

Image filtering



Overview of today's lecture

- Types of image transformations.
- Point image processing.
- Linear shift-invariant image filtering.
- Convolution.
- Image gradients.

Slide credits

Most of these slides were adapted directly from:

- Kris Kitani (15-463, Fall 2016).

Inspiration and some examples also came from:

- Fredo Durand (Digital and Computational Photography, MIT).
- Kayvon Fatahalian (15-769, Fall 2016).

Types of image transformations

What is an image?

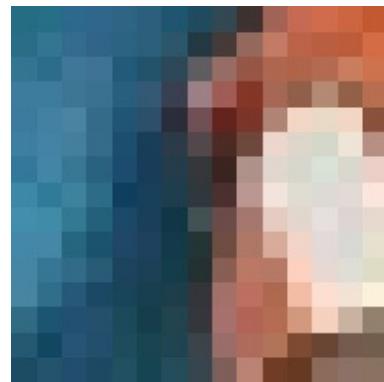


What is an image?



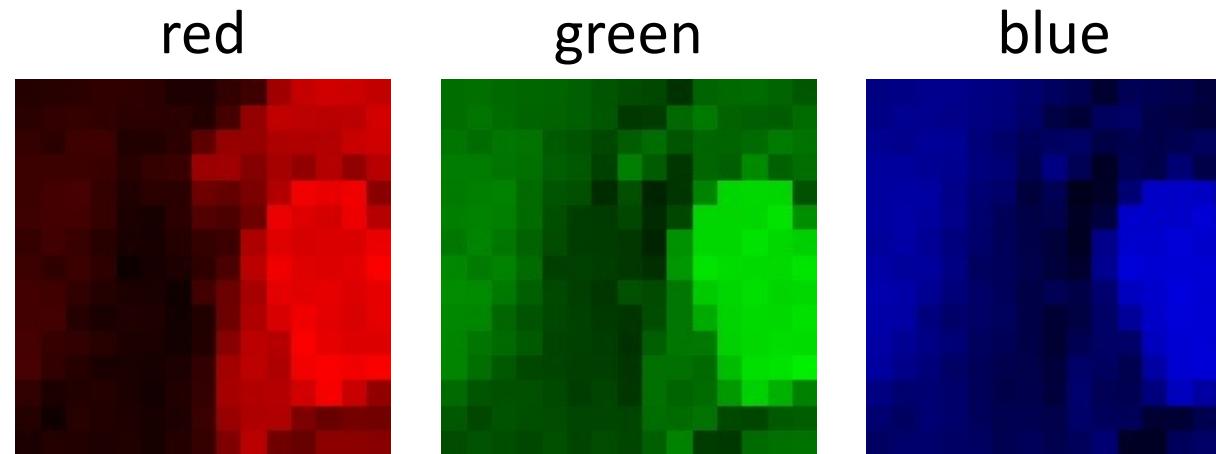
A (color) image
is a 3D tensor
of numbers.

What is an image?



color image patch

How many bits are
the intensity values?



actual intensity values per channel

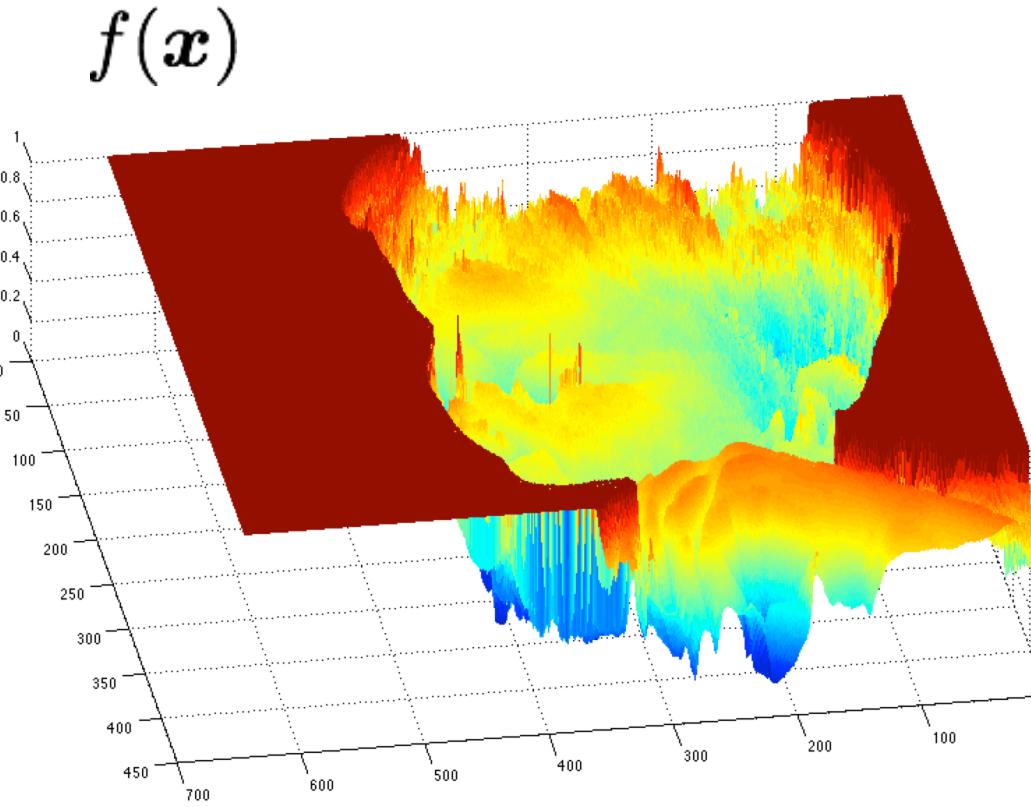
Each channel
is a 2D array of
numbers.

What is an image?



grayscale image

What is the range of
the image function f ?



domain $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

A (grayscale)
image is a 2D
function.

what types of image transformations can we do?



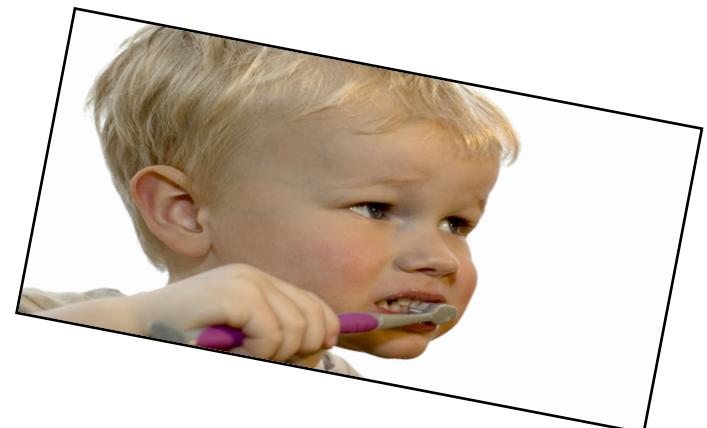
Filtering



changes pixel values



Warping



changes pixel locations

what types of image transformations can we do?

F



Filtering



$$G(\mathbf{x}) = h\{F(\mathbf{x})\}$$

G



changes *range* of image function

F

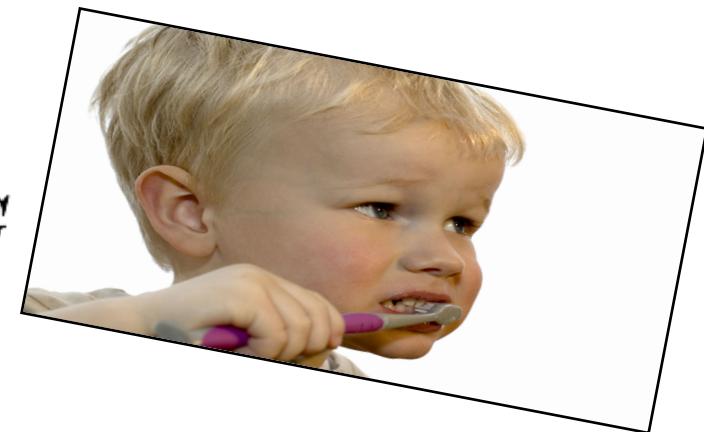


Warping



$$G(\mathbf{x}) = F(h\{\mathbf{x}\})$$

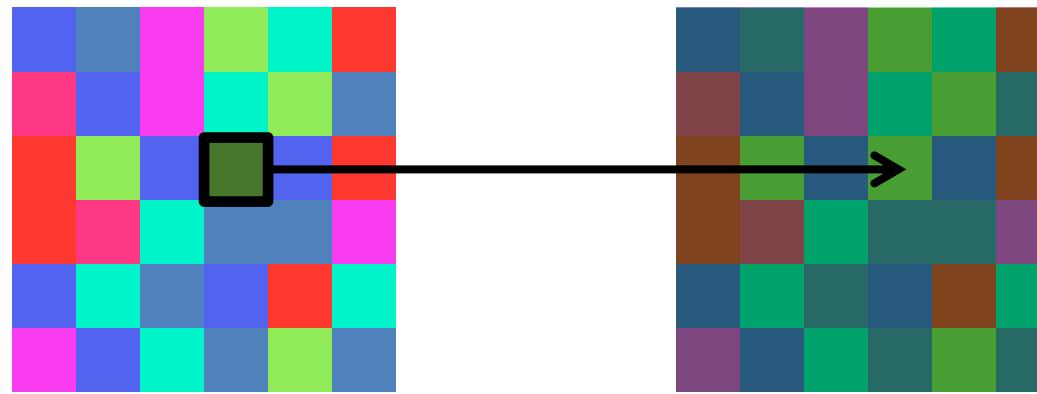
G



changes *domain* of image function

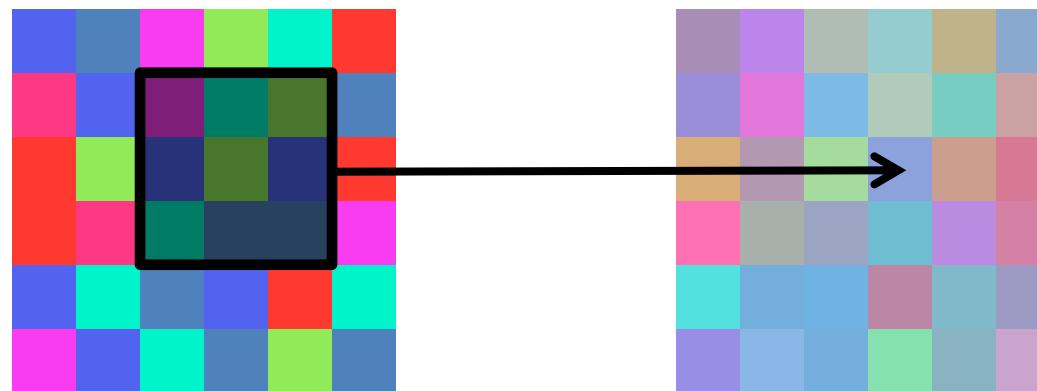
What types of image filtering can we do?

Point Operation



point processing

Neighborhood Operation



“filtering”

Point processing

Examples of point processing

original



darken



lower contrast



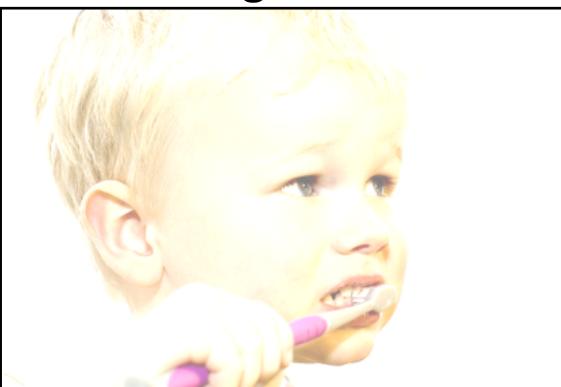
non-linear lower contrast



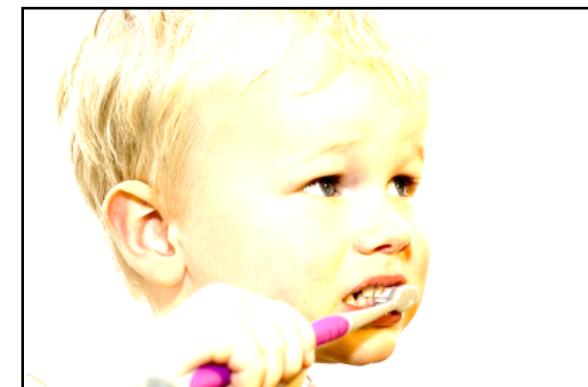
invert



lighten



raise contrast



non-linear raise contrast



How would you
implement these?

Examples of point processing

original



darken



lower contrast



non-linear lower contrast

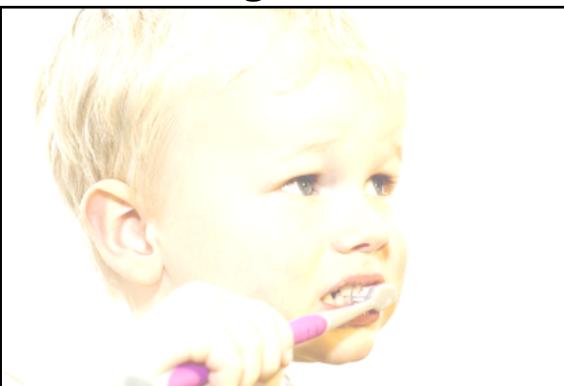


x

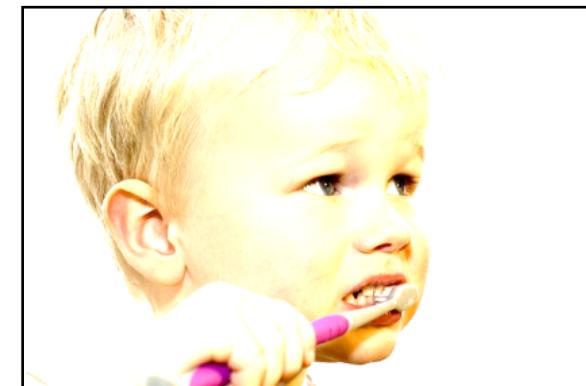
invert



lighten



raise contrast



non-linear raise contrast



How would you
implement these?

Examples of point processing

original



darken



lower contrast



non-linear lower contrast



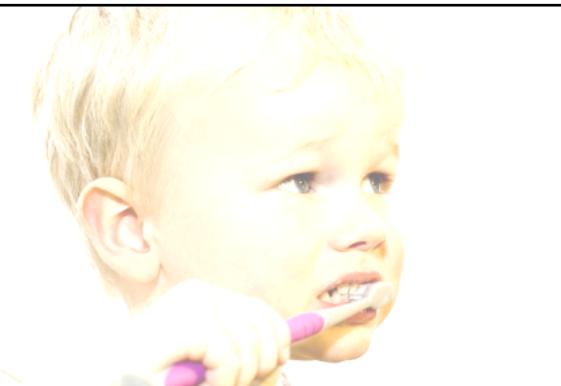
x

$x - 128$

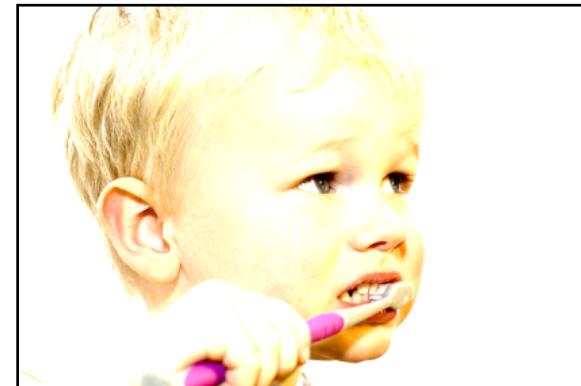
invert



lighten



raise contrast



non-linear raise contrast



How would you
implement these?

Examples of point processing

original



darken



lower contrast



non-linear lower contrast



x

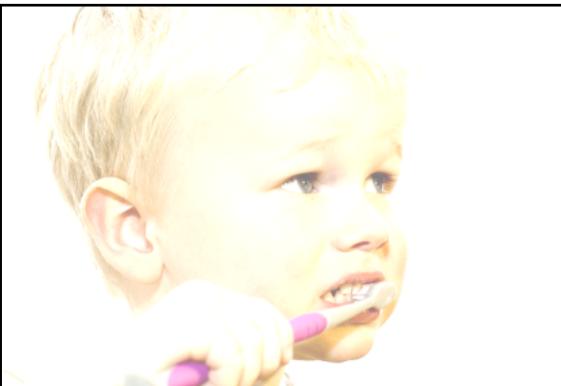
$x - 128$

$\frac{x}{2}$

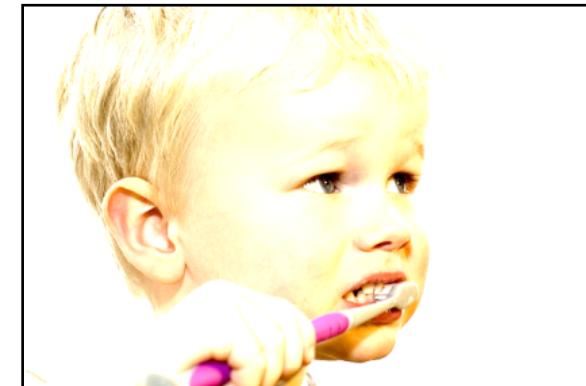
invert



lighten



raise contrast



non-linear raise contrast



How would you
implement these?

Examples of point processing

original



$$x$$

darken



$$x - 128$$

lower contrast



$$\frac{x}{2}$$

non-linear lower contrast

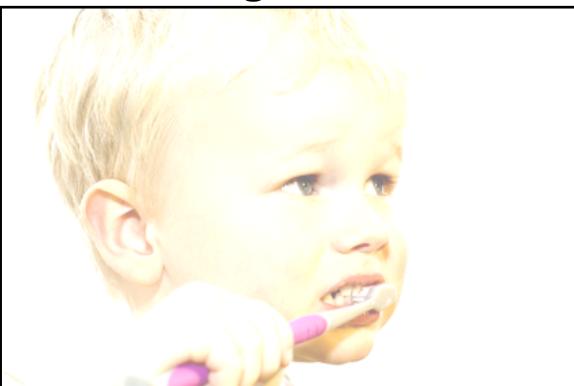


$$\left(\frac{x}{255}\right)^{1/3} \times 255$$

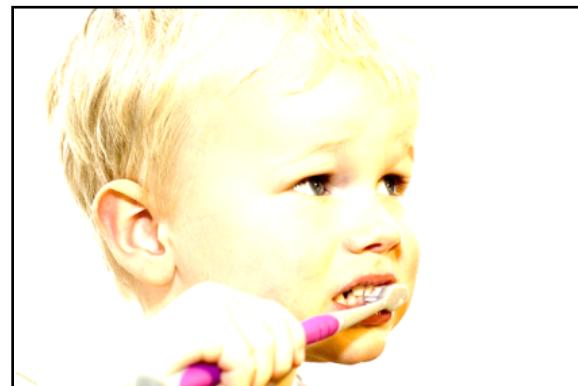
invert



lighten



raise contrast



non-linear raise contrast



How would you
implement these?

Examples of point processing

original



$$x$$

darken



$$x - 128$$

lower contrast



$$\frac{x}{2}$$

non-linear lower contrast



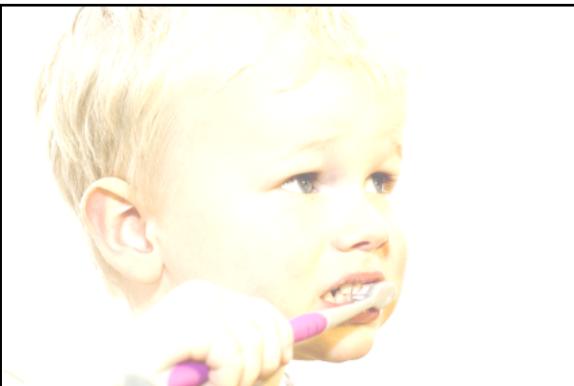
$$\left(\frac{x}{255}\right)^{1/3} \times 255$$

invert

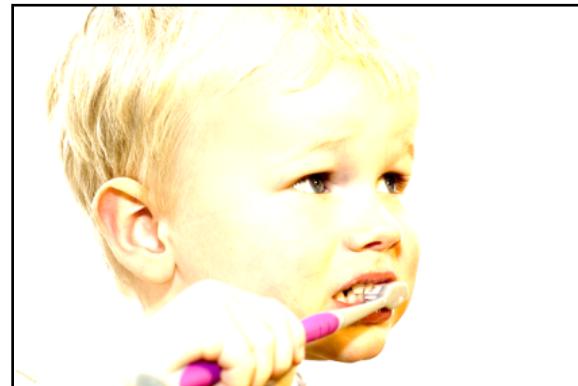


$$255 - x$$

lighten



raise contrast



non-linear raise contrast



How would you
implement these?

Examples of point processing

original



$$x$$

darken



$$x - 128$$

lower contrast



$$\frac{x}{2}$$

non-linear lower contrast



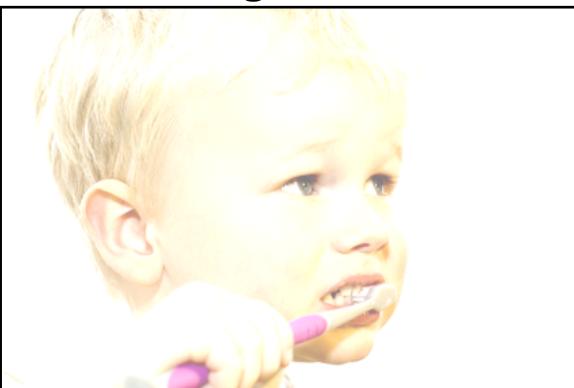
$$\left(\frac{x}{255}\right)^{1/3} \times 255$$

invert



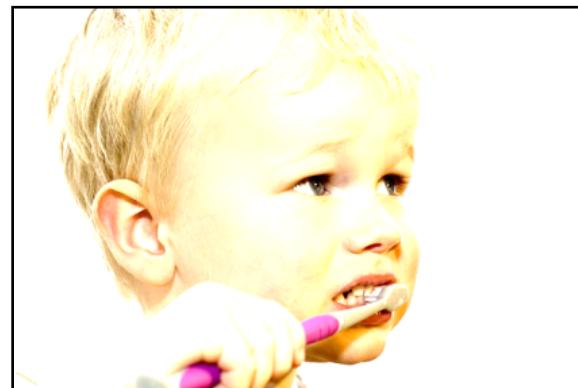
$$255 - x$$

lighten



$$x + 128$$

raise contrast



non-linear raise contrast



How would you
implement these?

