FILTERING IN THE FREQUENCY DOMAIN

Spatial Vs Frequency domain

- Spatial Domain (I)
  - – “Normal” image space
  - – Changes in pixel positions correspond to changes in the scene
  - – Distances in I correspond to real distances
- Frequency Domain (F)
  - – Changes in image position correspond to changes in the spatial frequency
  - – This is the rate at which image intensity values are changing in the spatial domain image I

The Fourier Series

- Periodic functions can be expressed as the sum of sines and/or cosines of different frequencies each multiplied by a different coefficient

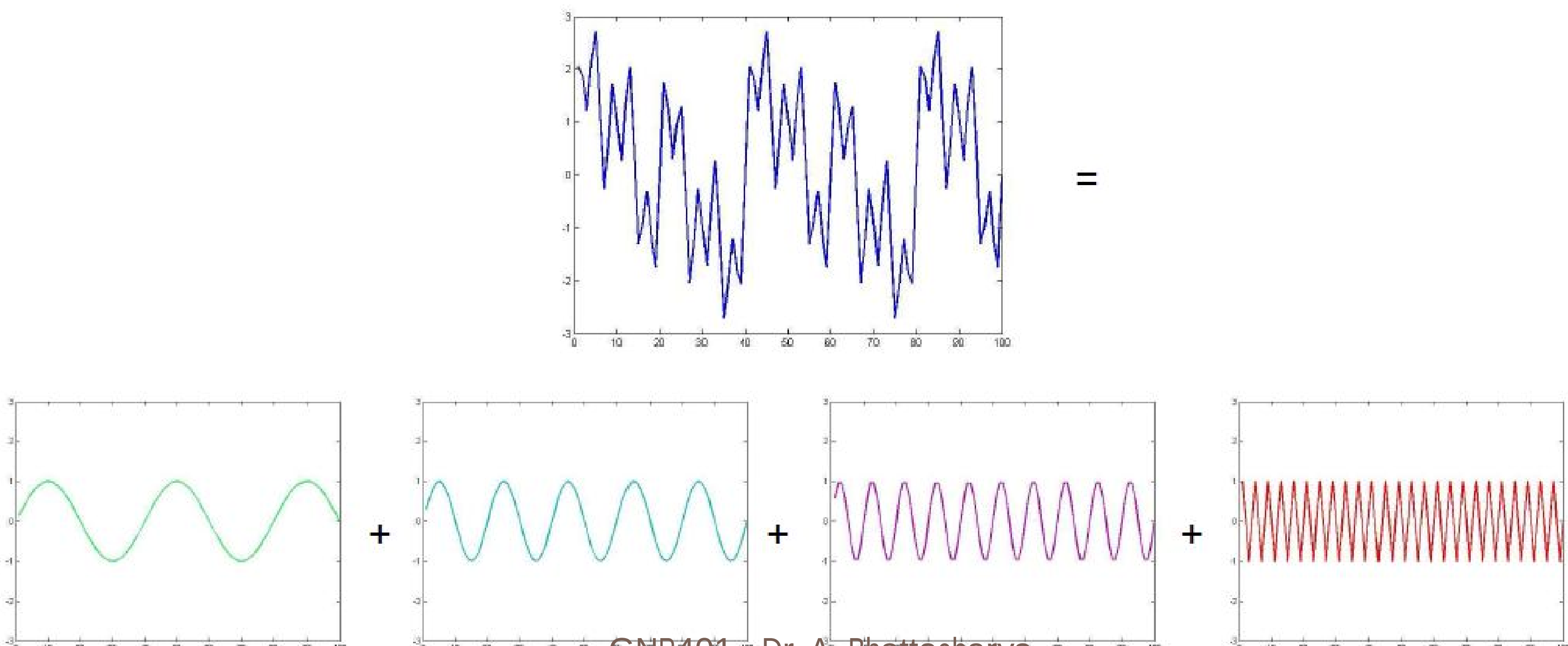


Image processing

- Spatial Domain (I)
  - – Directly process the input image pixel array
- Frequency Domain (F)
  - – Transform the image to its frequency representation
  - – Perform image processing
  - – Compute inverse transform back to the spatial domain

