LOGIC

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(*) Defination :- It is a branch of Mathematics is which matematical problems can be solved by using proper arguments.
These arguments are called Statement or Proposition

Propasition / Statement

Statement

A sentence will be a statement if it will satisfy the following 2 conditions :-

i) It should be gramatically correct
ii) It is either true or false but not both at the same time.

Example :-

(a) New Delhi is the capital of India
→ grammatically correct and always true. So, it is a statement

(b) $2+3=8$
→ grammatically correct and always false. So, it is a statement.

(c) Mahanadi is a long river
→ It is not a statement because though it is grammatically correct but ~~its is~~ its truth value varies from person to person.

(d) Rama is a rich man. → (Fuzzy statement)
→ It is not a statement.

Representation of a statement

p : Sky is blue (T)
q : $2+3=5$ (T)
r : $2\times5=15$ (F)

Every statement can be represented by an Alphabet.

Fuzzy Statement
The sentences whose ~~thruth~~ truth value varies from person to person are called fuzzy statement.
Example:-
(a) Rama is a good boy → It is a fuzzy statement.

Truth value of a Statement
T → always True
F → always False

If statement is true for everybody its truth value is taken as T. Similarly if a statement is always false then its truth value is taken as F.

Negation / Denial of a Statement ($\sim$ / $\neg$)

p: Sky is blue (T)
$\sim$p: sky is not blue (F)
Axiom of negation.

The negation of a statement P denoted by $\sim P$ / $\neg P$ is defined by it is another statement which is obtained from the statement P by changing P into its negative form.
In general the negation of a statement P can be written as it is not the case that P.

$\neg P$: It is not the case that P

⊙ Truth Table

p	$\sim$p
T	F
F	T

⊙ Properties of Negation

(1) $\sim(\sim p) = p$
p: Sky is blue
$\sim(\sim p)$: Sky is blue

Example :-
(a) $p : 2+3=8$
$\sim p : 2+3 \neq 8$
$\sim p$: It is not the case that $2+3=8$
(b) $q : 5>10$
$\sim q : 5 \leq 10$
$\sim q$: It is not the case that $5>10$

Axiom of Negation
If a statement be true then its negation is false & vice versa.

⊙ Truth Table of negation

p	$\sim$p
T	F
F	T

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($p \wedge q$) And/wh

LOGICAL OPERATION

Basically logical operations are used to combine 2 or more different statement into a single statement.

There are 4 types of logical operations

1. Conjuction ($\wedge$)
2. Disjuction ($\vee$)
3. Implication or conditional ($\Rightarrow$).
4. Double Implication or bi-conditional ($\Leftrightarrow$)

Conjuction

If p & q be any 2 statement then their conjuction denoted by $p \wedge q$ is defined as it is another statement which is obtained from p & q by using the connectives and/wh words.

Example :-

i) $p: 2+3=5$
q: Sky is blue.

So, $p \wedge q$: (stands for) $2+3=5$ and Sky is blue

ii) p : BBSR is the capital of Odisha
q : BBSR is the temple city.

So, $p \wedge q$: BBSR is the capital of Odisha which is the temple city.

⊙ Axiom of Conjuction

$p \wedge q$ is true, when both p & q are True but in all other cases it is false.

⊙ Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

⊙ Properties of Conjuction

i) It is idempotent

$$p \wedge p = p$$

ii) It is commutative

$$p \wedge q = q \wedge p$$

iii) It is associative

$$(p \wedge q) \wedge r = p \wedge (q \wedge r)$$

Equivalent statement

2 statements are said to be equivalent if their corresponding truth values are same.

If p & q are equivalent statements then we can write $p \equiv q$.

Ex Using Truth Table prove that $p \wedge q = q \wedge p$.

Ans

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

From the above table it is clear that $p \wedge q = q \wedge p$.

Quest → Using Truth Table, prove that
$(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$2^3 = 8$

Ans →

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

From the above table it is clear that $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

Disjuction

If p & q be any two statements then their disjuction denoted by $p \vee q$ is defined as it is another statement which is obtained from p & q by using the connectives or / either.... or...

Example :-

i) p : I will go to Puri by bus.
q : I will go to Puri by train.
$p \vee q$: I will go to Puri either by bus or by train.

ii) p : Sky is blue.
q : $2 \times 3 = 9$
$p \vee q$: Either sky is blue or $2 \times 3 = 9$.

Axiom of Disjunction

$p \vee q$ is false when both p & q are false. But in all other cases it is True.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Properties

(i) It is idempotant $\quad p \vee p = p$
(ii) Commutative $\quad p \vee q = q \vee p$
(iii) Associative $\quad (p \vee q) \vee r = p \vee (q \vee r)$

(iv) Conjuction is distributed over disjuction.

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(v) Disjuction is distributed over Conjuction.

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

H.W Verify all these by using Truth Table.

Q $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Q $(p \vee q) \vee r = p \vee (q \vee r)$

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Q $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

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Implication or Conditional

If p & q be any 2 statements. Then their implecation is denoted by $p \Rightarrow q$

In $p \Rightarrow q$,

p is called antecedent/hypothesis

q is called consequent/conclusion

Example

i) p: You will secure 95% mark in HSC examination.

q: I will give you a bike.

So,

$p \Rightarrow q$: If you will secure 95% mark in HSC examination then I will give you a bike.

ii> p : I will go to the market
q : I will buy a watch for you.
$p \Rightarrow q$: If I will go to the market then I will buy a watch for you.

Axiom of Implecation
$p \Rightarrow q$ is false when p is true & q is false.
But, in all other cases it is true.

Truth Table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	F

Languagically $p \Rightarrow q$ can be written in different forms which are given below.
i> If p then q
ii> If p, q.
iii> p is sufficient condition for q.
iv> q if p
v> q when p.
vi> Necessary condition for q is p
vii> q unless ~p
viii> p imples q
ix> q only if p
x> A sufficient condition q is p.
xi> q whenever p.
etc...

Biconditional or double Implecation
If p and q be any two statements then their double implecation is denoted by $p \Leftrightarrow q$.
Languagically $p \Leftrightarrow q$ can be written as
i> p if and only if q / p iff q
ii> p is necessary and sufficient condition for q.
example
p : $x+3=5$
q : $x^2=4$
$p \Leftrightarrow q$: $x+3=5$ iff $x^2=4$

Axiom of Double Implecation
$p \Leftrightarrow q$ is true only when both p & q have same truth values but in all other cases it is false.

Truth Table

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Some special operations in LOGIC:
In logic there are 3 special operations which are used on implication statement only. These special operations are converse, inverse & contrapositive.

i> Converse
The converse of $p \Rightarrow q$ is $q \Rightarrow p$

ii) Inverse

The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

iii) Contrapositive

The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

Example

Find the converse, inverse and contrapositive of the implecation statement -

"If 2 integers are odd then their sum is even"

Ans → Converse

If sum of 2 integers is even then these 2 integers are odd.

Inverse

If 2 integers are not odd then their sum is not even.

Contrapositive

If the sum of 2 integers is not even then they are not odd.

Exclusive or $\oplus$

If p & q be any 2 statements then their exclusive or is denoted by $p \oplus q$.

Axiom

$p \oplus q$ is true when out of p & q exactly 1 is true. But in all other cases it is false.

Truth Table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Tautology

A compound statement is said to be a Tautology if all its Truth values are true irrespective of the prime components.

Contradiction / Fallacy

A compound statement is said to be a contradiction if it is always false irrespective of the prime components.

p	$\sim p$	$p \wedge \sim p$	$p \vee \sim p$
T	F	F	T
F	T	F	T
		↓ Fallacy/ Contradiction	↓ Tautology

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Exclusive or / Inclusive or

Inclusive or
Let p & q be any 2 statements in case of a inclusive or of p & q we can take either p or q or both at the same time.

For examples
A student of +2 science can take either biology or mathematics. There are some students who takes only biology and some students who takes only mathematics and there are some students who takes both mathematics and biology.

Inclusive or is equivalent to disjunction or.

Exclusive or
In case of exclusive or of p & q either we can take p or q but both cannot be taken at the same time.

Example
A person is going to resturant, whenever he is going for registration. Then there are only 2 options for on non-veg. In this case the person can opt either veg or non-veg but he cannot opt both at the same time.

So, this is a case of exclusive or.

Precedence and Logical Operator
If in a compound statement more than one logical operations are present then we can operate this logical operations in a precedence ~~order~~ order.
The hirarchy of these logical operations are given below.

Hirarchy of Logical Operator
Order/Sequence
1 $\rightarrow$ $\sim$
2 $\rightarrow$ $\wedge$
3 $\rightarrow$ $\vee$
4 $\rightarrow$ $\Rightarrow$
5 $\rightarrow$ $\Leftrightarrow$

Representation of a statement in symbolically form.
Sometimes a compound statement is given at first we find out the primary statement from the given compound statement and then we represent it in symbolical form as given it in the compound statement.

Example
Represent the following statement in symbolical form:
Q) If I will go to Puri then I will go to sea beach & Konark temple.
Ans) Here the primary statements are
p : I will go to Puri
q : I will go to Sea Beach.
r : I will go to Konark temple.
Hence, the symbolical form of given statement is
$p \Rightarrow q \wedge r$

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Q) There a 2 resturants nearer to each other. The sign board of the 1^{st} resturant says that cheap food is not good. Similarly the sign bound of the ~~##~~ 2^{nd} resturant says that good food is not cheap. Whether the sign post of the resuturant saying same thing are not prove it logically.

Ans → Sign Board of 1^{st} resturant can be written as:-
If food is cheap then it is not good.

Sign Board of 2^{nd} resturant can be written as:-
If food is good then it is not cheap.

p : food is cheap
q : food is good

Hence, the symbolical form of the statement.

$p \Rightarrow \sim q$ — (I)

$q \Rightarrow \sim p$ — (II)

p	q	$\sim p$	$\sim q$	$p \Rightarrow \sim q$	$q \Rightarrow \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

From the above Truth Table it is clear that $p \Rightarrow \sim q$ & $q \Rightarrow \sim p$ are equivalent. Hence, both the sign board are saying same thing.

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Propositional Equivalence

(P) → T (Tautology)
→ F (Contradiction)
→ Contigency

If all the truth values of compound statement are true then it is called a Tautology.
If all the truth values of compound statement are false then it is called a Contradiction.
If some of the truth values are true and the remaining are false then it is called a contigency.

Example :- Identify the statement $p \wedge (\sim q \vee r)$

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

Contigency

Q) Prove De morgan's law using Truth T for 2 statements

$\sim (p \wedge q) = \sim p \vee \sim q$

$\sim (p \vee q) = \sim p \wedge \sim q$

Ans

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$p \vee q$	$\sim(p \vee q)$
T	T	F	F	T	F	F	T	F
T	F	F	T	F	T	T	T	F
F	T	T	F	F	T	T	T	F
F	F	T	T	F	T	T	F	T

same

$\sim p \wedge \sim q$
F
F
F
T

same.

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Logical Equivalent Statements

Q Prove that $p \Rightarrow q \equiv \sim p \vee q$

p	q	$\sim p$	$p \Rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Q Prove that $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Q Prove that $p \vee q \equiv \sim p \Rightarrow q$

p	q	$p \vee q$	$\sim p$	$\sim p \Rightarrow q$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

Q Show that $p \wedge q \equiv \sim(p \Rightarrow \sim q)$

p	q	$p \wedge q$	$\sim q$	$p \Rightarrow \sim q$	$\sim(p \Rightarrow \sim q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	F	F	T	F
F	F	F	T	T	F

Q Prove that $(p \Rightarrow q) \wedge (p \Rightarrow r) \equiv p \Rightarrow (q \wedge r)$

p	q	r	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (p \Rightarrow r)$	$q \wedge r$	$p \Rightarrow (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

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(8) Prove that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Some Basic Law of Logical Equivalence

(1) Identity Law

$P \wedge T = P$

$P \vee F = P$

'T' is the identity element is conjuction and 'F' is the identity element in disjuction

i.e. $P \wedge T = P$ $\quad$ $P \vee F = P$

(2) Domination Law

'F' is the domination on conjuction and 'T' is the domination over disjunction

i.e. $P \wedge F = F$

$P \vee T = T$

(3) Idempotent

$p \wedge p = p$

$p \vee p = p$

(4) Commutative

$p \wedge q = q \wedge p$

$p \vee q = q \vee p$

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(5) Law of double negation

$\sim(\sim p) = p$

(6) Associative Law

$p \wedge (q \wedge r) = (p \wedge q) \wedge r$

$p \vee (q \vee r) = (p \vee q) \vee r$

(7) Distributive Law

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(8) Demorgan's Law

$\sim(p \wedge q) = \sim p \vee \sim q$

$\sim(p \vee q) = \sim p \wedge \sim q$

(9) Absorption Law

$p \wedge (p \vee q) = p$

$p \vee (p \wedge q) = p$

(10) Negation Law

$p \wedge \sim p = F$

$p \vee \sim p = T$

★ NOTE

Basically Logical equivalents can be verified by using 2 different methods :-

(i) Using Truth Table

(ii) Using Laws of Logical equivalence.

Q Prove that $\sim(p \Rightarrow q)$ & $p \wedge \sim q$ are logically equivalent by using the laws of equivalence.

Ans
$\sim(p \Rightarrow q)$
$= \sim(\sim p \vee q)$ $\quad (\because p \Rightarrow q = \sim p \vee q)$
$= \sim(\sim p) \wedge \sim q$ $\quad (\because$ Demorgan's Law$)$
$= p \wedge \sim q$ (proved)

Q Show that $\sim(p \vee (\sim p \wedge q))$ & $\sim p \wedge \sim q$ are logically equivalent by using the Laws of equivalence.

Ans
$\sim(p \vee (\sim p \wedge q))$
$= \sim p \wedge \sim(\sim p \wedge q)$ $\quad (\because$ Demorgan's Law$)$
$= \sim p \wedge [\sim(\sim p) \vee \sim q]$ $\quad (\because$ Demorgan's Law$)$
$= \sim p \wedge [p \vee \sim q]$ $\quad (\because$ Law of double Negation$)$
$= (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ $\quad (\because$ Distributive Law$)$
$= F \vee (\sim p \wedge \sim q)$ $\quad ($Law of Negation$)$
$= \sim p \wedge \sim q$ $\quad ($Identity Law$)$
(Proved)

Q Show that $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology by using the law of equivalence.

Ans
$(p \wedge q) \Rightarrow (p \vee q)$
$\sim(p \wedge q) \vee (p \vee q)$ $\quad (\because p \Rightarrow q = \sim p \vee q)$
$(\sim p \vee \sim q) \vee (p \vee q)$ $\quad (\because$ Demorgan's Law$)$
$\sim p \vee \sim q \vee p \vee q$
$(p \vee \sim p) \vee (q \vee \sim q)$
$T \vee T$ $\quad (\because$ Law of negation$)$
T (Tautology)
Hence, the given statement is a tautology.

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Predicate & Quantifiers

PREDICATE

Every sentence has 2 parts such as Subject & Predicate

Example

$\underbrace{\text{BBSR}}_{\text{Subject}}\ \underbrace{\text{is the capital of Odisha}}_{\text{Predicate}}$

$\underbrace{\text{Two plus three}}_{\text{Subject}}\ \underbrace{\text{is greater than 7}}_{\text{Predicate}}$

Propositional Function

Generally every statement can be represented by an alphabet but if a sentence contains atleast ~~by~~ one variable then it can be represented as a fx^n of same variable.

$p(x): x > 10$
$q(x, y): x + 2 = y$
$r(x, y, z): x + y > z$

Hence whenever a sentence contains atleast one variable then it can be represented as a function of that variable and these fx^n's are called propositional functions.

Ex
$p(x)$, $q(x,y)$, $r(x,y,z)$ are propositional functions
propositional functions.

Quantification
Wherever in a propositional function we put the value of the variables then it is called a quantification and every quantification must be a statement with truth values true or false.

Ex

i) $p(x) : x > 7$

$p(8) : 8 > 7$

this a quantification which is always true

$p(5) : 5 > 7$

this a quantification which is always false.

ii) $p(x,y) : x+y > 5$

$p(1,3) : 1+3 > 5$ (F)

$p(2,5) : 2+5 > 5$ (T)

iii) $r(x,y,z) : x+y > z$

$r(1,2,5) : 1+2 > 5$ (F)

$r(2,3,1) : 2+3 > 1$ (T)

Types of Quantifications
Basically there are 2 types of quantifications:-

1. Universal Quantification ($\forall x\ p(x)$)
2. Existencial Quantification ($\exists x\ p(x)$)

Universal Quantifications ($\forall x\ p(x)$)
The Quantification which is true for all values of the variables; except few is called the Universal Quantifications.

The universal quantification can be expressed as $\forall x \in p(x)$

Ex

$p(x) : x+5 > x$

$p(1) : 6 > 1$

$p(2) : 2+5 > 2$

$p(3) : 3+5 > 3$

Here $p(x)$ is the universal quantification. So, it can be written as for $\forall x\ x+5 > x$

Existensial Quantification
The quantification which is true in very few cases of the variable x but it is false is most of the cases of the variable x. Then it is called exhistential quantification.

Basically existential quantification can be represented as $\exists x\ p(x)$.