Examples of point processing

original



$$x$$

darken



$$x - 128$$

lower contrast



$$\frac{x}{2}$$

non-linear lower contrast



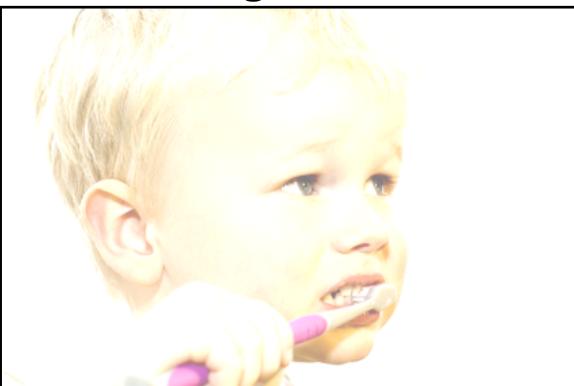
$$\left(\frac{x}{255}\right)^{1/3} \times 255$$

invert



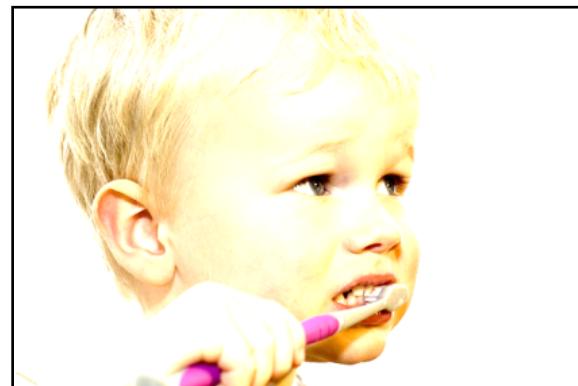
$$255 - x$$

lighten



$$x + 128$$

raise contrast



$$x \times 2$$

non-linear raise contrast



How would you
implement these?

Examples of point processing

original



$$x$$

darken



$$x - 128$$

lower contrast



$$\frac{x}{2}$$

non-linear lower contrast



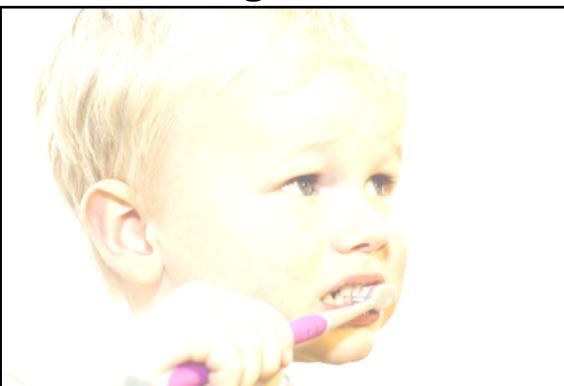
$$\left(\frac{x}{255}\right)^{1/3} \times 255$$

invert



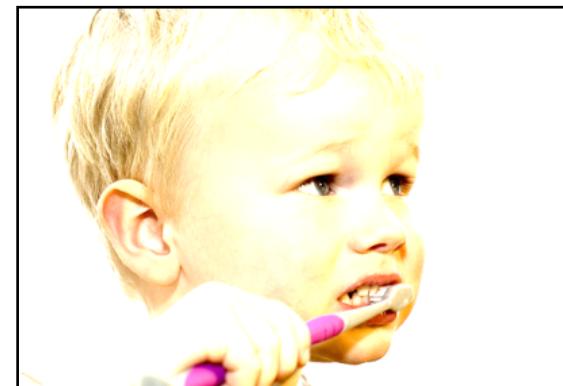
$$255 - x$$

lighten



$$x + 128$$

raise contrast



$$x \times 2$$

non-linear raise contrast



$$\left(\frac{x}{255}\right)^2 \times 255$$

Many other types of point processing



camera output



image after stylistic tonemapping

Many other types of point processing



Linear shift-invariant image filtering

Linear shift-invariant image filtering

- Replace each pixel by a *linear* combination of its neighbors (and possibly itself).
- The combination is determined by the filter's *kernel*.
- The same kernel is *shifted* to all pixel locations so that all pixels use the same linear combination of their neighbors.

Example: the box filter

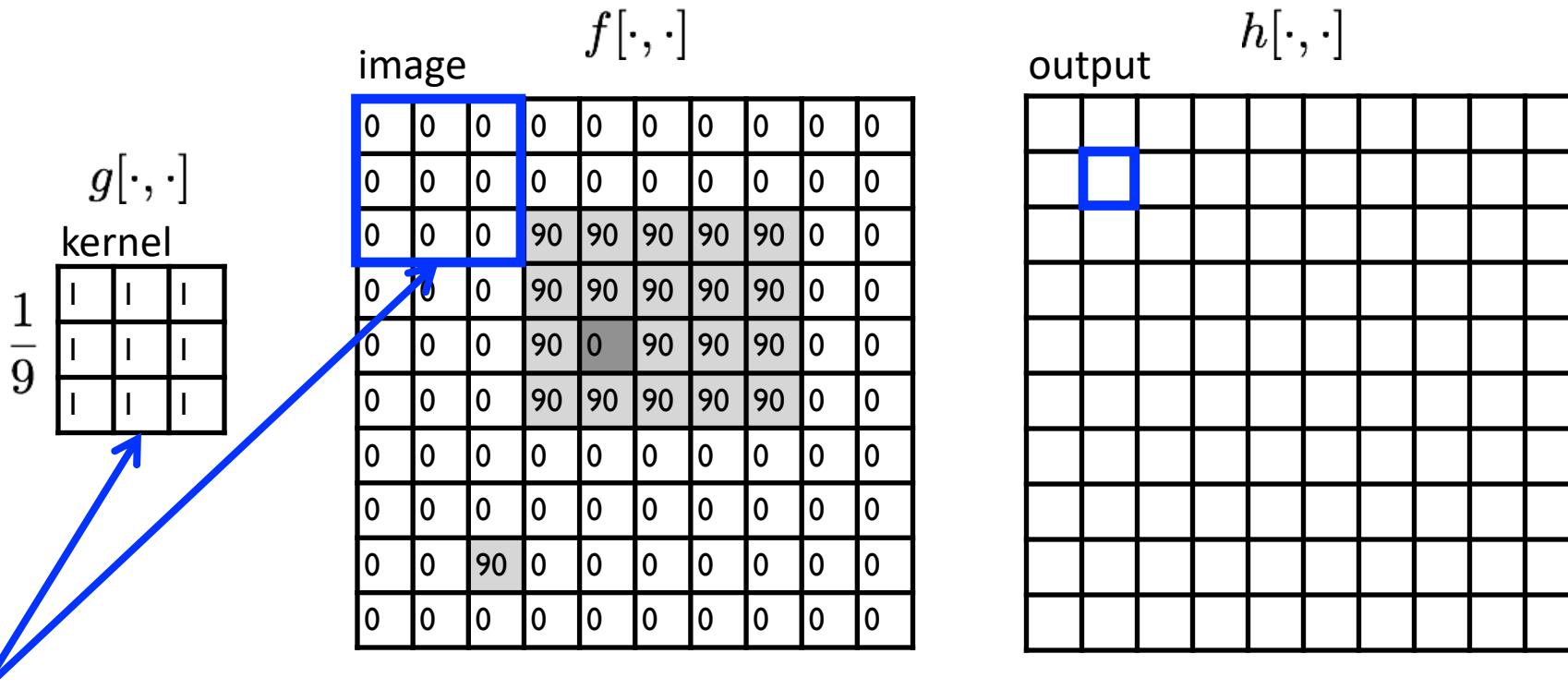
- also known as the 2D rect (not rekt) filter
- also known as the square mean filter

$$\text{kernel } g[\cdot, \cdot] = \frac{1}{9} \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

- replaces pixel with local average
- has smoothing (blurring) effect



Let's run the box filter



note that we assume that
the kernel coordinates
are centered

$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

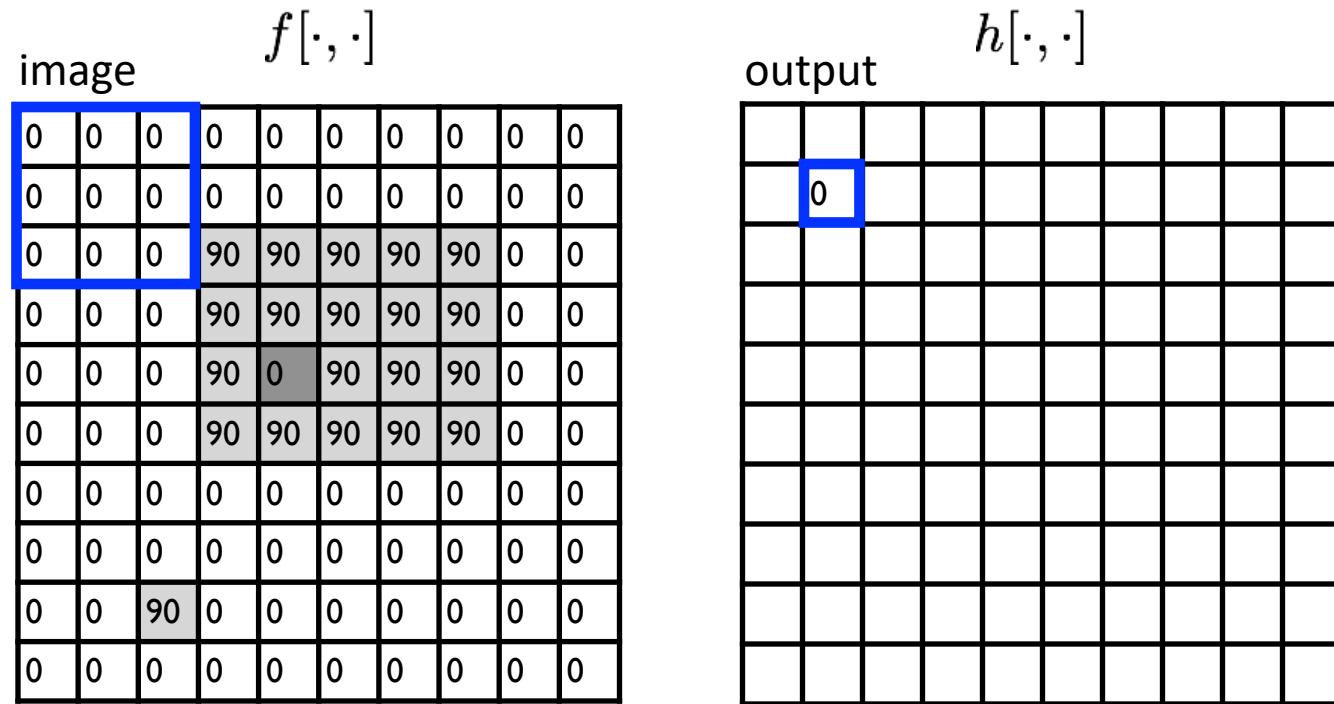
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1

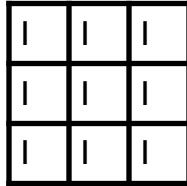


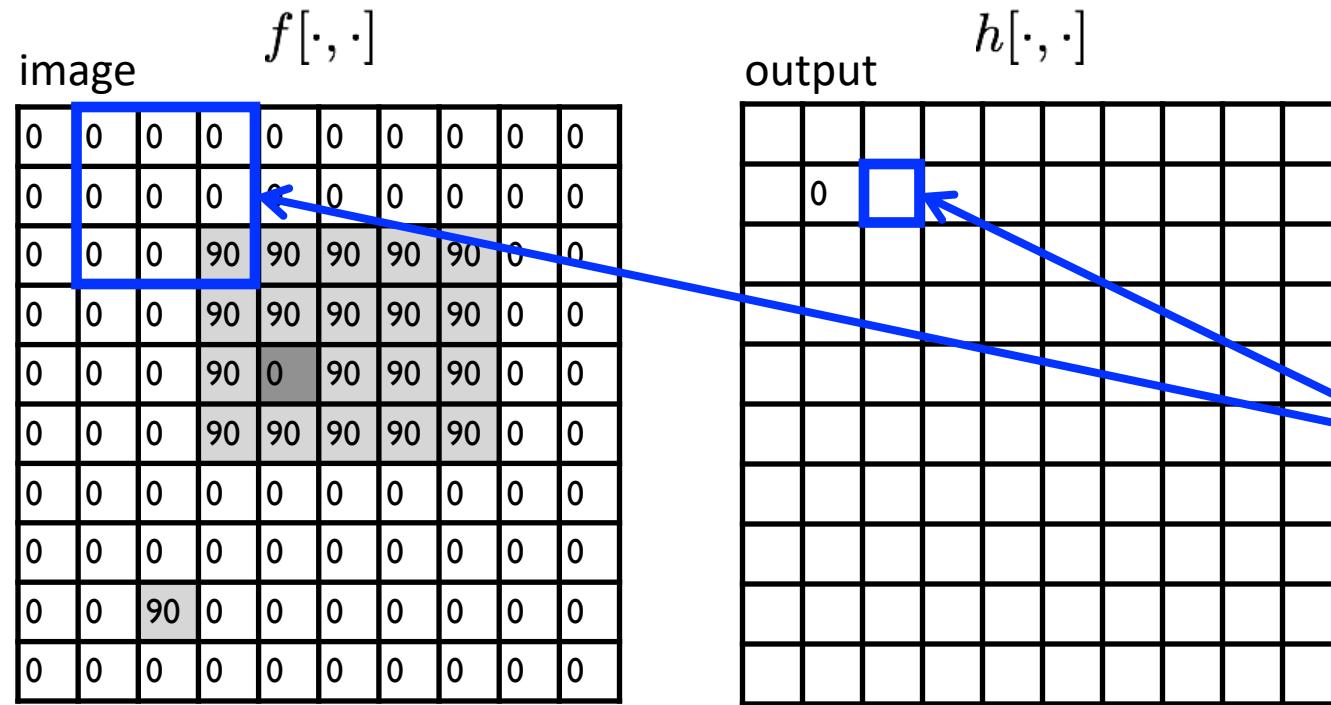
$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel
 $\frac{1}{9}$




shift-invariant:
as the pixel
shifts, so does
the kernel

$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

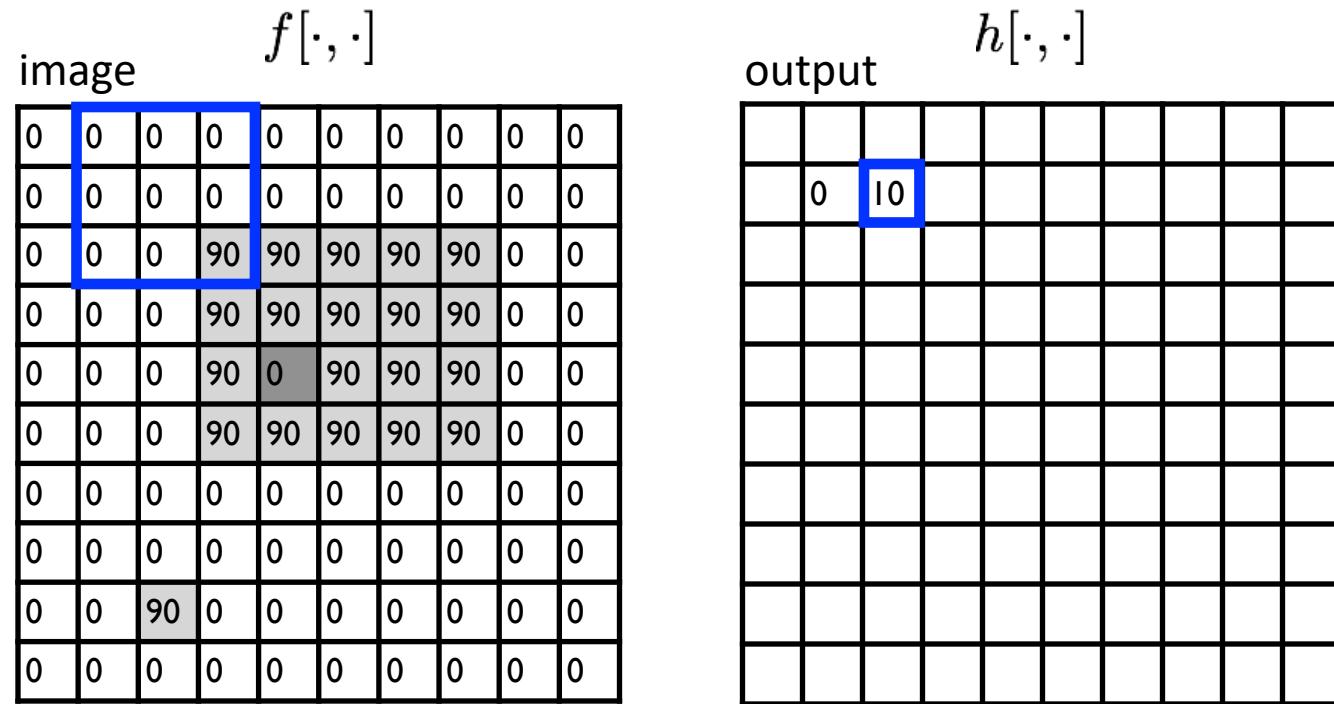
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

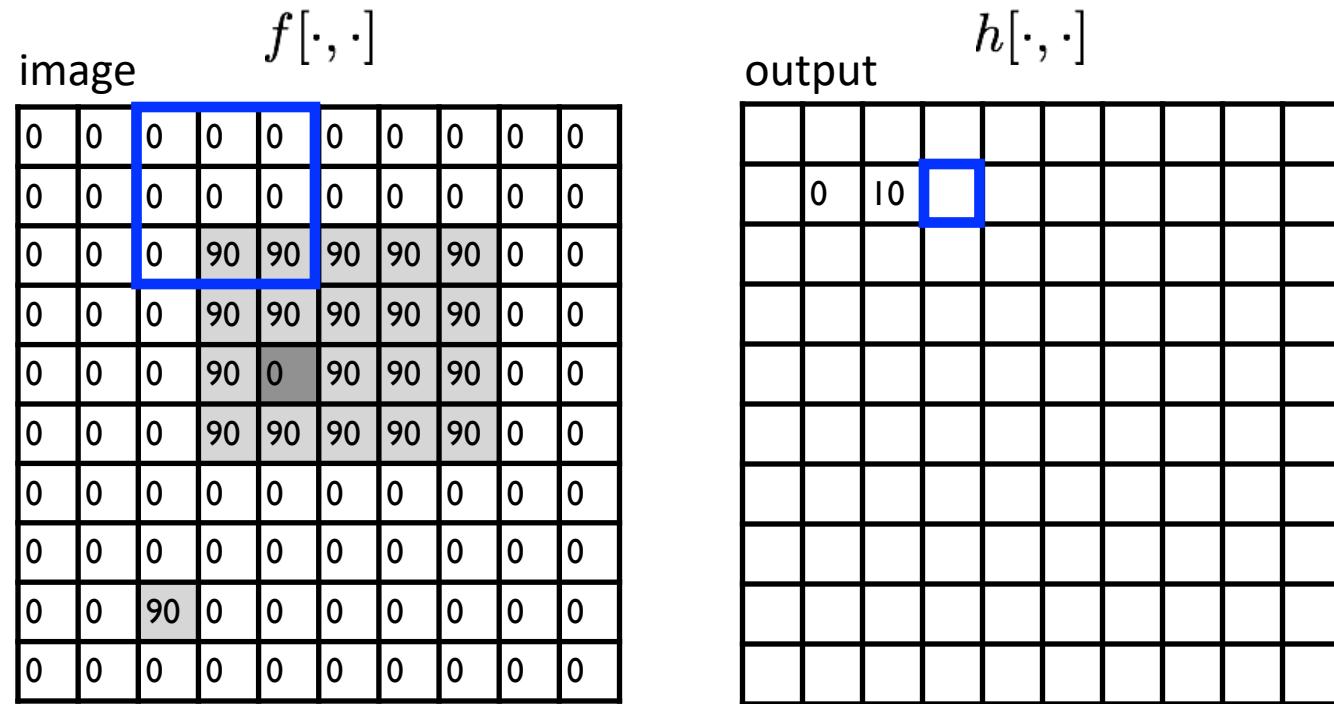
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

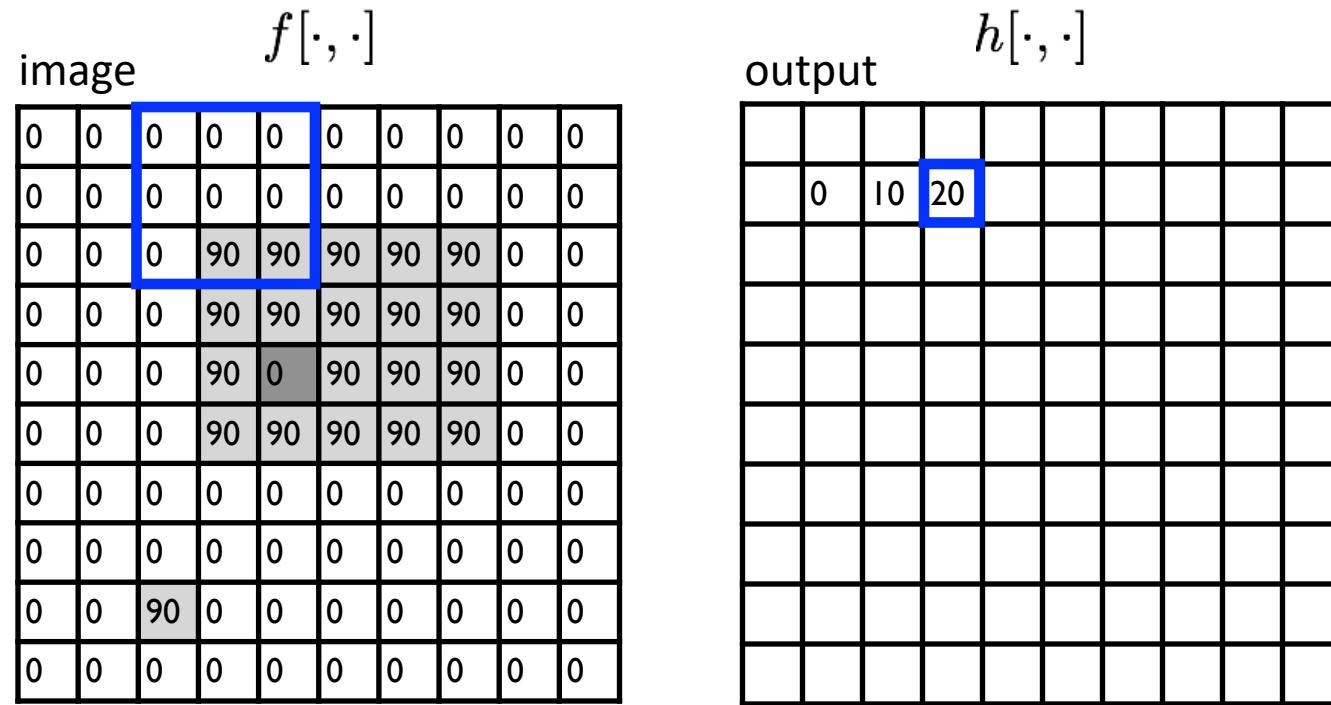
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

