GSVD Definition

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1 Definition

1.1 Our definition (Based on LAPACK 3.6.0)

The generalized singular value decomposition (GSVD) of an m-by-n real matrix A and p-by-n real matrix B is the following:

$$A = UCRQ^T \quad \text{and} \quad B = VSRQ^T$$

Specifically,

- U is an m-by-m orthogonal matrix.
- V is a p-by-p orthogonal matrix.
- Q is a n-by-n orthogonal matrix.
- R is a r-by-n matrix of structure $[0\ R_0]$ where R_0 is r-by-r and is a **nonsingular** upper triangular matrix. $r = rank([A; B])$
- C and S are m-by-r and p-by-r non-negative diagonal matrices and $C^TC + S^TS = I$.
- $C^TC = \text{diag}(\alpha_1^2, ..., \alpha_r^2)$, $S^TS = \text{diag}(\beta_1^2, ..., \beta_r^2)$, where $\alpha_i \in [0, 1]$ and $\beta_i \in [0, 1]$ for $i = 1, ..., r$. Further, the ratios α_i/β_i are called the **generalized singular values** of the pair A, B.

1.2 Other notable definition of GSVD

1.2.1 MATLAB 2019b

The generalized singular value decomposition (GSVD) of an m-by-n real matrix A and p-by-n real matrix B is the following:

$$A = UCX^T \quad \text{and} \quad B = VSX^T$$

- U is an m-by-m orthogonal matrix.
- V is a p-by-p orthogonal matrix.
- X is a n-by-q matrix where $q = min(m + p, n)$. If X is square, it's **nonsingular**.

- C is an m-by-q nonnegative diagonal matrix and S is a p-by-q nonnegative diagonal matrix and $C^TC + S^TS = I$.
- $C^TC = \text{diag}(\alpha_1^2, ..., \alpha_q^2)$, $S^TS = \text{diag}(\beta_1^2, ..., \beta_q^2)$, where $\alpha_i \in [0,1]$ and $\beta_i \in [0,1]$ for $i = 1, ..., q$. Further, the ratios α_i/β_i are called the **generalized singular values** of the pair A, B and are in non-decreasing order.

1.3 Analysis of definitions

We analyse the two major definitions of GSVD above.

- If $[A; B]$ doesn't have full rank, namely, $rank([A;B]) < min(m+p, n)$, Julia/LAPACK and MATLAB have **different** generalized singular values.

  By definition, Julia/LAPACK will produce r generalized singular values while MATLAB only create q generalized singular values where $q = \min(m+p, n)$.

  Example:

$$A = \begin{pmatrix} 113 & 735 & 1065 & 693 & 969 & 792 \\ 623 & 425 & 591 & 403 & 535 & 464 \end{pmatrix} \qquad B = \begin{pmatrix} 2253 & 1271 & 2476 & 1300 & 1676 & 2012 \\ 1828 & 986 & 2123 & 1059 & 1217 & 1854 \\ 2113 & 1361 & 2231 & 1335 & 1421 & 1722 \end{pmatrix}$$

  where $m = 2, p = 3, n = 6, rank([A;B]) = 4 < min(2+3, 6)$.

  Julia/LAPACK will output:

$$C = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.384923 & 0.0 & 0.0 \end{pmatrix} S = \begin{pmatrix} 0.0 & 0.922949 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

  Generalized singular values are: `Inf`, 0.417, 0, 0.

  MATLAB will output:

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \end{pmatrix}$$

  Generalized singular values are: 0, 0, 0, `Inf`, `Inf`.

- If $[A; B]$ has full rank, namely, $rank([A;B]) = min(m+p, n)$, two definitions can be interchangeable. [1]

X^T in MATLAB can be rewritten RQ^T in Julia/LAPACK.

Case 1: $min(m+p,n) = m+p$

$$A = \begin{pmatrix} 0.5377 & 0.8622 & -0.4336 & 2.7694 & 0.7254 & -0.2050 \\ 1.8339 & 0.3188 & 0.3426 & -1.3499 & -0.0631 & -0.1241 \\ -2.2588 & -1.3077 & 3.5784 & 3.0349 & 0.7147 & 1.4897 \end{pmatrix}$$

$$B = \begin{pmatrix} 1.4090 & 0.6715 & 0.7172 & 0.4889 & 0.7269 & 0.2939 \\ 1.4172 & -1.2075 & 1.6302 & 1.0347 & -0.3034 & -0.7873 \end{pmatrix}$$

where $rank([A;B]) = 5 = 2+3$.

1.4 Applications

1. Intersection of null spaces
2. Generalized Eigenvalue problem
3. Certain Least squares problem

References

[1] Alan Edelman and Yuyang Wang. The gsvd: Where are the ellipses?, matrix trigonometry, and more. *arXiv preprint arXiv:1901.00485*, 2019.