Example

$p(x) : x^2 - 7x + 12 = 0$

$(x-4)(x-3)$

$p(3) = 9 - 21 + 12 = 0$

$p(4) = 16 - 28 + 12 = 0$

$p(5) = 25 - 35 + 12 = 2 \neq 0$

It is clear that $p(x)$ is true when x is equal to 3 & 4. But for other values of x its is false. So, it is clear that $p(x)$ the statement $p(x)$ is true for only $x = 3$ & 4 but for other values of $p(x)$ it is false. So here $p(x)$ is an exhistential quantification.

$\exists x\ p(x)$

Q) Let $p(x) : x^2 > 0$

→ This universal quantification because it is true for all values of x.

(Q) $p(x) : x^2 < 10$
This is called existential quantification because it is satisfied for some value of x.

n-place or n-ary predicate

If in a propositional function n number of variables are present then it is called n-ary predicate.

$p(x) : x > 10$ — 1-ary predicate
$p(x, y) : x + y > 10$ — 2-ary predicate
$p(x_1, x_2, x_3, \dots, x_k)$ — k-ary predicate

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* NOTE
Let $p(x)$ be a proposition function. Let the domain of x be $x_1, x_2, x_3, \dots, x_n$. Here $p(x_i)$ is a quantification of this $p(x_1)$ is a statement whose truth value is either True or False. Similarly, $p(x_2), p(x_3), \dots, p(x_n)$ has statements whose truth values are either true or false.

It is clear that $p(x_1) \wedge p(x_2) \wedge p(x_3) \wedge \dots \wedge p(x_n)$ is also another statement. The truth value of this statement is T when all of $p(x_1), p(x_2), p(x_3), \dots, p(x_n)$ are true. It is also false when atleast one of them is false.

Similarly disjunction; $p(x_1) \vee p(x_2) \vee p(x_3) \vee \dots \vee p(x_n)$ is also another statement, the truth value of this statement is T when atleast any one out of $p(x_1), p(x_2), p(x_3) \dots p(x_n)$ are true.

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It is ~~also~~ false when ~~atleast one~~ all of them ~~is~~ are false.
~~Similarly disju~~

(Q) $p(x) = x^2 > 20$
$x = \{0, 1, 3, 3, 4, 5\}$
then find the truth value of each of the quantification and also find the quantification of:
i> $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$
ii> $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$
iii> $P(0) \vee P(1) \vee P(3) \vee P(4)$
v> $P(0) \vee P(1) \vee P(4) \vee P(5)$

$P(0) = 0^2 > 20$ (F)
$P(1) = 1 > 20$ (F)
$P(2) = 4 > 20$ (F)
$P(3) = 9 > 20$ (F)
$P(4) = 16 > 20$ (F)
$P(5) = 25 > 20$ (T)

Ans
i> Truth Values is False
ii> False
iii> False
iv> True

* NOTE 2
Let $p(x)$ be a proposition function where the domain of x is $x_1, x_2, x_3, \dots, x_n$. If $p(x) = x_1 \wedge x_2 \wedge x_3 \dots$ is true, then the proposition of $p(x)$ is called universal quantification and it is represented as $\forall x\, p(x)$.

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* NOTE 3

If $p(x)$ we any preposition, x_1, x_2, x_3 be any of its domain. Then,

i) If $p(x_1) \vee p(x_2) \vee p(x_3) \dots$ is false then in this case $p(x)$ is neither universal nor existential.

ii) If $p(x_1) \vee p(x_2) \vee p(x_3) \dots$ are true but all the individual then it is an existential quantification

Example:-

Let $p(x)$ stands for $x^3 > 9$

$p(x) : x^3 > 9$

where x is equal to 1, 2, 3, 4, 5.

$x = \{1, 2, 3, 4, 5\}$

Verify with $p(x)$ is existancial or universal quantification.

Ans → Existential.

$p(1) \quad 1^3 > 9 \quad (F)$

$p(5) \quad 5^3 > 9 \quad (T)$

Since $p(1)$ is false and $p(5)$ is true, so it is clear that this is existential quantification.

Negation of Universal & Exhistential quantification.

The negation of any universal quantification is an existential quantification, similarly the negation of any existential quantification is universal quantification.

Quantification	Existencial
$\forall x\, p(x)$	$\exists x \sim p(x)$
$\exists x (p(x))$	$\forall x \sim p(x)$

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Example

Q) Write the negation of the quantification all Natural numbers are integer.

Ans → All natural number are integer.

$p(x) : x$ is an integer; $x \in$ Natural

$\forall x\, p(x)$

$\sim \forall x\, p(x)$

$\Rightarrow \exists x \sim p(x)$

Let $p(x)$ stands for x is an integer. So, the given statement is an universal quantification.

The negation of this universal statement is an existance statement. "There exists x negation of x.

So the negation of the statement is

There exist some natural number which are not Integers

Q) Find the negation of:

"There is an honest Politician"

$p(x) : x$ is honest

Ans → The given statement is existencial quantification.

$\exists x\, p(x)$

$\forall x \sim p(x)$

Negation is → All politicians are not honest.

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(Q) Write the negation of :-
$\forall x (x^2 > x)$ & $\exists x (x^2 = 5)$

→ $\sim \forall x (x^2 > x)$ | $\sim \exists x (x^2 = 5)$
$\exists x (x^2 \leq x)$ | $\forall x (x^2 \neq 5)$

Logical equivalence using quantifiers:

Suppose p & q are 2 logical statements using quantifiers. These 2 statements ~~using quo~~ are logically equivalent iff their predicates are equivalent.
In other words 2 statements p & q are logically equivalent if the following 2 conditions are statisfied:

i) If p is true then q is true
ii) If q is true then p is true.

(Q) Prove that $\forall x (p(x) \wedge Q(x))$ & $\forall x P(x) \wedge \forall x Q(x)$ are Logical equivalent.

Ans: To prove this here 2 cases will arise,
Case 1 :- Let $\forall x (p(x) \wedge Q(x))$ is T
we have to show that $\forall x p(x) \wedge \forall x Q(x)$ is True.
Now,
$\forall x (P(x) \wedge Q(x))$ is T
$\Rightarrow P(x) \wedge Q(x)$ is T
$\Rightarrow P(x)$ is T & $Q(x)$ is T
$\Rightarrow \forall x P(x)$ is T & $\forall x Q(x)$ is T

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$\Rightarrow \forall x P(x) \wedge \forall x (Q(x))$ is True

Case 2 :- Let $\forall x p(x) \wedge \forall x Q(x)$ is true, we have to show that $\forall x (P(x) \wedge Q(x))$ is true.
$\forall (x) p(x) \wedge \forall x Q(x)$ is T
$\forall x p(x)$ is T & $Q(x)$ is T
$P(x)$ is T & $Q(x)$ is T
& $p(x) \wedge Q(x)$ is T
$\forall x (P(x) \wedge Q(x))$ is T

(Q) Prove that $\sim \forall x (p(x) \Rightarrow q(x))$ & $\exists x (p(x) \wedge \sim q(x))$ are logically equivalent.

Ans Case 1
Let $\sim \forall x (p(x) \Rightarrow q(x))$ is true then we have to show that $\exists x (p(x) \wedge \sim q(x))$
$\sim \forall x (p(x) \Rightarrow q(x))$ is T
$\Leftrightarrow \exists x \sim (\sim p(x) \vee Q(x))$ is T
$\Leftrightarrow \exists x (\sim (\sim p(x)) \wedge \sim Q(x))$ is T
$\Leftrightarrow \exists x (p(x) \wedge \sim Q(x))$ is T.

Translation from English into logical expression.

(Q) Every student in the class has studied calculus.
$p(x)$ = x is a student
$p(y)$ = x studied calculus
$\forall x \, p(x) \Rightarrow q(x)$

Here $p(x)$: x is a student of the class.
& $p(y)$: x has studied calculus.
So, the given sentence can be written in logical form as.
$\forall x \, p(x) \Rightarrow q(x)$

In other words this statement can be written as

$\forall x\ p(z)$

where, $p(z)$: x studied calculus.

Example

Let $p(x)$: x is a professor

$Q(x)$: x is ignorant

$R(x)$: x is vain

Express the following statements in logical expression form.

Ans

(i) No professor are ignorant $\Rightarrow \forall x\ p(x) \Rightarrow \sim Q(x)$

(ii) All ignorant people are vain $\Rightarrow \forall x\ Q(x) \Rightarrow R(x)$

(iii) No professors are vain $\Rightarrow \forall x\ p(x) \Rightarrow \sim R(x)$

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Q(16) Prove that

$p \Leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

Ans R.H.S

$(p \wedge q) \vee (\sim p \wedge \sim q)$

$= (p \vee (\sim p \wedge \sim q)) \wedge (q \vee (\sim p \wedge \sim q))$

$[(p \wedge q) \vee r) = p \vee r \wedge q \vee r]$

$= [(p \vee \sim p) \wedge (p \vee \sim q)] \wedge [(q \vee \sim p) \wedge (q \vee \sim q)]$

$= [T \wedge (p \vee \sim q)] \wedge [(q \vee \sim p) \wedge T]$ (Law of negation)

$= (p \vee \sim q) \wedge (q \vee \sim p)$ (Identity Law)

$= (\sim q \vee p) \wedge (\sim p \vee q)$

$= (q \Rightarrow p) \wedge (p \Rightarrow q)$

$= p \Leftrightarrow q$

(Ans)

$$\boxed{\begin{array}{l} p \Rightarrow q = \sim p \vee q \\ p \vee q = \sim p \Rightarrow q \end{array}}$$

Q(17) $\sim(p \Leftrightarrow q)$ and $p \Leftrightarrow \sim q$ are logically equivalent.

Ans $\sim(p \Leftrightarrow q) \equiv p \Leftrightarrow \sim q$.

L.H.S

$\sim(p \Leftrightarrow q)$ ~~$\equiv p \Leftrightarrow \sim q$~~

$\sim ((p \wedge q) \vee (\sim p \wedge \sim q))$ $[\because p \Leftrightarrow q = (p \wedge q) \vee (\sim p \wedge \sim q)]$

$\sim (p \wedge q) \wedge \sim (\sim p \wedge \sim q)$ (Demorgan's law)

$(\sim p \vee \sim q) \wedge (\sim(\sim p) \vee \sim(\sim q))$

$(\sim p \vee \sim q) \wedge (p \vee q)$

$(\sim p \vee \sim q) \wedge (q \vee p)$ (Commutative Law)

$(p \Rightarrow \sim q) \wedge (\sim q \Rightarrow p)$

$p \Leftrightarrow \sim q$ (Proved)

(R.H.S)

Q(18) P.T.

$p \Rightarrow q$ is logically equivalent to $\sim q \Rightarrow \sim p$.

Ans L.H.S

$p \Rightarrow q$

$\Rightarrow \sim p \vee q$ $(\because p \Rightarrow q = \sim p \vee q)$

$\Rightarrow q \vee \sim p$ (Commutative)

$\Rightarrow \sim q \Rightarrow \sim p$ (Proved)

R.H.S

Q(19) S.T.

$\sim p \Leftrightarrow q \equiv p \Leftrightarrow \sim q$

$$\boxed{\begin{array}{l} p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p) \\ p \Leftrightarrow q = (p \wedge q) \vee (\sim p \wedge \sim q) \end{array}}$$

Ans L.H.S

$\sim p \Leftrightarrow q$

$= (\sim p \wedge q) \vee (\sim(\sim p) \wedge \sim q)$

$= (\sim p \wedge q) \vee (p \wedge \sim q)$ $(\because \sim(\sim p) = p)$

$= (p \wedge \sim q) \vee (\sim p \wedge q)$

$= (p \wedge \sim q) \vee (\sim p \wedge \sim(\sim q))$

$= p \Leftrightarrow \sim q \quad [\because p \Leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)]$

(R.H.S)

Q(20) Show that

$\sim (p \oplus q) \equiv p \Leftrightarrow q$

p	q	$p \oplus q$	$\sim (p \oplus q)$	$p \Leftrightarrow q$
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

Equivalence

Hence,

$\sim (p \oplus q) \equiv p \Leftrightarrow q$

Q(21) P.T

$\sim (p \Leftrightarrow q) = \sim p \Leftrightarrow q$

Ans

$\Rightarrow \sim (p \Leftrightarrow q)$

$\Rightarrow \sim [(p \wedge q) \vee (\sim p \wedge \sim q)]$

$\Rightarrow \sim (p \wedge q) \wedge \sim (\sim p \wedge \sim q)$ [Demorgan's law]

$\Rightarrow (\sim p \vee \sim p) \wedge (p \vee q)$ [Demorgan's law]

$\Rightarrow (p \vee q) \wedge (\sim p \vee \sim q)$ [Commutative]

$\Rightarrow (p \vee q) \wedge (\sim q \vee \sim p)$ [Commutative]

$\Rightarrow \sim p \Rightarrow q \wedge (q \Rightarrow \sim p)$

$\Rightarrow \sim p \Leftrightarrow q$ $\quad (\because p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p))$

R.H.S

(Proved)

Q(22) P.T

$p \Rightarrow q \wedge (p \Rightarrow r) \equiv p \Rightarrow (q \wedge r)$

L.H.S

$p \Rightarrow q \wedge (p \Rightarrow r)$

$\Rightarrow (\sim p \vee q) \wedge (\sim p \vee r)$ $\quad (\because p \Rightarrow q = \sim p \vee q)$

$\Rightarrow \sim p \vee (q \wedge r)$ $\quad (\because p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r))$

$\Rightarrow p \Rightarrow (q \wedge r)$ $\quad (\because p \vee q = \sim p \Rightarrow q)$

R.H.S (Proved)

Dual of a Statement

p^*

$\wedge \rightarrow \vee$

$\vee \rightarrow \wedge$

$T \rightarrow F$

$F \rightarrow T$

Dual of any statement p denoted by p^* is another statement which is obtained from p by replacing conjunction by disjuction, $\vee$ by $\wedge$, T by F & F by T.

Example

Find the dual of the statement

i) $s = [(p \vee q) \wedge r] \Rightarrow (p \wedge q)$

$s^* = [(p \wedge q) \vee r] \Rightarrow (p \vee q)$

ii) $s = [(p \wedge T) \vee (q \wedge F)] \Rightarrow r$

$s^* = [(p \vee F) \wedge (q \vee T)] \Rightarrow r$

NAND $(p|q)$

If p & q be any 2 statements then p NAND q is denoted by $p|q$.

Axiom
p NAND q is False when both p & q are true but in all other cases it is true.

Truth Table

p	q	$p\|q$
T	T	F
T	F	T
F	T	T
F	F	T

NOR $(p\downarrow q)$

p NOR q is denoted by $p \downarrow q$

Axiom
$p \downarrow q$ is true when both p & q are false. But in rest of the cases it is false.

Truth Table

p	q	$p\downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Uniqueness Quantifier

$\exists! x\, p(x)$

The uniqueness is denoted by $\exists! x (p(x))$. Here $p(x)$ is true only for one value of x.

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Nested Quantifiers

Quantifier is said to be a nested quantifier if there is a scope of one quantifier within the other one.

Example

1. $\forall x\, \exists y\, (x+y=15)$ (y is in the scope of x)
This a nested quantifier where y depends upon the variable x.

2. $\forall x\, \forall y\, (x \cdot y = y \cdot x)$ (Here both the quantifier both are in the scope of each other)
In this case both x & y are within the scope of other one.

3. Translate the following quantifier into English $\forall x \forall y$ $(x+y=y+x)$ where x & y are any real number.
$\forall x \forall y\, (x+y=y+x)$

Ans → If x & y are any 2 real number then $x+y$ is always equal to $y+x$.

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Q) $\forall x \forall y\ p(x,y)$ where x & y are real numbers.
$p(x,y) = x \cdot y = y \cdot x$

Ans → It is clear that $p(x,y)$ is true for every values of x & y.
So, this quantifier is always true.

Q) Find the truth values of:
$\forall x \exists y\ p(x,y)$
where $p(x,y): x+y=10$
x & y are real number.

Ans → This quantifier is true when $p(x,y)$ is true and this false when $p(x,y)$ is false.

Determination of Truth or False of a quantifier of 2 variables:

Quantifier	True	False
$\forall x \forall y\ p(x,y)$ $\forall y \forall x\ p(x,y)$	T when $p(x,y)$ is T	False when $p(x,y)$ is F
$\forall x \exists y\ p(x,y)$	T when $\forall x \exists y$ $p(x,y)$ is true	F when $\forall x \exists y$ s.t. $p(x,y)$ is F.
$\exists x \forall y\ p(x,y)$	T when $p(x,y)$ is T	F when $p(x,y)$ is False
$\exists x \exists y\ p(x,y)$		

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Q) Example Let $Q(x,y): x+y=15$ where x & y are real numbers.
What are the Truth values of the quantification.
i) $\forall x \exists y\ Q(x,y)$
ii) $\exists x \forall y\ Q(x,y)$

Ans i) $\forall x \exists y\ (x+y=15) \rightarrow$ True
$\forall x \exists y\ Q(x,y) \equiv \forall x \exists y\ (x+y=15)$
It is clear that $\forall$ real value of x there exist only one real values of y. such that $x+y=15$. Hence this statement is always true.

ii) $\exists x \forall y\ Q(x,y) \equiv \exists x \forall y\ (x+y=15)$
Languagical meaning of this is there are some real numbers x, ~~st~~ such that if we add any real number y then the sum is 15.
It is not possible always therefore this statement is false.