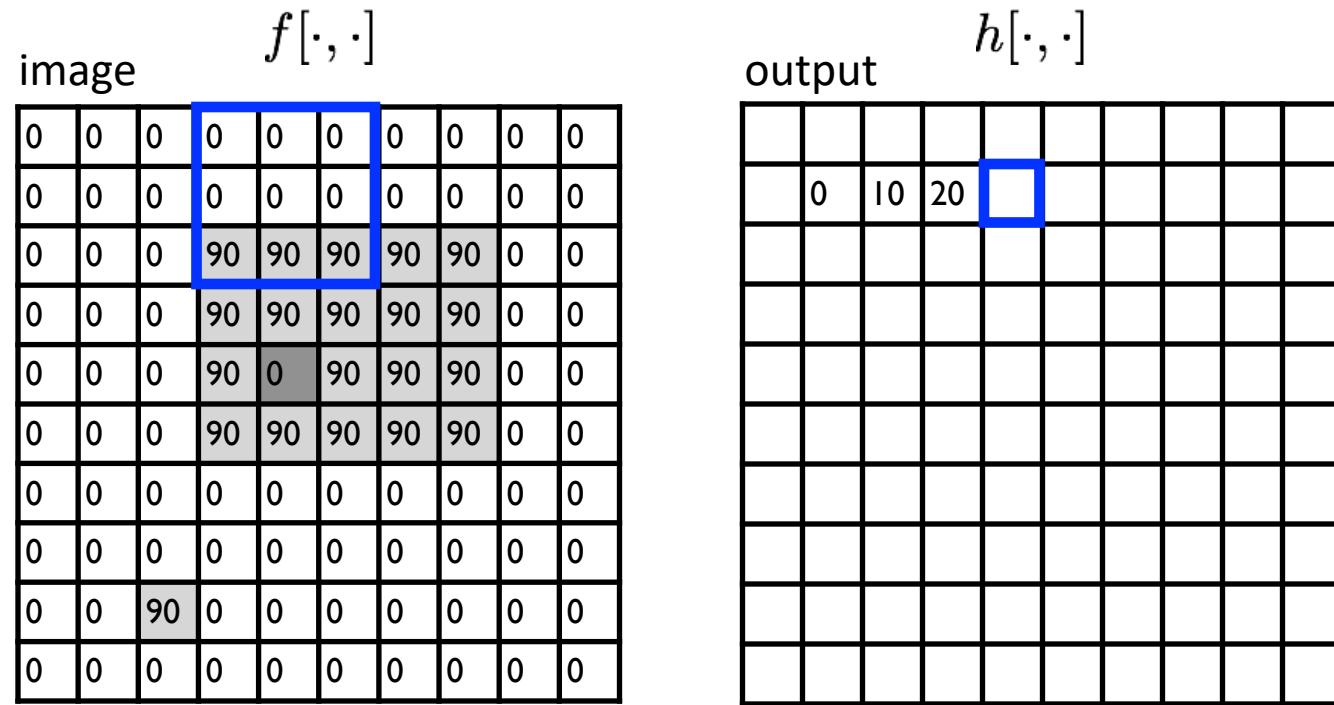
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output k, l filter image (signal)

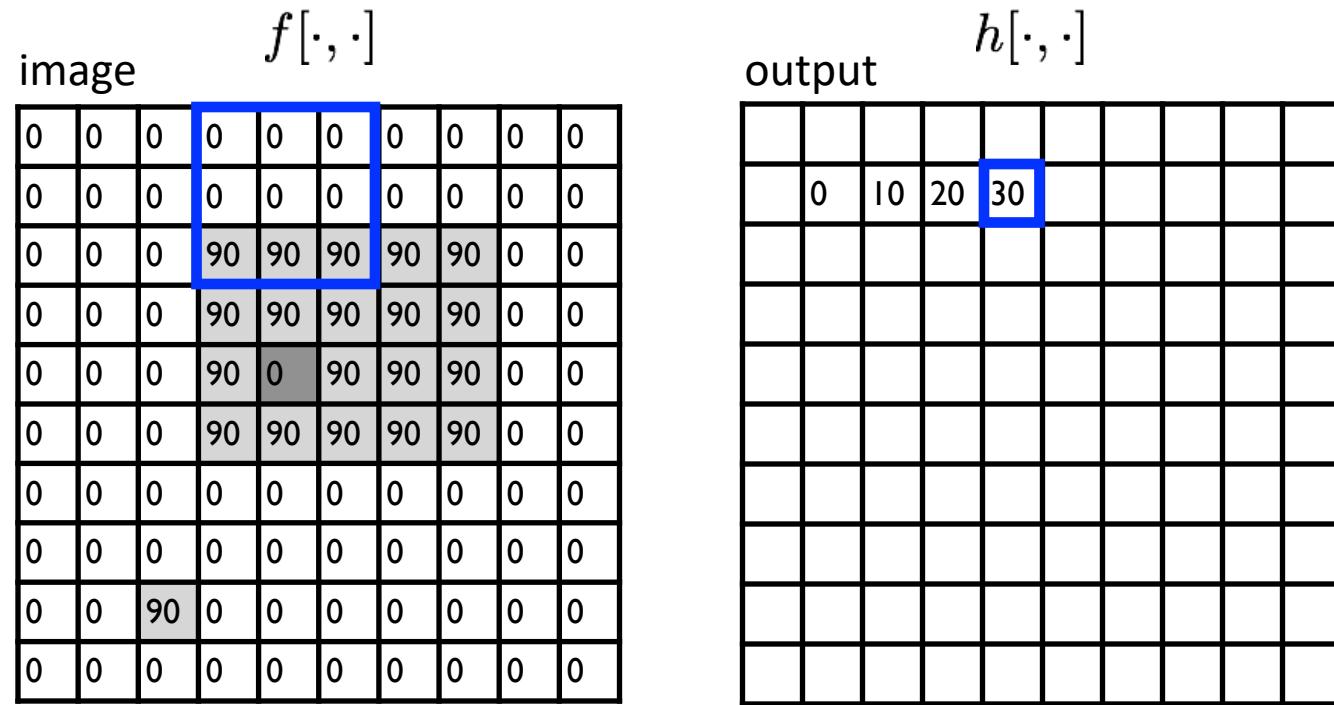
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

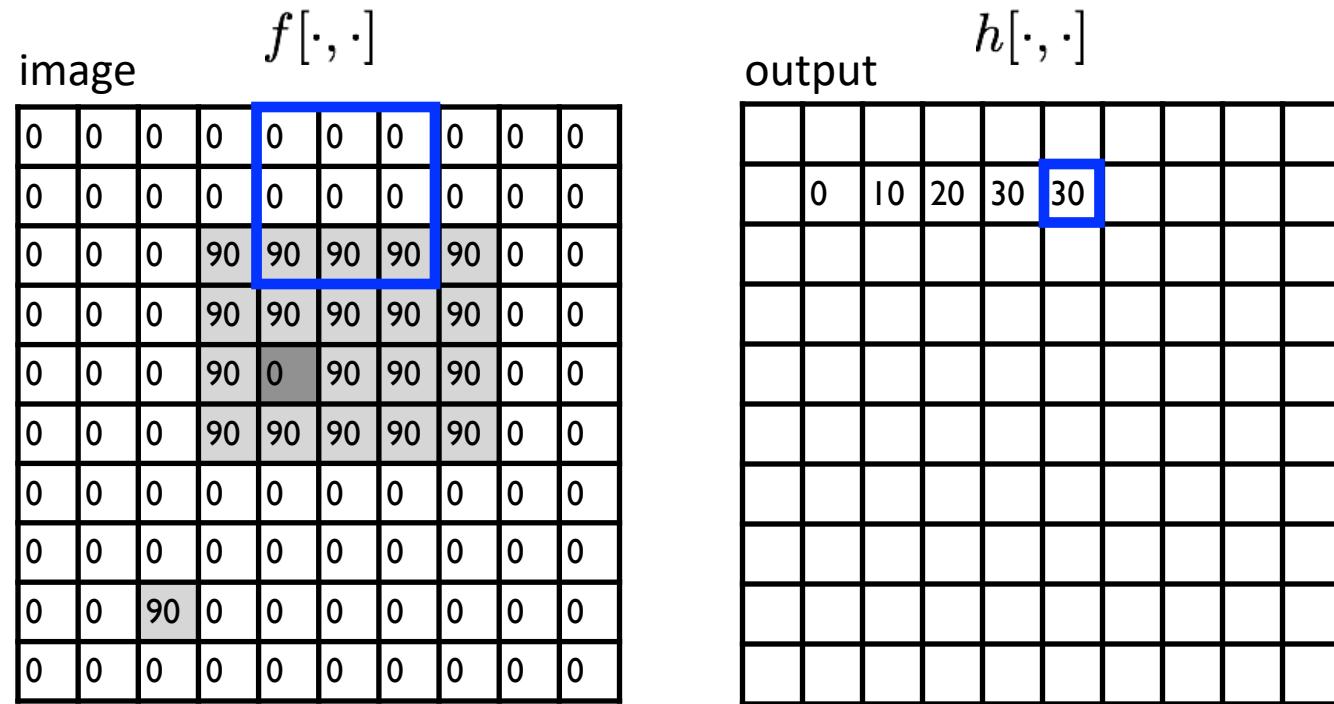
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

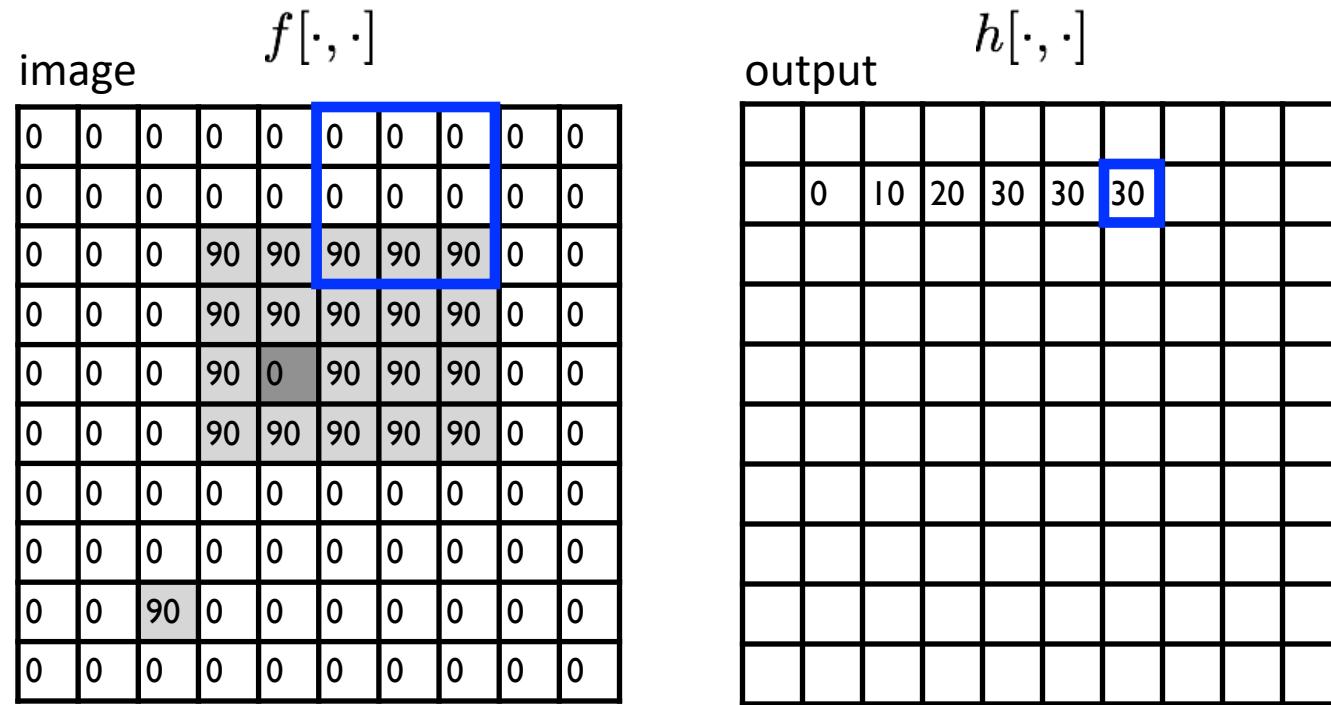
output filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output k, l filter image (signal)

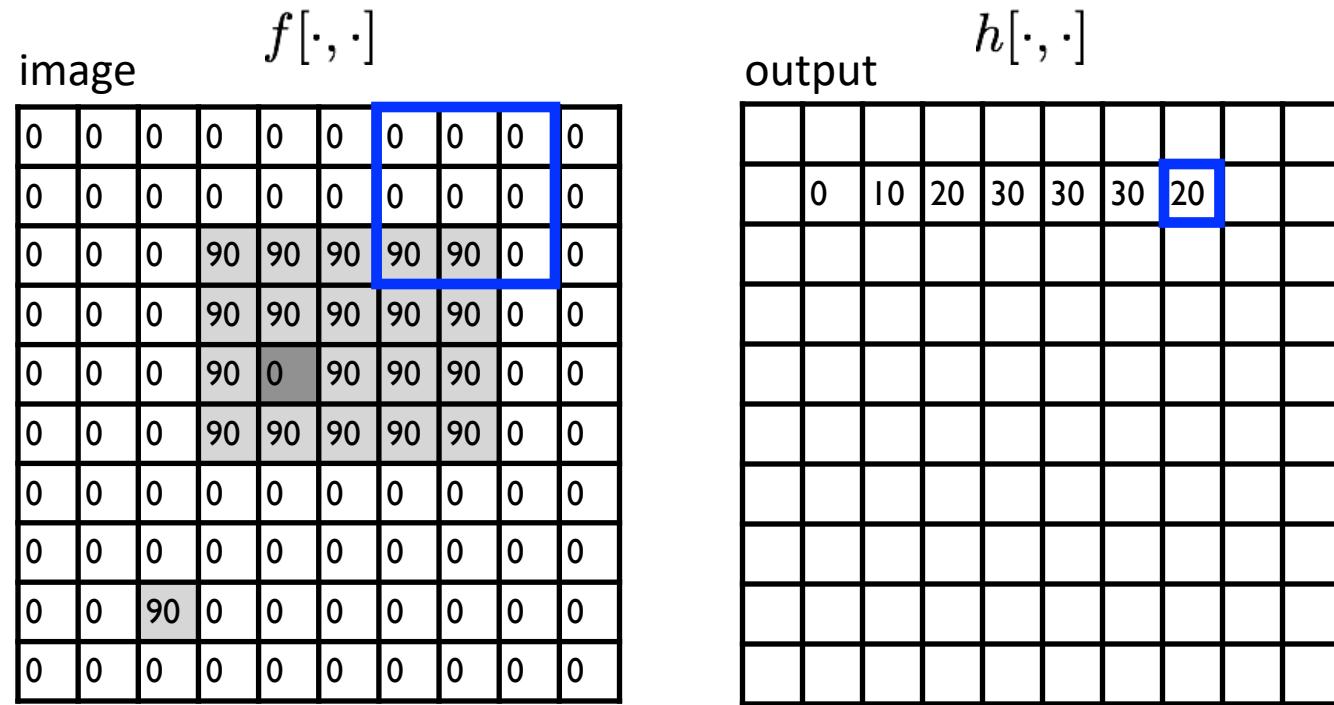
Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

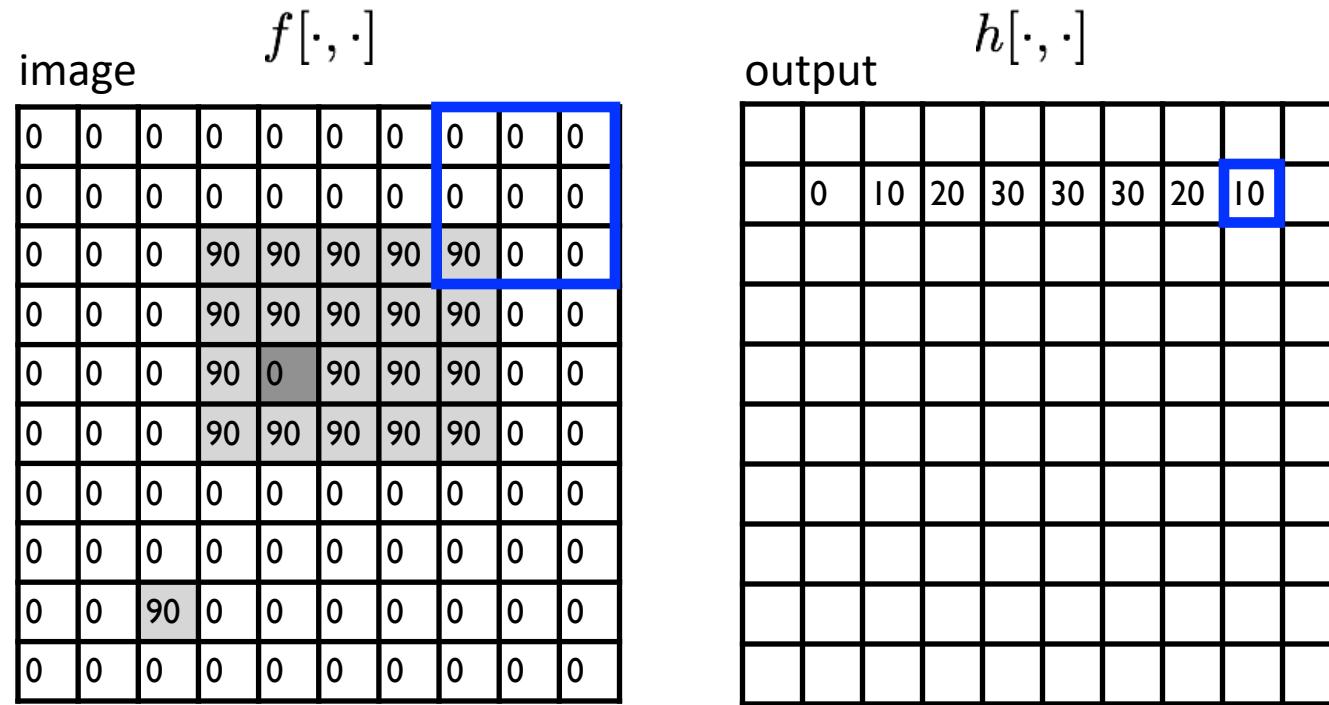
output filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9} \begin{matrix} | & | & | \\ | & | & | \\ | & | & | \end{matrix}$$



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output k, l filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

1	1	1
1	1	1
1	1	1

$$h[m, n] = \sum_{k, l} g[k, l] f[m + k, n + l]$$

output k, l filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

1	1	1
1	1	1
1	1	1

A 10x10 grid representing a heatmap. The values are as follows:

output									
	0	10	20	30	30	30	20	10	
	0	20	20						

The value at position (2, 3) is highlighted with a blue box.

