GSVD Definition

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1 Definition

The generalized singular value decomposition (GSVD) of an m-by-n matrix A and p-by-n matrix B is given as follows:

$$A = UCRQ^T, \quad B = VSRQ^T$$

- U is an m-by-m orthogonal matrix.
- V is a p-by-p orthogonal matrix.
- Q is a n-by-n orthogonal matrix.
- C and S are m-by-r and p-by-r, where $r = \text{rank}(\begin{pmatrix} A \\ B \end{pmatrix})$. Both are non-negative and diagonal and $C^TC + S^TS = I$.
- $C^TC = \text{diag}(\alpha_1^2, ..., \alpha_r^2)$, $S^TS = \text{diag}(\beta_1^2, ..., \beta_r^2)$, where α_i, $\beta_i \in [0, 1]$ for $i = 1, ..., r$. The ratios α_i/β_i are called the generalized singular values of the pair A, B, and are in non-increasing order. The first k values are infinite ($\beta_i = 0$), the next s values are finite and non-zero and the last $r - k - s$ values are zero ($\alpha_i = 0$). Here, $k = \text{rank}(\begin{pmatrix} A \\ B \end{pmatrix}) - rank(B)$ and $s = \text{rank}(A)$ + $\text{rank}(B)$ - $\text{rank}(\begin{pmatrix} A \\ B \end{pmatrix})$.
- R is a r-by-n matrix of structure $(0,\ R_0)$ where R_0 is r-by-r, upper triangular and nonsingular.

C and S have the following detailed structure:

(1) $m \geq r$

$$C = \begin{array}{c c} & \begin{array}{cc} k & l \end{array} \\ \begin{array}{c} k \\ l \\ m-k-l \end{array} & \begin{pmatrix} I & 0 \\ 0 & \Sigma_1 \\ 0 & 0 \end{pmatrix} \end{array}, \quad S = \begin{array}{c c} & \begin{array}{cc} k & l \end{array} \\ \begin{array}{c} l \\ p-l \end{array} & \begin{pmatrix} 0 & \Sigma_2 \\ 0 & 0 \end{pmatrix} \end{array}$$

Here, $l = \text{rank}(B)$, Σ_1 and Σ_2 are diagonal matrices and $\Sigma_1^2 + \Sigma_2^2 = I$, and Σ_2 is nonsingular. Also, $\alpha_1 = \cdots = \alpha_k = 1$, $\alpha_{k+i} = (\Sigma_1)_{ii}$ for $i = 1, \cdots, l$, $\beta_1 = \cdots = \beta_k = 0$, $\beta_{k+i} = (\Sigma_2)_{ii}$ for $i = 1, \cdots, l$.

(2) $m < r$

$$C = \begin{array}{c} \\ k \\ m-k \end{array} \!\!\begin{array}{c} \begin{array}{ccc} k & m-k & k+l-m \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & \Sigma_1 & 0 \end{pmatrix} \end{array}, \quad S = \begin{array}{c} \\ m-k \\ k+l-m \\ p-l \end{array} \!\!\begin{array}{c} \begin{array}{ccc} k & m-k & k+l-m \end{array} \\ \begin{pmatrix} 0 & \Sigma_2 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

Still, Σ_1 and Σ_2 are diagonal matrices and $\Sigma_1^2 + \Sigma_2^2 = I$, and Σ_2 is nonsingular. Also, $\alpha_1 = \cdots = \alpha_k = 1$, $\alpha_{k+i} = (\Sigma_1)_{ii}$ for $i = 1, \cdots, m-k$, $\alpha_{m+1} = \cdots \alpha_r = 0$, $\beta_1 = \cdots = \beta_k = 0$, $\beta_{k+i} = (\Sigma_2)_{ii}$ for $i = 1, \cdots, m-k$, $\beta_{m+1} = \cdots = \beta_r = 1$.