Q) Let $p(a,b,c): a+b=c$
where a,b,c all are real numbers. Write the truth values of nested quantification.
i) $\forall a \forall b \exists P(a,b,c)$
ii) $\exists c \forall a \forall b\ P(a,b,c)$

Ans i) $\forall a \forall b \exists c\ (a+b=c)$
For every real no. ~~a~~ a & b there exist another real no. c such that $a+b=c$. It is always True.

11) $\exists c \forall a \forall b\ (a+b=c)$

There is a real no. c which can be written as the sum of every 2 real numbers. It is not possible because, if c=7 then a & b cannot be written as 2 real number like a=7 & b=9. So, it is always false.

Translation of mathematical statements into the statements involving nested Quantification.

Some mathematical statements can be written in terms of nested quantifications. Some example related to these are given below.

Example

Translate the statement

Q(a) The sum of every 2 negative real number is always a negative real number.

Ans→

i) $\forall x\ (x<0)\ \forall y\ (y<0) \rightarrow (x+y<0)$

ii) $\forall x\ \forall y\ (x+y<0)$

These statement can be written in nested quantification form in 2 different ways as stated above i) & ii)

Q(b) For every real number there is an additive inverse.

Ans→

Nested Quantification form.

$\forall x\ \exists y\ (x+y=0)$

It is clear that $\forall x$ the addative inverse must exists and $x+y=0$. So, the nested quantification form of this is as stated above, where the domain of viscous is set of real number.

Q For every real number except 0 the multiplicative inverse exists.

Ans→ $\forall x\ (x \neq 0)\ \exists y\ (x \cdot y = 1)$

Translation from Nested Quantification into English.

Q Let c(x): x has a computer

c(x,y): x & y are friends

c(y): y has a computer

Translate the nested Quantification

$\forall x\ (c(x) \vee \exists y\ ((c(y) \wedge p(x,y)))$ where x & y are the students of particular class.

Ans: For every student x of the class, such that x has a computer or there exist another student y such that y has a computer & x & y are friends.

Q Let L(x,y): x likes y, where the domain of viscous is the set of all peoples of the world. Using quantities translate each of the following statement:

i) Everybody likes Happy.

ii) Everybody likes someone.

iii) There is somebody to whom everybody likes.

iv) Nobody likes anybody.

Ans→ i) $\forall x\ L(x, \text{Rama})$

ii) $\forall x\ \exists y\ L(x,y)$

iii) $\exists x\ \forall y\ (y,x)$

iv) $\forall x\ \forall y\ \sim L(x,y)$

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Translation from English statement to logical expression.

Example

Q) Express the statement if a person is female and is a parent then this person is someone's mother.

Ans→ Let $f(x)$: x is a female.

$p(x)$: x is a parent

$M(x,y)$: x is the mother of y.

So, the given statement can be expressed logically

$\forall x(f(x) \wedge p(x)) \rightarrow \exists y M(x,y)$

or

$\forall x \exists y\ (f(x) \wedge p(x)) \rightarrow M(x,y)$

Q) Express the statement, everyone has exactly one best friend in logical expression form.

Ans→ Let x & y $\in$ the set of human beings & $B(x,y)$ stands for y is the best friend of x. The logical expression of the given statement is $\boxed{\forall x \exists ! y\ B(x,y)}$

($\exists !$ = unique/only one)

Negation of Nested Quantifiers

In a nested quantifier whenever we find out the negation. Then we apply it in step by step procedure. At first this negation will be applied on the first quantifiers whereas the others remain same.

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In the 2nd stage we apply the negation on the 2nd quantifier. Whereas all other quantifiers remain unchanged. Repeating this process continuously untill the negation will be applied on the last quantifiers. Whatever quantifier is available at the end is called the negation of the given nested quantifiers.

Q) Write the negation of

$\forall x \exists y (x+y=15)$

Ans→ $\forall x \exists y\ (x+y=15)$

The negation of the given statement is

$\sim(\forall x \exists y\ (x+y=15))$

$\cong \exists x \sim(\exists y\ (x+y=15))$

$\cong \exists x \forall y \sim(x+y=15)$

$\cong \exists x \forall y (x+y \neq 15)$

Q) Write the negation of

$\forall x \exists y (x \cdot y = 5)$, x, y are any non-zero real number.

Ans→ $\sim(\forall x \exists y\ (x \cdot y = 5))$

$\cong \exists x \sim(\exists y\ (x \cdot y = 5))$

$\cong \exists x \forall y \sim(x \cdot y = 5)$

$\cong \exists x \forall y\ (x \cdot y \neq 5)$

Q) Write the negation of

$\exists w \forall a \exists f\ (p(f,a) \wedge q(w,f))$

$\cong \forall w \cdot \sim(\forall a \exists f (p(f,a) \wedge q(w,f)))$

$\cong \forall w \exists a \sim(\exists f (p(f,a) \wedge q(w,f)))$

$\equiv \forall \omega \exists a \forall\, [\sim(p(f,a) \wedge q(\omega,f))]$

$\equiv \forall \omega \exists a \forall\, [(\sim(p(f,a)) \vee (\sim q(\omega,f))]$

Q Find the negation of $\lim_{x \to a} f(x) = L$

Ans— $\forall \epsilon > a\ \exists\, \delta > 0$

We know that the limit of the function is

$\exists \epsilon \sim (\exists \delta \forall x (|x-a| < \delta \Rightarrow |f(x) - L| < f))$

$\Rightarrow \exists \epsilon \forall \delta \sim (\forall x (|x-a| < \delta \Rightarrow |f(x) - L| < \epsilon))$

$\Rightarrow \exists \epsilon \forall \delta \exists x \sim ((|x-a| < \delta \Rightarrow |f(x) - L| < \epsilon))$

$\Rightarrow \exists \epsilon \forall \delta \exists x \sim (\sim |x-a| < \delta) \vee |f(x)-L| < \epsilon)$

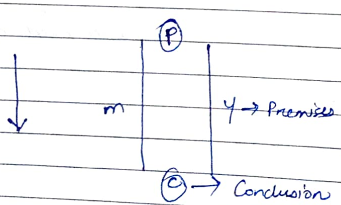
$\because (p \Rightarrow q \equiv \sim p \vee q)$

$\Rightarrow \exists \epsilon \forall \delta \exists x (\sim(\sim |x-a| < \delta) \wedge \sim(|f(x)-L| < \epsilon))$

$\Rightarrow$ $\because$ (Demorgans law)

$\Rightarrow \exists \epsilon \forall \delta \exists x (|x-a| < \delta \wedge |f(x)-L| \geq \epsilon)$

Rules of Inference



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Rules of Interferences

Let us consider a compound statement P. If some person are arguing over this compound statement. Then some person are in the favour of that statement and some are against.

After some arguments we must reach at a conclusion. But every intermedia value statement are called premises & the final statement is called conclusion.

~~All the intermediate statements are obtained from the hypothesis by using somes rules which called don't rule of secondarly~~

Rules of Interferences

All the intermediate statements are obtained from the hypothesis by using some rules which are called rules of inferences.

Argument

An argument is a sequence of propositions. In other words whenever 2 or more statements are combined which don't contain any variables then it is called an Argument

Example

If I will go to Puri or Konark then I will go to the sea beach.

* NOTE

If I will go to Puri or Konark then I willl go to the sea beach. (Conclusion is True)

* So (An argument is valid when the conclusion is True).

Propositional Argument

It is a sequence of statements of which atleast one of the statement much contain a variable.

Example

If $x+2=5$ or $x-1=2$ then $x^2=9$

* **NOTE**

A propositional argument will be valid if the conclusion is True.

Rules of Interference

① Law of Modus Ponens

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

This law can be written as

$$\boxed{p \wedge (p \to q) \equiv q}$$

→ Here, $\{p \wedge (p \to q)\} \to q$ is a Tautology.

(II) Law of Modus Tollens

$$\begin{array}{l} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$$

$$\left.\begin{array}{l} \sim q \\ p \to q \end{array}\right\} \text{conjuction}$$

$$\boxed{\sim q \wedge (p \to q) \equiv \sim p}$$

→ Here, $\{\sim q \wedge (p \to q)\} \to \sim p$ is a tautology.

(III) Hypothetical Syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

This can be written as

$$\boxed{(p \to q) \wedge (q \to r) \equiv p \to r}$$

→ Here, $\{(p \to q) \wedge (q \to r)\} \to p \to r$ is a Tautology.

(IV) Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

This can be written as

$$\boxed{(p \vee q) \wedge (\sim p) \equiv q}$$

→ Here $\{(p \vee q) \wedge (\sim p)\} \to q$ is a Tautology.

(V) Addition

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

This can be written as $\boxed{p \equiv p \vee q}$

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→ Here, $p \Rightarrow p \vee q$ is a Tautology.

(VI) Simplification

$$\boxed{\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}}$$

This can be written as $\boxed{p \wedge q \cong q}$

→ Here, $(p \wedge q) \rightarrow p$ is a Tautology.

(VII) Law of Conjunction (First Law)

$$\boxed{\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}}$$

$$\boxed{(p) \wedge (q) \cong p \wedge q}$$

(VIII) Law of Resolution

$$\boxed{\begin{array}{l} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}}$$

$$(p \vee q) \wedge (\sim p \vee r) \cong q \vee r$$

→ Here $\boxed{\{(p \vee q) \wedge (\sim p \vee r)\}} \Rightarrow q \vee r$ is a Tautology.

(Q) State which rule of inference is the basis of the following argument.
It is below freezing and raining now therefore it is below freezing now.

Ans ⇒ Let,
p : It is freezing now
q : It is raining now
So, the given statement can be represented symbolically as $p \wedge q$.
But here the conclusion is the statement p. Hence, the given statement is written as :- $\begin{array}{l} p \wedge q \\ \hline \therefore q \end{array}$

Hence, the Law of simplication is used in this :-

(Q) Which rule of inference is used, In the following statement
If I will go to New Delhi, then I will go to Agra, If I go to Agra then I will go to Taj Mahal. Therefore if I will go to New Delhi then I will go to Taj Mahal.

Ans Let,
p : I will go to New Delhi ; q : I will go to Agra
r : I will go to Taj Mahal

$$\boxed{\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore p \Rightarrow r \end{array}}$$ Law of Hypothetical Syllogism.

* NOTE
Sometimes a hypothetical statement is given by using this statement we can find out the conclusion by rule of Inference

(Q) Let the hypothesis be, It is not Sunny this afternoon and it is colder than yesterday, we will go swimming only if it is Sunny, if we don't go swimming then we will take a canoe trip. If we take a canoe trip then we will be home by sunset. Show that this hypothesis will give the conclusion we will be home by sunset.

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Ans Let,

p: It is sunny this afternoon
q: It is colder than yesterday
r: we will go swimming
s: we will take a canoe trip.
t: We will be home by sunset.

Here, the first hypothesis is
→ $\sim p \wedge q$
, the second hypothesis is
→ $r \to p$
Then, the third & fourth hypothesis are
→ $\sim r \to s$ & $s \to t$

Now,

i) $\sim p \wedge q$ (Hypothesis)
ii) $\sim p$ (Symplification R)
iii) $r \to p$ (Hypothesis)
iv) $\sim r$ (M.T)
v) $\sim r \to s$ (Hypothesis)
vi) s (M.P)
vii) $s \to t$ (Hypothesis)
viii) t (Conclusion)

Q) By using rules of inferences. Find out the conclusion of the hypothesis,
Candy works hard, if Candy works hard then he is a dull boy. If Candy is a dull boy, then he will not get the job.
Show that the conclusion is Candy will not get a job.

Ans Let,

p: Candy works hard
q: He is a dull boy
r: He will get the job.

1st Hypothesis = p
2nd Hypothesis = $p \Rightarrow q$
3rd Hypothesis = $q \Rightarrow \sim r$

Now,

i) p (Hypothesis)
ii) $p \Rightarrow q$ (Hypothesis)
iii) q (Modus Ponens)
iv) $q \Rightarrow \sim r$ (Hypothesis)
v) $\sim r$ (Modus Ponens)

Q) Show that the hypothesis,
If you send a mail, then I will finish writing the program, If you don't send the mail then I will go to sleep early. If I go to sleep early, then I will wave up feeling refreshed.
Find the conclusion of these hypothesis by using the rules of inferences.

Ans → Let,

p : If the mail is send
q ; I will finish writing the program
r : I will go to sleep early
s ; I will wake up feeling refreshed.

1st Hypothesis = $p \Rightarrow q$
2nd Hypothesis = $\sim p \Rightarrow r$
3rd Hypothesis = $r \Rightarrow s$

Now,

i) $p \Rightarrow q$ (Hypothesis)
ii) $\sim p \Rightarrow r$ (Hypothesis)

No such rule
So, we have to change $p \Rightarrow q$

i) $p \Rightarrow q$ (Hypothesis)
ii) $\sim q \Rightarrow \sim p$ (Contrapositive)
iii) $\sim p \Rightarrow r$ (Hypothesis)
iv) $\sim q \Rightarrow r$ (H.S)
v) $r \Rightarrow s$ (Hypothesis)
vi) $\sim q \Rightarrow s$ (H.S)

Hence the conclusion is, If I will finish writing the program then I will wake up feeling refreshed.

Rules for inferences using Quantified Statements

For quantifier statements basically there are four rules

i) Universal Instantitation } $\forall x$
ii) Universal Generalization
iii) Existential Instantitation } $\exists x$
iv) Existential Generalization

(i) Universal Instantitation:

If $p(x)$ be a statement which is true for every x, then for ~~are~~ any arbitrary element c from the domain of viscous of x $p(c)$ is statisfied. This rule is called Universal Instantitation.

Mathematically it is written as:

$$\frac{\forall x\ p(x)}{p(c) \text{ is satisfied}}$$

(ii) Universal Generalization

Suppose $p(x)$ be any statement, if $p(x)$ be true for every arbitrary value of c. where c belongs to the domain of viscous of x. Then $p(x)$ is satisfied for every values of x. Mathematically it is written as:

$$\frac{p(c),\ c \text{ is arbitrary}}{\forall x\ p(x)}$$

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Name of the rule	Rules of Inferences
(i) Universal Instantiation	$\forall x\, p(x)$ / $p(c)$, c is any arbitrary element
(i) Universal Generalization	$p(c)$, c is arbitrary / $\forall x\, p(x)$
(i) Existential Instantiation	$\exists x\, p(x)$ / $p(c)$, some element c.
(i) Existential Generalization	$p(c)$, for some element c / $\exists x\, p(x)$

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(Q) Show that the premises, everyone in the graph theory class has taken a course in ~~Discrete Mathematics.~~ Computer Science and Rama is a student in this class ⇒ Rama has taken a course in Computer Science will be the conclusion.

Ans→ Let,

G(x): x is in the class of Graph Theory.
C(x): x is in the class of Computer Science.

The symbolical form of both the statement are:

(i) $\forall x\, G(x) \rightarrow C(x)$
(ii) G(Rama).
$\overline{C(Rama)}$

Now,

(i) $\forall x\, G(x) \rightarrow C(x)$ (Hypothesis)
(ii) $G(Rama) \rightarrow C(Rama)$ (U. Instantiation)
(iii) G(Rama) (Hypothesis)
(iv) C(Rama) (Modus P)

Hence, Rama has taken a course in Computer Science (Proved)

(Q) Show that the premises, a student in this class has not read the book and everyone in this class has passed in the first examination ⇒ (the conclusion) ~~dosn~~ someone who passed in the first exam has not read the book.

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Ans→ Let,

$C(x)$: x is a student in the class

$B(x)$: x has read the book.

$P(x)$: x has passed the 1st exam.

The symbolical form of the given statements are:-

(i) $\exists x\ C(x) \wedge \sim B(x)$

(ii) $\forall x\ C(x) \Rightarrow P(x)$

Now,

(i)	$\exists x\ C(x) \wedge \sim B(x)$	(Hypothesis)
(ii)	$C(a) \wedge \sim B(a)$	(Ex. Instantiation)
(iii)	$C(a)$	(Simplification Rule)
(iv)	$\forall x\ C(x) \rightarrow P(x)$	
(v)	$C(a) \rightarrow P(a)$	(Univ. Instatiation)
(vi)	$P(a)$	((iii) & (v) using Modus Ponens)
(vii)	$\sim B(a)$	(Simplification R in (ii))
(viii)	$P(a) \wedge \sim B(a)$	
(ix)	$\exists P(x) \wedge \sim B(x)$	(Ex. Generalization)

Hence, the conclusion is someone who passed the 1st examination has not read the book.

(Proved)

(Article 1.5 Completed)

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Article 1.6

Introduction to Proof.

→ **Theorem**

A theorem is a statement which can be shown to be true.

→ **Proposition**

The less important theorems are called propositions.

→ **Proof**

It is a valid argument that establish the truth of the theorems.

→ **Axioms**

These are the statements which has no proofs, but it is assumed to be true always.