$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output k, l filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

1	1	1
1	1	1
1	1	1

$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

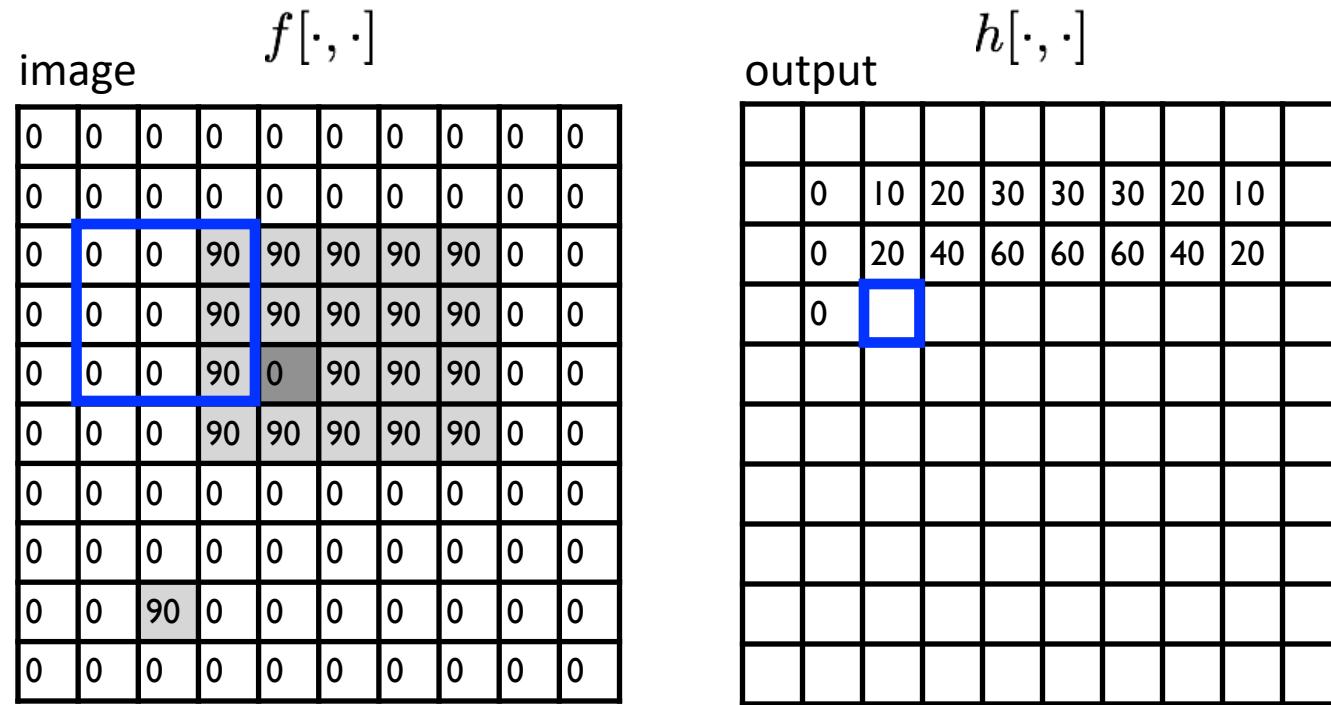
output k, l filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

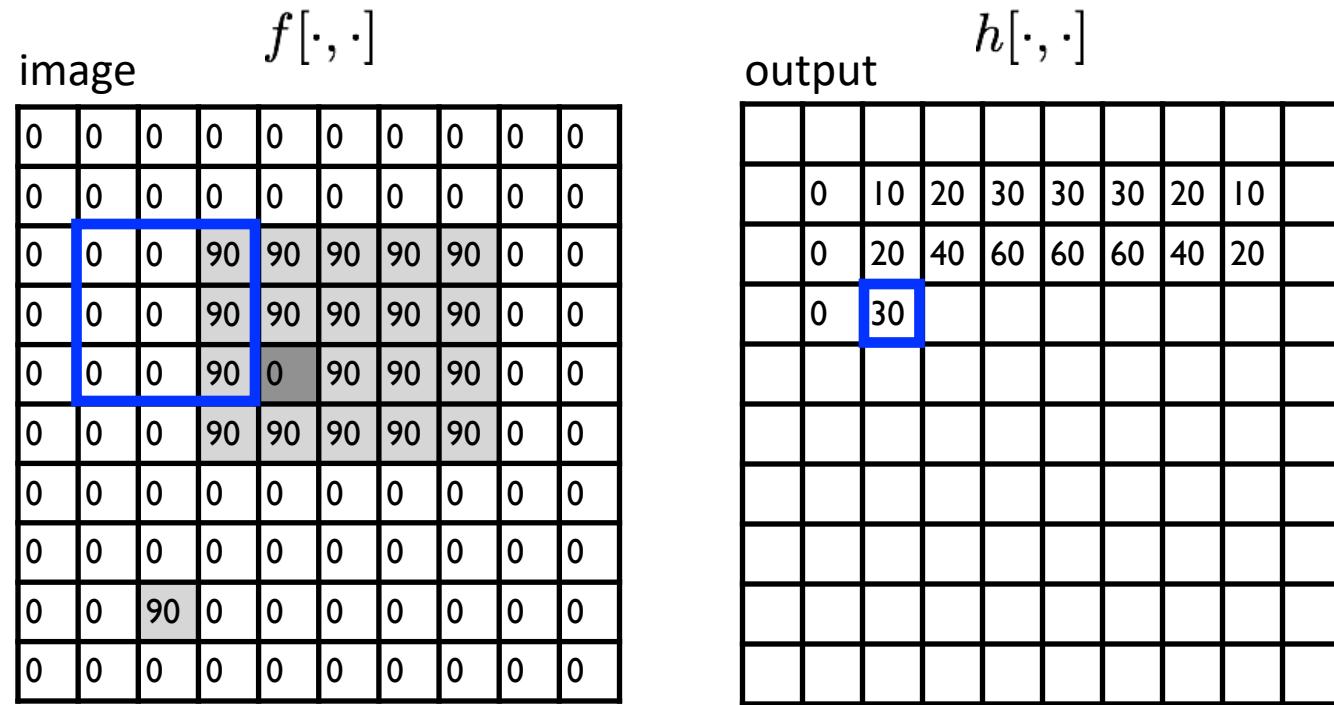
output k, l filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$



$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output k, l filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1

image	$f[\cdot, \cdot]$	output	$h[\cdot, \cdot]$
0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0	0 10 20 30 30 30 20 10	0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0	0 0 0 90 90 90 90 90 0 0	0 20 40 60 60 60 40 20	0 0 0 0 0 0 0 0 0 0
0 0 0 90 90 90 90 90 0 0	0 0 0 90 90 90 90 90 0 0	0 30 50 80 80 90 60 30	0 0 0 0 0 0 0 0 0 0
0 0 0 90 0 90 90 90 90 0 0	0 0 0 90 90 90 90 90 0 0	0 30 50 80 80 90 60 30	0 0 0 0 0 0 0 0 0 0
0 0 0 90 90 90 90 90 90 0 0	0 0 0 90 90 90 90 90 0 0	0 20 30 50 50 60 40 20	0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0	0 10 20 30 30 30 20 10	0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0	10 10 10 10 0 0 0 0	0 0 0 0 0 0 0 0 0 0
0 0 90 0 0 0 0 0 0 0	0 0 0 90 0 0 0 0 0 0	10 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0

$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

Let's run the box filter

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9}$$

1	1	1
1	1	1
1	1	1

image	$f[\cdot, \cdot]$	output	$h[\cdot, \cdot]$
0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0	0 10 20 30 30 30 20 10	0 20 40 60 60 60 40 20
0 0 0 90 90 90 90 90 0 0	0 0 0 90 90 90 90 90 0 0	0 30 50 80 80 90 60 30	0 30 50 80 80 90 60 30
0 0 0 90 90 90 90 90 0 0	0 0 0 90 0 90 90 90 0 0	0 20 30 50 50 60 40 20	0 20 30 50 50 60 40 20
0 0 0 90 0 90 90 90 90 0 0	0 0 0 90 90 90 90 90 0 0	0 10 20 30 30 30 20 10	0 10 10 10 10 0 0 0
0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0	10 10 10 10 0 0 0 0	10 10 10 10 0 0 0 0
0 0 90 0 0 0 0 0 0 0	0 0 90 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0

$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

... and the result is

$$g[\cdot, \cdot]$$

kernel

$$\frac{1}{9} \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

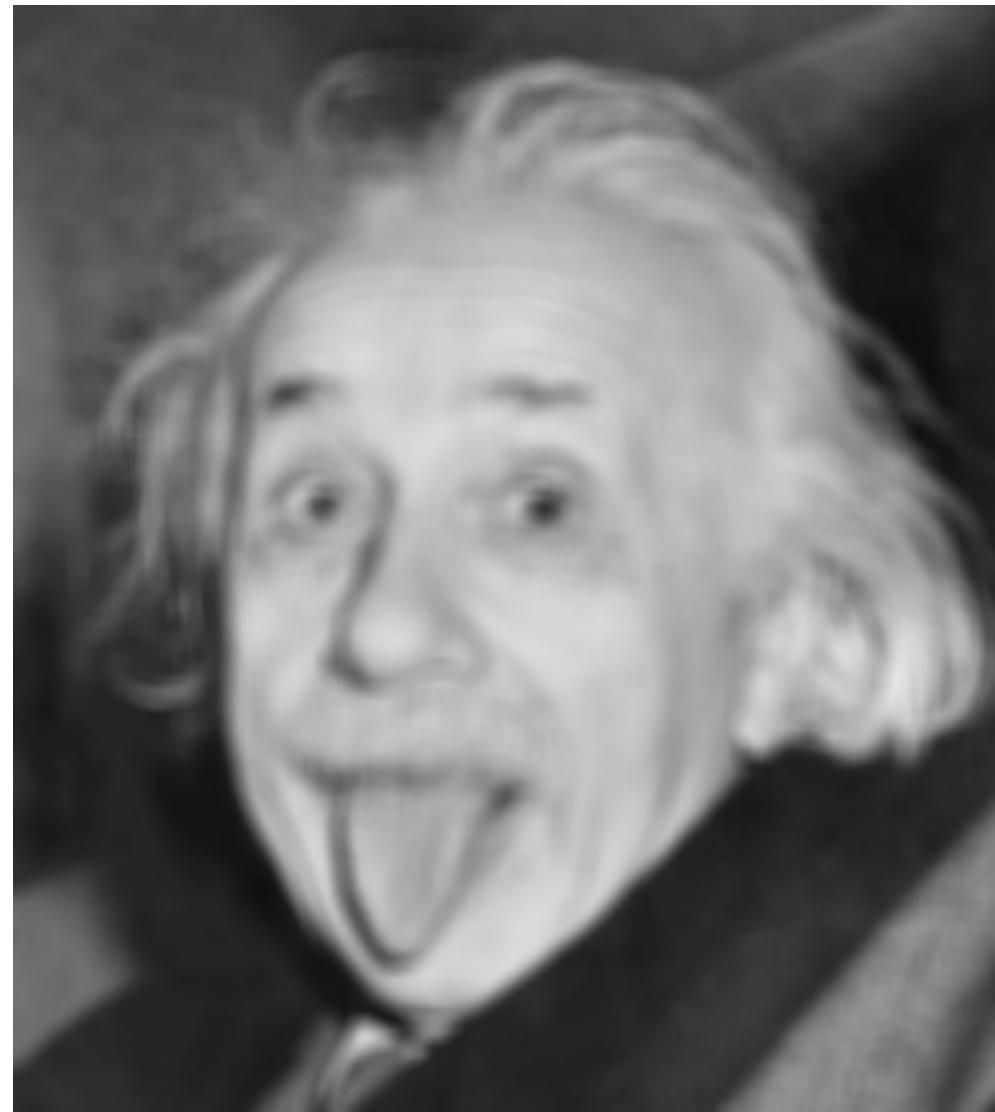
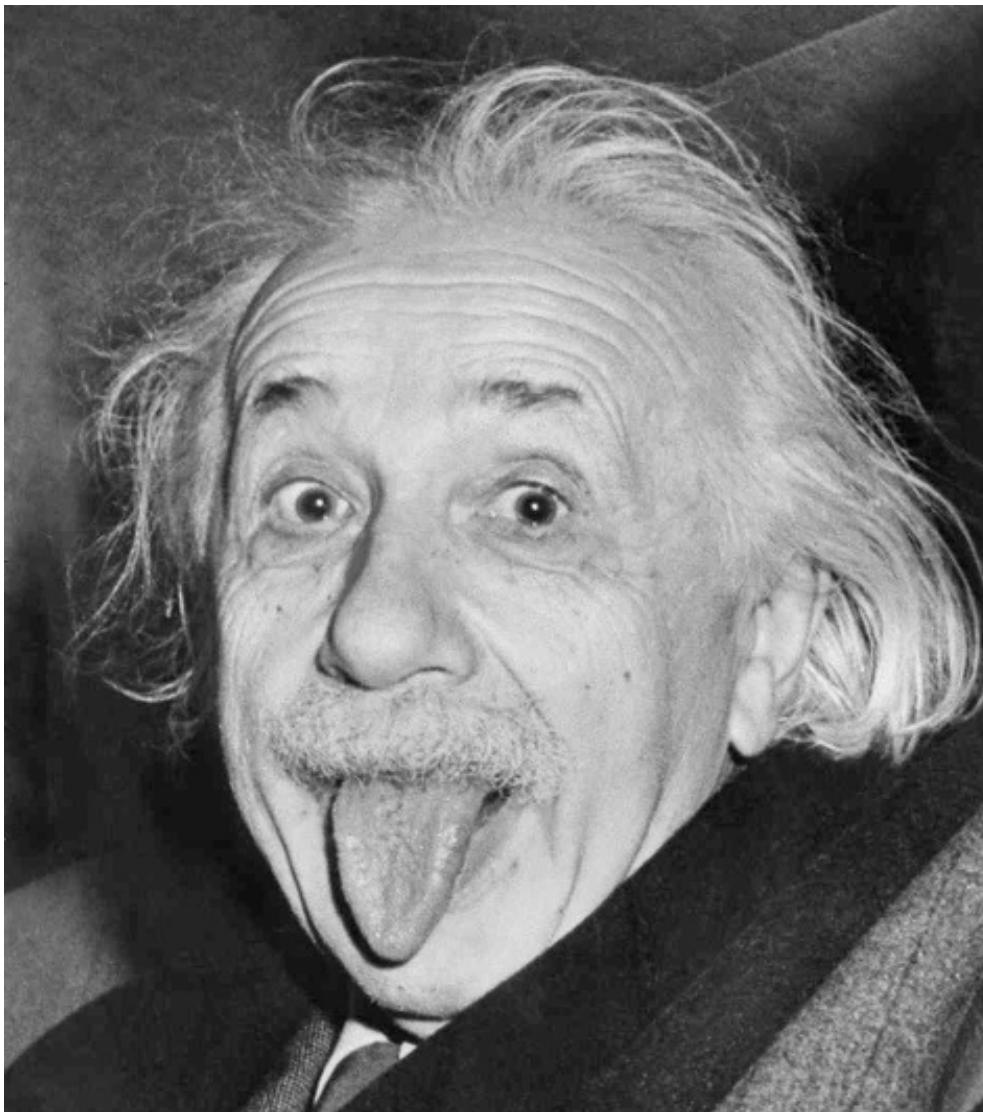
image	$f[\cdot, \cdot]$
0	0 0 0 0 0 0 0 0 0
0	0 0 0 0 0 0 0 0 0
0	0 0 0 90 90 90 90 90 0
0	0 0 0 90 90 90 90 90 0
0	0 0 0 90 0 90 90 90 0
0	0 0 0 90 90 90 90 90 0
0	0 0 0 0 0 0 0 0 0
0	0 0 0 0 0 0 0 0 0
0	0 0 90 0 0 0 0 0 0
0	0 0 0 0 0 0 0 0 0

output	$h[\cdot, \cdot]$
0	10 20 30 30 30 20 10
0	20 40 60 60 60 40 20
0	30 50 80 80 90 60 30
0	30 50 80 80 90 60 30
0	20 30 50 50 60 40 20
0	10 20 30 30 30 20 10
10	10 10 10 10 0 0 0
10	10 10 10 10 0 0 0

$$h[m, n] = \sum_{k,l} g[k, l] f[m + k, n + l]$$

output filter image (signal)

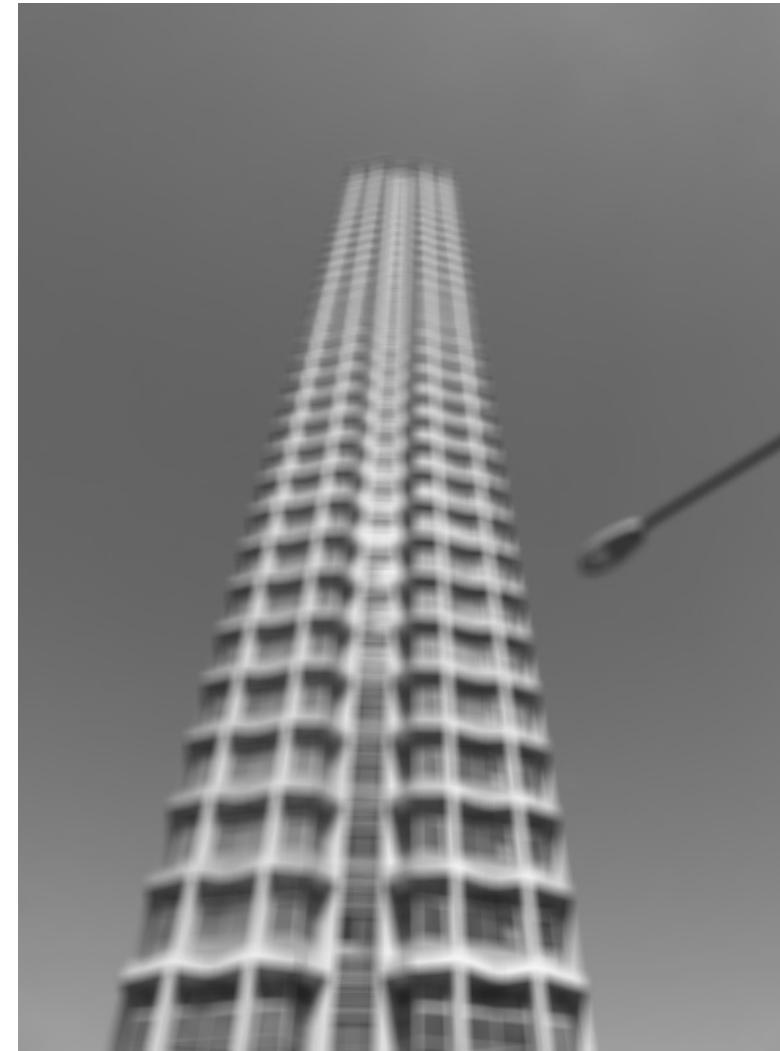
Some more realistic examples



Some more realistic examples



Some more realistic examples



Convolution

Convolution for 1D continuous signals

Definition of filtering as convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$

filtered signal filter input signal notice the flip

Convolution for 1D continuous signals

Definition of filtering as convolution:

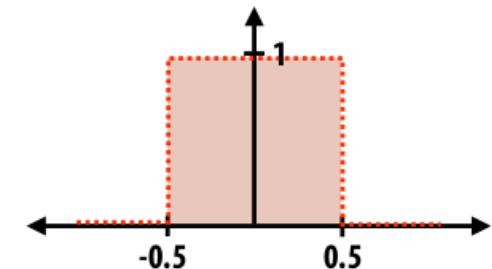
$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$

filtered signal filter input signal notice the flip

Consider the box filter example:

1D continuous
box filter

$$f(x) = \begin{cases} 1 & |x| \leq 0.5 \\ 0 & otherwise \end{cases}$$



filtering output is a
blurred version of g

$$(f * g)(x) = \int_{-0.5}^{0.5} g(x - y)dy$$

Convolution for 2D discrete signals

Definition of filtering as convolution:

$$(f * g)(x, y) = \sum_{i,j=-\infty}^{\infty} f(i, j)I(x - i, y - j)$$

filtered image filter input image notice the flip

Convolution for 2D discrete signals

Definition of filtering as convolution:

$$(f * g)(x, y) = \sum_{i,j=-\infty}^{\infty} f(i, j)I(x - i, y - j)$$

filtered image filter input image notice the flip

If the filter $f(i, j)$ is non-zero only within $-1 \leq i, j \leq 1$,
then

$$(f * g)(x, y) = \sum_{i,j=-1}^1 f(i, j)I(x - i, y - j)$$

The kernel we saw earlier is the 3x3 matrix representation of $f(i, j)$.

Convolution vs correlation

Definition of filtering as convolution:

$$(f * g)(x, y) = \sum_{i,j=-\infty}^{\infty} f(i, j)I(x - i, y - j)$$



notice the flip

Definition of filtering as correlation:

$$(f * g)(x, y) = \sum_{i,j=-\infty}^{\infty} f(i, j)I(x + i, y + j)$$



notice the lack of a flip

- Most of the time won't matter, because our kernels will be symmetric.
- Will be important when we discuss frequency-domain filtering (lectures 5-6).

Separable filters

A 2D filter is separable if it can be written as the product of a “column” and a “row”.

example:
box filter

1	1	1
1	1	1
1	1	1

=

1
1
1

*

1	1	1
---	---	---

row

column

What is the rank of this filter matrix?

Separable filters

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1	1	1
1	1	1

=

1
1
1

*

1	1	1
---	---	---

row

column

Why is this important?

Separable filters

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*

1	1	1
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column

2D convolution with a separable filter is equivalent to two 1D convolutions (with the “column” and “row” filters).

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2D convolution with a separable filter is equivalent to two 1D convolutions (with the “column” and “row” filters).

If the image has $M \times M$ pixels and the filter kernel has size $N \times N$:

- What is the cost of convolution with a non-separable filter?

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---	---	---

row

column

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If the image has $M \times M$ pixels and the filter kernel has size $N \times N$:

- What is the cost of convolution with a non-separable filter? $\longrightarrow M^2 \times N^2$
- What is the cost of convolution with a separable filter?

Separable filters

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1
1

*

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---	---	---

row

column

2D convolution with a separable filter is equivalent to two 1D convolutions (with the “column” and “row” filters).

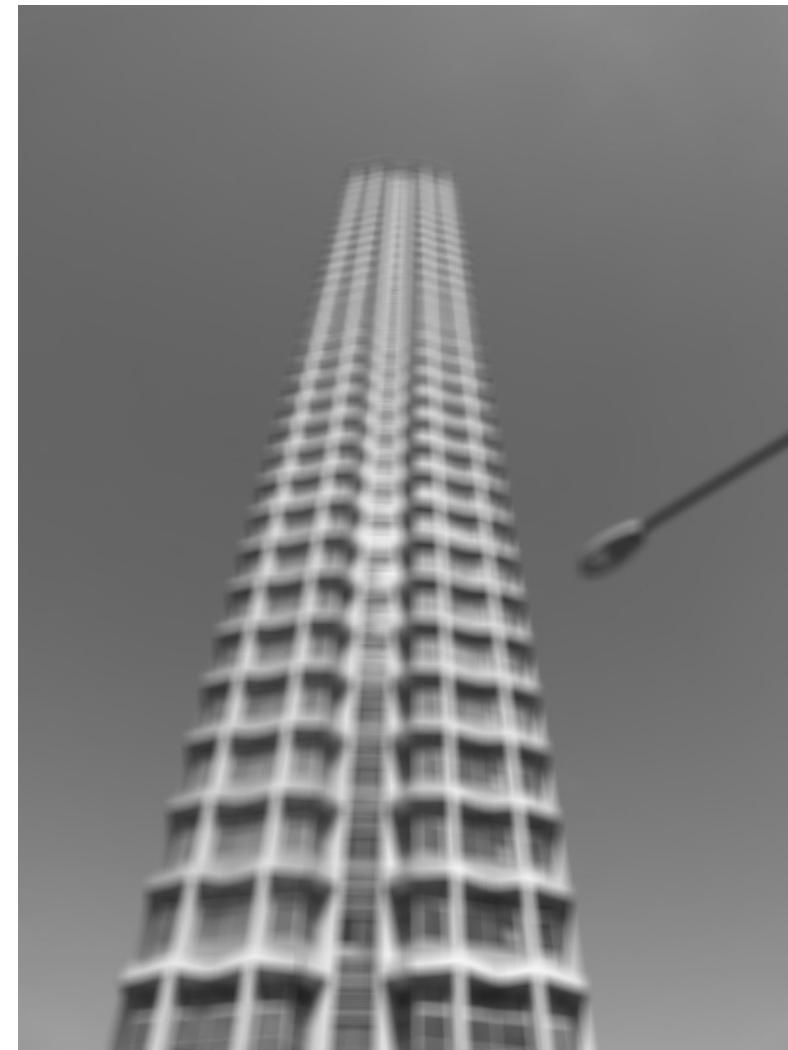
If the image has $M \times M$ pixels and the filter kernel has size $N \times N$:

- What is the cost of convolution with a non-separable filter? $\longrightarrow M^2 \times N^2$
- What is the cost of convolution with a separable filter? $\longrightarrow 2 \times N \times M^2$

A few more filters



original



3x3 box filter

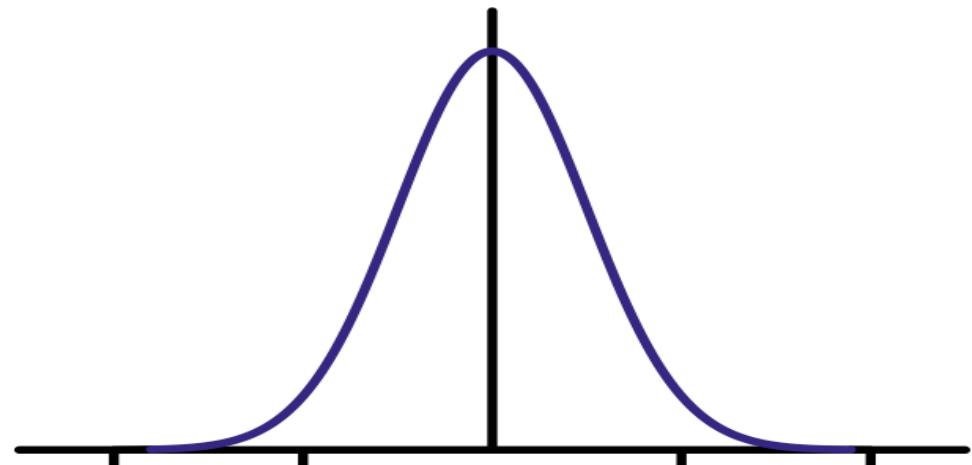
do you see
any problems
in this image?