2 Other notable definitions of GSVD

We list four major definitions of GSVD for further discussion, and they are ordered below:

2.1 Definition(1): Van Loan (1976) [1]

Given an m-by-n matrix A and a p-by-n matrix B with $m \geq n$ and $r = \text{rank}(\begin{pmatrix} A \\ B \end{pmatrix})$, the generalized singualr value decomposition of A and B is:

$$A = UCX^{-1}, \quad B = VSX^{-1}$$

$$C = \begin{array}{c} \\ q \\ r-q \\ m-r \end{array} \!\!\begin{array}{c} \begin{array}{ccc} q & r-q & n-r \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & \Sigma_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad S = \begin{array}{c} \\ q \\ r-q \\ p-r \end{array} \!\!\begin{array}{c} \begin{array}{ccc} q & r-q & n-r \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

- U is an m-by-m orthogonal matrix.
- V is a p-by-p orthogonal matrix.
- X is an n-by-n nonsingular matrix.
- C and S are m-by-n and p-by-n, and $q = max\{r-p, 0\}$. $\alpha_1 = \cdots = \alpha_q = 1$, $\Sigma_1 = \text{diag}(\alpha_{q+1}, \cdots, \alpha_r)$, $\beta_1 = \cdots = \beta_q = 0$, $\Sigma_2 = \text{diag}(\beta_{q+1}, \cdots, \beta_r)$. $\Sigma_1^2 + \Sigma_2^2 = I$.

2.2 Definition(2): Paige (1981) [2]

Given an m-by-n matrix A and a p-by-n matrix B with $r = rank(\begin{pmatrix} A \\ B \end{pmatrix})$, the generalized singular value decomposition of A and B is below:

$$A = UC(W^T R,\ 0)Q^T, \quad B = VS(W^T R,\ 0)Q^T$$

$$
C = \begin{array}{c} \\ k \\ s \\ m-k-s \end{array}
\begin{array}{c} \begin{array}{ccc} k & s & r-k-s \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & \Sigma_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad
S = \begin{array}{c} \\ p-r+k \\ s \\ r-k-s \end{array}
\begin{array}{c} \begin{array}{ccc} k & s & r-k-s \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & I \end{pmatrix} \end{array}
$$

- U is an m-by-m orthogonal matrix.
- V is a p-by-p orthogonal matrix.
- W is an r-by-r orthogonal matrix.
- R is an r-by-r nonsingular matrix, its singular values are equal to the nonzero singular values of $\begin{pmatrix} A \\ B \end{pmatrix}$.

  $rank(R) = rank(\begin{pmatrix} A \\ B \end{pmatrix})$.
- Q is an n-by-n orthogonal matrix.
- C and S are m-by-r and p-by-r. k = rank($\begin{pmatrix} A \\ B \end{pmatrix}$)-rank($B$), s = rank(A) + rank(B) - rank($\begin{pmatrix} A \\ B \end{pmatrix}$). $\alpha_1 = \cdots = \alpha_k = 1$, $\Sigma_1 = \text{diag}(\alpha_{k+1}, \cdots, \alpha_{k+s})$, $\alpha_{k+s+1} = \cdots = \alpha_r = 0$, $\beta_1 = \cdots = \beta_k = 0$, $\Sigma_2 = \text{diag}(\beta_{k+1}, \cdots, \beta_{k+s})$, $\beta_{k+s+1} = \cdots = \beta_r = 1$. α_i/β_i are called the "non-trivial" generalized singular values of matrix pair A, B. $\Sigma_1^2 + \Sigma_2^2 = I$.

2.3 Definition(3): MATLAB 2019b

The generalized singular value decomposition of an m-by-n matrix A and a p-by-n matrix B is the following:

$$A = UCX^T, \quad B = VSX^T$$

- U is an m-by-m orthogonal matrix.
- V is a p-by-p orthogonal matrix.
- X is an n-by-q matrix where $q = min\{m+p, n\}$.
- C is an m-by-q matrix and S is a p-by-q. Both are nonnegative, diagonal and $C^TC + S^TS = I$.
- $C^TC = \text{diag}(\alpha_1^2, \cdots, \alpha_q^2)$, $S^TS = \text{diag}(\beta_1^2, \cdots, \beta_q^2)$, where α_i, $\beta_i \in [0,1]$ for $i = 1, \cdots, q$. The ratios α_i/β_i are called the generalized singular values of the pair A, B and are in non-decreasing order.

