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Counting Principle

Counting Principle

Product — Addition

Product
n—m
n×m

Addition
n+m

Basically there are 2 types of counting principle:-
① Product rule
② Addition rule

Product Rule :-
If an event occurs in m different ways and for each occurence of this event another event occurs in n different ways then the total no. of ways in which both the event occurs is $n \times m$

Addition Rule :-
If an even occurs in m different ways an another event occurs in n different ways. Without depending upon the 1st event. Then the total number of ways in which both the event occurs is $n+m$.

Q How many 3 digit no. can be formed by using the digit 1,2,3,4,5.
① without Repetation
② With Repetation.

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Ans ① Without Repeatation

3 4 5ways

$3 \times 4 \times 5 = 60$

② With Repetation.

5 ways 5 ways 5 ways

$5 \times 5 \times 5 = 125$

Q How many 3 digited even numbers can be form by using the digit 1,2,3,4,5,6,7
① Without Repetation
② With Repetation.

Ans ① $5 \times 6 \times 3 = 90$
② $7 \times 7 \times 3 = 147$.

Q How many 3 digited even no. can be formed by using 0,1,2,3,4,5,6.
① without Repetation
② with Repetation.

Ans ① without Rep.
In general | 5 | 6 | 4 | $= 5 \times 6 \times 4 = 120$
fixed 0 ← | 0 | 5 | 3 | $= 1 \times 5 \times 3 = 15$
$120 - 15 = 105$ (Ans)

② with R.
| 6 | 7 | 4 |
$6 \times 7 \times 4 = 168$ (Ans)

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Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion for two events A × B is :

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Similarly, the Principle of Inclusion and Exclusion for three events is :

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

Example

Q How many binary bit strings of length 9 that ends with either 00 or starts with 1.

Ans

Three Cases :

Case 1 :

1								
	2	2	2	2	2	2	2	2

Remaining 8 can be filled with 0 or 1 in 2^8 ways.

So, no. of string = $2^8 = 256$

Case 2 :

	0	0
2^7		

Remaining 7 can be filled with 0 or 1 in 2^7 ways = 128 ways.

Case 3 :

1		0	0
	2^6		

Let, 1st be 1 and last 2 digits 0, 0.

So remaining can be filled in 2^6 ways = 64 ways

So, required strings = 256 + 128 − 64
= 320

Q A computer password has length 6 to 8 characters where each character is an upper case of letter or a digit. If each password contains atleast 1 digit, then how many passwords can be formed.

Ans

6 → | | | | | | |

$36^6 - 26^6$

(Total no. of possible combinatn) (Combinations possible with only alphabets i.e. no digits)

7 → | | | | | | | |

$36^7 - 26^7$

8 → $36^8 - 26^8$

Required number = $36^6 + 36^7 + 36^8 - 26^6 - 26^7 - 26^8$

Q (i) How many diff. 3 letter initials can people have.
(ii) If repetition is not allowed
(iii) starting with 4.

Ans

(i)

26	26	26

→ 26^3

(ii)

26	25	24

→ $26 \times 25 \times 24$

(iii)

4		
1	26	26

→ 26^2

Q How many binary bit strings are there of length 8.

Ans → 2^8 (Ans)

Q How many binary bit strings are there of length 6 or less.

Ans → $2^6+2^5+2^4+2^3+2^2+2^1$

$= 64+32+16+8+4+1$

$= 125$ (Ans)

Q How many binary strings of length 7 can be formed which is starting with 1 and ending with 0.

→ | 1 | | | | | | 0 |

2 2 2 2 2

$2^5 = 32$

Q How many string of three decimal digit.
(i) do not contain the same digit three times.
(ii) Begin with an odd digit.
(iii) Have exactly two digit that are 4.

→ (i) | | | |

10 10 10

$= 10^3 = 1000$

Same 3: | | | | $= 10^1 = 10$

Req^d = $1000 - 10 = 990$ (Ans)

(ii) | | | |

5 10 10

$= 10 \times 10 \times 5$

$= 500$ (Ans)

(iii) | 4 | 4 | | | 4 | | 4 | | | 4 | 4 |

1 1 9   1 9 1   9 1 1

$9 + 9 + 9 = 27$ (Ans)

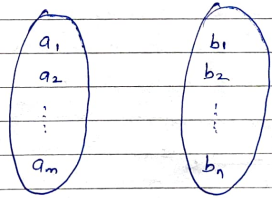
Q How many different functions can be formed from set A to set B.

$|A| = 7$, $|B| = 4$

Ans → $A \to B$ $= |B|^{|A|}$

Q If $|A| = m$ & $|B| = n$, then find the total no. of function from $|A|$ to $|B|$. Also find the no. of one-one functions from $A \to B$ when
i) $m < n$ ii) $m = n$ iii) $m > n$

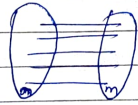
Ans $A \to B$ $= n^m$

i) 

$m < n$

∴ one-one possible

$C(n, m)$

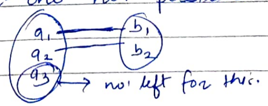
ii) 

$m = n$

∴ one-one possible

$C(n, m)$

iii) one-one not possible (∵ $m > n$)



Greater Integer $f x^n$

$$\lfloor x \rfloor = \begin{cases} x, & x \in Z \\ k, & x \notin Z,\ k < x < k+1 \\ & (k \in Z) \end{cases}$$

$\lfloor 3 \rfloor = 3$

$\lfloor 2.5 \rfloor = 2$

$\lfloor -3.4 \rfloor = -4$

Smallest Integer $f x^n$

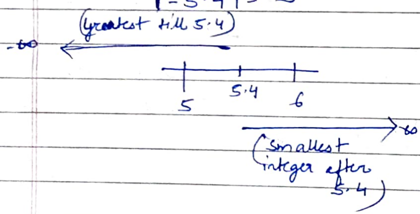
$$\lceil x \rceil = \begin{cases} x, & x \in Z \\ k+1, & x \notin Z,\ k < x < k+1 \\ & (k \in Z) \end{cases}$$

$\lceil 3 \rceil = 3$

$\lceil -5 \rceil = -5$

$\lceil 7.4 \rceil = 8$

$\lceil -5.4 \rceil = -5$



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Pigeon-hole Principle

If there are n pigeons and m no' of holes, where $n > m$ then at least one of the hole must be occupied by more than one no' of pigeon.