Frequencies in an Image

- Any spatial or temporal signal has an equivalent frequency representation
- What do frequencies mean in an image ?
  - – High frequencies correspond to pixel values that change rapidly across the image (e.g. text, texture, leaves, etc.)
  - – Strong low frequency components correspond to large scale features in the image (e.g. a single, homogenous object that dominates the image)
- We will investigate Fourier transformations to obtain frequency representations of an image

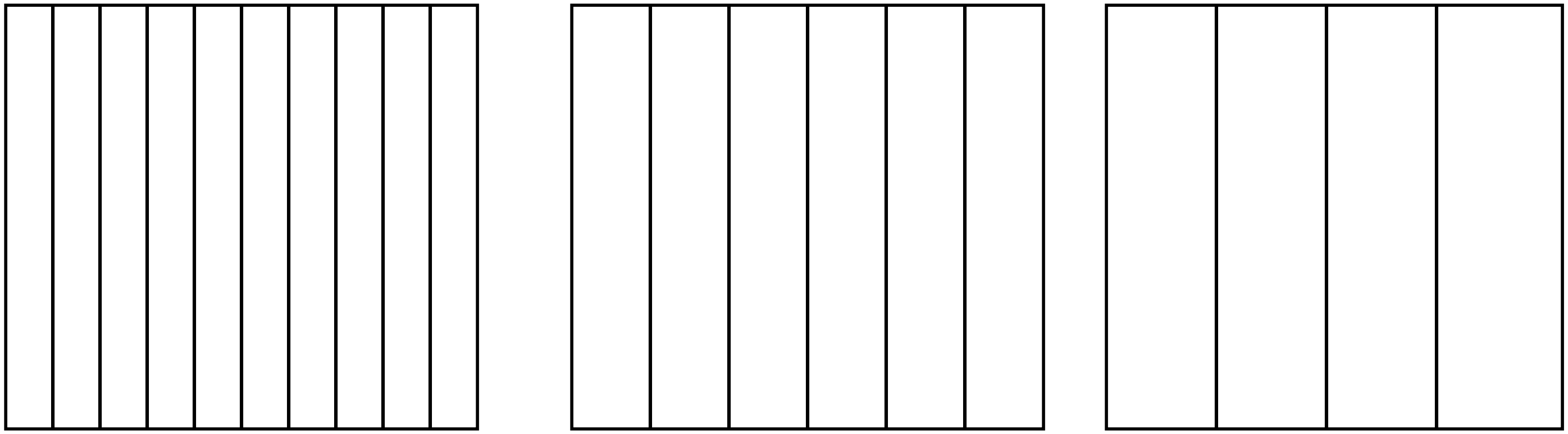
Properties of a Transform

A transform maps image data into a different mathematical space via a transformation equation

Most of the discrete transforms map the image data from the spatial domain to the frequency domain, where ***all*** the pixels in the input (spatial domain) contribute to ***each*** value in the output (frequency domain)

Spatial Frequency

- Rate of change



- Faster the rate of change over distance, higher the frequency

Image Transforms

- **Image transforms** are used as tools in many applications, including enhancement, restoration, correlation and SAR data processing
- **Discrete Fourier transform** is the most important transform employed in image processing applications
- Discrete Fourier transform is generated by sampling the **basis functions** of the continuous transform, i.e., the sine and cosine functions

Concept of Fourier Transform

The Fourier transform decomposes a complex signal into a weighted sum of sinusoids, starting from zero-frequency to a high value determined by the input function

The lowest frequency is also called the ***fundamental*** frequency

Frequency Decomposition

The base frequency or the fundamental frequency is the lowest frequency. All multiples of the fundamental frequency are known as ***harmonics***.