→ **Lemma**

A Lemma is the less importance theorem which is used to proof other reasults.

Different method of Proof

⊙ **Direct Method**

In this case to proof $p \Rightarrow q$ we start from p and reach at the point q. Or in other words to prove $p \Rightarrow q$ we assume that p is true and by using this assumtion we verify that q is true.

Example

(Q) Using direct method prove that if n is an even natural number then n^2 is also an even natural number.

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Ans **Proof**

Let, n is an even natural number.

$\Rightarrow n = 2k$, $(k \in N)$

$\Rightarrow n^2 = 4k^2$ (Squaring both sides)

But,

$k \in N$

$\Rightarrow k^2 \in N$

$\Rightarrow 2k^2 \in N$

$\Rightarrow 2(2k^2)$ is even natural number

$\Rightarrow 4k^2$ is an even natural number

$\Rightarrow n^2$ is an even natural number

(Proved)

(8) If n is an odd natural number then n^2 is an odd natural number. Prove it by direct method.

Ans **Proof**

Let, n is an odd natural number.

$\Rightarrow n = 2k+1$, $(k \in N)$

$\Rightarrow n^2 = 4k^2 + 1 + 4k$ (Squaring both side)

$\Rightarrow n^2 = 4k^2 + 4k + 1$

$\Rightarrow n^2 = 2(2k^2 + 2k) + 1$ —— (1)

$k \in N$

$2k^2 + 2k \in N$

$2(2k^2 + 2k)$ is even natural number

$2(2k^2 + 2k) + 1$ is an odd natural number.

So, n^2 is odd natural number.

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(2) # Method of Contraposition

It is one of the indirect method for proving any statement.

In this method to prove any statement $p \Rightarrow q$ we always prove its contrapositive that means $\sim q \Rightarrow \sim p$ by direct method.

Example

(Q) In n is an integer and $3n+2$ is an odd integer. Then n is odd. By using method contraposition prove it.

Ans - Let, p : $n \in Z$ and $3n+2$ is odd.

q : n is odd.

So, the given statement is

$p \Rightarrow q$

To, prove this by contraposition method. We have to prove $\sim q \Rightarrow \sim p$ by direct method. That means we have to P.T.

If n is even then $3n+2$ is even.

Let,

n is even

$\Rightarrow n = 2k, k \in Z$

Now, $3n+2 = 3(2k)+2$

So, $3n+2 = 2(3k+1)$

$\therefore 3n+2 = 2 \times \text{something} = \text{even}$.

$\sim q \Rightarrow \sim p$ is verified

Hence, $p \Rightarrow q$ is proved by contraposition method.

(Q) Using contraposition method P.T.,

If $n = ab$, a & b are +ve integers then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Ans If $a > \sqrt{n}$ or $b > \sqrt{n}$ then $n \neq ab$.

~~Let,~~

$a > \sqrt{n}$ or $b > \sqrt{n}$

$\Rightarrow a \cdot b > \sqrt{n} \times \sqrt{n}$

$\Rightarrow a \cdot b > n$

$\Rightarrow a \cdot b \neq n$

Since, $\sim q \Rightarrow \sim p$ is ~~true~~ proved. So by contrapositive method $p \Rightarrow q$ is true.

③ • Vacuous Method

We know that $p \Rightarrow q$ is true when p is false. Therefore in Vacuous Method to prove $p \Rightarrow q$ is valid it is enough to show that p is false.

④ • Trivial Method

We know that $p \Rightarrow q$ is true when q is true. So, in this method to prove $p \Rightarrow q$ as valid. It is enough to show that q is true.

Example

Q Let P_n be a statement

P_n: If $n > 1$ then $n^2 \geq n$ where $n \in \mathbb{Z}$

Verify the statement P_0 by using vacuous method.

Ans → Let $p: n > 1$

& $q: n^2 \geq n$

So, the given statement in symbolical form is

$p \Rightarrow q$

Now, P_n: If $n > 1$ then $n^2 \geq n$, $n \in \mathbb{Z}$

P_0: If $0 > 1$ then $0^2 \geq 0$

$\therefore p: 0 > 1$

$q: 0^2 \geq 0$

But, p is alway False. By using vacuous method. So, the given statement is true.

Q Let P_n: If a & b are +ve integers and $a \geq b$ then $a^n \geq b^n$. Prove that P_0 is true by using Trivial method.

Ans Let $p: a \geq b$

$q: a^n \geq b^n$

So, $P_n: p \Rightarrow q$

P_0: If $a \geq b$ then $a^0 \geq b^0$

Here,

$p: a \geq b$

$q: a^0 \geq b^0$

Since, $a^0 = b^0 = 1$ always. So, it is clear that q is true. So, by trivial method $p \Rightarrow q$ is true.

Rational Number

If p & q be any ~~xy~~ two integers ~~then~~ then p/q is called a rational number provided that $q \neq 0$.

Irrational Numbers

The real numbers which are not rational are called irrational number.

Ex → $\sqrt{2}, \sqrt{3}, \sqrt{5}$.

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Using direct method, ~~show that~~ prove that,
ⓐ The sum of 2 rational number is rational
ⓑ The subtraction of 2 rational number is rational
ⓒ The product of 2 rational number is ~~of~~ rational
ⓓ The division of 2 rational number is rational

~~Let, r & q ∈ Z & where~~

Ans Let, $r, q \rightarrow Q$ (rational number)

$r = a/b \quad s = c/d$

$a, b, c, d \in Z \quad b, d \neq 0$ → (Integer)

$r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in Q$

Hence, sum of two Rational number is Rational N.

5 Method of Contradiction

Suppose we have to prove one of the statement $p \Rightarrow q$ is true.

In contradiction method initially we assume that let p & $\sim q$ are true. If we start the proof ~~this~~ by using this assumtion then ultimately we reach at a wrong conclusion.

This wrong conclusion ~~arises~~ arises due to the wrong assumption.

So, $p \Rightarrow q$ is true.

Q Using contradiction method. P.T. $\sqrt{3}$ is an irrational number.

Ans→ If possible let $\sqrt{3}$ is a rational number.

So, $\sqrt{3} = p/q$ and $q \neq 0$, $(p, q) = 1$ (G.C.D)

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$\Rightarrow 3 = p^2/q^2$

$\Rightarrow p^2 = 3q^2$ —— (1)

$\Rightarrow p^2$ is a multiple of 3

$\Rightarrow p$ is a multiple of 3

Let $p = 3m, \quad m \in Z$

$p^2 = 9m^2$

$\Rightarrow 3q^2 = 9m^2$ (By — (1))

$\Rightarrow q^2 = 3m^2$

$\Rightarrow$ ~~q² is a~~ q^2 is a multiple of 3

$\Rightarrow q$ is a multiple of 3.

Since p & q are multiple of 3 so the G.C.D of p and q maybe 3 or multiple of 3. This is a contradiction which contradicts our assumption that G.C.D of $(p, q) = 1$. This contradiction arises due to the wrong assumption that $\sqrt{3}$ is a rational number. Hence $\sqrt{3}$ is an irrational number.

Proof ~~of~~ of Equivalence

Sometimes in a given question more than 1 conclusions are given. Let the conclusions are $P_1, P_2, P_3, \dots P_n$. To prove these conclusions are equivalent. It is enough to show that

$P_1 \rightarrow P_2, \; P_2 \rightarrow P_3, \; P_3 \rightarrow P_4, \; \dots, \; P_{n-1} \rightarrow P_n, \; P_n \rightarrow P_1$

⑧ If n is an integer then show that the following statement are equivalent.
i) n is an odd integer.
ii) $(n+1)$ is an even integer.
iii) n^2 is an odd integer.

Ans → Let us assume that,
P_1: n is an odd integer
P_2: $(n+1)$ is an even integer
P_3: n^2 is an odd integer
To prove these 3 states are equivalent. It is enough to show that.
$P_1 \to P_2$, $P_2 \to P_3$ & $P_3 \to P_1$

$P_1 \to P_2$
Let n is odd.
we have to show $n+1$ is even.
Since n is odd,
$\Rightarrow n = 2k+1$
$\Rightarrow n+1 = 2k+1+1$
$\Rightarrow n+1 = 2(k+1)$ $(k \in Z)$
$\Rightarrow n+1$ is even

$P_2 \to P_3$

Let $n+1$ is even.

$\Rightarrow n+1 = 2k$, $k \in Z$
$\Rightarrow n = 2k-1$
$\Rightarrow n^2 = (2k-1)^2$
$\Rightarrow n^2 = 4k^2+1-4k$
$\Rightarrow n^2 = 4k^2-4k+1$
$\Rightarrow n^2 = 2(2k^2-2k)+1$
$\Rightarrow n^2 = 2m+1$
n^2 is odd.

$P_3 \to P_1$
Let n^2 is odd. We have to prove that n is an odd integer.
If possible, Let n is even integer.

$n = 2k$, $k \in Z$
$n^2 = 4k^2$
$n^2 = 2(2k^2)$
n^2 is even

This is a contradiction which contradicts our assumption that n^2 is odd. This contradiction arises due to the wrong ~~assump~~ assumption n is even. So, n is odd.

⑧ Find the mistakes in the following proof.

A→ ~~$y = 3x$~~
① $x = y$
② $\Rightarrow x^2 = xy$ (By multiplying x b/s)
③ $\Rightarrow x^2 - y^2 = xy - y^2$ (By subtracting y^2 both sides)
④ $\Rightarrow (x-y)(x+y) = y(x-y)$
⑤ $\Rightarrow x+y = y$ (Cancellation law)
⑥ $\Rightarrow 2y = y$ $(\because x = y)$
⑦ $\Rightarrow 2 = 1$ (Cancelation law)

Ans Here the mistake occurs in step 5 because zero cannot be cancelled in cancellation law.

Chapter - 2

METHOD OF INDUCTION

It is a process by using which we can prove any mathematical statement P_n, where $n \in N$.
That means method of induction is applicable ~~oly~~ only on the set of natural number.

Let, P_n be any statement, where $n \in q$.
If P_n be true for $n = 1$. Then assuming the truth of P_n for $n = k$. If we verify that P_n is true for $n = k+1$. Then P_n is true for every natural number n.

NOTE :-
Generally there are 3 steps in Induction.

Step 1 → In this step we show that P_n is true for $n = 1$

Step 2 → Let assume that P_n is true for $n = k$, where $k < 1$

Step 3 → we have to show that P_n is true for $n = k+1$.

Examples

(1) Using Induction, prove that $1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Answer → Let $P_n : 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1 :
We have to show that P_n is true for $n = 1$.

L.H.S

$1^2 = 1$

R.H.S

$$\frac{(1+1)(2+1)}{6}$$

$$= \frac{1\times 2\times 3}{6}$$

$$= 1$$

$LHS = R.H.S$

Hence P_n is true ~~for~~ $n = 1$

Step 2 :
Assume that P_n is true for $n = k$
i.e.

$$1^2+2^2+3^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6} \quad ---(1)$$

Step 3 :
We have to show that P_n is true for $n = k+1$.

L.H.S

$$1^2+2^2+3^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$1^2+2^2+3^2+\dots+k+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

[By eq^n - (1)]

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

$$= (k+1)\left[\frac{2k^2+4k+3k+6}{6}\right]$$

$$= (k+1)\left[\frac{2k(k+2)+3(k+2)}{6}\right]$$

$$= (k+1)\left[\frac{(2k+3)(k+2)}{6}\right]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$= R.H.S$

So, P_n is true for $n = k+1$

Hence, by Induction P_n is true for all $n \in N$.

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Q Using Induction prove that

Ans $P_n : \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

(i) L.H.S $\frac{1}{2}$, R.H.S $1 - \frac{1}{2^1} = 1 - \frac{1}{2} = \frac{1}{2}$

L.H.S = R.H.S

(ii) $\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ — (i)

(iii) $\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$

L.H.S

$$\underbrace{\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k}} + \frac{1}{2^{k+1}}$$

$$1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \quad (\text{By eq}-(i))$$

$$1 - \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$$

$$= 1 - \frac{2-1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^{k+1}}$$

$= R.H.S$

(Q) Find a formulae for

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\dots+\frac{1}{n(n+1)}$$

by induction and prove it by using method of Induction.

Ans Here,

$$T_1 = \frac{1}{1\cdot2} = \frac{1}{2}$$

$$T_1+T_2 = \frac{1}{2}+\frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$T_1+T_2+T_3 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{6+2+1}{12} = \frac{3}{4}$$

So, it is clear that

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\dots+\frac{1}{n(n+1)} = \frac{n}{n+1}$$

Let,

$$P_n : \frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{n(n+1)} = \frac{n}{n+1}$$

(1) L.H.S $\frac{1}{1\cdot2}$; R.H.S $\frac{1}{1+1}$

$= \frac{1}{2}$ $\qquad = \frac{1}{2}$

L.H.S = R.H.S

(2) $$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{k(k+1)} = \frac{k}{k+1} \quad \text{—(i)}$$

(3) $$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\underbrace{\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\dots+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2} \quad \text{(By using (i))}$$

L.H.S

$$\frac{1}{k+1}\left(k+\frac{1}{k+2}\right) = \frac{1}{k+1}\left(k+\frac{1}{k+2}\right)$$

$$= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$= \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$

$$= \frac{(k+1)}{(k+2)} \quad \text{R.H.S}$$

(Q) Using induction prove that

$$1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1}, \quad r \neq 1$$

Ans

(1) L.H.S $\cancel{1}$ $\qquad$ R.H.S $\cancel{\frac{r^{1+1}-1}{1-1}} = \cancel{\frac{r^2-1}{r-1}} = r$

P.T.O

~~L.H.S = R.H.S~~

Ans Let,
$P_n : 1+x^1+x^2+ \dots + x^n = \frac{x^{n+1}-1}{x-1}, x \neq 1$

(I) L.H.S

$1+x^1$ (Because P_n has $n+1$ terms)

$= 1+x^1$

R.H.S

$\frac{x^2-1^2}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = (x+1)$

L.H.S = R.H.S

(II) $1+x+x^2+x^3+ \dots +x^k = \frac{x^{k+1}-1}{x-1}$ — (1)

(III) $\underbrace{1+x+x^2+ \dots +x^k}+x^{k+1} = \frac{x^{k+2}-1}{x \cancel{+1-1}}$

$\cancel{x^{k+1}} \frac{x^{k+1}-1}{x-1} + x^{k+1} = \frac{x^{k+2}-1}{x}$

L.H.S

$\frac{x^{k+1}-1}{x-1} + x^{k+1}$

$\Rightarrow \frac{x^{k+1}-1+x^{k+1}(x-1)}{x-1}$

$\Rightarrow \frac{\cancel{x^{k+1}}-1+x^{k+2}-\cancel{x^{k+1}}}{x-1}$

$\Rightarrow \frac{x^{k+2}-1}{x-1}$ (R.H.S)

(Q) Using Induction prove that

$3^{2n}-8n-1$; is divisible by 64.

Ans Let,
$P_n : 3^{2n}-8n-1$ is divisible by 64.

(I) We have to show P_n is true for ~~when~~ $n=1$

when $n=1$, $3^{2n}-8n-1$
$\Rightarrow 3^2-8-1$
$\Rightarrow 9-9$
$\Rightarrow 0$

Since 0 is divisible by 64. so P_n for $n=1$.

(II) Assume than P_n is true for $n=k$

$3^{2k}-8k-1$ is divisible by 64.
So, there exist an integer m such that,
$\Rightarrow 3^{2k}-8k-1 = 64m$
$\Rightarrow 3^{2k} = 64m+8k+1$ — (1)

(III) We have to show that P_n is true for $n=k+1$

$3^{2(k+1)} - 8(k+1) - 1$ ~~= 64m~~ is divisible by 64.

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L.H.S

$3^{2(k+1)} - 8(k+1) - 1$

$= 3^{2k+2} - 8k - 8 - 1$

$= 3^{2k}\, 3^2 - 8k - 9$

$= 9(64m + 8n + 1) - 8k - 9$

$= 64\times 9m + 72n + \cancel{9} - 8k - \cancel{9}$

$= 64\times 9m + 64k$

$= 64(2m + k)$

So, $3^{2(k+1)} - 8(k+1) - 1$ is divisible by 64.

so,

P_n is true for all values.

(Q) If a set has n number of elements then prove that it has 2^n number of subsets by using Induction.

Ans The above statement can be written mathematically as

If $|A| = n$ then $|P(A)| = 2^n$

Let,

P_n : If $|A| = n$ then $|P(A)| = 2^n$

(i) We show that P_n is true for $n=1$

Let,

$|A| = 1$

$A = \{a\}$

$P(A) = \{\phi, \{a\}\}$

$|P(A)| = 2$

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(ii) Let P_n be true for $n=k$

If $|A| = k$ then $|P(A)| = 2^k$ —— (i)

(iii) We have to show that P_n is true for $n=k+1$

If $|A| = k+1$ then $|P(A)| = 2^{k+1}$

Let, $|A| = k+1$

$\Rightarrow A = \{a_1, a_2 \cdots, a_k, a_{k+1}\}$

$\Rightarrow A = B \cup \{a_{k+1}\}$

$(\because$ when $B = \{a_1, a_2, a_3 \cdots, a_k\})$

B has 2^k subsets. (Since, B has k elements)

Again, $\{a_{k+1}\}$ has 2 subsets.