2.4 Definition(4): Edelman (2019) [3]

The generalized singular value decomposition of an m-by-n matrix A and a p-by-n matrix B is the following:

$$A = UCH, \quad B = VSH$$

$$C = \begin{array}{r} \\ k \\ s \\ m-k-s \end{array}\!\!\begin{array}{c} \begin{array}{ccc} k & s & r-k-s \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & \Sigma_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad S = \begin{array}{r} \\ p-r+k \\ s \\ r-k-s \end{array}\!\!\begin{array}{c} \begin{array}{ccc} k & s & r-k-s \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & I \end{pmatrix} \end{array}$$

- U is an m-by-m orthogonal matrix.
- V is a p-by-p orthogonal matrix.
- C is an m-by-r matrix and S is an n-by-r matrix where $r = \text{rank}(\begin{pmatrix} A \\ B \end{pmatrix})$. $C^TC + S^TS = I$. $k = \text{rank}(\begin{pmatrix} A \\ B \end{pmatrix})$-rank$(B)$, $s = \text{rank}(A) + \text{rank}(B)$ - rank$(\begin{pmatrix} A \\ B \end{pmatrix})$. $\alpha_1 = \cdots = \alpha_k = 1$, $\Sigma_1 = \text{diag}(\alpha_{k+1}, \cdots, \alpha_{k+s})$, $\alpha_{k+s+1} = \cdots = \alpha_r = 0$, $\beta_1 = \cdots = \beta_k = 0$, $\Sigma_2 = \text{diag}(\beta_{k+1}, \cdots, \beta_{k+s})$, $\beta_{k+s+1} = \cdots = \beta_r = 1$. $\Sigma_1^2 + \Sigma_2^2 = I$.
- H is an r-by-n matrix and has full row rank.

3 Discussion and comments

I. Our formulation always reveals the rank of $\begin{pmatrix} A \\ B \end{pmatrix}$.

From our decomposition, we can immediately know the rank of $\begin{pmatrix} A \\ B \end{pmatrix}$ from the number of columns of C or S.

We can also gain rank$(\begin{pmatrix} A \\ B \end{pmatrix})$ from Definition(1), (2) and (4). However, we cannot obtain such information from Definition(3).

II. We can get the common nullspace of A and B from our formulation.

If we rewrite our formulation of GSVD as:

$$A(Q_1,\ Q_2) = UC(0,\ R_0), \quad B(Q_1,\ Q_2) = VS(0,\ R_0)$$

where Q_1 is n-by-$(n-r)$, Q_2 is n-by-r and R_0 is r-by-r. Then, we have $\text{null}(A) \cap \text{null}(B) = \text{span}\{Q_1\}$. In other words, Q_1 is the orthonormal basis of the common nullspace of A and B.

We can also get the common nullspace of A and B from Definition(2) and (4).

- If we rewrite the GSVD of Definition(2) as:

$$A(Q_1,\ Q_2) = UC(W^T R,\ 0), \qquad B(Q_1,\ Q_2) = VS(W^T R,\ 0)$$

  where Q_1 is n-by-r, Q_2 is n-by-$(n-r)$. Then, we have $\text{null}(A) \cap \text{null}(B) = \text{span}\{Q_2\}$.

- In Definition(4), $\text{null}(A) \cap \text{null}(B) = \text{null}(H)$. Alternatively, if we do RQ factorization on H, namely, $H = (0,\ R_0)Q^T$, where R_0 is an r-by-r upper triangular matrix and Q is an n-by-n orthgonal matrix, then $\text{null}(A) \cap \text{null}(B) = \text{span}\{Q(:, 1 : n-r)\}$.