A given signal can be constructed back from its frequency decomposition by a weighted addition of the fundamental frequency and all the harmonic frequencies

Different forms of Fourier Transform

Continuous Fourier Transform

$$F(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-j2\pi ux}dx$$

Fourier Series

$$f(x) = a_0 + \sum_{n=-\infty}^{+\infty} a_n \cos(2\pi nx) + b_n \sin(2\pi nx)$$

$$where$$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(2\pi nx)dx$$

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(2\pi nx)dx$$

Continuous Fourier Transform

- In the continuous domain, the basis functions of the Fourier transform are the complex exponentials $e^{-j2\pi ux}$
- These functions extend from $-\infty$ to $+\infty$
- These are continuous functions, and exist everywhere

Real and Imaginary Parts of Fourier Transform

$$F(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-j2\pi ux}dx$$

$$F(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\cos(2\pi ux)dx - \frac{1}{2\pi}j\int_{-\infty}^{+\infty} f(x)\sin(2\pi ux)dx$$

Real part **Imaginary Part**

The Discrete Fourier Transform

- Since we are dealing with images, we will be more interested in the *discrete* Fourier Transform (DFT)
- For a function *f(x)*, *x=0,1,...,M-1* we have

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}, \quad u = 0,\ldots,M-1$$

iDFT

$$f(x) = \sum_{x=0}^{M-1} F(u)e^{j2\pi ux/M}, \quad x = 0,\ldots,M-1$$

- For discrete functions, the DFT and iDFT always exist

The Discrete Fourier Transform

- Recall Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

from which we obtain

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left(\cos\frac{2\pi ux}{M} - j\sin\frac{2\pi ux}{M}\right)$$

for u = 0,…,M-1

- Each term is composed of ALL values of *f(x)*
- The values of *u* are the *frequency domain*
- Each *F(u)* is a *frequency component* of the transform

The 2-D Discrete Fourier Transform

- Since our images are nothing more than 2-D discrete functions, we are interested in the 2-D DFT

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

for *u=0,…,M-1* and *v=0,…,N-1* and the iDFT is defined as

$$f(x,y) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

The 2-D Discrete Fourier Transform

- Since the values of the FT are complex numbers, it is sometimes more convenient to express F(u) in terms of polar coordinates

$$F(u,v) = |F(u,v)| e^{-j\phi(u,v)}$$

where the magnitude or *spectrum* is denoted by

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{\frac{1}{2}}$$

and the phase angle by

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

The 2-D Discrete Fourier Transform

DFT (Continued)

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

- At *u*=*v*=*0*, the FDT reduces to

$$F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

- This is nothing more than the average grayscale level of the image
- This is often referred to as the *DC Component* (0 frequency)

The 2-D Discrete Fourier Transform

DFT (Continued)

- The FT has the following translation property

$$f(x,y)e^{-j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

which for u_0=M/2 and v_0=N/2 we see that

$$e^{-j2\pi(ux/M+vy/N)} = e^{-j\pi(x+y)} = (-1)^{x+y}$$

$$\Rightarrow F(f(x,y)(-1)^{x+y}) = F(u-M/2, v-N/2)$$

- In image processing, it is common to multiply the input image by $(-1)^{x+y}$ prior to computing *F(u,v)*
- This has the effect of centering the transform since F(0,0) is now located at u=M/2, v=N/2

Properties of the Fourier Transform

Properties of the Fourier Transform

- The FT is a *linear* operator

$$F(af+bg)=aF(f)+bF(g)$$

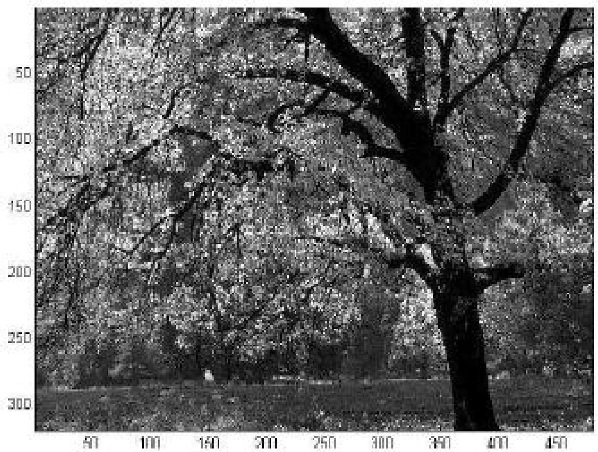
- Some other useful properties include

Discrete	$\Leftrightarrow$	Periodic
Real	$\Leftrightarrow$	Symmetric
f+g	$\Leftrightarrow$	F+G
f*g	$\Leftrightarrow$	FG