But it is clear that the union of every subset of B with the subset of set $\{a_k + 1\}$ gives a subset of A.

So, the number of subsets of A is ~~the~~,

$A = B \times \{a_k + 1\}$

$|P(A)| = |P(B)| \times |P\{a_k + 1\}|$

$= 2^k \times 2$

$= 2^{k+1}$

(Q) Using Induction prove that n is less than 2^n for $\forall n$.

$n < 2^n, \forall n$.

Ans Let,

$P_n : n < 2^n, \forall n$.

(i) P_n is $n=1$

$1 < 2^1$

when $n=1$, $1 < 2^1$ so, P_n is true for $n=1$.

(ii) Let, P_n is true for $n=k$
So,
$$k < 2^k \quad \text{——— (i)}$$

(iii) We have to show that ~~Let~~, P_n is true for $n=k+1$
So,
$$k+1 < 2^{k+1}$$

Now,
$$k < 2^k$$
$$k+1 < 2^k+1 \quad \text{——— (ii)}$$
We know that $(k>1)$
$$1 < 2^k$$
$$1+2^k < 2^k+2^k$$
$$1+2^k < 2 \cdot 2^k$$
$$1+2^k < 2^{k+1} \quad \text{——— (iii)}$$

from (ii) & (iii)
$$x+1 < 2^k+1 < 2^{k+1}$$
$$\Rightarrow x+1 < 2^{k+1}$$

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(9) If $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then prove that $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

Ans → Let

P_n: If $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

Step 1

When $n=1$ $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$$A^1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Statement is true for $n=1$.

Step 2 :-
Assume that P_n is true for $n=k$.

$$A^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \quad \text{——— (1)}$$

Step 3 :
We have to show that P_n is true for $n=k+1$

$$A^{k+1} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$$

$$L.H.S = A^{k+1} = A^k \cdot A^1$$
$$= \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$
$$= \begin{pmatrix} a^k \cdot a + 0 \cdot 0 & a^k \cdot 0 + 0 \cdot b \\ 0 \cdot a + b^k \cdot 0 & 0 \times 0 + b^k \cdot b \end{pmatrix}$$
$$= \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$$

Hence by Induction P_n is true for all N.

(8) Using induction Prove that $2^n > n^2 \mid n > 4$.

Ans → Let P_n: $2^n > n^2 \quad \mid n > 4$

Step 1 :-
We have to show that P_n is true for $n=5$.

L.H.S $2^5 = 32$

R.H.S $5^2 = 25$

$\because 32 > 25$

$\Rightarrow 2^n > n^2$ when $n=5$

$\Rightarrow 2^n > n^2$ when $n=5$

Step2 :-

Assume that P_n is true for $n=k$.

$P_k : 2^k > k^2$ —— (1)

Step3 :-

We have to show that it is true for $n=k+1$.

$P_{k+1} : 2^{k+1} > (k+1)^2$

As from (i) $2^k > k^2$

$2^k.2 > k^2.2$ $\quad$ $2^k.2 > k^2+1+2k$.

$2^k.2 > 2k^2$

$2^k.2 > k^2+k^2$

L.H.S

$2^{k+1} = 2^k.2^1 > k^2.2$ (By (1))

$2^{k+1} > 2k^2$ $\quad \forall k>2$

$2k^2 > (k+1)^2$ —— (ii)

$2^{k+1} > 2k^2 > (k+1)^2$

$\Rightarrow 2^{k+1} > (k+1)^2$

It is true for $n=k+1$

Hence it is true for all N.

*** (8) Using induction prove that $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2^n} \geq 1+\frac{n}{2}$

Ans Let P_n: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2^n} \geq 1+\frac{n}{2}$.

Step 1 :-

We have to show that P_n is true for $n=1$

L.H.S $P_1 = 1+\frac{1}{2} = \frac{3}{2}$ $\qquad$ R.H.S $1+\frac{n}{2} = 1+\frac{1}{2} = \frac{3}{2}$

L.H.S = R.H.S

$\therefore P_n$ is true for $n=1$

Step 2 :-

Assume that P_n is true for $n=k$

$P_k : 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{2^k} \geq 1+\frac{k}{2}$ —— (i)

Step3 :-

We have to show that P_n is true for $n=k+1$

$P_{k+1} : 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^{k+1}} \geq 1+\frac{(k+1)}{2}$

L.H.S

$\cancel{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\dots+\frac{1}{2^{k+1}}}$

$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\dots+\frac{1}{2^{k+1}}$

$\Rightarrow 1+\frac{k}{2}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\dots+\frac{1}{2^{k+1}}$ (by (1))

$\Rightarrow 1+\frac{k}{2}+\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}+\dots+\frac{1}{2^{k+1}}$ $\left(\frac{1}{2^k+1} > \frac{1}{2^{k+1}}\right.$

$\left.\Rightarrow \frac{1}{2^k+2} > \frac{1}{2^{k+1}}\right)$

$\Rightarrow 1+\frac{k}{2}+2^k.\frac{1}{2^{k+1}}$

$\Rightarrow 1+\frac{k}{2}+\cancel{2^k}.\frac{1}{\cancel{2^k}.2}$

$\Rightarrow 1+\frac{k}{2}+\frac{1}{2}$

$\Rightarrow 1+\frac{(k+1)}{2}$ (Ans)

Q) Using induction prove that $4^{n+1} + 5^{2n-1}$ ~~$4^{k+1}+5^{2k-1}$~~ is divisible by 21. $4^{n+1}+5^{2n-1}$

Ans Let P_n : ~~$4^{k+1}+5^{2k-1}$~~ is divisible by 21.

Step 1

We ~~kno~~ have to show that P_n is true for $n=1$.

L.H.S

$4^2 + 5^{2-1}$

$= 16+5$

$= 21$

$\therefore$ 21 is divisible 21

So, P_n is true for $n=1$.

Step 2

Assume that statement is true for $n=k$.

$4^{k+1} + 5^{2k-1} = 21m \quad (m \in Z)$

$\Rightarrow 4^{k+1} = 21m - 5^{2k-1}$

Step 3

We have to show that P_n is true for $n=k+1$.

$P_{k+1} : 4^{(k+1)+1} + 5^{2(k+1)-1} = 21m$

L.H.S $= 4^{k+2} + 5^{2k+1}$

$= 4^{k+1} 4 + 5^{2k+1}$

$= (21m - 5^{2k-1}) \cdot 4 + 5^{2k+1}$ (from (1))

$= 4 \cdot 21m - 4 \cdot 5^{2k-1} + 5^{2k-1} + 2$

$= 4 \cdot 21m - 4 \cdot 5^{2k-1} + 5^{2k-1} \cdot 25$

$= 4 \cdot 21m + 5^{2k-1}(21)$

$= 21(4m + 5^{2k-1})$

P_n is true for all N.

Q) Using induction prove that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Ans Let $P_n : (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Step 1

We have to show P_n is true for $n=1$

L.H.S,

$P_1 = (\cos\theta + i\sin\theta)$

R.H.S

$\cos\theta + i\sin\theta$

L.H.S = R.H.S

Step 2

Assume that it is true for $n=k$.

$P_k : (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ — (1)

Step 3:

We have to ~~kno~~ show that is is true for $n=k+1$

$P_{k+1} = (\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$.

L.H.S

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k (\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

L.H.S = R.H.S

Hence by induction ~~it~~ P_n is true for all N.

Q) Prove that $(A_1 \cup A_2 \cup A_3 \cup \dots A_n)' = A_1' \cap A_2' \cap A_3' \cap \dots A_n'$

Ans Let $P_n : (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n)' = A_1' \cap A_2' \cap A_3' \cap \dots \cap A_n'$

Step 1 :-
We have to show P_n is true for $n=1$.

L.H.S $= A_1'$

R.H.S $= A_1'$

L.H.S = R.H.S

P_n is true for $n=1$

Step 2 :-
Assume that P_n be true for $n=k$.

$P_k : (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k)' = A_1' \cap A_2' \cap A_3' \dots A_k'$ — (1)

Step 3 :-
We have to show that P_n is true for $n=k+1$

L.H.S

$= (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_{k+1})'$

$= (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k \cup A_{k+1})'$

$= (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_k)' \cap (A_{k+1})'$ (By demorgan's Law)

$= A_1' \cap A_2' \cap \dots \cap A_n' \cap A_{k+1}'$ (By (1))

= R.H.S

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Strong Induction

Let P_n be any statement where $n \in N$. To prove the validity of the statement P_n by using Strong Induction, we have to follow the following 3 steps.

Step I – We have to show that P_n is true for $n=1$

Step II – Let us assume that P_n is true for all natural numbers which are less than or equal to k.

Step III → We have to show that P_n is true for $n=k+1$.

* NOTE
~~of~~ In the procedure of induction and strong induction the deviation occurs in step 2 only whereas the first & 3rd step are same in both cases

Example

Using strong induction prove that any integer greater than 1 can be expressed as a product of prime factors.

Ans
P_n : If $n \in Z$ & $n > 1$ then n can be expressed as a product of prime factors

Step (1) We have to show that P_n is true for $n=2$

$n=2$

~~that~~ Since $n=2$ is a Prime Number. so, It is clear that P_n is true for $P_n = 2$

Step (2) Let us assume that P_n is true for all nautral numbers $\leq K$,

Step (3) We have to show that P_n is true for $n=k+1$. That means we have to show that $k+1$ can be extressed as

as product of Prime Number.
To prove this 2 cases will arise.

Case-I

$k+1 \rightarrow$ Prime

Let $k+1$ is a Prime Number.
Since, $k+1$ is a Prime number, so it is clear that $k+1$ is expressed as a Product of Prime No.

Case-II

Let $k+1$ is a composite number. since, $k+1$ is a composite no. So, it can be expressed as a product of Natural Number.

$k+1 = a \cdot b \quad , \quad a, b < k$

Since, a & b are less than k.
So, by step(2) it can be expressed as a product of Prime Factors.

Since, a & b ~~are~~ both can be expressed as a product of Prime factors. so, $a \cdot b$ can be expressed as a product of Prime Factors.

$k+1$ can be expressed as a Prime Factors.

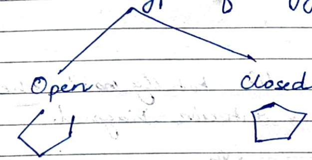
Some Basic Definations regarding Geometrical figures

→ Polygon

A many sided figure is called a Polygon

→ Types of Polygons

There are 2 types of Polygon

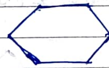


→ Simple Polygon

A polygon in which no 2 non-adjacent edges ~~touches~~ ~~or~~ intersects each other is called a simple Polygon.

Example

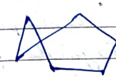
Pentagon, Hexagon



Non-Simple

A polygon is which non-adjacent edges intersects each other is called a non-simple polygon.

Example



→ Diagonal

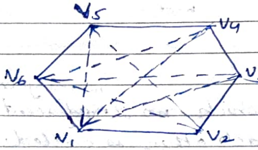
By joining any 2 non-adjacent vertices of a Polygon a diagonal is obtained. But generally there a 2 types of diagonals:

i) Interior Diagonal

ii) Exterior Diagonal.

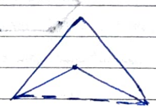
Interior Diagonal

The diagonal which lies totally inside the polygon is called an Interior Diagonal.



Exterior Diagonal

The diagonal which doesnot lies inside the Polygon is called exterior diagonal.



Triangulation

It is a process of breaking a simple polygon into triangles by joining the diagonals provided that non of the diagonal intersect other diagonal.

Example

(Q) Using strong induction prove that a simple polygon with n sides where n is an integer with $n \geq 3$ can be triangulated into $n-2$ triangles.

Ans → P_n : $n \to n-2$, $n \geq 3$

Let P_n: If a simple polygon contains n number of vertices then it can be tranculated into $(n-2)$ no. of triangles.

Step ①

We have to show that P_n is true for $n=3$.

If $n=3$ then the polygon contains 3 vertices which is a triangle.

Since, it has no diagonals. So, it can be triangulated into only 1 triangle.

So, P_n is true for $n=3$.

Step ②

Assume that P_n is true for $n \leq k$. If a simple polygon contain n number of vertices where $n \leq k$ then it can be triangulated into $n-2$ no. of triangles.

Step ③

We have to show that P_n is true for $n=k+1$.

That means if a simple polygon contains $k+1$ no. of vertices. Then it can triangulated into $k-1$ no. of time.

Let us consider a simple polygon with $k+1$ no. of vertices. By joining any one diagonal. The polygon will be divided in another 2 simple polygons. Say, P_1 & P_2. Let the no. of vertices in P_1 is s.

& the no. of vertices in P_2 is $k+1-s+2$
$= k-s+3$

Since, P_1 & P_2 contains less than k no. of vertices so by step 2. P_1 can be triangulated into $s-2$ triangle. Similarly, ~~the no. of~~ P_2 can be triangulated into $k-s+3-2$ no. of triangles. That is $k-s+1$.
So, the total no. of triangle in the polygon is
$\cancel{k}.\ \cancel{s}-2+k\cancel{-s}+1$
$k-1$

So, the no. of triangles is $(k-1)$
So, P_n is true for $n=k+1$

Reacurrence Reinforcement

$f: N \rightarrow R$

Sequence:-
A sequence is a set of finite or infinite elements provided that all the terms of the sequence will satisfy a particular relation.

For example,
Let a_n be a sequence. where $a_n = n^2+1$

$a_n = n^2+1$
$a_1 = 1^2+1 = 2$
$a_2 = 2^2+1 = 5$
$a_3 = 3^2+1 = 10$
$a_4 = 4^2+1 = 17$.

Q Write the n^{th} term of the sequence
$(1, 3, 5, 7, \ldots\ldots)$

→ $t_n = 2n+1/2n-1$

Recurrence:-
The Recurrence relation for the sequence n is defined as it is an equation in which n is related with its predecessors, $a_1, a_2, a_3, \ldots, a_{n-1}$

Ex
Q Find the recurrence relation for the sequence
$(a_n) = 5^n$

→ Here, $a_n = 5^n$
$a_{n-1} = 5^{n-1}$

$\Rightarrow a_{n-1} = \frac{5^n}{5^1}$

$\Rightarrow 5a_{n-1} = 5^n$
$\Rightarrow \boxed{5a_{n-1} = a_n}$

Q A person is depositing

Find the recurrence relation of this and find the n term after 8 years.

Q) Find the first 5 terms of the ~~recurrance rela~~ sequence whose recurrance relation is $a_n = 5a_{n-1}$, $\underbrace{a_0 = 3}_{\text{Initial condition}}$.

→ $a_1 = 5a_0$
$a_1 = 5 \times 3 = 15$

$a_2 = 5a_1 = 5 \times 15 = 75$
$a_3 = 5a_2 = 5 \times 75 = 375$
$a_4 = 5a_3 = 5 \times 375 = 1875$
$a_5 = 5a_4 = 5 \times 1875 = 9375$

Q) Verify wheater the sequence $a_n = 3n$ is solution of the recurrance relation: $a_n = 2a_{n-1} - a_{n-2}$ or not. Also verify weather $a_n = 2^n$ & $a_n = 6$ as solution of the above recurrance relation.

→ $a_n = 2a_{n-1} - a_{n-2}$ — (1)

$a_n = 3n$

$a_{n-1} = 3(n-1)$; $a_{n-2} = 3(n-2)$

R.H.S

$a_n = 2(3(n-1)) - 3(n-2)$
$= 2(3n-3) - 3n + 6$
$= 6n - \cancel{6} - 3n + \cancel{6}$
$= 3n$ L.H.S

So, $a_n = 3n$ is a solution.

$a_n = 2^n$
$a_{n-1} = 2^{n-1}$ $a_{n-2} = 2^{n-2}$

R.H.S

$2a_{n-1} - a_{n-2}$
$= 2 \cdot 2^{n-1} - 2^{n-2}$
$= \left(\frac{2^n}{2} - \frac{2^n}{2^2} \right)$

$$= 2^n\left(1 - \frac{1}{2^2}\right)$$

$$= 2^n\left(1 - \frac{1}{4}\right)$$

$$= 2^n \frac{3}{4}$$

$$= \frac{3 \cdot 2^n}{2^2}$$

$$= 3 \cdot 2^{n-2}$$

$$= 3 \cdot a_{n-2} \neq \text{L.H.S}$$

So, $a_n = 3n$ is not a solution of this equation.

25/11/21

Q// From recurrence relationship and give the initial condition for the no. of bit strings of length n which doesnot contain 2 consecutive zeroes.
Also find out, how many bit strings are there of length 6.

⇒ Let a_n be the number of bit strings of length n in which 2 consecutive zeros are not present.
Now,
a_1 is the no. of bit strings of length 1 in which 2 consecutive zeros are not present.
Here the bit strings are zero and one.
So, $a_1 = 2$.
Again a_2 is the number of bit strings of length 2 in which 2 consecutive zeros are not present so, here the strings are (1,0), (0,1), (1,1).
~~Then~~ To find a_3, 2 cases will arise. So

Case 1
Let the string ends with 1 in this case there is no restriction for the preceding place of 1.
So, the preciding 2 places can be filled with a_2 ways.

