	0.6699
0.	0.	0.	0.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	0.	0.

```
COLUMNS 9 THRU 12
 0.      0.      0.      0.
 0.      0.      0.      0.
 0.      0.      0.      0.
 0.      0.      0.      0.
 0.      0.      0.      0.
 0.      0.      0.      0.
0.3834   0.      0.      0.
 0.     0.0541   0.      0.
```

We see that dim $\mathcal{N}_c(A) \cap \mathcal{N}_c(B) = 2 = n_1 - r_1$ which corresponds to two L_0-blocks in the KCF of $A - \lambda B$. Further $A - \lambda B$ has $r_1 = 4$ $J_k(0)$-blocks or L_k-blocks of order $\geqq 1$. Notice that the four zero columns of $V^H BX$ correspond to the original column nullity of $B(= \dim \mathcal{N}_c(B))$. The zeros and ones displayed above are all accurate to the rounding level.

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