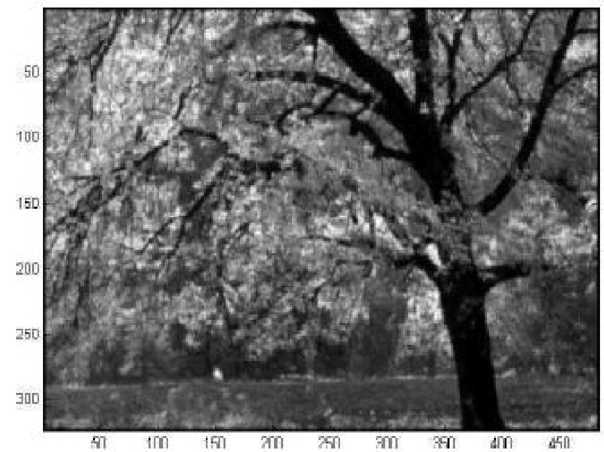
Basis for filtering in the frequency domain!

Filtering Example
Smooth an Image with a Gaussian Kernel

- Traditionally, we would just convolve the image with the a gaussian kernel



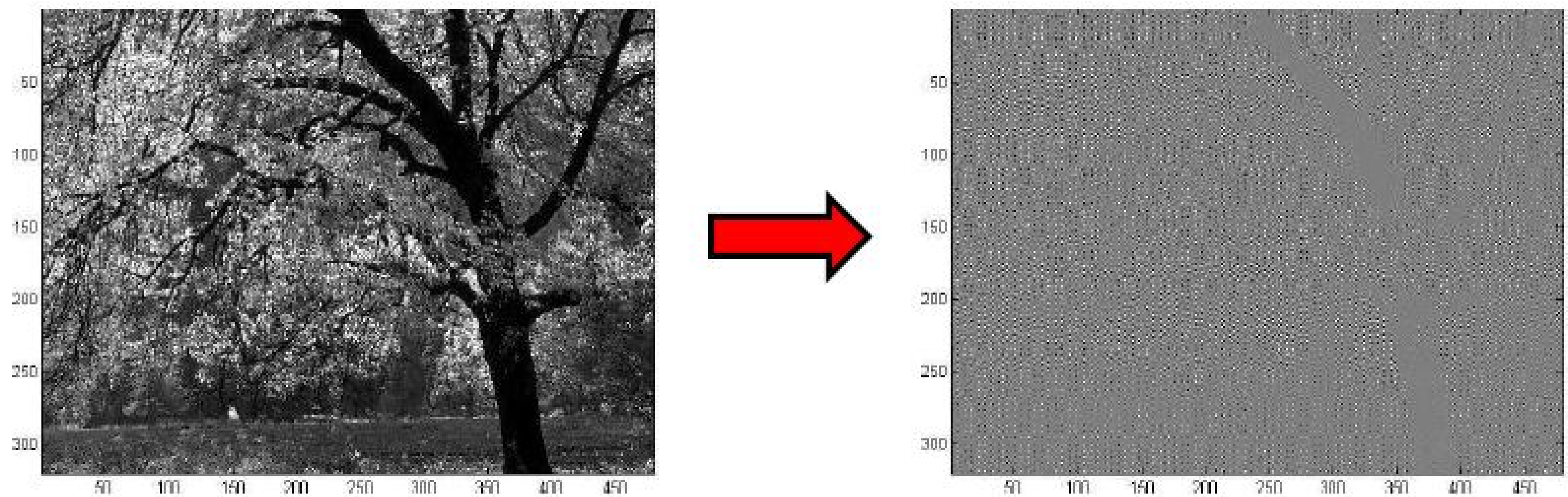
$$* \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} / 256 \quad =$$



- Instead, we will perform multiplication in the frequency domain to achieve the same effect