Case 2
Let the string ends with 0. In this case there is some restriction for the preceding place of 1. So the 2nd place is alway must have 1. So the remaining 1 place can be filled with a_1 ways.

So $a_3 = a_1 + a_2$

a_3 - - - - 1 $\quad$ a_3 ways

a_1

a_2 - - - 10 $\quad$ a_2 ways

$$a_4 = a_2 + a_3$$
$$a_5 = a_3 + a_4$$
$$a_6 = a_4 + a_5$$

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$

Article - 6.1 (Completed)

Article 6.2

Recurrence Relation

We know that, the Rec. Relation for a sequence n is relation of a_n with its predecessor.

Ex

$5a_{n-1} + \cancel{6a_{n-2}}\ 6a_{n-2}$

$a_n = 2a_{n-1} + 3a_{n-2} + 2^n$

Types of recurrence relation :-

① **Linear** - A recurrence relation is said to be linear if in more of the term two or more a_i's are not multiplied. Otherwise the equation is non-linear.

$a_n - 6a_{n-1} + 3a_{n-2} = 0$ (Linear)

$a_n - 6a_{n-1} + 3a_{n-2} = 0$ (Non-Linear)

$a_n - 6a_{n-1}a_{n-2} + 5a_{n-3} = 0$ (Non-linear)

② **Homogeneous** - A recurrence relation is said to be homogeneous if every non-zero terms contain $1\ a_i$.

ex

$a_n - 5a_{n-2} + 6a_{n-3} = 0$ (Homogeneous)

$a_n + 6a_{n-1} + 5a_{n-2} = 3^n$ (non-Homogeneous)

Degree - Degree of a recurrence relation is the difference between the highest suffix of A_i and lowest suffix of a_i.

example

$a_n - 3a_{n-1} + 5a_{n-2} + 6a_{n-3} = 0$

degree $= n-(n-3) = 3$

∴ Hence the degree is 3.

Solution of homogeneous Recurrence Relation

Let us consider a homogeneous recurrence relation.

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$

Let $a_n = \alpha^n$ be the solution.

Substituting this in the given equation.

$\alpha^n - c_1\alpha^{n-1} - c_2\alpha^{n-2} - c_3\alpha^{n-3} = 0$

$\Rightarrow \alpha^n - c_1\dfrac{\alpha^n}{\alpha} - c_2\dfrac{\alpha^n}{\alpha^2} - c_3\dfrac{\alpha^n}{\alpha^3} = 0$

$\Rightarrow \alpha^n\left(1 - \dfrac{c_1}{\alpha} - \dfrac{c_2}{\alpha^2} - \dfrac{c_3}{\alpha^3}\right) = 0$

$\Rightarrow \alpha^n\left(\dfrac{\alpha^3 - c_1\alpha^2 - c_2\alpha - c_3}{\alpha^3}\right) = 0$

$\Rightarrow \dfrac{\alpha^3 - c_1\alpha^2 - c_2\alpha - c_3}{\alpha^3} = 0$

$\Rightarrow \boxed{\alpha^3 - c_1\alpha^2 - c_2\alpha - c_3 = 0}$ This is characteristic equation.

Solving the characteristic equation we get different values of α. Basing upon this α we can find out the solution in 3 different cases.

Case 1

P.T.O.

classmate
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$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$

$a_n = \alpha^n$

$a_n - C_1 a_{n-1} - C_2 a_{n-2} - \cdots - C_k a_{n-k} = 0$

$\alpha^k - C_1 \alpha^{k-1} - C_2 \alpha^{k-2} \cdots - C_k$

$\Downarrow$

(i) $-\ \alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_k$

$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_k \alpha_k^n$

(ii) – Real and ~~distinct~~ equal

$a_n = (A_1 + A_2 n + A_3 n^2 + \cdots + A_k n^{k-1}) \alpha^n$

(iii) $a_n = A_1 (\alpha + i\beta)^n + A_2 (\alpha - i\beta)^n$

Solving the characteristic eq^n we get different values of α.

→ Basing upon these α we can find out the solution in 3 different cases.

Case I

If all the roots of the characteristic eq^n are real and distinct.

Say = $\alpha_1, \alpha_2 \cdots \alpha_k$

then the sol^n is

$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_k \alpha_k^n$

where $A_1, A_2, A_3, \cdots, A_k$ are any constant

Case II

If all the roots are real and equal.

say = $\alpha, \alpha, \alpha, \alpha, \cdots$ (k times)

then the sol^n is

$a_n = (A_1 + A_2 n + A_3 n^2 + \cdots + A_k n^{k-1}) \alpha^k$

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Case III

If the roots are complex conjugate

say = $\alpha + i\beta$ & $\alpha - i\beta$

Then the sol^n is

$a_n = A_1 (\alpha + i\beta)^n + A_2 (\alpha - i\beta)^n$

Example

Q Solve the recurrance relation :-

$a_n - 7a_{n-1} + 12 a_{n-2} = 0$

Ans → $\alpha^2 - 7\alpha + 12 = 0$

This characteristic ~~eq~~ relation is homogenius ~~eq~~ rel^n.

$\alpha^2 - 3\alpha - 4\alpha + 12 = 0$

$\Rightarrow \alpha(\alpha - 3) - 4(\alpha - 3) = 0$

$\Rightarrow (\alpha - 3)(\alpha - 4) = 0$

$\Rightarrow \alpha = 3, \alpha = 4$

* Hence, both the roots are real and distinct.

$a_n = A_1 3^n + A_2 4^n$

This is the solution of the required R.R.

Q Solve the recurrance relation :-

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

where the initial conditions are.

$a_0 = 1, \quad a_1 = 3$.

Ans → $\alpha^2 - 5\alpha + 6 = 0$

$\Rightarrow \alpha^2 - 3\alpha - 2\alpha + 6 = 0$

$\Rightarrow (\alpha - 3)(\alpha - 2) = 0$

$\Rightarrow \alpha = 3, 2$ (Real and distinct)

$a_n = A_1 2^n + A_2 3^n$ —— (i)

Putting $n=0$ in eq^n -(i), we get.

$a_0 = A_1 + A_2$

$A_1 + A_2 = 1$ —— (ii)

Putting $n=1$ in eq^n -(i), we get

$a_1 = 2A_1 + 3A_2$

$\Rightarrow 2A_1 + 3A_2 = 3$ —— (iii)

(iii) $\times 1$ $= \cancel{2A_1} + 3A_2 = 3$

(ii) $\times 2$ $= \cancel{2A_1} + 2A_2 = 2$

$\quad - \quad - \quad -$

$A_2 = 1$ ✓

$\Rightarrow A_1 = 0$ ✓

$a_n = A_1 2^n + A_2 3^n$

$a_n = 3^n$

Q) Solve the recurrence relation:

$a_n - 6a_{n-1} = -11a_{n-2} + 6a_{n-3}$

Ans → $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

This is Homogeneous recurrence relation.

$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$

$\alpha^3 - \alpha^2 - 5\alpha^2 + 5\alpha + 6\alpha - 6 = 0$

$\alpha^2(\alpha-1) - 5\alpha(\alpha-1) + 6(\alpha-1) = 0$

$(\alpha-1)(\alpha^2 - 5\alpha + 6) = 0$

$(\alpha-1)(\alpha-2)(\alpha-3) = 0$

$\alpha = 1, 2, 3$

Q) Solve the recurrence relation:

$a_n = 4a_{n-1} + a_{n-2} = 0$

Ans → $\alpha^2 - 4\alpha + 1 = 0$

$\alpha = \dfrac{4 \pm \sqrt{16-4}}{2}$

$= 2 \pm \sqrt{3}$

$2+\sqrt{3}$, $2-\sqrt{3}$

Q) Solve the recurrence relation:

$a_n = 4a_{n-1} - 4a_{n-2}$

Ans → $a_n - 4a_{n-1} + 4a_{n-2} = 0$

$\alpha^2 - 4\alpha + 4 = 0$

$(\alpha-2)^2 = 0$

$\alpha = 2, 2$

$a_n = (A_1 + A_2 n)2^n$

Q) Solve the recurrence relation:

$a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$.

Ans. $\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0$

$\Rightarrow (\alpha-1)^3 = 0$

$\Rightarrow \alpha = 1, 1, 1$

$a_n = (A_1 + A_2 n + A_3 n^2)1^n$

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Q) $a_n - 2a_{n-1} + 4a_{n-2} = 0$

$\alpha^2 - 2\alpha + 4 = 0$

$\alpha = \frac{2 \pm \sqrt{4 - 4\cdot 4}}{2}$

$= 1 \pm \sqrt{3}i$

So, the solⁿ is $a_n = A_1(1+\sqrt{3})^n + A_2(1-\sqrt{3})^n$

Q) Solve the recurrance relation:

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

where initial conditions are $a_0 = 1,\ a_1 = -2,\ a_2 = -1$

Ans → $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$

$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$

$(\alpha + 1)^3 = 0$

$\alpha = -1, -1, -1$

$a_n = (A_1 + A_2 n + A_3 n^2)(-1)^n$ — (I)

$a_0 = A_1 (-1)^0$

$\boxed{1 = A_1}$

$a_0 = 1$

$a_1 = (A_1 + A_2 + A_3)(-1)$

$A_1 = 1$

$-2 = -1 - A_2 - A_3$

$-1 = -A_2 - A_3$

$\boxed{A_2 + A_3 = 1}$

$a_2 = (A_1 + A_2 2 + A_3 4)$

$\Rightarrow -1 = A_1 + 2A_2 + 4A_3$

$\Rightarrow -2 = 2A_2 + 4A_3$

$\Rightarrow \boxed{-1 = A_2 + 2A_3}$ — (III)

$A_2 + A_3 = 1$

$A_2 + 2A_3 = -1$

$-A_3 = 2$

$A_2 = 3$

Substituting the values of A_1, A_2, A_3 in (I)

$a_n = (1 + 3n - 2n^2)(-1)^n$

This is the required solⁿ of the homogeneous Reccurance Relation.

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Solution of Non-Homogeneous Recurrance Relation:-

The standard form of a non-homogeneous fxⁿ is:

$a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = f(n)$

This non-homogeneous eqⁿ has 2 types of solⁿ.

i) Homogeneous solⁿ $a_n^{(h)}$

ii) Particular solⁿ $a_n^{(p)}$

→ Homogeneous solⁿ

The homogeneous eqⁿ of the given non-homogeneous is obtained by putting $f(n) = 0$.

So, the homogeneous eqⁿ is

Solving this we get the homogeneous solution, which is denoted by $a_n^{(h)}$

→ Particular solⁿ

Depending upon the value of $f(n)$ the particular solution can be determined in different cases.

Let $f(n)$ is a polynomial of degree 'k'.

Let $f(n) = B_0 + B_1 n + B_2 n^2 + \cdots + B_k n^k$

since $f(n)$ is a polynomial of degree k so the particular solution must be a polynomial of degree k.

Let $a_n(p) = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_n n^k$ — (2)

so,

$a_{n-1} = \alpha_0 + \alpha_1 (n-1) + \alpha_2 (n-1)^2 + \cdots + \alpha_n (n-1)^k$

$a_{n-2} = \alpha_0 + \alpha_1 (n-2) + \alpha_2 (n-2)^2 + \cdots + \alpha_n (n-2)^k$

substituting this in equation (1) and comparing different power of n in both sides we get some equations. On solving this equation with $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k$. Putting this in equations we get $a_n(p)$.

So, the total solution is $a_n = a_n^{(h)} + a_n^{(p)}$

example

Solve the non-homo. Reccurance Relation:

$a_n + 5a_{n-1} + 6a_{n-n} = 3n^2$

(P.T.O)

Q $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

The Homogeneous eqn is

$a_n + 5a_{n-1} + 6a_{n-2} = 0$

Ch. eqn

$\alpha^2 + 5\alpha + 6 = 0$

$\Rightarrow \alpha^2 + 3\alpha + 2\alpha + 6 = 0$

$\Rightarrow (\alpha+3)(\alpha+2) = 0$

$\alpha = -2, -3$

$a_n^{(h)} = A_1(-2)^n + A_2(-3)^n$

(Both the roots are real and distinct).

Particular solution –

$f(n) = 3n^2$

$a_n^{(p)} = \alpha_0 + \alpha_1 n + \alpha_2 n^2$ — (1)

$a_{n-1} = \alpha_0 + \alpha_1(n-1) + \alpha_2(n-1)^2$

$a_{n-2} = \alpha_0 + \alpha_1(n-2) + \alpha_2(n-2)^2$

substituting this in the given equation:

$\alpha_0 + \alpha_1 n + \alpha_2 n^2 + 5\alpha_0 + 5\alpha_1(n-1) + 5\alpha_2(n-1)^2 + 6\alpha_0 + 6\alpha_1(n-2) + 6\alpha_2(n-2)^2 = 3n^2$

$\Rightarrow 2\alpha_0 + \alpha_1 n + \alpha_2 n^2 + 5\alpha_1 n - 5\alpha_1 + 5\alpha_2(n^2+1-n) + 6\alpha_1 n - 12\alpha_1 + 6\alpha_2(n^2+4-4n) = 3n^2$

$\Rightarrow 12\alpha_0 - 5\alpha_1 + 5\alpha_2 - 12\alpha_1 + 24\alpha_2 + n(\alpha_1 + 5\alpha_1 - 10\alpha_2 + 6\alpha_1 - 24\alpha_2) + n^2(\alpha_2 + 5\alpha_2 + 6\alpha_2) = 3n^2$

$\Rightarrow 12\alpha_0 - 17\alpha_1 + 29\alpha_2 + n(10\alpha_1 - 34\alpha_2) + 12\alpha_2 n^2 = 3n^2$

Comparing the constant term in both sides we get,

$12\alpha_0 - 17\alpha_1 + 29\alpha_2 = 0$

$\Rightarrow 12\alpha_0 - 17 \times \frac{17}{24} + 29 \times \frac{1}{4} = 0$

$\Rightarrow 12\alpha_0 = 289/_{24} - 29/_{4}$

$\Rightarrow 12\alpha_0 = \frac{289 - 174}{24}$

$\Rightarrow \boxed{\alpha_0 = \frac{115}{288}}$

$12\alpha_1 - 34\alpha_2 = 0$

$\Rightarrow 12\alpha_1 - \frac{17}{2} = 0$

$\Rightarrow \boxed{\alpha_1 = \frac{17}{2 \times 12} = \frac{17}{24}}$

$12\alpha_2 = 3$

$\boxed{\alpha_2 = 1/4}$

So, the particular solution is

$a_n^{(P)} = \alpha_0 + \alpha_1 n + \alpha_2 n^2$

$= \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$

Total solution :-

$a_n^{(h)} + a_n^{(P)}$

$= A_1(-2)^n + A_2(-3)^n + \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$

Q Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 5n$

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

$\alpha^2 - 5\alpha + 6 = 0$

$\alpha^2 - 3\alpha - 2\alpha + 6 = 0$

$(\alpha - 3)(\alpha - 2) = 0$

$\alpha = 3, 2.$

both the roots are real and different

$a_n^{(h)} = A_1(2)^n + A_2(3)^n$

Particular Solution :-

$f(n) = 5n$

$a_n(P) = \alpha_0 + \alpha_1 n \quad \text{—— (1)}$

$a_{n-1} = \alpha_0 + \alpha_1(n-1)$

$a_{n-2} = \alpha_0 + \alpha_1(n-2)$

Substituting this in the given equation

$\Rightarrow \alpha_0 + \alpha_1 n - 5\alpha_0 - 5\alpha_1(n-1) - 5\alpha_2(n-1)^2 + 6\alpha_0 + 6\alpha_1(n-1) + 6\alpha_2(n-2)^2 = 5n$

$\Rightarrow \alpha_0 + \alpha_1 n - 5\alpha_0 - 5\alpha_1(n-1) + 6\alpha_0 + 6\alpha_1(n-2) = 5n$

$\Rightarrow \alpha_0 + \alpha_1 n - 5\alpha_0 - 5\alpha_1 n + 5\alpha_1 + 6\alpha_0 + 6\alpha_1 n - 12\alpha_1 = 5n$

$\Rightarrow 2\alpha_0 - 7\alpha_1 + n(2\alpha_1) = 5n$

$\Rightarrow 2\alpha_0 - 7\alpha_1 = 0$

$2\alpha_1 = 5$

$\alpha_1 = 5/2$

$2\alpha_0 - 7 \times \frac{5}{2} = 0 \quad \Rightarrow 2\alpha_0 = \frac{35}{2}$

$\Rightarrow \alpha_0 = \frac{35}{4}$

$a_n^{(P)} = \frac{35}{4} + \frac{5}{2}n$

So, the Total solution

$a_n^{(h)} + a_n^{(P)}$

$= A_1(2)^n + A_2(3)^n + \frac{35}{4} + \frac{5}{2}n$

If $f(n) = (\alpha_0 + \alpha_1 n + \alpha_2(n^2) + \dots + \alpha_k n^k)\beta^n$ where β is not a characteristic root of the given equation then the particular solution is.