III. We can solve the generalized eigenvalue problem ($A^T Ax = \lambda B^T Bx$) from our formulation.

If we let $X = Q \begin{array}{c} \begin{array}{cc} n-r & r \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & R_0^{-1} \end{pmatrix} \end{array}$, then

$$X^T A^T AX = \begin{array}{c|cc} & n-r & r \\ \hline n-r & 0 & 0 \\ r & 0 & C^T C \end{array}, \quad X^T B^T BX = \begin{array}{c|cc} & n-r & r \\ \hline n-r & 0 & 0 \\ r & 0 & S^T S \end{array}$$

Thus, we know the "non-trivial" eigenpairs of the generalized eigenvalue problem:

$$A^T AX_{i+n-r} = \lambda_i B^T BX_{i+n-r}, \quad i = 1, \cdots, r$$

$\lambda_i = (\alpha_i/\beta_i)^2$ are eigenvalues, where α_i/β_i is the generalized singular value of A and B. X_{i+n-r} denotes the $(i+n-r)th$ column of X and are the corresponding eigenvectors.

We can solve the generalized eigenvalue problem from Definition(1), (2) and (4).

- In Definition(1),

$$\begin{aligned} X^T A^T AX &= X^T (UCX^{-1})^T (UCX^{-1})X \\ &= X^T (X^{-1})^T C^T U^T UCX^{-1}X \\ &= X^T (X^T)^{-1} C^T C \\ &= C^T C \end{aligned}$$

  Similarly, $X^T B^T BX = S^T S$. Therefore, the first r quotients of the diagonal entries of $C^T C$ and $S^T S$ are the "non-trivial" eigenvalues of the generalized eigenvalue problem and the first r columns of X are the corresponding eigenvectors.

- In Definition(2),

  If we let $X = Q \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{pmatrix} R^{-1}W & 0 \\ 0 & I \end{pmatrix} \end{array}$, then

$$\begin{array}{cc} & \begin{array}{cc} r & n-r \end{array} \\ X^T A^T A X = \begin{array}{c} r \\ n-r \end{array} & \begin{pmatrix} C^T C & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{cc} r & n-r \end{array} \\ X^T B^T B X = \begin{array}{c} r \\ n-r \end{array} & \begin{pmatrix} S^T S & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

Thus, we know that the "non-trivial" eigenvalues of the generalized eigenvalue problem are the square of the generalized singular values and and the first r columns of X are the corresponding eigenvectors.

- In Definition(4),

  If we do RQ factorization on H, namely, $H = (0,\ R_0)Q^T$, and let $X = Q \begin{array}{c} \begin{array}{cc} n-r & r \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & R_0^{-1} \end{pmatrix} \end{array}$, then the "non-trivial" eigenvalues of the generalized eigenvalue problem are the square of the generalized singular values and and the last r columns of X are the corresponding eigenvectors.

IV. Two special cases of the generalized singular value decomposition.

- When B is square and nonsingular, the generalized singular value decomposition of A and B is equivalent to the singular value decomposition of AB^-1, regardless of how the GSVD is defined.
- No matter how we fomulate GSVD, if the columns of $(A^T,\ B^T)^T$ are orthonormal, then the generalized singular value decomposition of A and B is equivalent to the Cosine-Sine decomposition (CSD) of $(A^T,\ B^T)^T$, namely:

$$A = UCQ^T, \quad B = VSQ^T$$

  where U is m-by-m, V is p-by-p and Q is n-by-n and all of them are orthogonal matrices.

References

[1] Charles F Van Loan. Generalizing the singular value decomposition. *SIAM Journal on Numerical Analysis*, 13(1):76–83, 1976.

[2] Christopher C Paige and Michael A Saunders. Towards a generalized singular value decomposition. *SIAM Journal on Numerical Analysis*, 18(3):398–405, 1981.

[3] Alan Edelman and Yuyang Wang. The gsvd: Where are the ellipses?, matrix trigonometry, and more. *arXiv preprint arXiv:1901.00485*, 2019.