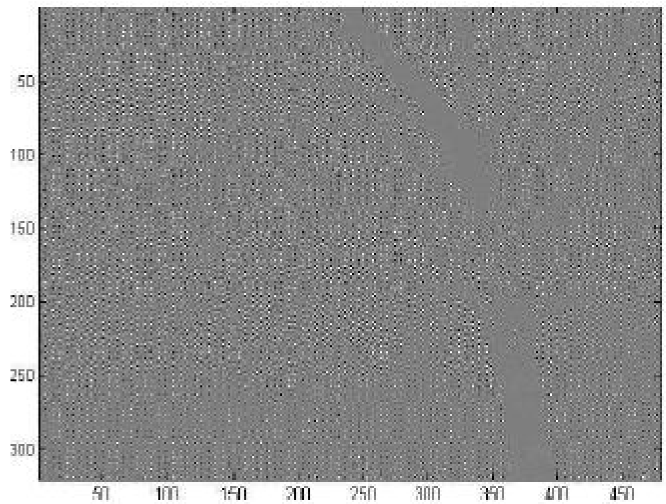
Filtering Example
Smooth an Image with a Gaussian Kernel

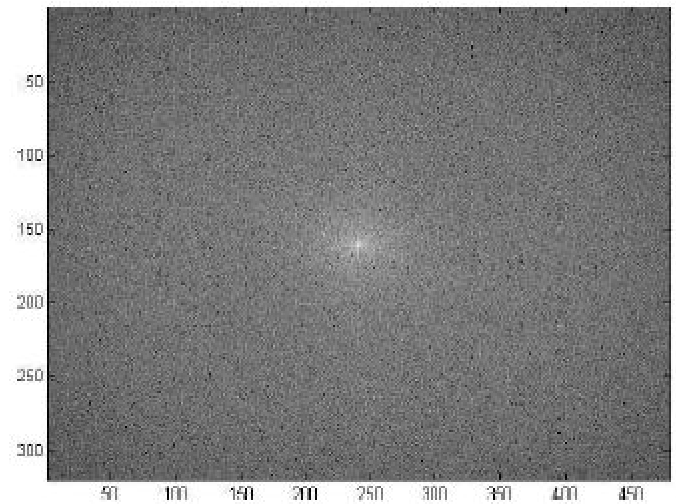
1. Multiply the input image by $(-1)^{x+y}$ to center the transform