The Gaussian filter

- named (like many other things) after Carl Friedrich Gauss
- kernel values sampled from the 2D Gaussian function:

$$f(i, j) = \frac{1}{2\pi\sigma^2} e^{-\frac{i^2+j^2}{2\sigma^2}}$$

- weight falls off with distance from center pixel
- theoretically infinite, in practice truncated to some maximum distance



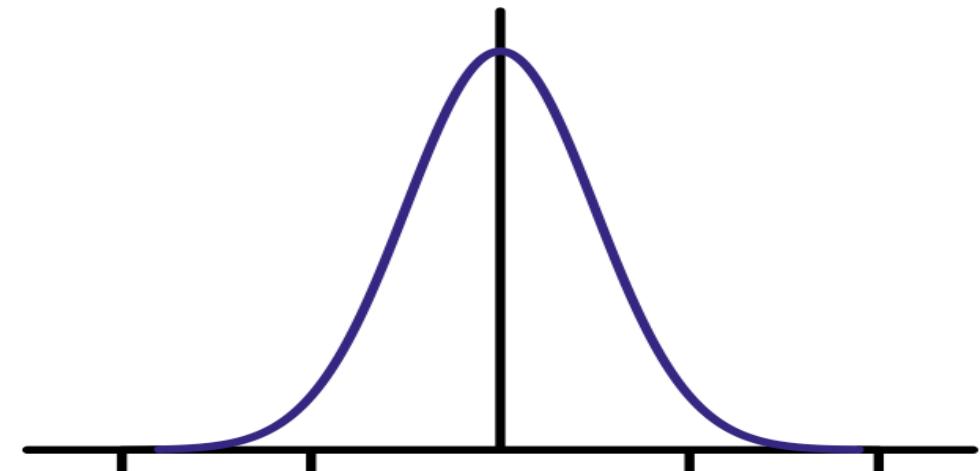
Any heuristics for selecting where to truncate?

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Is this a separable filter?

kernel $\frac{1}{16}$

1	2	1
2	4	2
1	2	1

Any heuristics for selecting where to truncate?

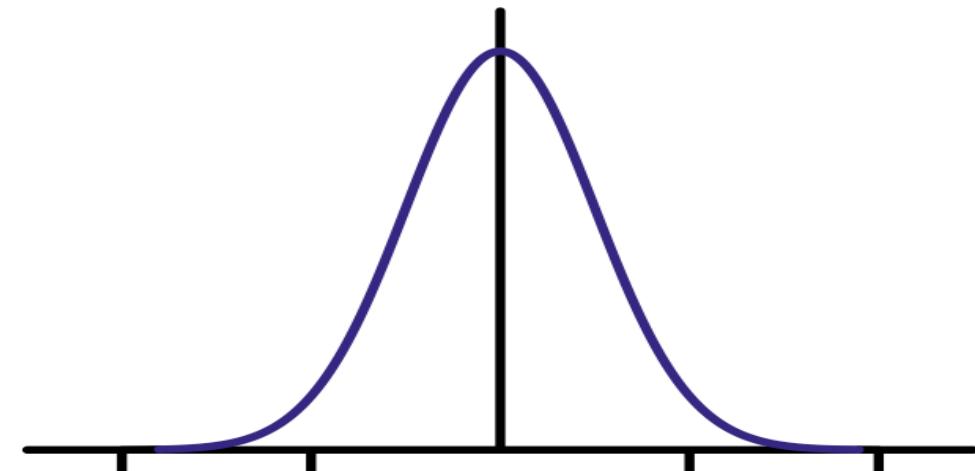
- usually at $2-3\sigma$

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Is this a separable filter? Yes!

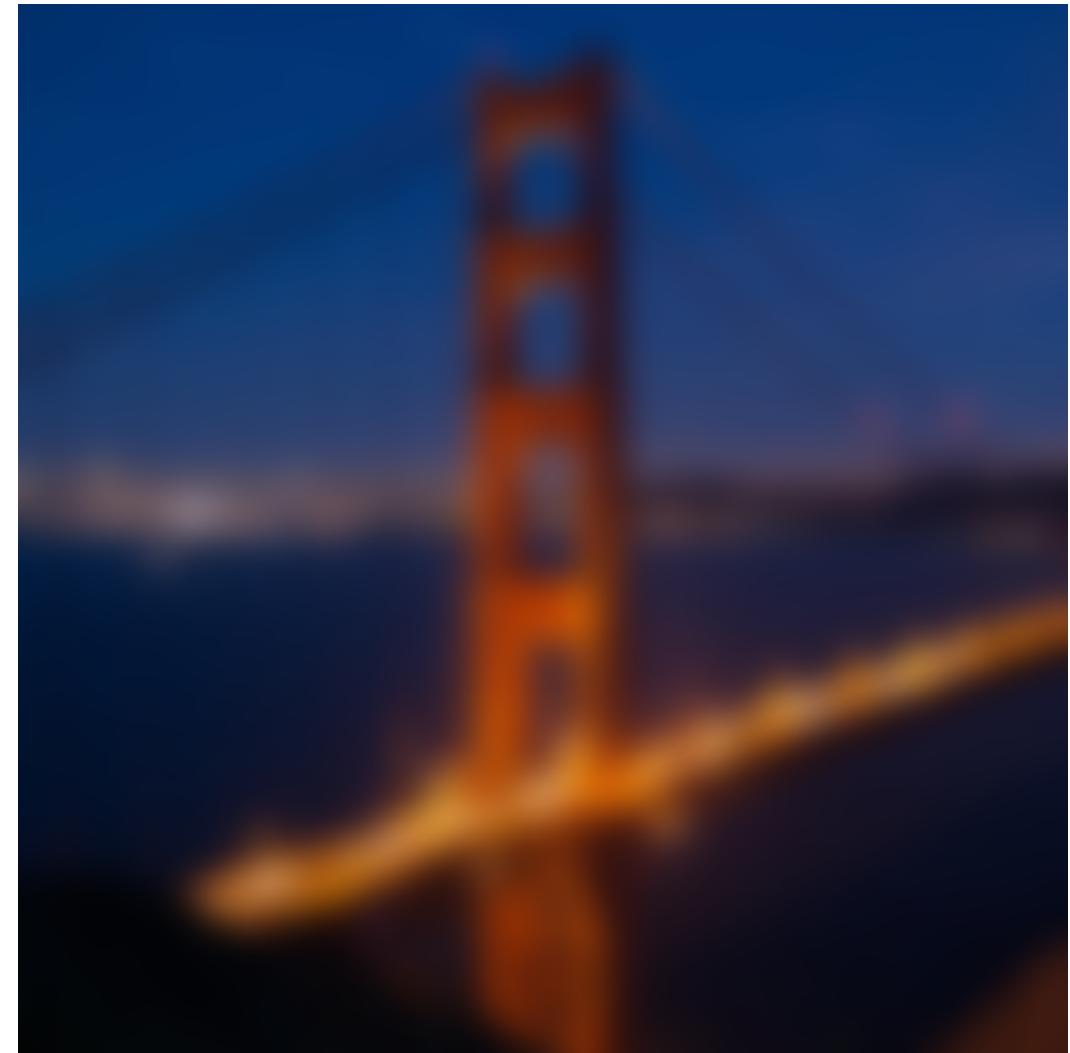
kernel $\frac{1}{16}$

1	2	1
2	4	2
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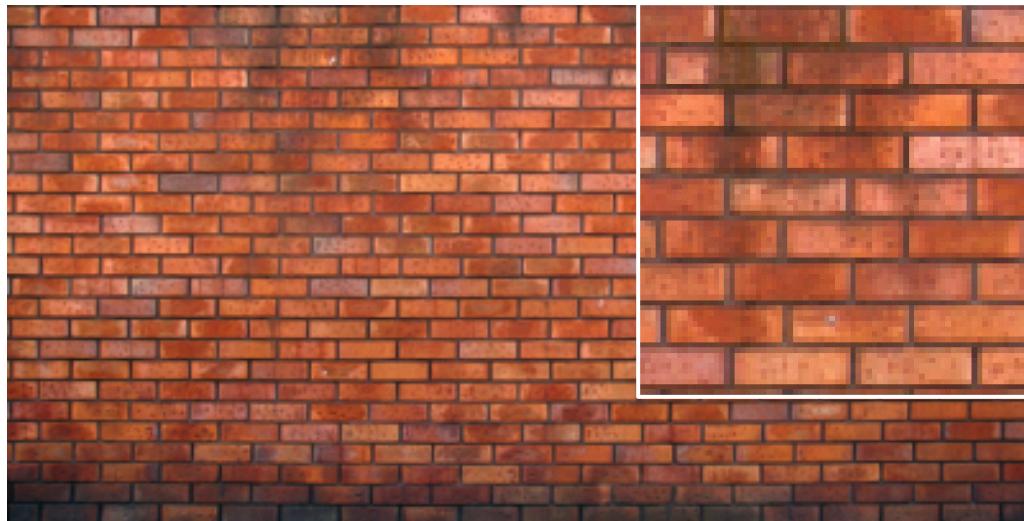
Any heuristics for selecting where to truncate?

- usually at $2-3\sigma$

Gaussian filtering example

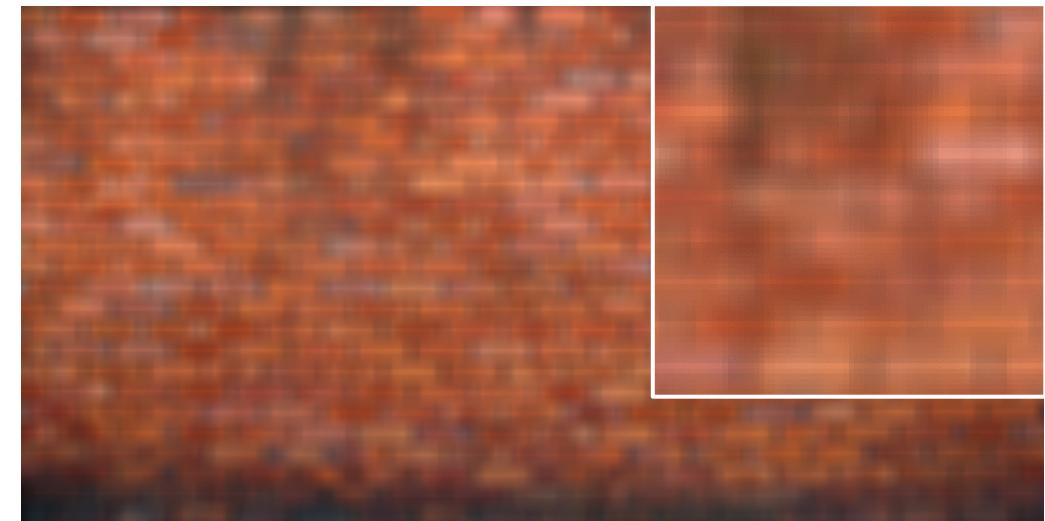
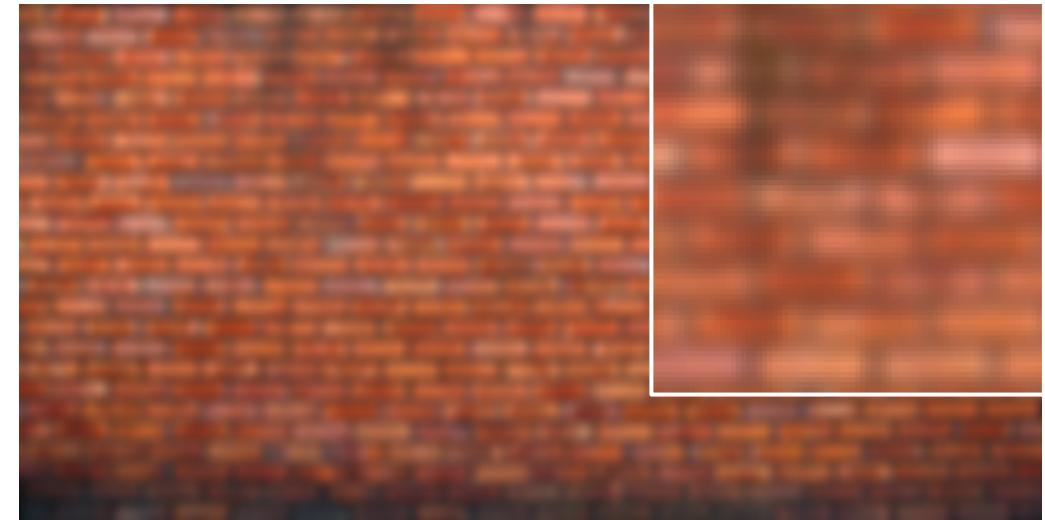


Gaussian vs box filtering

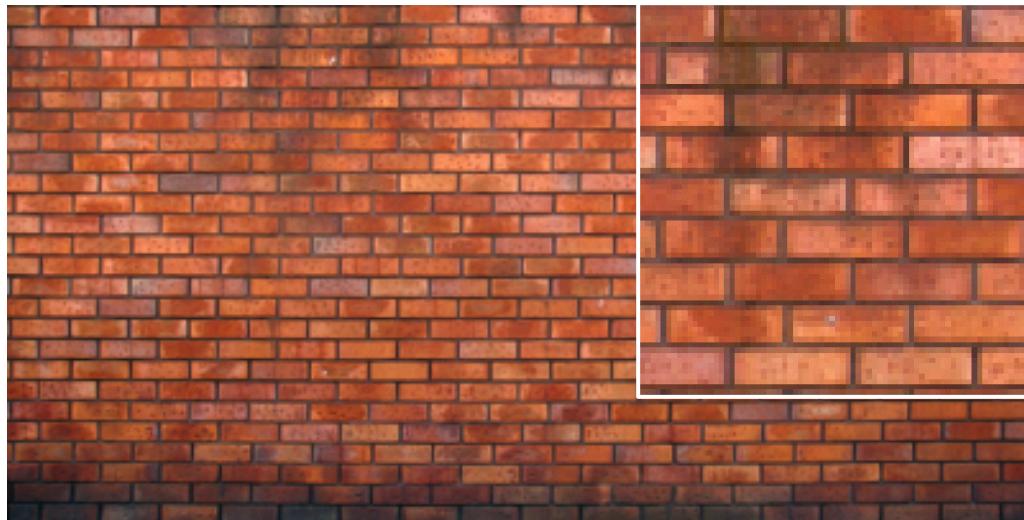


original

Which blur do you like better?



Gaussian vs box filtering

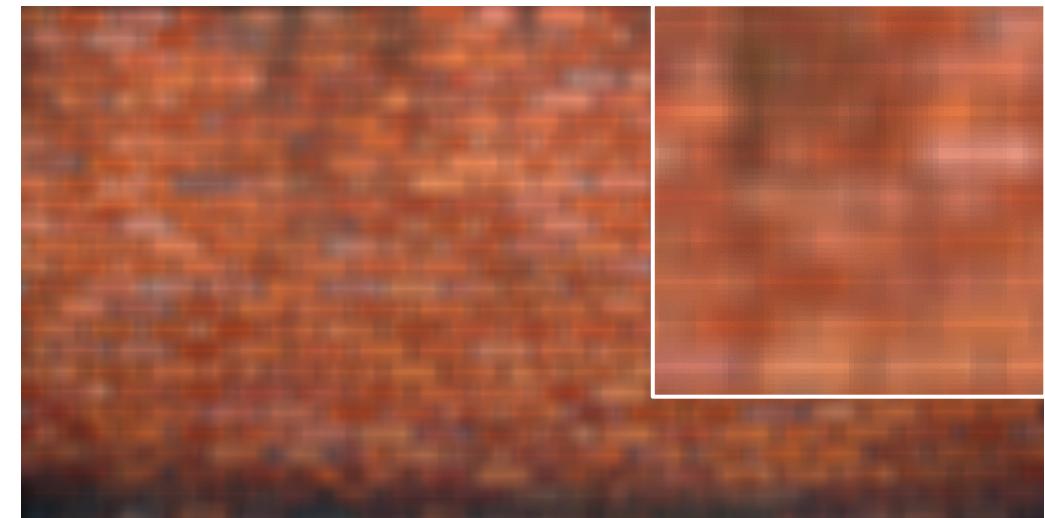


original

Which blur do you like better?

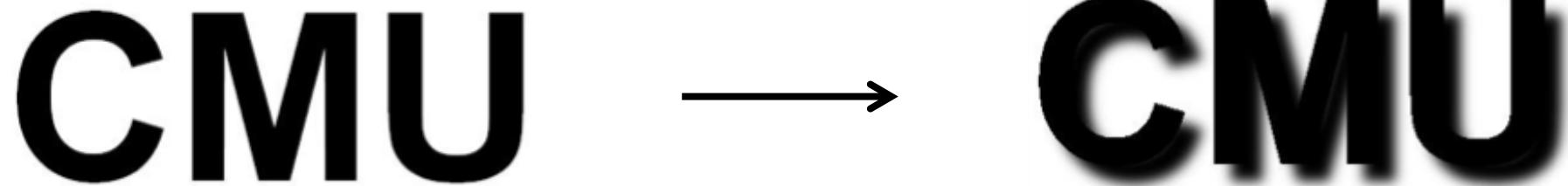


7x7 Gaussian

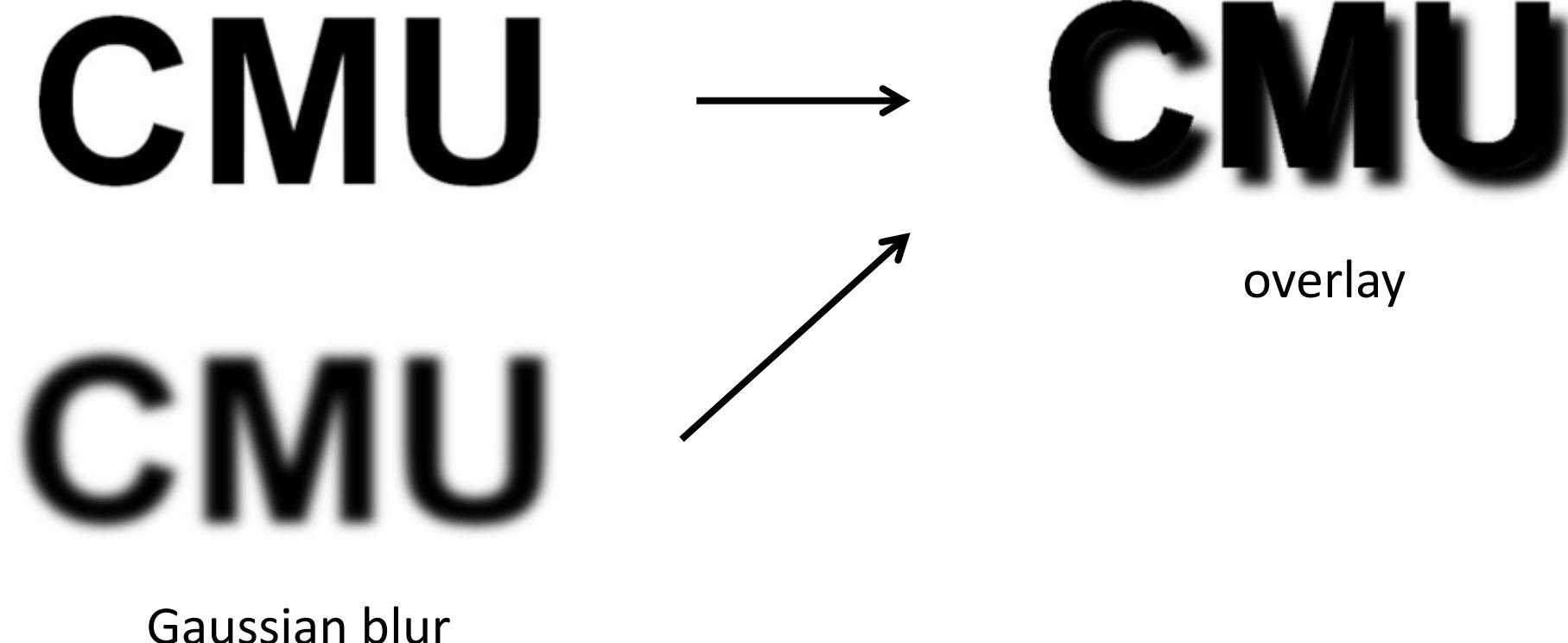


7x7 box

How would you create a soft shadow effect?



How would you create a soft shadow effect?



Other filters

input



filter

0	0	0
0	1	0
0	0	0

output

?

Other filters

input



filter

0	0	0
0	1	0
0	0	0

output



unchanged

Other filters

input



filter

0	0	0
0	1	0
0	0	0

output



unchanged

input



filter

0	0	0
0	0	1
0	0	0

output

?

