GSVD Algo

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1 Algorithm

Our algorithm consists of three routines. The first one is to reduce matrices A and B to triangular form while revealing the rank of B and $[A; B]$, which is called pre-processing. The second one is to to further reduce the two triangular matrices obtained from the first routine to one triangular matrix via QR decomposition in a Givens rotation fashion. The third one is to compute the Cosine-Sine decomposition (CSD) of two orthogonal matrices we get from QR decomposition.

We present our algorithm based on the two decompositions above.

1.1 Pre-processing

Given that $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times n}$, $\text{rank}(B) = l$ and $\text{rank}([A; B]) = k + l = r$, here's a detailed derivation of pre-processing procedure.

Step 1: QR decomposition with column pivoting of B, and determine the effective rank of $B = l$.

$$BP_1 = Q_1 \cdot \begin{array}{c c} & \begin{array}{cc} l & n-l \end{array} \\ \begin{array}{c} l \\ p-l \end{array} & \begin{bmatrix} B_{11}^{(1)} & B_{12}^{(1)} \\ 0 & 0 \end{bmatrix} \end{array}$$

Step 2: RQ decomposition of $\begin{bmatrix} B_{11}^{(1)} & B_{12}^{(1)} \end{bmatrix}$ when $p \geq l$ and $n \neq l$.

$$BP_1 = Q_1 \cdot \begin{array}{c c} & \begin{array}{cc} n-l & l \end{array} \\ \begin{array}{c} l \\ p-l \end{array} & \begin{bmatrix} 0 & B_{12}^{(2)} \\ 0 & 0 \end{bmatrix} \end{array} \cdot Q_2$$

$$BP_1 Q_2^T = Q_1 \cdot \begin{array}{c c} & \begin{array}{cc} n-l & l \end{array} \\ \begin{array}{c} l \\ p-l \end{array} & \begin{bmatrix} 0 & B_{12}^{(2)} \\ 0 & 0 \end{bmatrix} \end{array}$$

$$Q_1^T BP_1 Q_2^T = \begin{array}{c c} & \begin{array}{cc} n-l & l \end{array} \\ \begin{array}{c} l \\ p-l \end{array} & \begin{bmatrix} 0 & B_{12}^{(2)} \\ 0 & 0 \end{bmatrix} \end{array}$$

Step 3: View $AP_1Q_2^T$ as block matrix and apply QR with column pivoting of $A_{11}^{(1)}$ when $A_{11}^{(1)}$ is not empty, and determine the effective rank of $A_{11}^{(1)} = k$.

$$AP_1 Q_2^T = \begin{array}{c c} & \begin{array}{cc} n-l & l \end{array} \\ m & \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \end{bmatrix} \end{array}$$

$$A_{11}^{(1)} P_2 = Q_3 \cdot \begin{array}{c c} & \begin{array}{cc} k & n-l-k \end{array} \\ \begin{array}{c} k \\ m-k \end{array} & \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ 0 & 0 \end{bmatrix} \end{array}$$

Step 4: RQ decomposition of $\begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} \end{bmatrix}$ when $n - l \geq k$.

$$A_{11}^{(1)} P_2 = Q_3 \cdot \begin{array}{c c} & \begin{array}{cc} n-l-k & k \end{array} \\ \begin{array}{c} k \\ m-k \end{array} & \begin{bmatrix} 0 & A_{12}^{(3)} \\ 0 & 0 \end{bmatrix} \end{array} \cdot Q_4$$

View $A_{12}^{(1)}$ as block matrix, we have:

$$Q_3^T A P_1 Q_2^T P_2 Q_4^T = \begin{array}{c} \\ k \\ m-k \end{array} \begin{array}{c} \begin{array}{ccc} n-l-k & k & l \end{array} \\ \begin{bmatrix} 0 & A_{12}^{(3)} & A_{13}^{(3)} \\ 0 & 0 & A_{23}^{(3)} \end{bmatrix} \end{array}$$

$$Q_1^T B P_1 Q_2^T P_2 Q_4^T = \begin{array}{c} \\ l \\ p-l \end{array} \begin{array}{c} \begin{array}{cc} n-l & l \end{array} \\ \begin{bmatrix} 0 & B_{12}^{(2)} \\ 0 & 0 \end{bmatrix} \end{array}$$

Step 5: QR decomposition of $A_{23}^{(3)}$ when $m \geq k$.

$$A_{23}^{(3)} = Q_5 \cdot \tilde{A}_{23}^{(3)}$$

$$Q_5^T Q_3^T A P_1 Q_2^T P_2 Q_4^T = \begin{array}{c} \\ k \\ m-k \end{array} \begin{array}{c} \begin{array}{ccc} n-l-k & k & l \end{array} \\ \begin{bmatrix} 0 & A_{12}^{(3)} & A_{13}^{(3)} \\ 0 & 0 & \tilde{A}_{23}^{(3)} \end{bmatrix} \end{array}$$

Let $U = Q_3 Q_5$, $Q = P_1 Q_2^T P_2 Q_4^T$, and $V = Q_1$, we have:

$$\begin{array}{c} \\ m \\ p \end{array} \begin{array}{c} n \\ \left[\begin{array}{c} U^T A \\ \hline V^T B \end{array}\right] \end{array} Q = \begin{array}{c} \\ k \\ m-k \\ l \\ p-l \end{array} \begin{array}{c} \begin{array}{ccc} n-l-k & k & l \end{array} \\ \left[\begin{array}{ccc} 0 & A_{12}^{(3)} & A_{13}^{(3)} \\ 0 & 0 & \tilde{A}_{23}^{(3)} \\ \hline 0 & 0 & B_{12}^{(2)} \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

1.2 QR Decomposition by Givens rotation

1.3 CS Decomposition

1.4 The Complete Algorithm

1. Preprocess A and B: get $U, V, Q, R_{23}^{(a)}, R_{13}^{(b)}, k, l$.

2. Compute the QR decomposition of $[R_{23}^{(a)}; R_{13}^{(b)}]^T$, obtain $[Q_1^T; Q_2^T]^T$ and R'_{23}.

3. Compute the CSD of Q_1 and Q_2, get $U_1, V_1, Z_1, C_1 and S_1$.

4. Set C and S with C_1 and S_1.

5. $U[1:m, k+1:t] = U[1:m, k+1:t]U_1$, $t = min\{m, k+l\}$.

6. $V[1:p, 1:l] = V[1:p, 1:l]V_1$.

7. $T = Z_1^T R'_{23}$.

8. Compute the RQ decomposition of T such that $T = R_{23}Q_3$.

9. $R_{13}^{(a)} = R_{13}^{(a)} Q_3^T$.

10. $Q[1:n, n-l+1:n] = Q[1:n, n-l+1:n]Q_3^T$.