GSVD Def

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1 Definition

1.1 Our definition

The generalized singular value decomposition (GSVD) of an m-by-n matrix A and p-by-n matrix B is the following, where $m + p \geq n$:

$$A = UCRQ^T \quad \text{and} \quad B = VSRQ^T$$

where U, V and Q are orthogonal matrices. Let $k + l$ be the effective numerical rank of the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$, then R is a $(k+l)$-by-n matrix of structure $[0 \ R_0]$ where R_0 is $(k+l)$-by-$(k+l)$ and is nonsingular upper triangular matrix, C and S are m-by-$(k+l)$ and p-by-$(k+l)$ non-negative "diagonal" matrices and satisfy $C^TC + S^TS = I$. The nonzero elements of C are in **non-increasing** order while the nonzero elements of S are in **non-decreasing** order.

1.2 Other notable definition of GSVD

1.2.1 Julia 1.0

"Compute the generalized SVD of A and B, returning a GeneralizedSVD factorization object F, such that $A = F.U * F.D_1 * F.R_0 * F.Q^T$ and $B = F.V * F.D_2 * F.R_0 * F.Q^T$.

The entries of $F.D1$ and $F.D2$ are related, as explained in the LAPACK documentation for the generalized SVD and the xGGSVD3 routine which is called underneath (in LAPACK 3.6.0 and newer)."

1.2.2 LAPACK 3.6.0

"The generalized (or quotient) singular value decomposition of an m-by-n matrix A and a p-by-n matrix B is given by the pair of factorizations:

$$A = U\Sigma_1[0, R]Q^T \quad \text{and} \quad B = V\Sigma_2[0, R]Q^T$$

Σ_1 is m-by-r, Σ_2 is p-by-r. (The integer r is the rank of $\begin{pmatrix} A \\ B \end{pmatrix}$, and satisfies $r \leq n$.) Both are real, non-negative and diagonal, and $\Sigma_1^T\Sigma_1 + \Sigma_2^T\Sigma_2 = I$. Write $\Sigma_1^T\Sigma_1 = diag(\alpha_1^2, ..., \alpha_r^2)$ and $\Sigma_2^T\Sigma_2 = diag(\beta_1^2, ..., \beta_r^2)$, where α_i and β_i lie in the interval from 0 to 1. The ratios α_1/β_1,...,α_r/β_r are called the generalized singular values of the pair A, B. If $\beta_i = 0$, then the generalized singular value α_i/β_i is infinite."

1.2.3 MATLAB 2019b

"$[U, V, X, C, S] = gsvd(A, B)$ returns unitary matrices U and V, a (usually) square matrix X, and non-negative diagonal matrices C and S so that

$$A = U * C * X^T$$
$$B = V * S * X^T$$
$$C^T * C + S^T * S = I$$

A and B must have the same number of columns, but may have different numbers of rows. If A is m-by-p and B is n-by-p, then U is m-by-m, V is n-by-n, X is p-by-q, C is m-by-q and S is n-by-q, where $q = min(m + n, p)$.

The nonzero elements of S are always on its main diagonal. The nonzero elements of C are on the diagonal $diag(C, max(0, q - m))$. If $m \geq q$, this is the main diagonal of C.

$sigma = gsvd(A, B)$ returns the vector of generalized singular values, $sqrt(diag(C^T * C)./diag(S^T * S))$. The vector sigma has length q and is in **non-decreasing** order, where $\mathbf{q = min(m{+}n,p)}$. In other words, The nonzero elements of S are in **non-increasing** order while the nonzero elements of C are in **non-decreasing** order."

It's interesting to notice that MATLAB has a compact form of products for gsvd.

1.3 Differences of definitions and reasoning

We analyse the two major definitions of GSVD.

1.3.1 Examples to illustrate

Case 1: $m \geq n, p \geq n$.

Input matrix:

$$A = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix} \qquad B = \begin{pmatrix} 17 & 24 & 1 \\ 23 & 5 & 7 \\ 4 & 6 & 13 \\ 10 & 12 & 19 \\ 11 & 18 & 25 \end{pmatrix}$$

where $m = 4, p = 5, n = 3$.

The outputs produced by MATLAB and Julia are as follows. Subscript m denotes outputs from MATLAB while subscript j denotes outputs from Julia.

$$U_m = \begin{pmatrix} 0.5234 & 0.7015 & -0.4560 & 0.1614 \\ -0.8182 & 0.2552 & -0.4846 & 0.1749 \\ 0.0662 & -0.1911 & -0.5133 & -0.8340 \\ 0.2286 & -0.6374 & -0.5420 & 0.4977 \end{pmatrix} \qquad U_j = \begin{pmatrix} -0.45599 & 0.701479 & 0.537499 & 0.105333 \\ -0.484645 & 0.255184 & -0.795176 & 0.260184 \\ -0.5133 & -0.19111 & -0.0221448 & -0.836367 \\ -0.541955 & -0.637405 & 0.279822 & 0.47085 \end{pmatrix}$$