Other filters

input



filter

0	0	0
0	1	0
0	0	0

output



unchanged

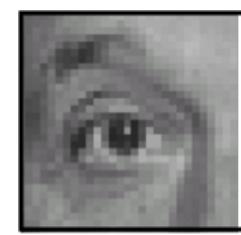
input



filter

0	0	0
0	0	1
0	0	0

output



shift to left
by one

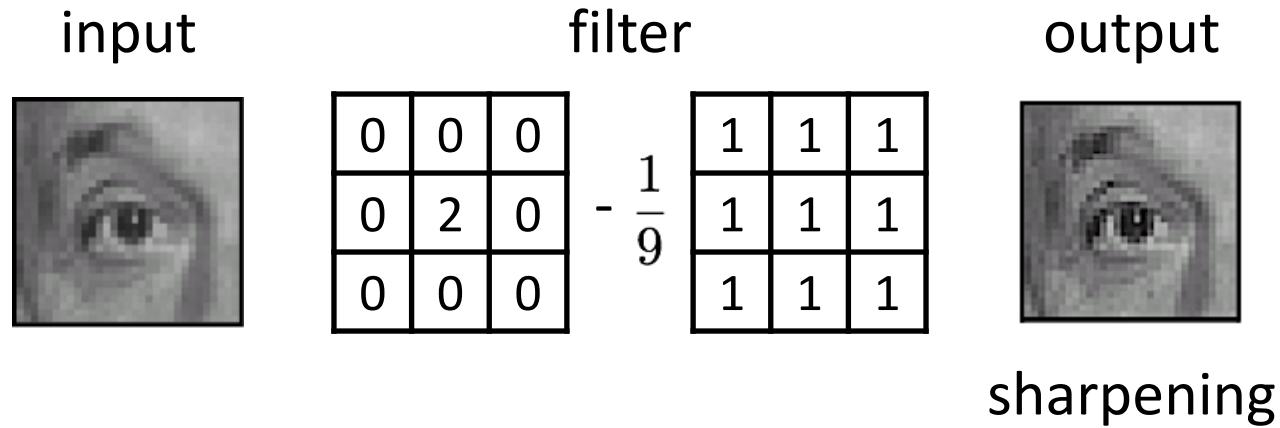
Other filters

input filter output


$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 2 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} - \frac{1}{9} \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

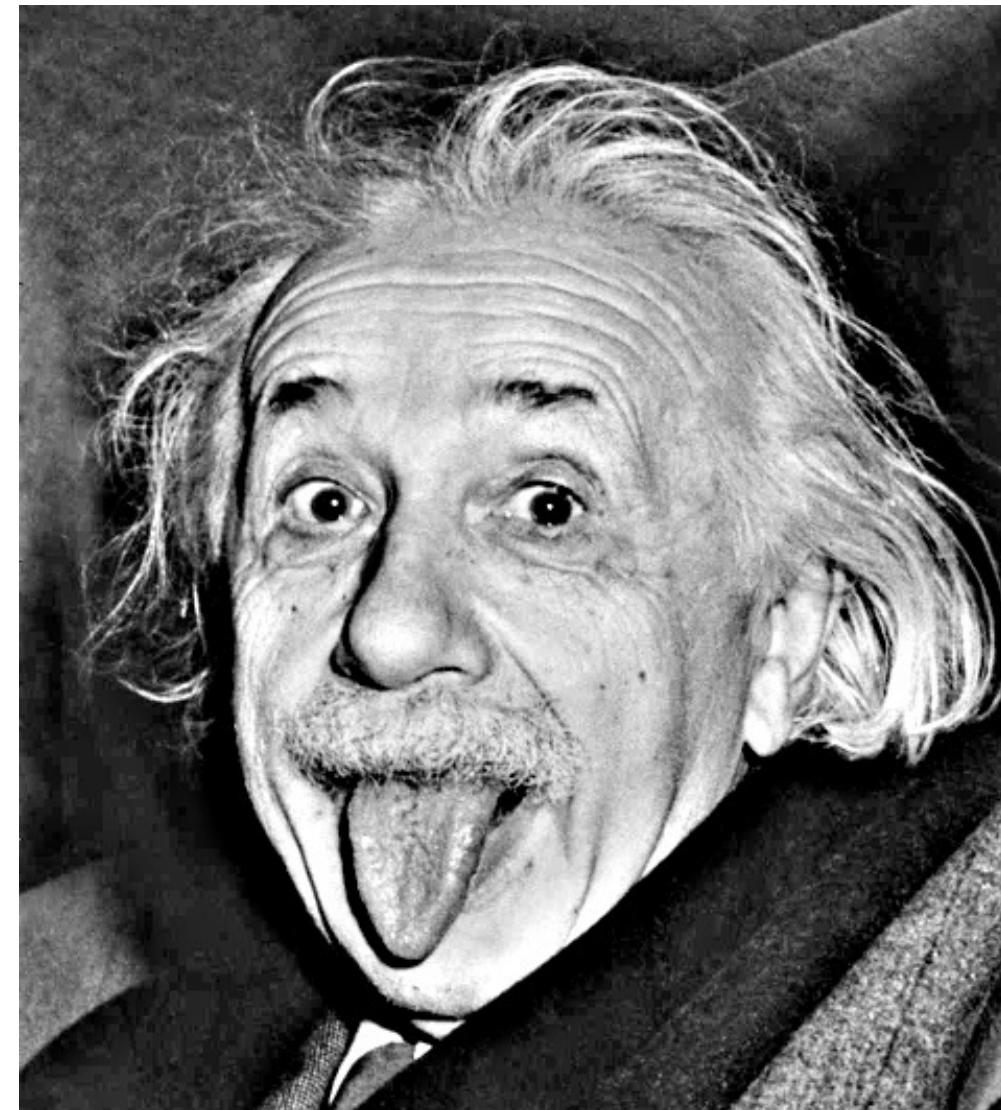
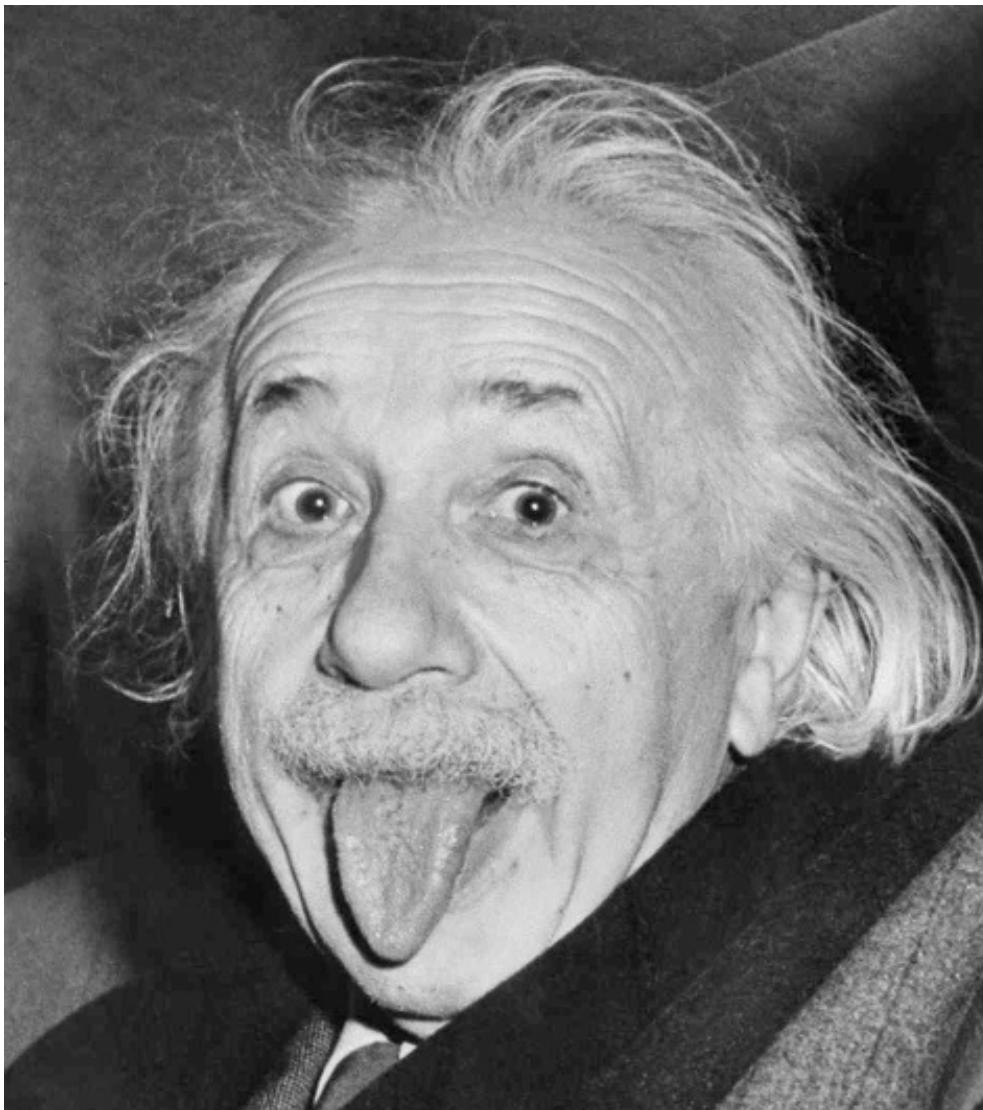
?

Other filters



- do nothing for flat areas
- stress intensity peaks

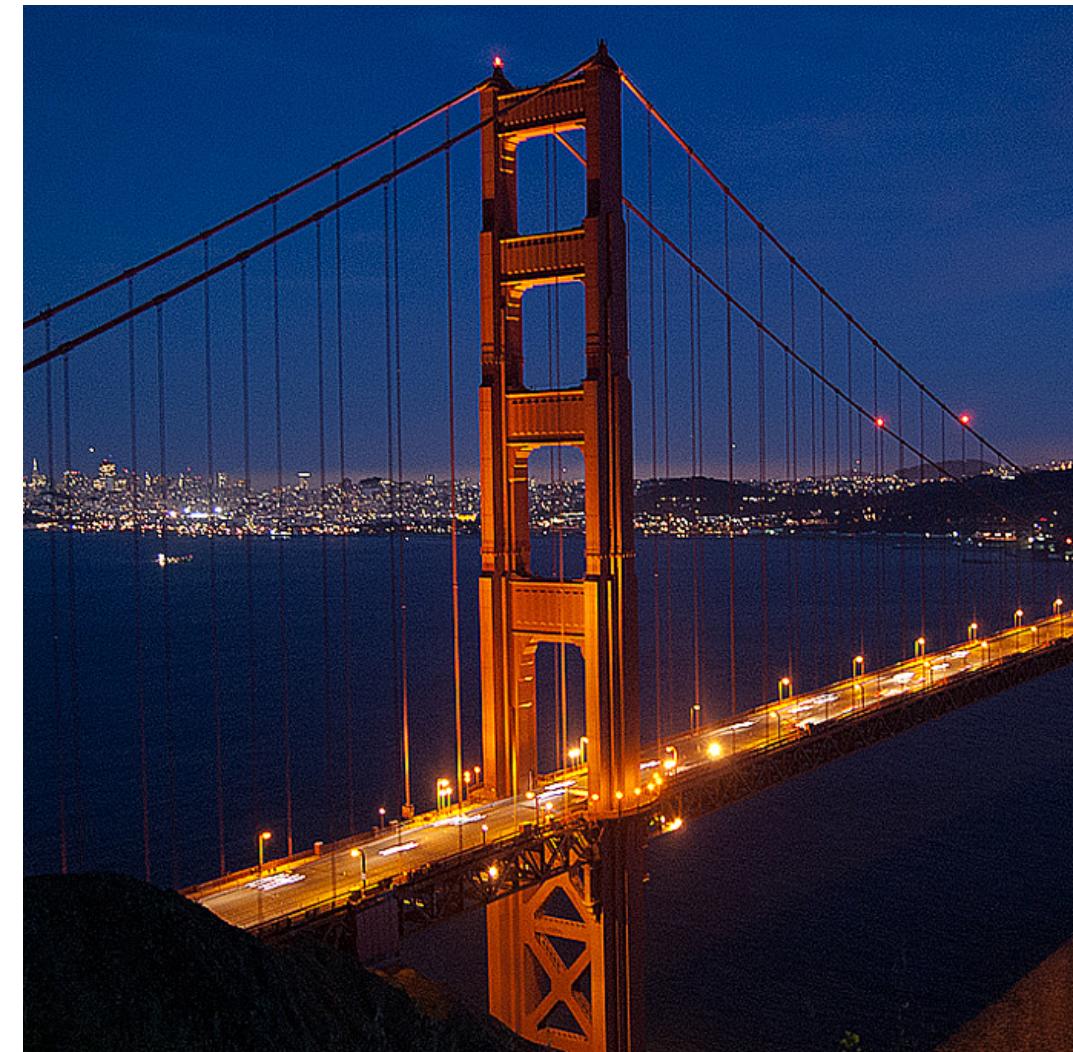
Sharpening examples



Sharpening examples



Sharpening examples



Sharpening examples



do you see
any problems
in this image?

Do not overdo it with sharpening



original



sharpened



oversharpened

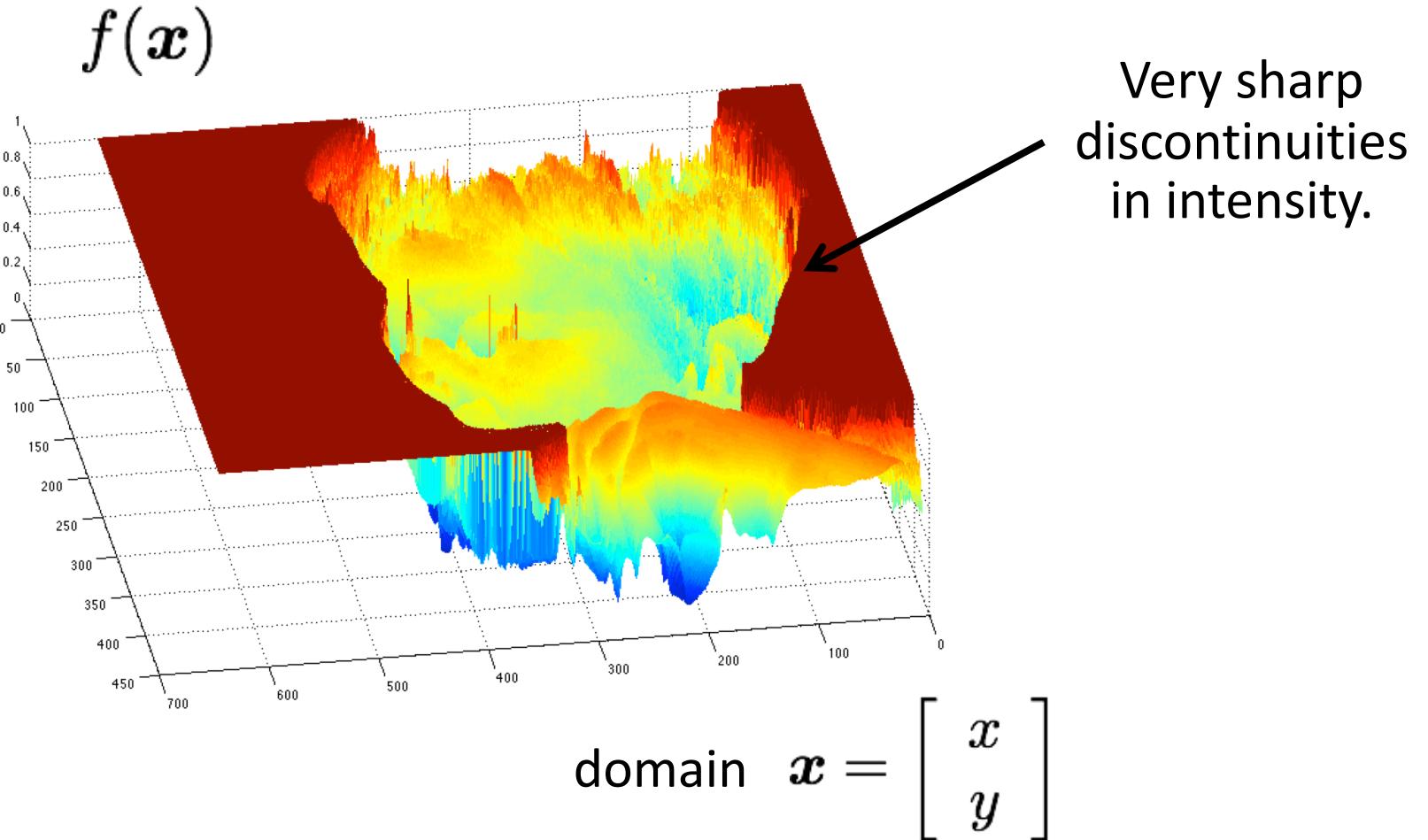
What is wrong in this image?

Image gradients

What are image edges?



grayscale image



Detecting edges

How would you go about detecting edges in an image (i.e., discontinuities in a function)?

Detecting edges

How would you go about detecting edges in an image (i.e., discontinuities in a function)?

- ✓ You take derivatives: derivatives are large at discontinuities.

How do you differentiate a discrete image (or any other discrete signal)?

Detecting edges

How would you go about detecting edges in an image (i.e., discontinuities in a function)?

- ✓ You take derivatives: derivatives are large at discontinuities.

How do you differentiate a discrete image (or any other discrete signal)?

- ✓ You use finite differences.

Finite differences

High-school reminder: definition of a derivative using forward difference

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Finite differences

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Alternative: use central difference

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + 0.5h) - f(x - 0.5h)}{h}$$

For discrete signals: Remove limit and set $h = 2$

$$f'(x) = \frac{f(x + 1) - f(x - 1)}{2}$$

What convolution kernel does this correspond to?

Finite differences

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<table border="1"><tr><td>-1</td><td>0</td><td>1</td></tr></table>	-1	0	1	?
-1	0	1		
<table border="1"><tr><td>1</td><td>0</td><td>-1</td></tr></table>	1	0	-1	?
1	0	-1		

Finite differences

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1D derivative filter

1	0	-1
---	---	----

The Sobel filter

1	0	-1
2	0	-2
1	0	-1

Sobel filter

=

1
2
1

*

1	0	-1
---	---	----

1D derivative
filter

What filter
is this?

The Sobel filter

1	0	-1
2	0	-2
1	0	-1

Sobel filter

=

1
2
1

Blurring

1	0	-1
---	---	----

1D derivative
filter

Does this filter return large responses on vertical or horizontal lines?

The Sobel filter

Horizontal Sober filter:

$$\begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 0 & -1 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline \end{array}$$

What does the vertical Sobel filter look like?

The Sobel filter

Horizontal Sober filter:

$$\begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 0 & -1 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline \end{array}$$

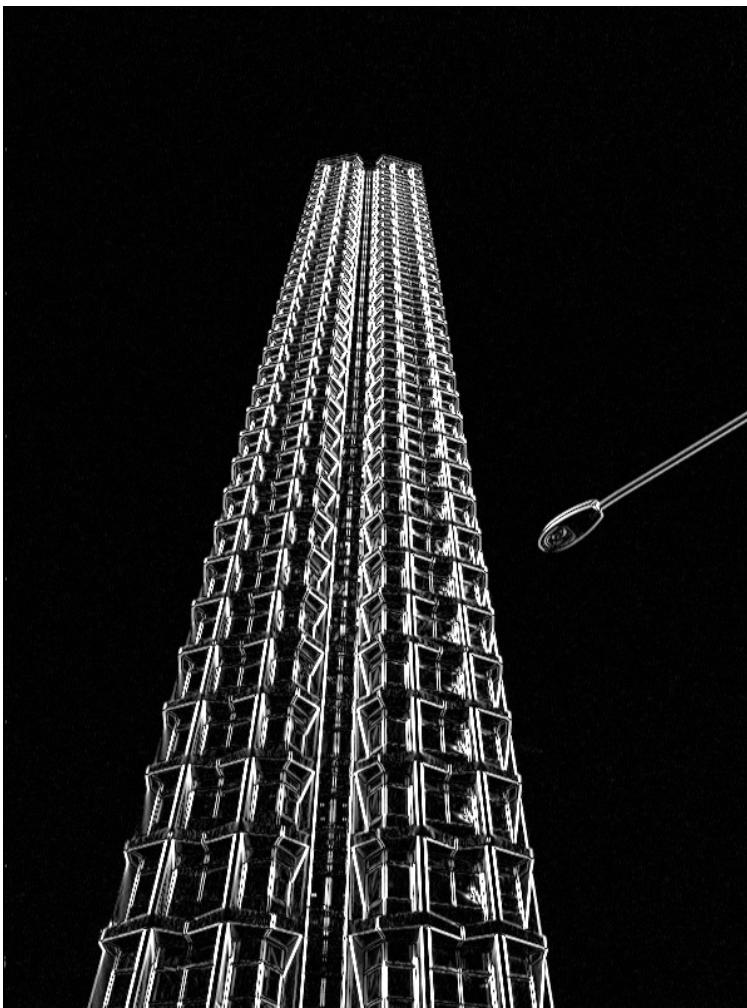
Vertical Sobel filter:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -2 & -1 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline -1 \\ \hline \end{array} * \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline \end{array}$$

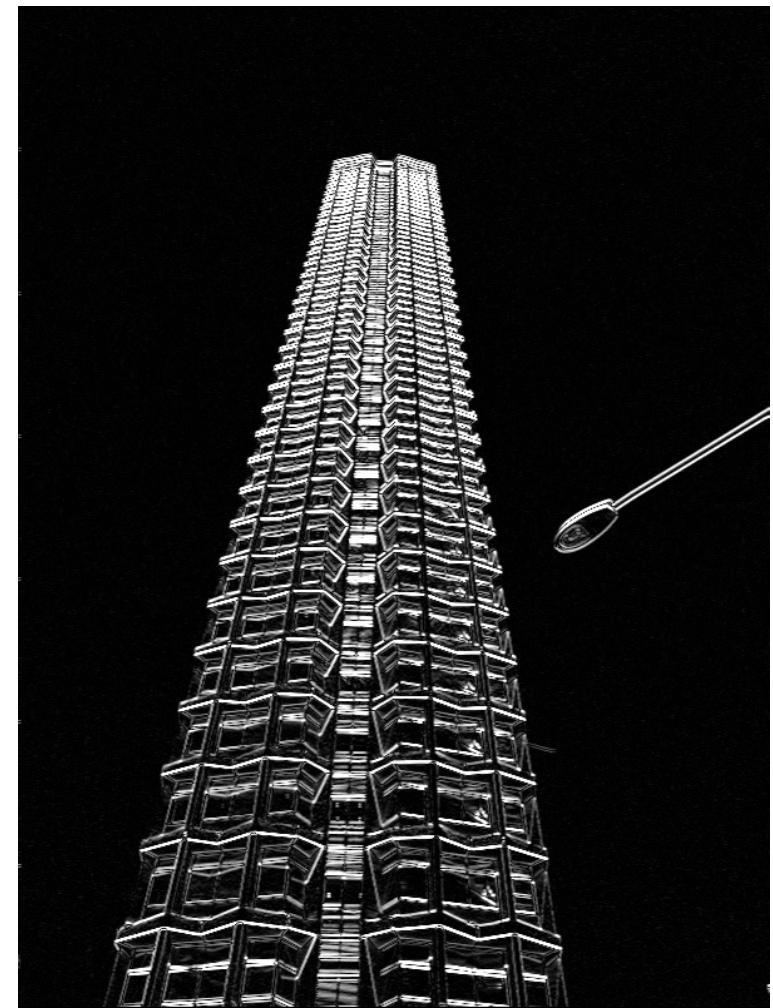
Sobel filter example



original



which Sobel filter?