Filtering Example
Smooth an Image with a Gaussian Kernel

2. Compute the DFT *F(u,v)* of the resulting image





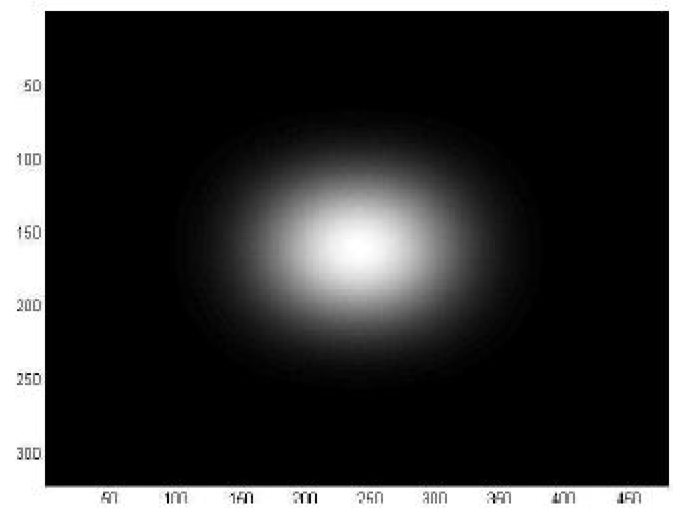
log transform

Filtering Example
Smooth an Image with a Gaussian Kernel

3. Multiply F(u,v) by a filter G(u,v)

$$\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} / 256$$

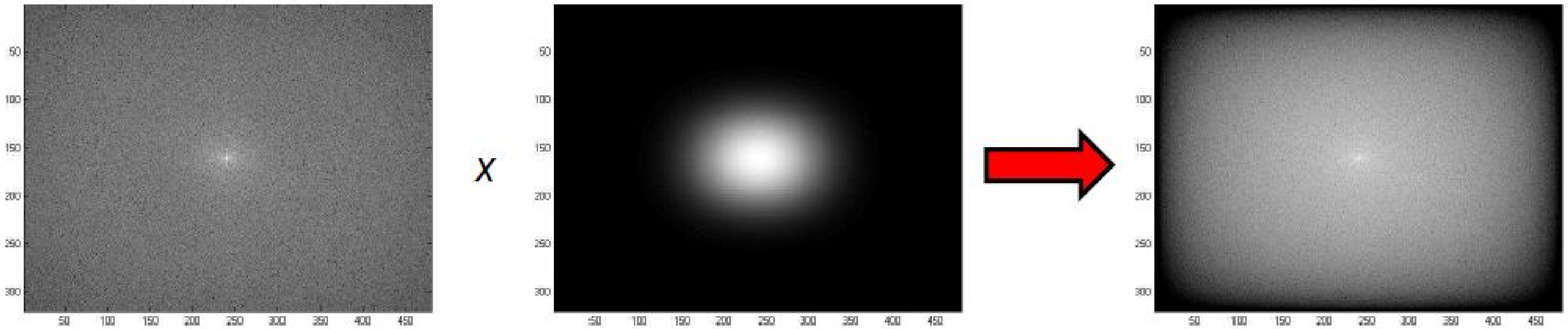
g(x,y)



G(u,v)

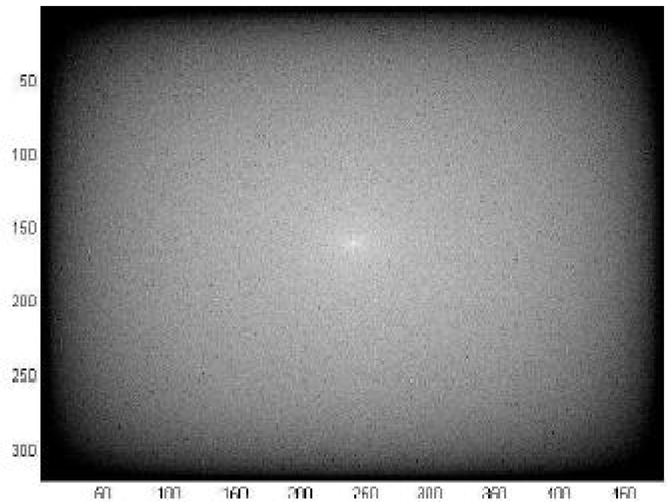
Filtering Example
Smooth an Image with a Gaussian Kernel

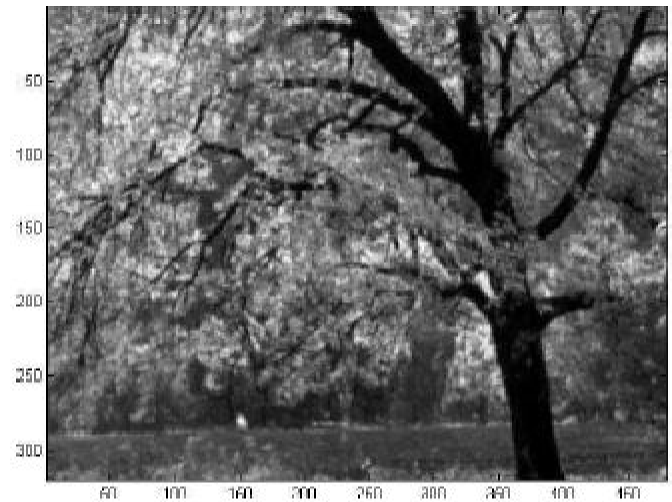
3. Multiply F(u,v) by a filter G(u,v)



Filtering Example
Smooth an Image with a Gaussian Kernel

4. Computer the inverse DFT transform h*(x,y)
5. Obtain the real part h(x,y) of 4
6. Multiply the result by $(-1)^{x+y}$





The Fourier Transform

- Functions that are NOT periodic BUT with finite area under the curve can be expressed as the integral of sines and/or cosines multiplied by a weight function