$$V_m = \begin{pmatrix} -0.8165 & -0.5613 & 0.0400 & -0.0252 & -0.1268 \\ 0.5443 & -0.7835 & 0.2772 & -0.0914 & 0.0685 \\ 0.1361 & -0.0359 & -0.3750 & -0.5616 & -0.7240 \\ 0.1361 & -0.1978 & -0.4935 & 0.7759 & -0.3109 \\ 0.0000 & -0.1754 & -0.7331 & -0.2711 & 0.5987 \end{pmatrix}$$

$$V_j = \begin{pmatrix} 0.0400256 & -0.561261 & 0.816497 & -0.0305609 & -0.125633 \\ 0.277172 & -0.783473 & -0.544331 & -0.0884322 & 0.0723129 \\ -0.374988 & -0.0358936 & -0.136083 & -0.591744 & -0.699583 \\ -0.493547 & -0.19778 & -0.136083 & 0.762108 & -0.343464 \\ -0.733055 & -0.17536 & 8.24241e-17 & -0.245512 & 0.609593 \end{pmatrix}$$

$$C_m = \begin{pmatrix} 0.0000 & 0 & 0 \\ 0 & 0.0603 & 0 \\ 0 & 0 & 0.5677 \\ 0 & 0 & 0 \end{pmatrix} \qquad C_j = \begin{pmatrix} 0.567693 & 0.0 & 0.0 \\ 0.0 & 0.0603455 & 0.0 \\ 0.0 & 0.0 & 3.86959e-17 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}$$

$$S_m = \begin{pmatrix} 1.0000 & 0 & 0 \\ 0 & 0.9982 & 0 \\ 0 & 0 & 0.8232 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S_j = \begin{pmatrix} 0.82324 & 0.0 & 0.0 \\ 0.0 & 0.998178 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{pmatrix}$$

$$X^T = \begin{pmatrix} 0.5443 & -14.4248 & 7.3485 \\ -31.6694 & -23.1751 & -14.6808 \\ -9.0419 & -23.1050 & -37.1682 \end{pmatrix} \qquad R * Q^T = \begin{pmatrix} -9.04186 & -23.105 & -37.1682 \\ -31.6694 & -23.1751 & -14.6808 \\ -0.544331 & 14.4248 & -7.34847 \end{pmatrix}$$

Case 2: $m \geq n, p < n$.

Input matrix:

$$A = \begin{pmatrix} 1 & 6 & 11 & 16 \\ 2 & 7 & 12 & 17 \\ 3 & 8 & 13 & 18 \\ 4 & 9 & 14 & 19 \\ 5 & 10 & 15 & 20 \end{pmatrix} \qquad B = \begin{pmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \end{pmatrix}$$

where $m = 5, p = 3, n = 4$.

$$U_m = \begin{pmatrix} 0.1687 & 0.6003 & -0.6325 & -0.4472 & 0.1056 \\ 0.1678 & -0.7651 & -0.3162 & -0.4472 & 0.2939 \\ -0.2298 & -0.1180 & 0.0000 & -0.4472 & -0.8563 \\ -0.7185 & 0.1300 & 0.3162 & -0.4472 & 0.4085 \\ 0.6118 & 0.1527 & 0.6325 & -0.4472 & 0.0483 \end{pmatrix}$$

$$U_j = \begin{pmatrix} -0.447214 & -0.632456 & 0.0273945 & 0.530756 & 0.342852 \\ -0.447214 & -0.316228 & 0.406068 & -0.727935 & -0.0722408 \\ -0.447214 & 4.87701e-15 & -0.840587 & -0.168879 & -0.254743 \\ -0.447214 & 0.316228 & 0.35339 & 0.398539 & -0.645199 \\ -0.447214 & 0.632456 & 0.0537336 & -0.0324808 & 0.629331 \end{pmatrix}$$

$$V_m = \begin{pmatrix} -0.8993 & -0.1569 & 0.4082 \\ 0.2828 & 0.5033 & 0.8165 \\ -0.3336 & 0.8497 & -0.4082 \end{pmatrix} \qquad V_j = \begin{pmatrix} 0.408248 & -0.0777357 & -0.909555 \\ 0.816497 & 0.476683 & 0.325739 \\ -0.408248 & 0.875631 & -0.258076 \end{pmatrix}$$

$$C_m = \begin{pmatrix} 0.0000 & 0 & 0 & 0 \\ 0 & 0.0000 & 0 & 0 \\ 0 & 0 & 0.4146 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_j = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.414632 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.94235e-15 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.06327e-16 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

$$S_m = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0.9100 & 0 \end{pmatrix} \qquad S_j = \begin{pmatrix} 0.0 & 0.909989 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

$$X^T = \begin{pmatrix} -15.9769 & -1.0227 & -1.8712 & -13.4314 \\ 7.6533 & 11.1708 & 9.6608 & 12.1832 \\ 7.6267 & 7.6267 & 7.6267 & 7.6267 \\ -6.7082 & -17.8885 & -29.0689 & -40.2492 \end{pmatrix} \qquad R{*}Q^T = \begin{pmatrix} -6.7082 & -17.8885 & -29.0689 & -40.2492 \\ 7.62671 & 7.62671 & 7.62671 & 7.62671 \\ 9.02033 & 11.2175 & 9.78741 & 13.3105 \\ -15.2469 & -0.0425091 & -1.01973 & -12.3152 \end{pmatrix}$$

Case 3: $m < n, p < n$.