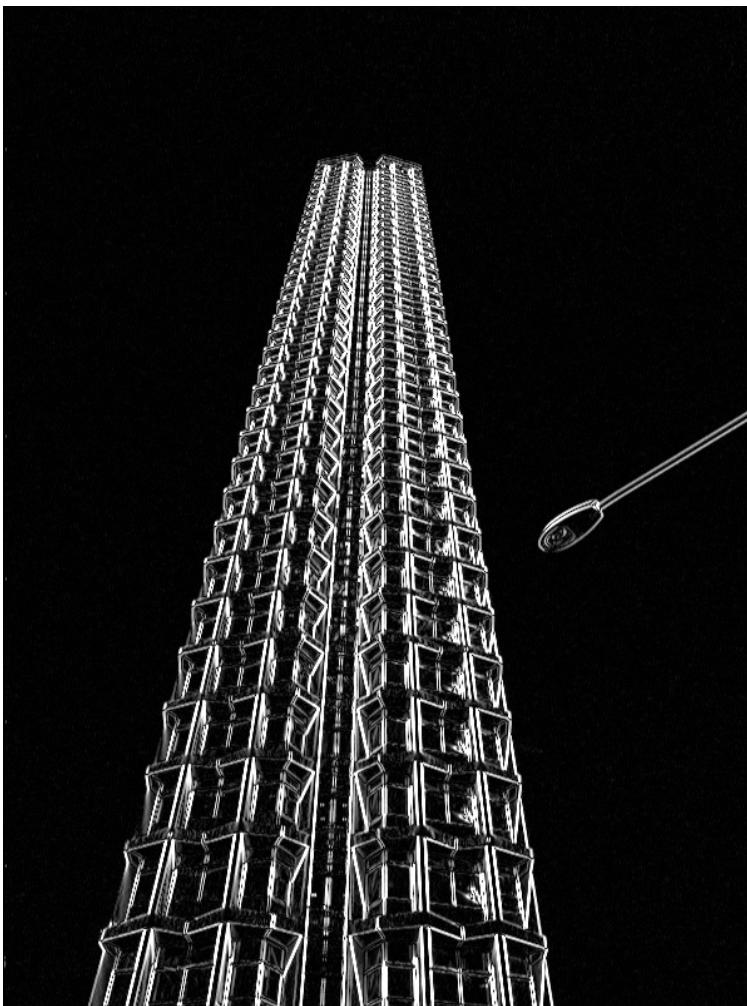


which Sobel filter?

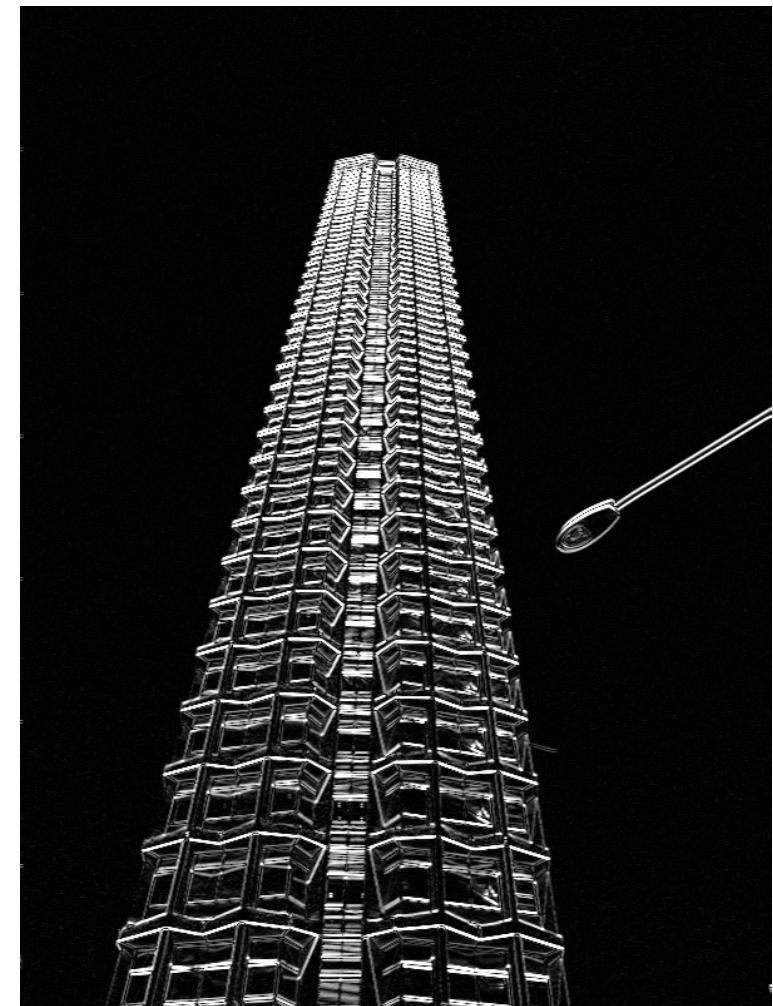
Sobel filter example



original

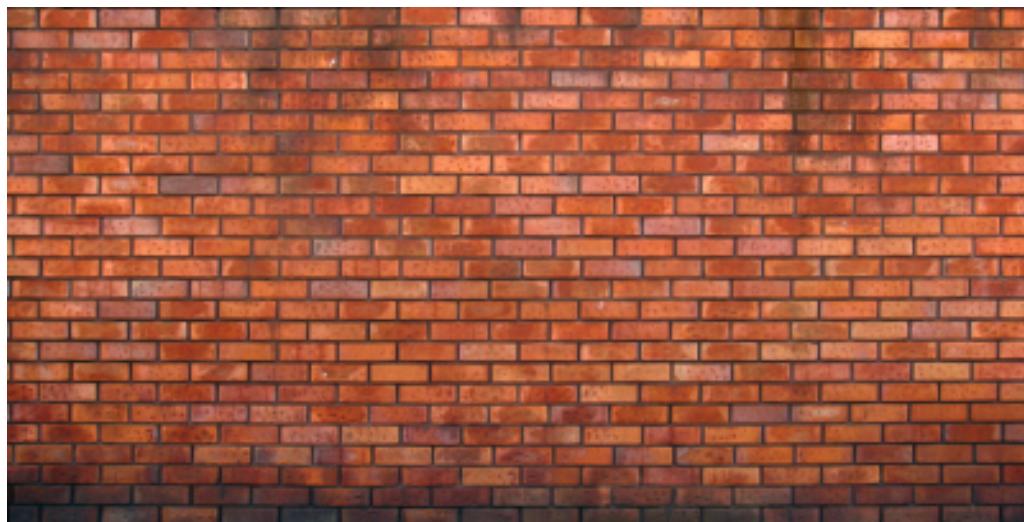


horizontal Sobel filter

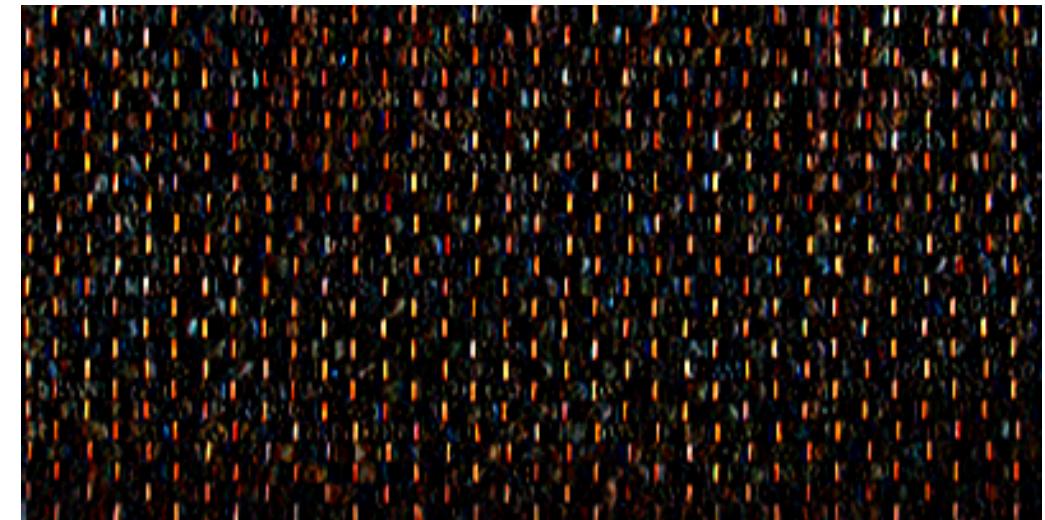


vertical Sobel filter

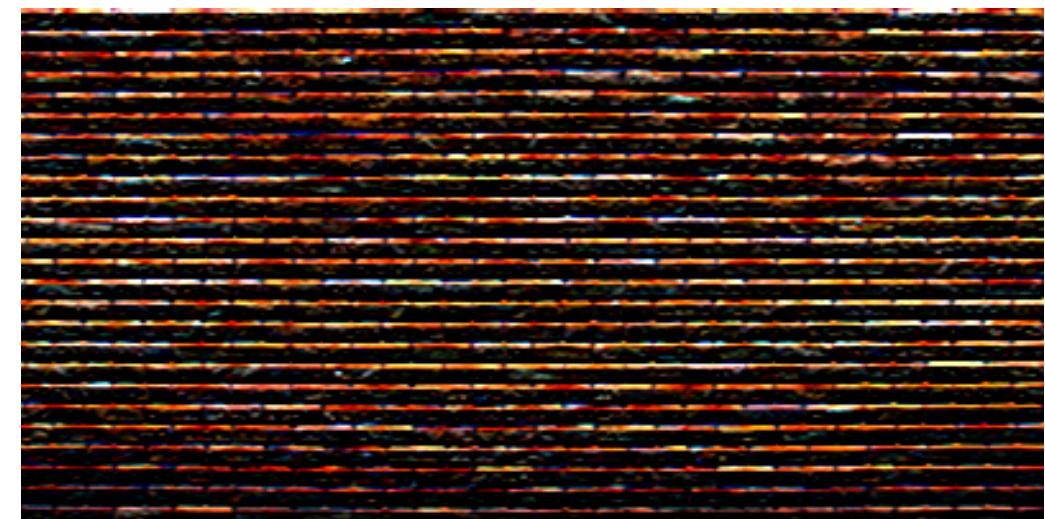
Sobel filter example



original



horizontal Sobel filter



vertical Sobel filter

Several derivative filters

Sobel

1	0	-1
2	0	-2
1	0	-1

1	2	1
0	0	0
-1	-2	-1

Scharr

3	0	-3
10	0	-10
3	0	-3

3	10	3
0	0	0
-3	-10	-3

Prewitt

1	0	-1
1	0	-1
1	0	-1

1	1	1
0	0	0
-1	-1	-1

Roberts

0	1
-1	0

1	0
0	-1

- How are the other filters derived and how do they relate to the Sobel filter?
- How would you derive a derivative filter that is larger than 3x3?

Computing image gradients

1. Select your favorite derivative filters.

$$\mathbf{S}_x = \begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline 2 & 0 & -2 \\ \hline 1 & 0 & -1 \\ \hline \end{array} \quad \mathbf{S}_y = \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -2 & -1 \\ \hline \end{array}$$

Computing image gradients

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2. Convolve with the image to compute derivatives.

$$\frac{\partial f}{\partial x} = \mathbf{S}_x \otimes f$$

$$\frac{\partial f}{\partial y} = \mathbf{S}_y \otimes f$$

Computing image gradients

1. Select your favorite derivative filters.

$$\mathbf{S}_x =$$

1	0	-1
2	0	-2
1	0	-1

$$\mathbf{S}_y =$$

1	2	1
0	0	0
-1	-2	-1

2. Convolve with the image to compute derivatives.

$$\frac{\partial \mathbf{f}}{\partial x} = \mathbf{S}_x \otimes \mathbf{f}$$

$$\frac{\partial \mathbf{f}}{\partial y} = \mathbf{S}_y \otimes \mathbf{f}$$

3. Form the image gradient, and compute its direction and amplitude.

$$\nabla \mathbf{f} = \left[\frac{\partial \mathbf{f}}{\partial x}, \frac{\partial \mathbf{f}}{\partial y} \right]$$

gradient

$$\theta = \tan^{-1} \left(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x} \right)$$

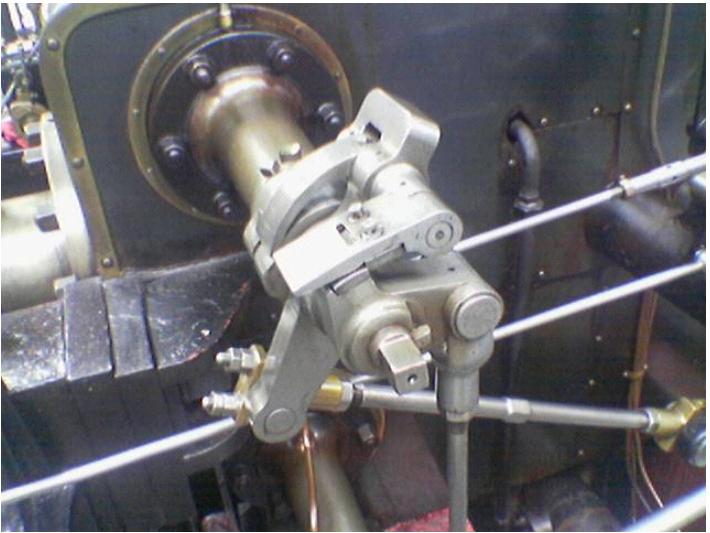
direction

$$\|\nabla f\| = \sqrt{\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2}$$

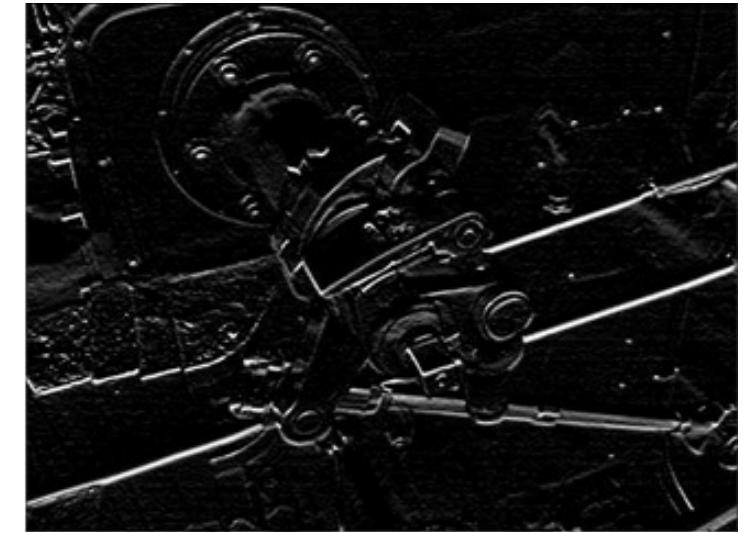
amplitude

Image gradient example

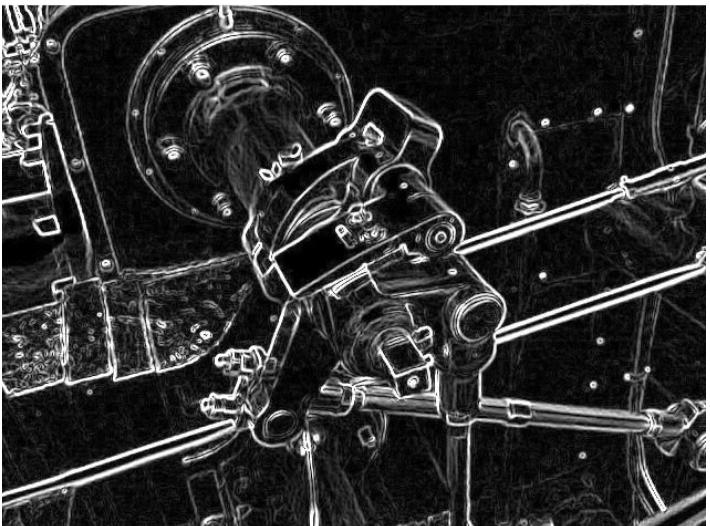
original



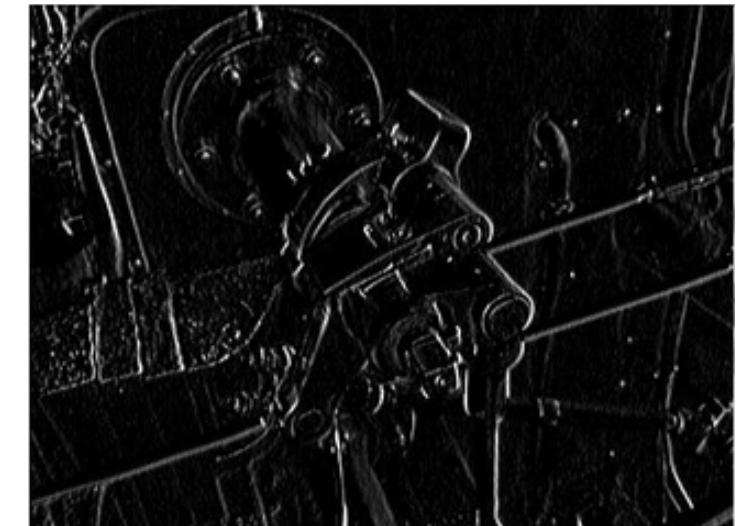
vertical derivative



gradient amplitude



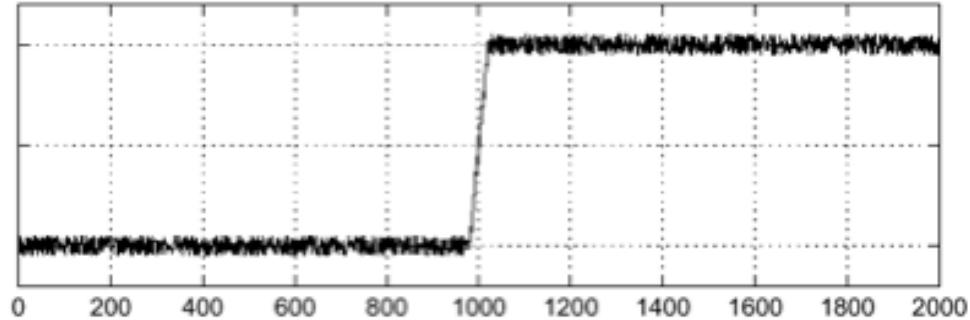
horizontal derivative



How does the gradient direction relate to these edges?