$a_n^{(P)} = (Y_0 + Y_1 n + Y_2 + \dots + Y_k n^k)\beta^n$

By using the procedure of case 1 we get the value of $Y_0, Y_1, Y_2, \dots, Y_k$.

Putting this we get the particular solution as well as Total solution.

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(8) Solve the recurrance relation.

$a_n + a_{n-1} = 5 \cdot 2^n$

Ans → H.S

$a_n + a_{n-1} = 0$ H.S (Homog. Soln)

Ch

$\alpha + 1 = 0$

$\alpha = -1$

$(-1)^n$

P.S

$f(n) = 5n\,2^n$ P.S (Particular Soln)

$\beta = 2$ is n of a ch root

$a_n^{(P)} = (\alpha_0 + \alpha_1 n)2^n$

$a_{n-1} = (\alpha_0 + \alpha_1(n-1))2^{n-1}$

Substituting the given eqn we get,

$(\alpha_0 + \alpha_1 n)2^n + (\alpha_0 + \alpha_1(n-1))2^{n-1} = 5n\,2^n$

$\Rightarrow (\alpha_0 + \alpha_1 n)2^n + (\alpha_0 + \alpha_1(n-1))\frac{2^n}{2} = 5n2^n$

$\Rightarrow \cancel{2^n}\left[\alpha_0 + \alpha_1 n + \frac{(\alpha_0 + \alpha_1(n-1))}{2}\right] = 5n\cancel{2^n}$

$\Rightarrow \frac{3\alpha_0}{2} + \alpha_1\left(n + \frac{(n-1)}{2}\right) = 5n$

$\Rightarrow \frac{3\alpha_0 - \alpha_1}{2} + \frac{3\alpha_1 n}{2} = 5n$

Comparing the Coefficients

$\frac{3\alpha_1}{2} = 5$

$\alpha_1 = \frac{10}{3}$

$\frac{3\alpha_0 - \alpha_1}{2} = 0$

$3\alpha_0 - \alpha_1 = 0$

$3\alpha_0 = \alpha_1 = \frac{10}{3}$

$\alpha_0 = \frac{10}{9}$

Particular solⁿ

$a_n^{(P)} = (\alpha_0 + \alpha_1 n) 2^n$

$= \left(\frac{10}{9} + \frac{10}{3}n\right) 2^n$

Total solⁿ

$a_n = a_n^{(h)} + a_n^{(P)}$

$\therefore \Rightarrow a_n = A(-1)^n + \left(\frac{10}{9} + \frac{10}{3}n\right) 2^n$ (Ans)

(5) $a_n - 5a_{n-1} + 6a_{n-2} = (n+1)5^n$

Ans H.S $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Ch (Characteristic eqⁿ)

$\alpha^2 - 5\alpha + 6 = 0$

$\alpha^2 - 3\alpha - 2\alpha + 6 = 0$

$\Rightarrow \alpha(\alpha - 3) - 2(\alpha - 3) = 0$

$\Rightarrow (\alpha - 2)(\alpha - 3)$

$\alpha = 2, 3$

Homog. Solⁿ $\Rightarrow a_n^{(h)} = A_1(3)^n + A_2(3)^n$

Particular solⁿ

$f(n) = (n+1)5^n$

Hence $\beta = 5 \rightarrow$ Not a characteristic root of the given eqⁿ.

$a_n^{(P)} = (\alpha_0 + \alpha_1 n)5^n$ ($\because$ F(n) polynomial of d-1)

$a_{(n-1)} = [\alpha_0 + \alpha_1(n-1)]5^{n-1}$

$a_{(n-2)} = [\alpha_0 + \alpha_1(n-2)]5^{n-2}$

Substituting this in the given eqⁿ we get,

$(\alpha_0 + \alpha_1 n)5^n - 5[\alpha_0 + \alpha_1(n-1)]\frac{5^n}{5} + 6(\alpha_0 + \alpha_1(n-2))\frac{5^n}{5^2}$

$= (n+1)5^n$

$\Rightarrow \cancel{5^n}(\alpha_0 + \alpha_1 n) - \frac{\cancel{5}}{\cancel{5}}[\alpha_0 + \alpha_1(n-1) + \frac{6}{25}(\alpha_0 + \alpha_1(n-2))]$

$= (n+1)\cancel{5^n}$

$\Rightarrow [\cancel{\alpha_0} + \cancel{\alpha_1 n}) - \cancel{\alpha_0} - \cancel{\alpha_1 n} + \alpha_1 + \frac{6}{25}\alpha_0 + \frac{6}{25}\alpha_1 n - \frac{12}{25} = (n+1)$

Comparing constant in both sides.

$$\alpha_1 + \frac{6}{25}\alpha_0 - \frac{12}{25}\alpha_1 = 1$$

$$\Rightarrow \frac{13}{25}\alpha_1 + \frac{6\alpha_0}{25} = 1$$

$$\Rightarrow 13\alpha_1 + 6\alpha_0 = 25 \quad -(1)$$

(H)

$$\frac{6}{25}\alpha_1 = 1$$

$$\boxed{\alpha_1 = 25/6}$$

$$6\alpha_0 = 25 - 13\alpha_1 = 25 - 13\left(\frac{25}{6}\right) = \frac{150-325}{6} = \frac{-175}{6}$$

$$\boxed{\alpha_0 = \frac{-175}{36}}$$

$$\therefore a_n^{(P)} = (\alpha_0 + \alpha_1 n)5^n$$

$$= \left(\frac{-175}{36} + \frac{25}{6}n\right)5^n$$

Total sol^n:

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$\therefore a_n = A_1(2)^n + A_2(3)^n + \left(\frac{-175}{36} + \frac{25}{6}n\right)5^n \quad \text{(Ans)}$$

Case - III

If $f(n) = (a_0 + a_1 n + a_2 n^2 + \dots + a_k n^k)\beta^n$

Where β is a characteristic root of the homogeneous equation with multiplicity m(repeated). Then the particular solution is:

$$a_n^{(P)} = n^m[\alpha_0 + \alpha_1 n + \alpha_2 n + \dots + \alpha_k n^k]\beta^n$$

(9) Solve the rec. rel^n $a_n - 2a_{n-1} = 3n \cdot 2^n$

Ans Homog sol^n

$$a_n - 2a_{n-1} = 0$$

Characteristic eq^n

$$\alpha - 2 = 0$$

$$\alpha = 2.$$

H. sol^n $a_n^{(h)} = A(2)^n$

Par. sol^n :-

$$F(n) = 3n \cdot 2^n$$

Here $\beta = 2 \rightarrow$ is a characteristic root of multiplicity 1 $(\because \alpha = 2)$

$$a_n^{(P)} = n^2[\alpha_0 + \alpha_1 n]2^n$$

$$a_{(n-1)} = (n-1)[\alpha_0 + \alpha_1(n-1)]2^{n-1}$$

Substituting these in the given eq^n.

$$a_n - 2a_{n-1} = 3n2^n$$

we get,

$$\Rightarrow n[\alpha_0 + \alpha_1 n]2^n - \cancel{2}(n-1)[\alpha_0 + \alpha_1(n-1)]\frac{2^n}{\cancel{2}} = 3n2^n$$

$$\Rightarrow \cancel{2^n}[\alpha_0 n + \alpha_1 n^2 - \alpha_0(n-1) - \alpha_1(n-1)^2] = 3n\cancel{2^n}$$

$$\Rightarrow \cancel{\alpha_0 n} + \alpha_1 n^2 - \cancel{\alpha_0 n} + \alpha_0 - \alpha_1(n^2 + 1 - 2n) = 3n$$

$$\Rightarrow \cancel{\alpha_1 n^2} + \alpha_0 - \cancel{\alpha_1 n^2} - \alpha_1 + 2n\alpha_1 = 3n$$

$$\Rightarrow \alpha_0 - \alpha_1 + 2\alpha_1 n = 3n$$

Equating the coefficient of n.

$$2\alpha_1 = 3$$

$$\alpha_1 = 3/2$$

$\alpha_0 - \alpha_1 = 0$
$\Rightarrow \alpha_0 = \alpha_1 = 3/2$

The p. solⁿ is
$a_n^{(p)} = n[\alpha_0 + \alpha_1 n]2^n$
$= n\left[3/2 + 3/2 n\right]2^n$

$\therefore$ Total solⁿ $= a_n = a_n^{(h)} + a_n^{(p)}$
$\therefore a_n = A(2)^n + n\left[3/2 + 3/2 n\right]2^n$ (Ans)

(Q) Solve $a_n - 4a_{n-1} + 4a_{n-2} = n2^n$

Ans: Homg solⁿ
H. seqⁿ $= a_n - 4a_{n-1} + 4a_{n-2} = 0$

Ch. eqⁿ $\Rightarrow \alpha^2 - 4\alpha + 4 = 0$
$\Rightarrow \alpha^2 - 2\alpha - 2\alpha + 4 = 0$
$\Rightarrow \alpha(\alpha-2) - 2(\alpha-2) = 0$
$\Rightarrow (\alpha-2)(\alpha-2) = 0$
$\Rightarrow (\alpha-2)^2 = 0$
$\Rightarrow \alpha = 2$

$a_n^{(h)} = (A_1 + A_2 n)2^n$

P. solⁿ
$F(n) = n \cdot 2^n$
$\beta = 2 \rightarrow$ is a char. root with multiplicity 2
$(\because \alpha = 2, 2)$

$\therefore a_n^{(p)} = n^2[\alpha_0 + \alpha_1 n]2^n$
$a_{n-1} = (n-1)^2[\alpha_0 + \alpha_1(n-1)]2^{n-1}$
$a_{n-2} = (n-2)^2[\alpha_0 + \alpha_1(n-2)]2^{n-2}$

Substituting in the given eqⁿ.
~~$\Rightarrow n^2[\alpha_0+\alpha_1 n]$~~
$\Rightarrow n^2[\alpha_0 + \alpha_1 n]2^n - 4\cdot(n-1)^2[\alpha_0 + \alpha_1(n-1)]\frac{2^n}{2}$

$+ 4\cdot(n-2)^2[\alpha_0 + \alpha_1(n-2)]\frac{2^n}{4} = n\cdot 2^n$

$\Rightarrow 2^n[n^2(\alpha_0 + \alpha_1 n) - 2(n-1)^2(\alpha_0 + \alpha_1(n-1)) + (n-2)^2[\alpha_0 + \alpha_1(n-2)]] = n\cdot 2^n$

$\Rightarrow n^2\alpha_0 + \alpha_1 n^3 - 2\alpha_0(n^2 + 1 - 2n) - 2\alpha_1(n^3 - 3n^2 + n - 1)$
$+ \alpha_0(n^2 + 4 - 4n) + \alpha_1(n^3 - 6n^2 + 12n) - 8) = n$

Comparing the coefficients.

n^3 // $\alpha_1 - 2\alpha_1 + \alpha_1 = 0$. ($\alpha_1$ canot be found from here)

n^2 // $\alpha_0 - 2\alpha_0 + 6\alpha_1 + \alpha_0 - 6\alpha_1 = 0$

n // $4\alpha_0 - 6\alpha_1 - 4\alpha_0 + 12\alpha_1 = 1$

$6\alpha_1 = 1$
$\alpha_1 = 1/6$

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Generating Function

The generating function for a sequence a_0, a_1, a_2, a_3 is $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$$\boxed{G(x) = \sum_{k=0}^{\infty} a_k x^k}$$

Example

Find the generating function of

1, 1, 1, 1, 1, ...

Here the generating function is

$\Rightarrow G(x) = 1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 + \dots$

$\Rightarrow G(x) = 1 + x + x^2 + x^3 + \dots$

$\Rightarrow G(x) = \dfrac{1}{1-x}$

Q) Find the generating function of $a_n = 3^n$.

$1, 3, 3^2, 3^3, 3^4, \dots$

Here, the generating function

$G(x) = 1 + 3\cdot x + 3^2 x^2 + 3^3 x^3$

$\Rightarrow G(x) = \sum_{k=0}^{\infty} 3^k x^k$

$\Rightarrow G(x) = \dfrac{1}{1-3x}$

Q) Find the generating function for the sequence.

$1, 5, 5^2, 5^3, 5^4, \dots, 5^k$.

Ans $\rightarrow$ $G(x) = 1 + 5x + 5^2 x^2 + 5^3 x^3 + \dots + 5^k x^k$

$G(x) = 1 + 5x + (5x)^2 + (5x)^3 + \dots + (5x)^k$

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$\Rightarrow G(x) = \dfrac{1 - (5(x))^{k+1}}{1-5x}$

$\Rightarrow G(x) = \dfrac{1-(5x)^{k+1}}{1-5x}$

Addition & Product of two Generating Function

$f(x) = \sum_{k=0}^{\infty} a_k x^k$

$g(x) = \sum_{j=0}^{\infty} b_k x^k$

Let $f(x)$ & $g(x)$ be 2 generating functions whose series form is given above.

Addition :-

$$f(x) + g(x) = \sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k$$

$$= \sum_{k=0}^{\infty} a_k x^k + b_k x^k$$

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

Product :-

$$f(x) \times g(x) = \sum_{k=0}^{\infty} a_k x^k \times \sum_{k=0}^{\infty} b_k x^k$$

$$= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots)$$

$$= a_0 b_0 + (a_0 b_0 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \dots$$

$$f(x) \times g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$$

Q. Express the function $\frac{1}{(1-x)^2}$ is terms of a series.

Soln

$$\frac{1}{(1-x)^2}$$

Here $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $a_n = 1$

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , b_n = 1$$

$$f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{j-k}\right) x^k$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (1 \cdot 1) x^k$$

$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (1)\right) x^k$$

$$f(x) \cdot g(x) = \sum_{k=0}^{\infty} (k+1) x^k$$

Extended Binomial Coefficient

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + \cdots + {}^nC_n b^n$$

$${}^nC_r = C(n,r) = \binom{n}{r}$$

$n < 0$
$k \in \mathbb{Z}^+$

$$\binom{n}{k} = \begin{cases} \frac{n(n-1)(n-2)(n-3)\cdots(n-(k-1))}{\lfloor k}, & k > 0 \\ 1, & k = 0 \end{cases}$$

If n be any $-ve$ integer or fraction or k be any $+ve$ integer then the extended binomial coefficient of n and denoted by $\binom{n}{k}$ is defined as

$$\binom{n}{k} = \begin{cases} n(n-1)(n-2)\cdots(n-(k-1)), & k > 0 \\ 1, & k = 0 \end{cases}$$

Evaluate

i) $$\binom{-5}{4} = \frac{(-5)(-6)(-7)(-8)}{4!} = 70$$

ii) $$\binom{1/2}{3} = \frac{(1/2)(1/2 - 1)(1/2 - 2)}{3!}$$

$$= \frac{\frac{1}{2} \times \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{}$$

$$= \frac{3/8}{3 \times 2 \times 1}$$

$$= \frac{3}{8} \times \frac{1}{3 \times 2} = \frac{1}{16} \text{ (Ans)}$$

Relation between Extended Binomial Coefficient and combination where n is a $-ve$ integer.

Ans > If n is any $-ve$ integer and r be any natural number

$$\binom{-n}{r} = (-1)^r \, C(n+r-1, r)$$

L.H.S $\binom{-n}{r}$

$$= \frac{(-n)(-n-1)(-n-2)(-n-3)\cdots(-n-(r-1))}{r!}$$

$$= (-1)^r \frac{n(n+1)(n+2)(n+3)\cdots(n+r-1)}{r!}$$

$$= \frac{(-1)^r n(n+1)(n+2)\cdots(n+r-1)}{r!}$$

$$= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!}$$

$$= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n(n-1)(n-2)\cdots 1}{r!\; 1\cdot 2\cdots(n-1)}$$

$$= (-1)^r \frac{n+r-1}{r!\,(n-1)!}$$

$$= (-1)^n \cdot C(n+r-1, r)$$

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Extended Binomial Theorem:-

If the power of a binomial is any -ve integer or fraction then it's expandation is called extended binomial theorem.

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k$$

$$\boxed{(a+b)^n = \sum_{k=0}^{\infty} (-1)^k C(n+k-1, k)\, a^{n-k} b^k}$$

$$\left(\because \binom{n}{k} = (-1)^k C(n+k-1, k)\right)$$

Generating function & their corresponding sequence:-

G.F		Sequence
$(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k} x^k$	$\rightarrow$	$\binom{n}{k}$
$(1+ax)^n = \sum_{k=0}^{\infty}\binom{n}{k}(ax)^k = \sum_{k=0}^{\infty}\binom{n}{k}a^k x^k$	$\rightarrow$	$\binom{n}{k}a^k$
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$	$\rightarrow$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$	$\rightarrow$	a^k
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k$	$\rightarrow$	$k+1$

Q. Find the no. of solutions of the equation $e_1+e_2+e_3=17$, where e_1, e_2, e_3 are +ve integer ($\in Z^+$) $2 \leq e_1 \leq 5$, $3 \leq e_2 \leq 6$, $4 \leq e_3 \leq 7$.