Input matrix:

$$A = \begin{pmatrix} 0.53767 & -2.2588 & 0.31877 & -0.43359 & 3.5784 & -1.3499 \\ 1.8339 & 0.86217 & -1.3077 & 0.34262 & 2.7694 & 3.0349 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.7254 & -0.20497 & 1.409 & -1.2075 & 0.48889 & -0.30344 \\ -0.063055 & -0.12414 & 1.4172 & 0.71724 & 1.0347 & 0.29387 \\ 0.71474 & 1.4897 & 0.6715 & 1.6302 & 0.72689 & -0.78728 \end{pmatrix}$$

where $m = 2, p = 3, n = 6$.

$$U_m = \begin{pmatrix} 0.8698 & -0.4935 \\ -0.4935 & -0.8698 \end{pmatrix} \qquad U_j = \begin{pmatrix} -0.669592 & -0.742729 \\ -0.742729 & 0.669592 \end{pmatrix}$$

$$V_m = \begin{pmatrix} 0 & 0 & 1.0000 \\ -0.7295 & 0.6839 & 0 \\ 0.6839 & 0.7295 & 0 \end{pmatrix} \qquad V_j = \begin{pmatrix} -0.56117 & -0.721976 & 0.40477 \\ 0.333328 & -0.644747 & -0.687891 \\ 0.757615 & -0.251102 & 0.602468 \end{pmatrix}$$

$$C_m = \begin{pmatrix} 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix} \qquad C_j = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

$$S_m = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \end{pmatrix} \qquad S_j = \begin{pmatrix} 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

$$X^T = \begin{pmatrix} 0.5348 & 1.1094 & -0.5746 & 0.5917 & -0.2577 & -0.7528 \\ 0.4783 & 1.0019 & 1.4592 & 1.6799 & 1.2380 & -0.3734 \\ 0.7254 & -0.2050 & 1.4090 & -1.2075 & 0.4889 & -0.3034 \\ -0.4373 & -2.3901 & 0.9226 & -0.5462 & 1.7457 & -2.6717 \\ -1.8604 & 0.3648 & 0.9801 & -0.0840 & -4.1746 & -1.9735 \end{pmatrix}$$

$$R * Q^T = \begin{pmatrix} -1.72211 & 0.872117 & 0.757821 & 0.0358548 & -4.45298 & -1.35023 \\ 0.828622 & 2.25498 & -1.11239 & 0.551456 & -0.803412 & 3.03476 \\ 0.113407 & 1.20226 & 0.190442 & 2.15175 & 0.621247 & -0.328218 \\ -0.66254 & -0.146045 & -2.09961 & 0.0 & -1.20261 & 0.227292 \\ 0.767603 & 0.899925 & 0.0 & 0.0 & -0.0759444 & -0.799285 \end{pmatrix}$$

Case 4: $m < n, p < n$ and $r < min(m + p, n)$.

Input matrix:

$$A = \begin{pmatrix} 113 & 735 & 1065 & 693 & 969 & 792 \\ 623 & 425 & 591 & 403 & 535 & 464 \end{pmatrix}$$

$$B = \begin{pmatrix} 2253 & 1271 & 2476 & 1300 & 1676 & 2012 \\ 1828 & 986 & 2123 & 1059 & 1217 & 1854 \\ 2113 & 1361 & 2231 & 1335 & 1421 & 1722 \end{pmatrix}$$

where $m = 2, p = 3, n = 6, r = 4$.

$$C_m = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad C_j = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.384923 & 0.0 & 0.0 \end{pmatrix}$$

$$S_m = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & \end{pmatrix} \qquad S_j = \begin{pmatrix} 0.0 & 0.922949 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

$$X^T = \begin{pmatrix} 2482 & 1420.9 & 2799.7 & 1479.1 & 1657.7 & 2360.9 \\ 2253 & 1271 & 2476 & 1300 & 1676 & 2012 \\ -1283 & -897.5 & -1283 & -846.2 & -867.4 & -910.3 \\ -543.9 & -351.3 & -523.5 & -329.9 & -477.9 & -375 \\ -1153.7 & -772.9 & -1099.8 & -730.6 & -998.4 & -837.8 \end{pmatrix}$$

$$R * Q^T = \begin{pmatrix} -1251.21 & -835.082 & -1193.92 & -788.846 & -1084.53 & -903.762 \\ -643.587 & -398.148 & -625.979 & -370.877 & -574.944 & -417.081 \\ -3537.92 & -2062.96 & -3909.13 & -2109.85 & -2455.06 & -3207.14 \\ 112.348 & 221.659 & 2.05099e-16 & 158.308 & 20.1259 & -129.56 \end{pmatrix}$$