Fourier Transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux}\, dx$$

Fourier Synthesis

Fourier Analysis

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux}\, du$$

Inverse Fourier Transform

Properties of the Fourier Transform

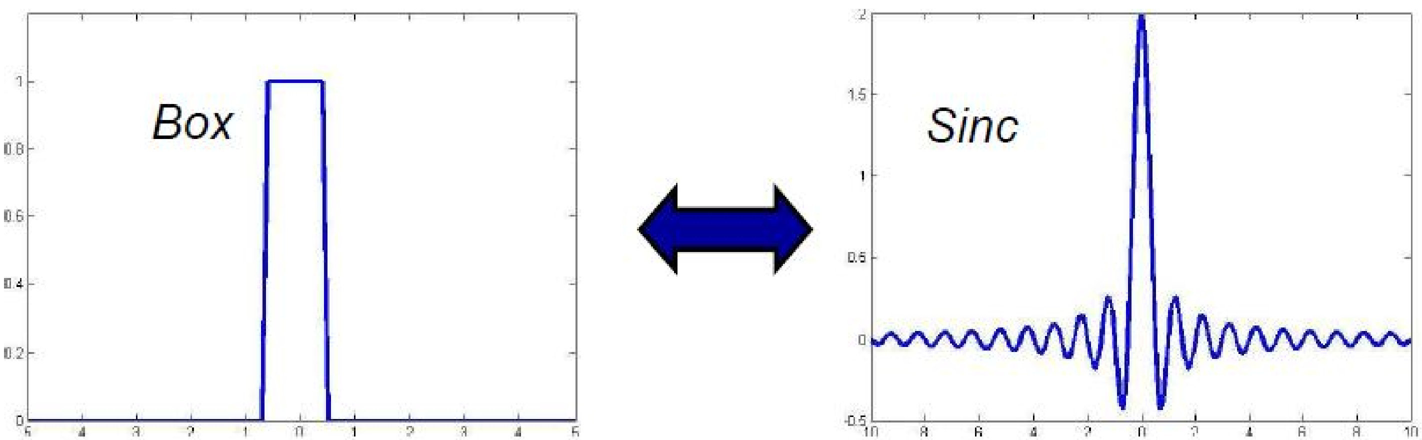
- The FT is a *linear* operator

$$F(af+bg)=aF(f)+bF(g)$$

- Some other useful properties include

Discrete	$\Leftrightarrow$	Periodic
Real	$\Leftrightarrow$	Symmetric
f+g	$\Leftrightarrow$	F+G
f*g	$\Leftrightarrow$	FG

Some Fundamental Transform Pairs

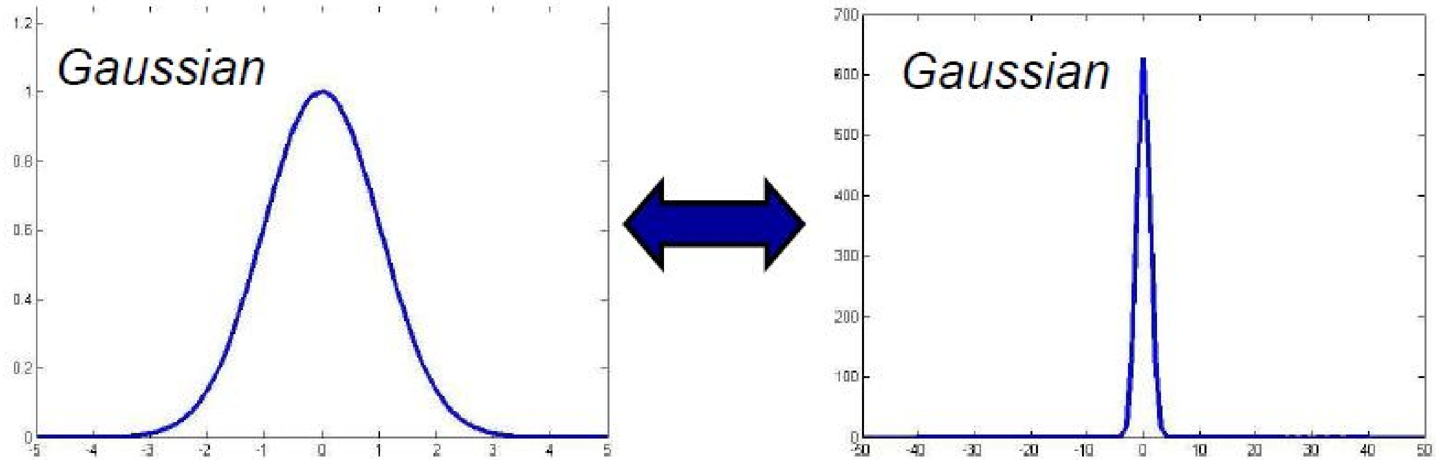


$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi xu} dx \quad = \int_{-1}^{1} e^{-j2\pi xu} dx$$