Ans Here the eqn is $e_1+e_2+e_3=17$
The G.Fxn for e_1, where $2 \leq e_1 \leq 5$ is
$x^2+x^3+x^4+x^5$
The G.Fxn for $e_2 \rightarrow x^3+x^4+x^5+x^6$
The G.Fxn for $e_3 \rightarrow x^4+x^5+x^6+x^7$

Now, The total generating function is
$= (x^2+x^3+x^4+x^5)(x^3+x^4+x^5+x^6)(x^4+x^5+x^6+x^7)$
$= x^2(1+x+x^2+x^3) \cdot x^3(1+x+x^2+x^3) x^4(1+x+x^2+x^3)$
$= x^9(1+x+x^2+x^3)^3$
$= x^9[(1+x)^3+(x^2+x^3)^3+3(1+x)^2(x^2+x^3)+$
$3(1+x)(x^2+x^3)^2]$ $(\because (a+b))$
$= x^9[(1+x)^3+x^6+x^9+3x^7+3x^8+3(1+x)^2(x^2+x^3)$
$+3(1+x)(x^2+x^3)^2]$

In this ~~expression~~ expression of x^{17} is 3. So, the no. of solution is 3.

Q. In how many ways 7 identical objects can be distributed among 3 children such that each child get atleast 2 object but not more than 4 object.

Ans Here the G.fxn for each of the children is $x^2+x^3+x^4$, since there are 3 children so the total G.fxn as $(x^2+x^3+x^4)^3$.

The coefficient of x^7 in the G.fxn will give the solutions.
Now,
$(x^2+x^3+x^4)^3$
$= [x^2(1+x+x^2)]^3$
$= x^6(1+x+x^2)^3$
$= x^6[(1+x)^3+3(1+x)^2x^2+3(1+x)x^4+x^6]$
$= x^6[1+3x+3x^2+x^3(1+x^2)+3x^4(1+x)+x^6]$

∴ Hence
In 3 diff. ways (Ans)

Q. Using G.Fxn find the no. of r combination from a set of n elements where integration of elements is allowed.

Ans No. of objects = n, 1, 2, 3, ..., n
(r combination)
for 1st element (1), the G.Fxn is
$1 \rightarrow 1+x+x^2+\dots$
Similarly for $2 \rightarrow 1+x+x^2 \dots$
$3 \rightarrow 1+x+x^2 \dots$

total $G(n) = (1+x+x^2+\dots+)^3 = \left(\frac{1}{1-x}\right)^n = (1-x)^{-n}$

$= \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k$

$= \sum_{k=0}^{\infty} (-1)^k \, c(-n+k-1, k)(-x)^k$

$= \sum_{k=0}^{\infty} c(-n+k-1, k)\, x^k$

∴ So the total no. of r combination is
$c(-n+r-1, r)$ (Ans)

Solution of rec rel^n using G.Fx^n :-

Ex :- Solve the rec rel^n $a_n = 3a_{n-1}$, using G.Fx^n, where the initial cond^n is $a_0 = 2$, $(n = 1, 2, \dots)$

Ans → The expression is

$a_n = 3a_{n-1}$

$\Rightarrow a_n - 3a_{n-1} = 0$

$\Rightarrow a_n x^n - 3a_{n-1} x^n = 0$ (multiply x^n both sides)

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow G(x) - 2 - 3\sum_{n=1}^{\infty} a_{n-1} x^{n-1} x = 0$$

$$\because G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$$
$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n = G(x) - a_0$$
$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n = G(x) - 2$$

$$\Rightarrow G(x) - 2 - 3x\sum_{k=0}^{\infty} a_k x^k = 0$$

$\Rightarrow G(x) - 2 - 3x\,G(x) = 0$

$\Rightarrow G(x) - 3x\,G(x) = 2$

$\Rightarrow G(x)(1 - 3x) = 2$

$$\Rightarrow G(x) = \frac{2}{(1-3x)}$$

$\Rightarrow G(x) = 2\sum 3^n x^n \Rightarrow G(x) = \sum 2\,3^n x^n$ $\left[\because \frac{1}{1-ax} = \sum_{n=0}^{\infty} (ax)^n\right]$

$\therefore a_n = 2 \cdot 3^n$ (Ans)

Q Solve the rec rel^n $a_n = 5a_{n-1} + 2$, $n = 1, 2, \dots$: $a_0 = 1$?

Ans → $a_n = 5a_{n-1} + 2$

$a_n - 5a_{n-1} = 2$

$\Rightarrow a_n x^n - 5a_{n-1} x^n = 2x^n$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 5\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 2x^n$$

$$\Rightarrow G(x) - 1 - 5\sum_{n=1}^{\infty} a_{n-1} x^{n-1} x = 2\sum_{n=1}^{\infty} x^n$$

$$\Rightarrow G(x) - 1 - 5x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 2(x + x^2 + \dots)$$

$\Rightarrow G(x) - 1 - 5x\,G(x) = 2x(1 + x + x^2 + \dots)$

$$\Rightarrow G(x) - 1 - 5(x)\,G(x) = \frac{2x}{1-x}$$

$$\Rightarrow G(x)(1 - 5x) = \frac{2x}{1-x} + 1 = \frac{2x + 1 - x}{1 - x}$$

$$\Rightarrow G(x) = \frac{1+x}{(1-x)(1-5x)} \quad \text{--- (1)}$$

$$\frac{1+x}{(1-x)(1-5x)} = \frac{A}{(1-x)} + \frac{B}{1-5x}$$

$$A = \left.\frac{1+x}{1-5x}\right]_{x=1} = \frac{2}{-4} = \frac{-1}{2}$$

$$B = \left.\frac{1+x}{1-x}\right]_{x=1/5} = \frac{1 + 1/5}{1 - 1/5} = \frac{6/5}{4/5} = \frac{3}{2}$$

Now,

$$G(x) = \frac{1+x}{(1-x)(1-5x)}$$

$$\Rightarrow G(x) = \frac{-1/2}{1-x} + \frac{3/2}{1-5x}$$

$$\Rightarrow G(x) = -\frac{1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} 5^n x^n$$

$$\left(\frac{1}{1-a^n} = \sum_{n=0}^{\infty} a^n x^n \right)$$

$\therefore a_n = -\frac{1}{2} + \frac{3}{2} 5^n$ (Ans)

Q Solve the rec. rel^n $a_n = 8a_{n-1} + 10^{n-1}$, $a_1 = 9$

Ans: Putting $n=1$, in the given eq^n,

$a_1 = 8a_0 + 10^0$

$a_1 = 8a_0 + 1$

$a_0 = 1$

$\therefore a_0 = 1$, $a_1 = 9$,

Now

$a_n - 8a_{n-1} = 10^{n-1}$

$\Rightarrow a_n x^n - 8a_{n-1}x^n = 10^{n-1}x^n$

$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 8\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} 10^{n-1}x^n$

$\Rightarrow G(x) - 1 - 8x \times G(x) = \sum_{n=1}^{\infty} 10^{n-1} x^{n-1} x$

$(\because a_0 = 1)$

$$\left(\begin{aligned} &\because x + 10x^2 + 10^2x^3 + \dots \\ &= x(1 + 10x + (10x)^2 + \dots) \\ &= \frac{x}{1-10x} \end{aligned} \right)$$

$$\Rightarrow G(x)(1-8x) - 1 = \frac{x}{1-10x}$$

$$\Rightarrow G(x)(1-8x) = \frac{x}{1-10x} + 1$$

$$\Rightarrow G(x)(1-8x) = \frac{1-9x}{1-10x}$$

$$\Rightarrow G(x) = \frac{1-9x}{(1-10x)(1-8x)}$$

$$\Rightarrow \frac{1-9x}{(1-10x)(1-8x)} = \frac{A}{1-10x} + \frac{B}{1-5x}$$

$$A = \left.\frac{1-9x}{1-8x}\right]_{x=1/10} = \frac{1}{2}$$

$$B = \left.\frac{1-9x}{1-10x}\right]_{x=1/8} = \frac{1}{2}$$

$$G(x) = \frac{1/2}{(1-10x)} + \frac{1/2}{(1-8x)}$$

$$G(x) = \frac{1}{2}\sum(10x)^n + \frac{1}{2}\sum(8x)^n$$

$$= \frac{1}{2}\sum 10^n x^n + \frac{1}{2} 8^n x^n$$

$\therefore a_n = \frac{1}{2}(10)^n + \frac{1}{2}(8^n)$ (Ans)

Q Solve $a_n = a_{n-1} + 2a_{n-2} + 2^n$, $a_0 = 4$, $a_1 = 12$

Ans: $a_n - a_{n-1} - 2a_{n-2} = 2^n$

$\Rightarrow a_n x^n - a_{n-1}x^n - 2a_{n-2}x^n = 2^n x^n$

$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1}x^n - 2\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2}^{\infty} (2x)^n$

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$$\Rightarrow G(x) - 4 - 12x = \sum_{n=2}^{\infty} a_{n-1} x^{n-1} x^1 - 2\sum_{n=2}^{\infty} a_{n-2} x^{n-2} x^2$$

$$= \sum_{n=2}^{\infty} (2x)^n$$

$n-1 = k$

$n = k+1$

$n = 1,\ k = 0$

$n = \infty,\ k = \infty$

$$\Rightarrow G(x) - 4 - 12x - x\sum_{k=1}^{\infty} a_k x^k - 2x^2 G(x) = (2x)^2 + (2x)^3 + \dots$$

$$\Rightarrow G(x) - 4 - 12x - x(G(x) - 4) - 2x^2 G(x) = 4x^2(1 + 2x + (2x)^2 + \dots)$$

$$\Rightarrow G(x) - 4 - 12x - xG(x) + 4x - 2x^2 G(x) = \frac{4x^2}{1-2x}$$

$$\Rightarrow G(x)(1 - x - 2x^2) = \frac{4x^2}{1-2x} + 4 + 8x$$

$$= \frac{4x^2 + 4 - \cancel{8x} + \cancel{8x} - 16x^2}{1-2x}$$

$$\Rightarrow -G(x)(2x^2 + x - 1) = \frac{-12x^2 + 4}{1-2x}$$

$$\Rightarrow G(x)(2x^2 + 2x - x - 1) = \frac{(12x^2 - 4)}{1-2x}$$

$$\Rightarrow G(x) = \frac{12x^2 - 4}{(1-2x)(1+x)(2x-1)}$$

$$G(x) = \frac{4 - 12x^2}{(1-2x)^2(1+x)}$$

$$\frac{4 - 12x^2}{(1-2x)^2(1+x)} = \frac{A}{(1-2x)} + \frac{B}{(1-2x)^2} + \frac{C}{1+x}$$

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$$B = \left.\frac{4 - 12x^2}{1+x}\right]_{x = 1/2}$$

$$= \frac{4-3}{3/2}$$

$$= \frac{2}{3}$$

$$C = \left.\frac{4 - 12x^2}{(1-2x)^2}\right]_{x=-1} = \frac{-8}{9}$$

<u>for A</u>

$$\frac{4 - 12x^2}{(1-2x)^2(1+x)} = \frac{A(1-2x)(1+x) + B(1+x) + C(1-2x)^2}{(1-2x)^2(1+x)}$$

$$\Rightarrow 4 - 12x^2 = A(1 - x - 2x^2) + B(1+x) + C(1 + 4x^2 - 4x)$$

<u>x^2 comparing coefficent</u>

$-12 = -2A + 4C$

$2A = 12 + 4C$

$= 12 + 4\left(-\frac{8}{9}\right)$

$= \frac{108 - 32}{9} = \frac{76}{9}$

Now,

$A = 38/9$

$$G(x) = \frac{38/9}{1-2x} + \frac{2/3}{(1-2x)^2} - \frac{8/9}{1+x}$$

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$$G(x) = \frac{38}{9}\sum(2x)^n + \frac{2}{3}\sum_{n=0}^{\infty} C(n+1,n)\,a^n x^n$$

$$-\frac{8}{9}\sum_{n=0}^{\infty}(-1)^n x^n$$

$$\left(\frac{1}{1-x} = 1-x+x^2\cdots = \sum(-1)^k x^k\right)$$

$$a_n = \frac{38}{9}2^n + \frac{2}{3}C(n+1,n)\,a^n - \frac{8}{9}(-1)^n \text{ (Ans)}$$

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Prove of Identities using $g \cdot Fx^n$

Whenever we prove the identities using generating fx^n. Basically we choose such type of generating fx^n which are given below.

$$\boxed{\begin{aligned}(1+x)^n &= C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n \\ (1-x)^n &= C_0 - C_1x + C_2x^2 - C_3x^3 + \dots + (-1)^nC_nx^n\end{aligned}}$$

* NOTE

The identities can be verified by using the help of derivatives, integrations and by comparing the coefficient.

$C_1 + 2C_2 + 3C_3 + 4C_4 + \dots + nC_n$

$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

Diff.

$n(1+x)^{n-1} = C_1 + C_2 2x + C_3 3x^2 + \dots + C_n n x^{n-1}$

$x = 1$

We know that

differentiating both sides we get

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Q) Evaluate $C_1 + 2C_2 + 3C_3 + 4C_4 + \dots + nC_n$

Ans we know that,

$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

diff. both sides w.r.t. x,

$n(1+x)^{n-1} = C_1 + C_2x + C_3 3x^2 + \dots + C_n n x^{n-1}$

Putting $x = 1$

$n2^{n-1} = C_1 + C_2 2 + C_3 3 + \dots + C_n n$

$\therefore \Rightarrow C_1 + 2C_2 + 3C_3 + \dots + nC_n = n2^{n-1}$ (Ans)

Q) Evaluate $C_2 + 2C_3 + 3C_4 + \dots + (n-1)C_n$

Ans we know that,

$\Rightarrow (1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

$\Rightarrow (1+x)^n - C_0 = C_1x + C_2x^2 + \dots + C_nx^n$

$\Rightarrow \dfrac{(1+x)^n - 1}{x} = \dfrac{C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n}{x}$ (Divide x both sides)

$\Rightarrow \dfrac{(1+x)^n - 1}{x} = C_1 + C_2x + C_3x^2 + \dots + C_nx^{n-1}$

Diff.^{n} both sides,

$\Rightarrow \dfrac{x(n(1+x)^{n-1}) - (1+x)^n + 1}{x^2} = C_2 + C_3 2x + C_4 3x^2 + \dots + C_n(n-1)x^{n-2}$

$\Rightarrow n2^{n-1} - 2^n = C_2 + 2C_3 + 3C_4 + \dots + (n-1)C_n$

$\Rightarrow 1^2C_1 + 2^2C_2 + 3^2C_3 + \dots + n^2C_n$

$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

Diff^n

$n(1+x)^{n-1} = C_1 + C_2 \cdot 2x + C_3 \cdot 3x^2 + C_4 4x^3 + \dots + C_n n x^{n-1}$

Multiply x

$nx(1+x)^{n-1} = C_1x + C_2 2x^2 + C_3 3x^3 + \dots + C_n n x^n$

Diffⁿ

$$n[(1+x)^{n-1} + x(n-1)(x+1)^{n-2}] = C_1 + C_2 2^2 x + C_3 3^2 x^2 + \dots + C_n n^2 x^{n-1}$$

$x=1$

$$n[2^{n-1} + (n-1)2^{n-2}] = C_1 + 2^2 C_2 + 3^2 C_3 + \dots + n^2 C_n$$

Q Evaluate (By integration)

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

Ans We know that,

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$$

Integrating w.r.t. x

$$\Rightarrow \int (1+x)^n dx = C_0 \int dx + C_1 \int x\,dx + C_2 \int x^2 dx + \dots + C_n \int x^n dx$$

$$\Rightarrow \frac{(1+x)^{n+1}}{n+1} + c = C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \dots + C_n \frac{x^{n+1}}{n+1} \quad \text{—(1)}$$

$x=0$

$$\Rightarrow \frac{1}{n+1} + c = 0 \Rightarrow c = -\frac{1}{n+1}$$

Putting $x=1$, in eqⁿ (1)

$$\frac{2^{n+1}}{n+1} + c = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

$$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

Q Evaluate (By coeff. Comparision)

Prove that $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = {}^{2n}C_n$

Ans We know that,

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n \quad \text{—(1)}$$

&

$$(x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots + C_n \quad \text{—(2)}$$

By multiplying both eqⁿ (1) & (2)

$$(1+x)^{2n} = (C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n) \times (C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots + C_n)$$

$${}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_{2n} x^{2n} = (C_0 x^n + C_1 x^{n-1} + \dots + C_n)$$

Comparing x^n in both sides

$$\therefore {}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 \quad \text{(proved)}$$

Q Evaluate $C_0 C_1 + C_1 C_2 + C_2 C_3 + \dots + C_{n-1} C_n = {}^{2n}C_{n-1}$?

Ans We know that,

$$(1+x)^n = C_0 + C_1 x + \dots + C_n x^n$$

$$(x+1)^n = C_0 x^n + \dots + C_n$$

By multiplying both

$$(1+x)^{2n} = (C_0 + \dots + C_n x^n) \times (C_0 x^n + \dots + C_n)$$

$${}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_{2n} x^{2n} = (C_0 + \dots + C_n x^n) \times (C_0 x^n + \dots + C_n)$$

By comparing coeff. of x^{n-1}, we get,

$$\therefore {}^{2n}C_{n-1} = C_0 C_1 + C_1 C_2 + C_2 C_3 + \dots + C_{n-1} C_n$$

(Proved)