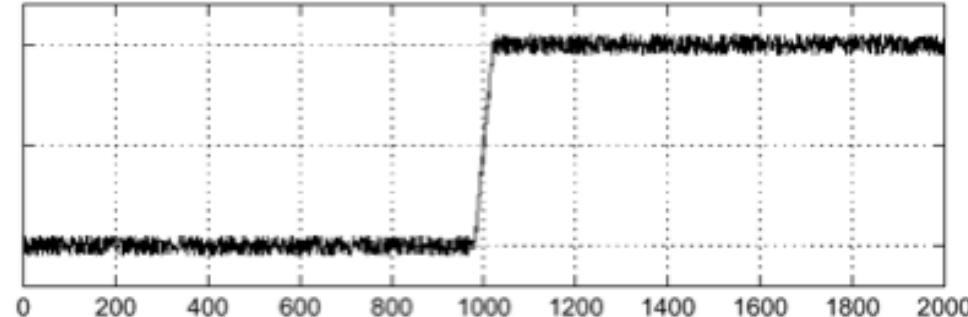
How do you find the edge of this signal?

intensity plot



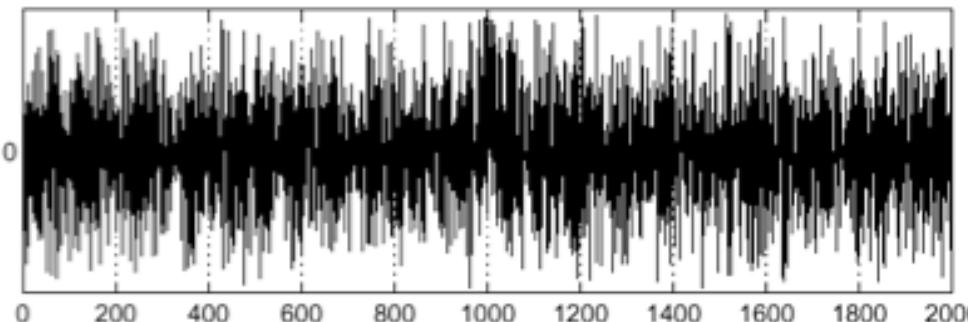
How do you find the edge of this signal?

intensity plot



Using a derivative filter:

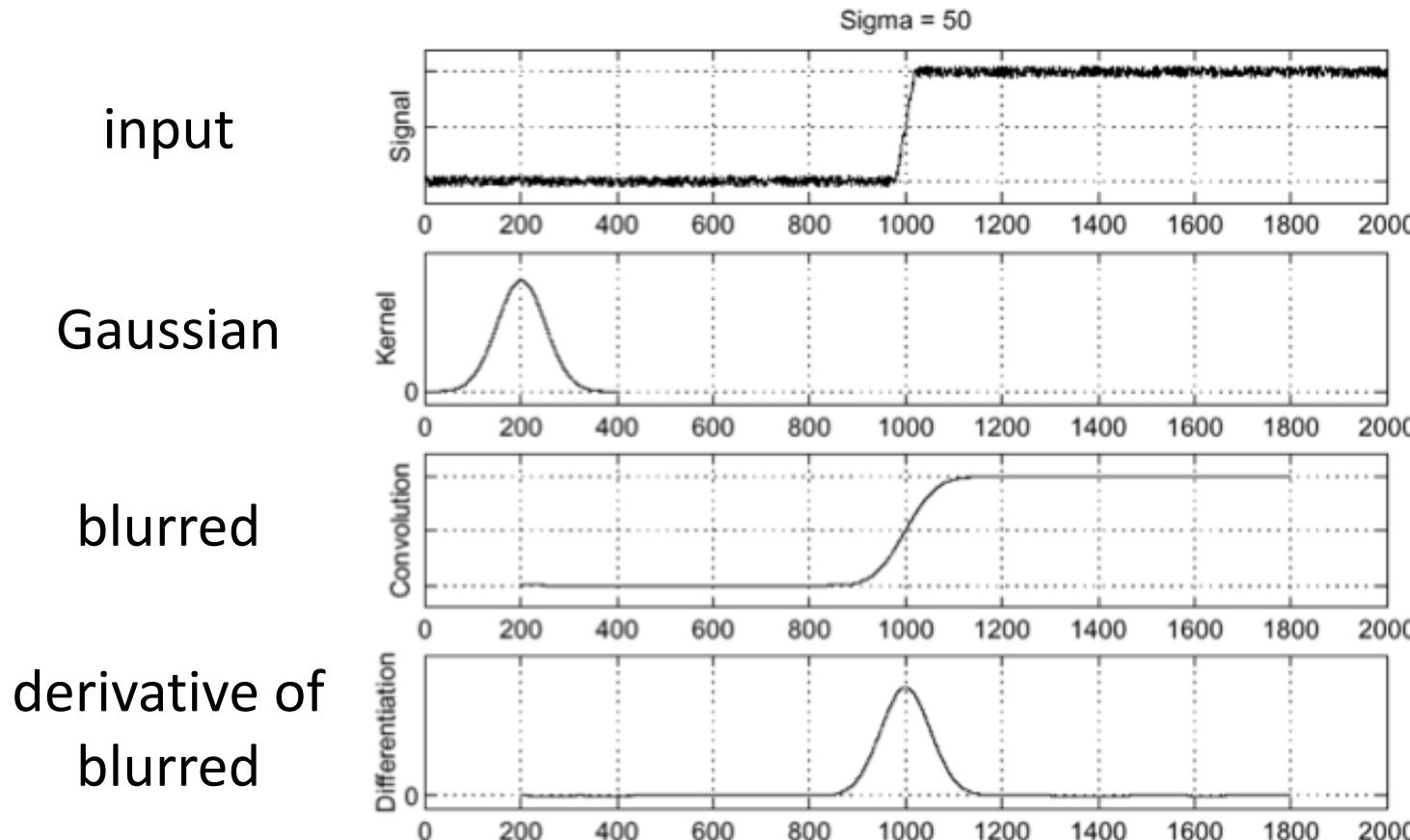
derivative plot



What's the
problem here?

Differentiation is very sensitive to noise

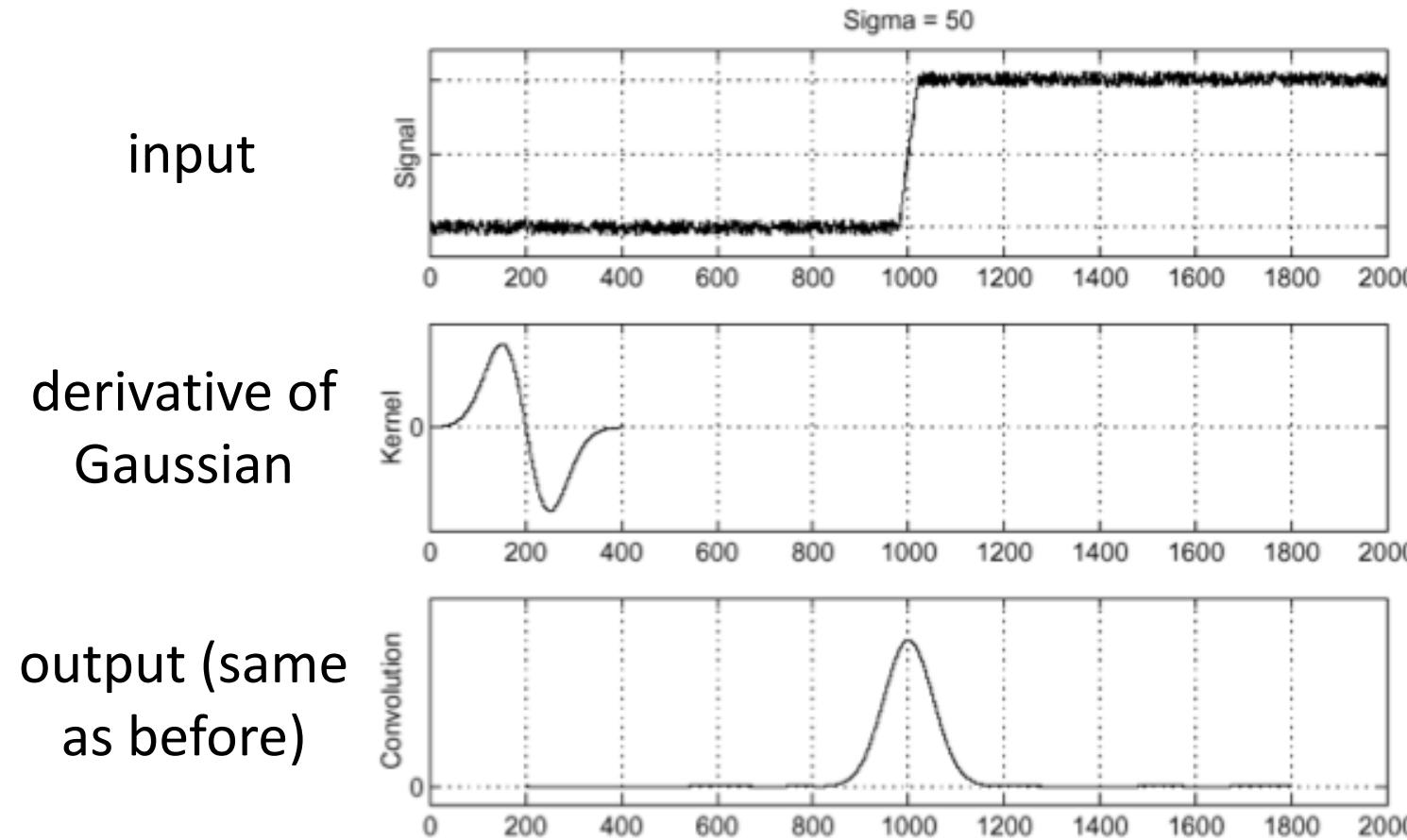
When using derivative filters, it is critical to blur first!



How much
should we blur?

Derivative of Gaussian (DoG) filter

Derivative theorem of convolution: $\frac{\partial}{\partial x}(h \star f) = (\frac{\partial}{\partial x}h) \star f$



- How many operations did we save?
- Any other advantages beyond efficiency?

Laplace filter

Basically a second derivative filter.

- We can use finite differences to derive it, as with first derivative filter.

first-order
finite difference

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + 0.5h) - f(x - 0.5h)}{h}$$



1D derivative filter

1	0	-1
---	---	----

second-order
finite difference

$$f''(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}$$



Laplace filter
?

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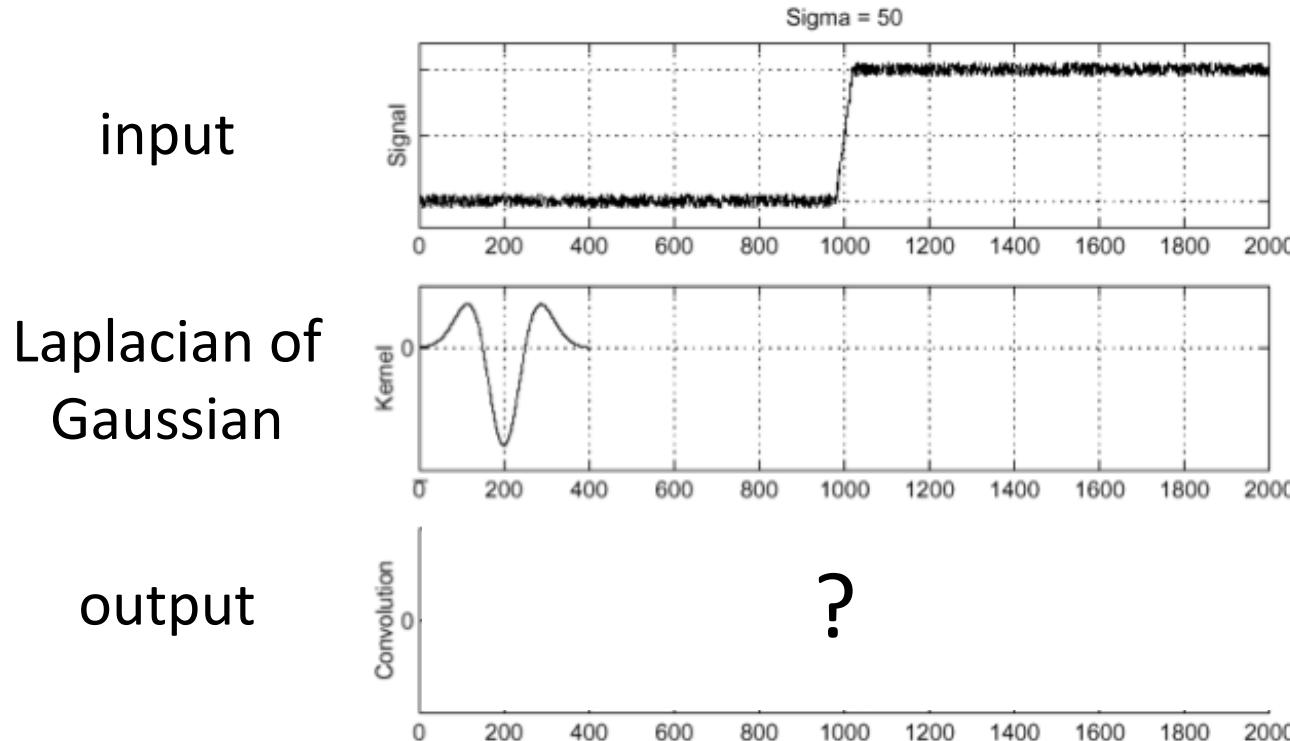


Laplace filter

1	-2	1
---	----	---

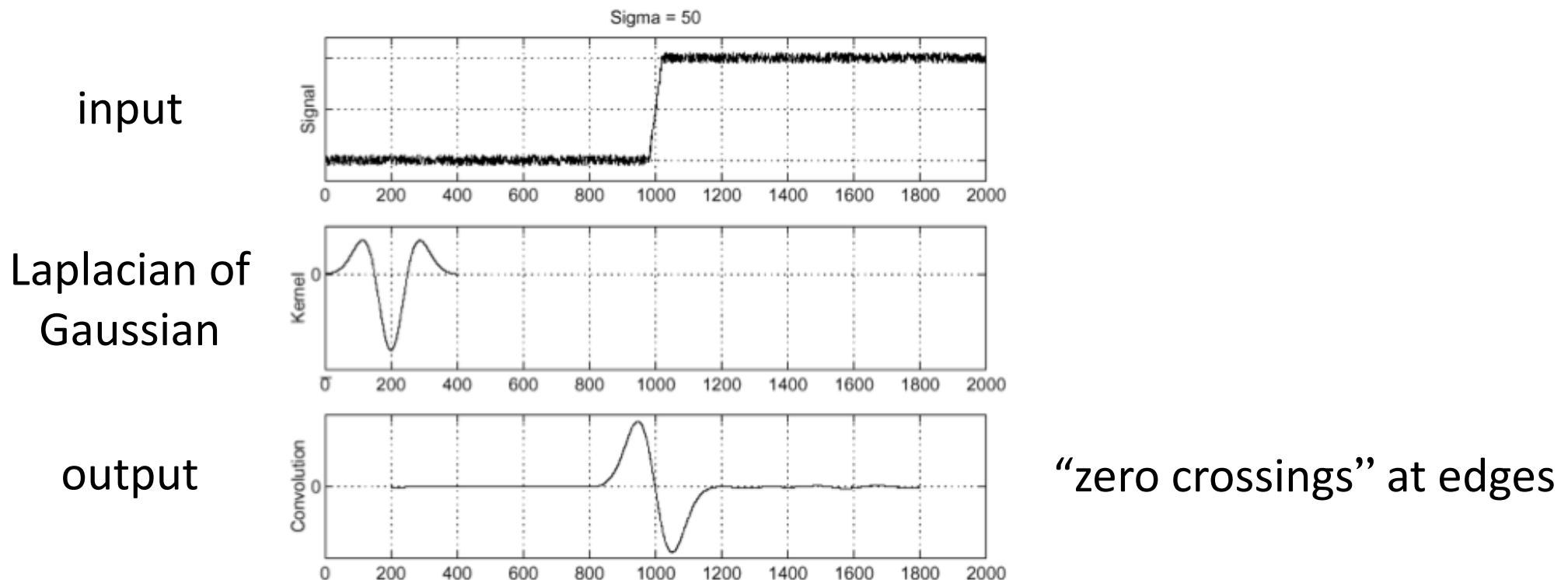
Laplacian of Gaussian (LoG) filter

As with derivative, we can combine Laplace filtering with Gaussian filtering



Laplacian of Gaussian (LoG) filter

As with derivative, we can combine Laplace filtering with Gaussian filtering



Laplace and LoG filtering examples



Laplacian of Gaussian filtering



Laplace filtering

Laplacian of Gaussian vs Derivative of Gaussian

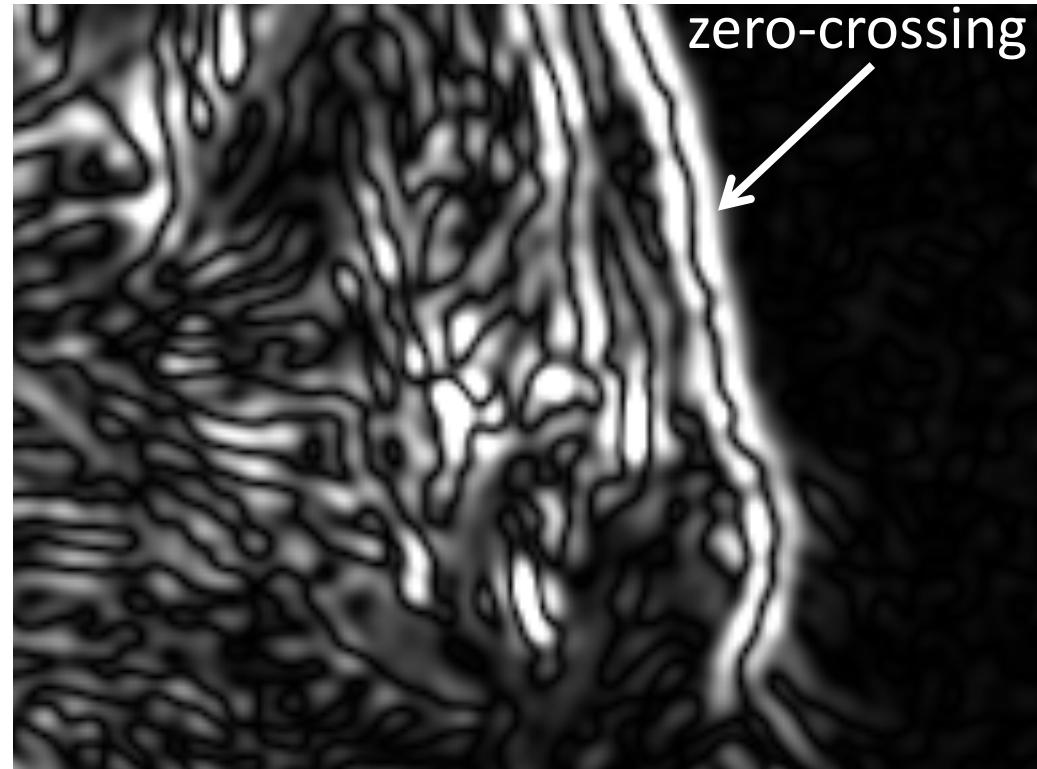


Laplacian of Gaussian filtering

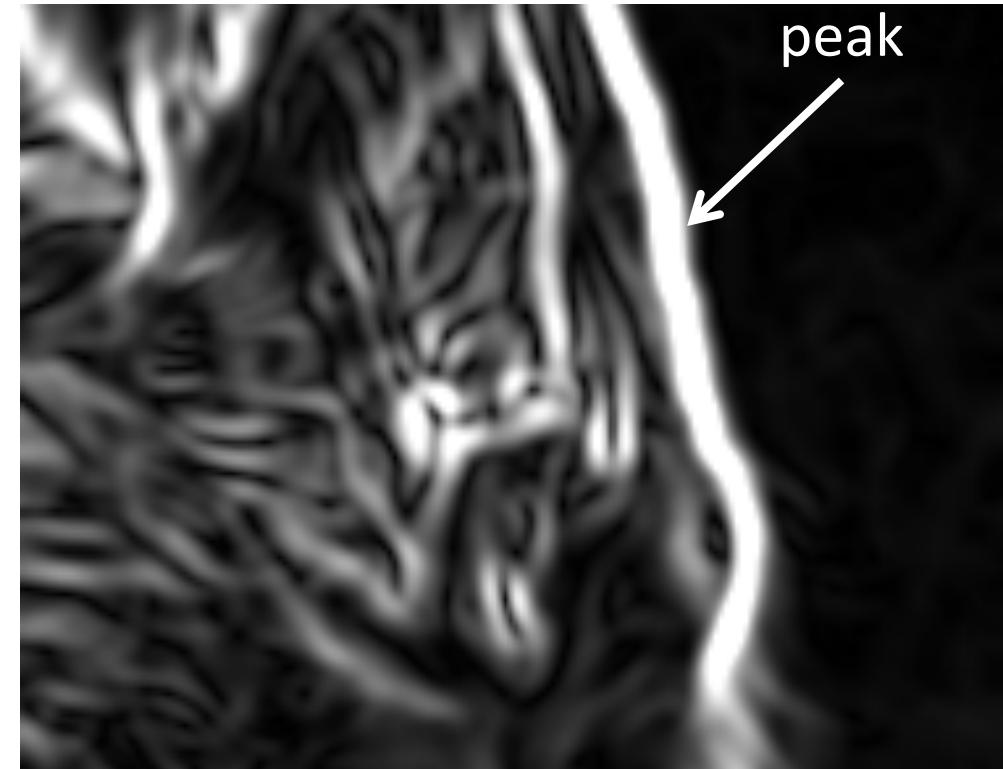


Derivative of Gaussian filtering

Laplacian of Gaussian vs Derivative of Gaussian



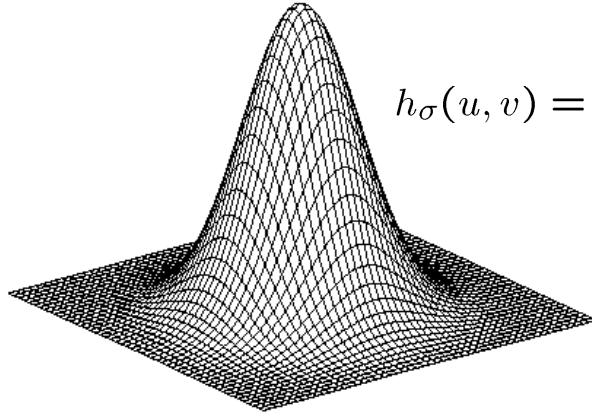
Laplacian of Gaussian filtering



Derivative of Gaussian filtering

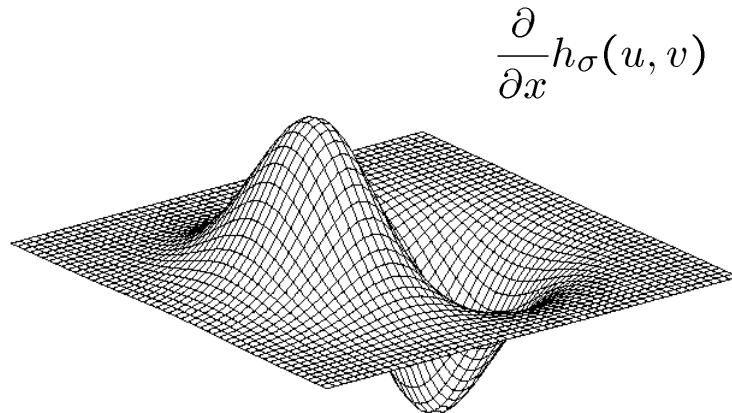
Zero crossings are more accurate at localizing edges (but not very convenient).

2D Gaussian filters



Gaussian

$$h_\sigma(u, v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}}$$

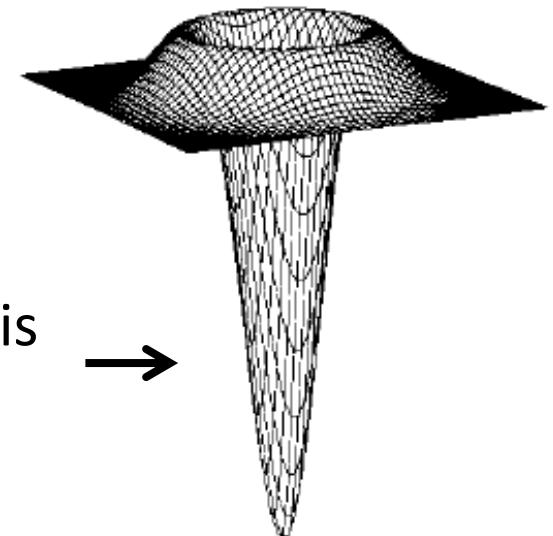


Derivative of Gaussian

$$\frac{\partial}{\partial x} h_\sigma(u, v)$$



$$\nabla^2 h_\sigma(u, v)$$



Laplacian of Gaussian

how does this relate to this
lecture's cover picture?