$$= \frac{-1}{j2\pi u}\left(e^{j2\pi u} - e^{-j2\pi u}\right)$$

$$= \frac{\sin 2\pi u}{\pi u}$$

Some Fundamental Transform Pairs



$$F(u)=\int_{-\infty}^{\infty} f(x)e^{-j2\pi xu}dx \quad =\int_{-\infty}^{\infty} e^{-ax^2}\left(\cos 2\pi ux - j\sin 2\pi ux\right)dx$$

$$=\int_{-\infty}^{\infty} e^{-ax^2}\cos 2\pi ux dx - j\int_{-\infty}^{\infty} e^{-ax^2}\sin 2\pi ux dx$$

$$=\sqrt{\frac{\pi}{a}}e^{-\pi^2u^2/a}$$

Example

Given $f(n)$ = [3,2,2,1], corresponding to the brightness values of one row of a digital image. Find $F(u)$ in both rectangular form, and in exponential form

Example Contd.

$$F(0) = \frac{1}{4}[3+2+2+1] = 2$$

$$F(1) = \frac{1}{4}[3 + 2e^{-j2\pi 1.1/4} + 2e^{-j2\pi 2.1/4} + 1.e^{-j2\pi 3.1/4}] =$$

$$\frac{1}{4}[3+2-2j+j] = \frac{1}{4}[1-j]$$

Example Contd.

$$F(2) = \frac{1}{4}[3 - 2 + 2 - 1] = \frac{1}{2}$$

$$F(3) = \frac{1}{4}[3 + 2e^{-j2\pi 3.1/4} + 2e^{-j2\pi 3.2/4} + 1.e^{-j2\pi 3.3/4}] =$$

$$\frac{1}{4}[3 + 2j - 2 - j] = \frac{1}{4}[1 + j]$$

Therefore F(u) = [2 ¼ (1-j) ½ ¼ (1+j)]

Magnitude-Phase Form

$F(0) = 2 = 2 + j0 \Rightarrow$ Mag=sqrt$(2^2 + 0^2) = 2$; Phase=$\tan^{-1}(0/2) = 0$

$F(1) = ¼ (1-j) = ¼ - j ¼ \Rightarrow$ Mag= ¼ sqrt$(1^2 + (-1)^2) = 0.35$;

Phase = $\tan^{-1}(-(1/4) / (1/4)) = \tan^{-1}(-1) = -\pi/4$

$F(2) = ½ = ½ + j0 \Rightarrow$ Mag = sqrt$((½)^2 + 0^2) = ½$

Phase = $\tan^{-1}(0 / (1/2)) = 0$

$F(3) = ¼ (1+j) = ¼ - j ¼ \Rightarrow$ Mag= ¼ sqrt$(1^2 + (-1)^2) = 0.35$;

Phase = $\tan^{-1}((1/4) / (1/4)) = \tan^{-1}(1) = \pi/4$

Fourier Transform Calculation

Given $f(n) = [\ 3 \quad 2 \quad 2 \quad 1\ ]$

$F(u) = [2 \quad \frac{1}{4}(1-j) \quad \frac{1}{2} \quad \frac{1}{4}(1+j)\ ]$

In phase magnitude form,

$M(u) = [2 \quad 0.35 \quad \frac{1}{2} \quad 0.35\ ]$

$\Phi(u) = [0 \quad -\pi/4 \quad 0 \quad \pi/4\ ]$

Calculate the above for $f(n) = [\ 0 \quad 0 \quad 4 \quad 4 \quad 4 \quad 4 \quad 0 \quad 0]$

Plot $f(n)$, $F(u)$, $M(u)$ and $\Phi(u)$ graphically