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Generalized pigeon-hole Principle :-

If there are m no. of pigeons and n no. of holes then atleast one of the hole must be occupied by $\left\lceil \frac{m}{n} \right\rceil$ no' of pigeons, this statement is called the generalized pigeon hole principle.

Example

Q) There are 100 persons atleast how many person will have birthday on same month.

Ans → 100 person.
12 month,

$\left\lceil \frac{100}{12} \right\rceil = 9$ (Ans)

∴ At least 9 person will have birthday in same month.

Q) How many cards will be selected from a pack of cards to guarantee that

(i) at least 3 cards of same suits are choosen

(ii) atleast 3 cards of the same denomination must be present.

Ans → i) It is clear that in a pack of cards. There are 4 suits, if we select two from each suit then we have 8 cards but out of these 8 cards if we select ~~only~~ any 3 cards then they do not have same suit. i.e. No possibility of selecting 3 cards of same suit.

If we select one more card then the total no of cards is 9, this additional cards may be of any of the suit now, if we select 3 cards out of these 9 cards then there is a possibility of selecting 3 cards of same suits. So the answer is 9 cards.

ii) 13 no denomination is there, if we take 2 cards of each - then we have 26 cards if we select any 3 cards then there is no posibility of selecting 3 cards of same denomination so one more cards, we have to select then the total no. of cards is 27, Now if we select, 3 cards out of these 27 cards then there is a possibility of selecting 3 cards of same denomination.

∴ So the ans is 27 cards.

Q) How many cards will be selected from a pack of cards, such that atleast 3 spades can be guarently selected from it.

Ans →

D	S	H	C
↓	↓	↓	↓
13	3	13	13

$13 + 3 + 13 + 13 = 42$ (Ans)

Since here we select 3 spades so the possibility is 13 (diamonds) + 13 (heart) + 13 (clubs) + 3 (spades) = 42 cards.

Q) A box contains 10 red balls and 10 blue balls. A person select balls are in random w/o observing it's colour.

i) How many balls must be select to be sure of having atleast 3 balls of the same colour.

Ans (i) $\lceil \frac{N}{2} \rceil \geq 3$

Now

$\frac{N}{2} = 2$

$N = 4 + 1 = 5$ (Ans)

ii) How many will be selected so that atleast 3 blue balls can be selected from it.

$10 + 3 = 13$ (Ans)

Q) Show that among any group of 5 integers may or maynot be consecutive, there are 2 with the same remainder when divides by 4.

Ans Let there are 5 no. i.e.

x_1, x_2, x_3, x_4, x_5

If any no. is divided by 4 then there ~~are~~ are 4 possibility of remainder i.e. 0, 1, 2, 3. If the 4 no's gives different remainders then the 5th number must repeat one of the remainder.

Q How many no. must be selected from the set {1,2,3,4,5,6} to guarantee that at least one pair of these numbers add up to 7.

Ans : In this case the set can be divided into another 2 set as even and odd i.e.

even , e = {2,4,6}　　 (c = {(1,6),(3,4),(5,2)})

odd , o = {1,3,5}

If we select all members from any one group and one from other group then definitely the sum may be 7. So 3+1 = 4)

Q How many no. must be selected from the set {1,3,5,7,9,11,13,15} to guarantee that atleast one pair of these numbers add upto 16.

Ans → set {1,3,5,7,9,11,13,15}

s = {(1,15), (3,13), (5,11), (7,9)}

∴ 4 + 1 = 5 (Ans)

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Permutation and Combination

Permutation

It is an arrangement of elements by taking sum or all at a time out of a given number of distinct element.

Permutation of n distinct element by taking r at a time is denoted by $P(n,r)$ where

$$P(n,r) = \frac{\lfloor n}{\lfloor n-r}$$

Types of Permutation

2 types

1. Linear Permutation
2. Circular Permutation.

Linear Permutation.

The arrangement in which the starting point and ending point are different is called a Linear Permutation.

If n number are arranged on a line then the total number of ~~element~~ arrangement is $\lfloor n$.

Circular Permutation

The arrangement in which the starting point and ending point are same is called a circular Permutation.

If n distinct object are arranged in a circle then the no. of arrangement is $\lfloor (n-1)$.

Basic formula of Permutation

① Permutation of n different objects by taking r at a time in which one particular thing always occurs is

$P(n-1, r-1) \times P(r, 1)$

② Permutation of n different objects by taking r at a time in which k particular thing always occurs is

$\boxed{P(n-k, r-k) \times P(r-k+1, k)}$

(Generalized)

③ Permutation of n different objects by taking r at a time in which 1 particular thing never occurs is

$P(n-1, r)$

④ Permutation of n different objects by taking r at a time in which k particular thing never occurs is

$\boxed{P(n-k, r)}$ (Generalized)

⑤ Permutation of n objects out of which k are of one type, m are of second type and rest are of third type is:

$$\boxed{\frac{n!}{k!\ m!\ (n-k-m)!}}$$ (Generalized)

Combination

It is the selection or grouping of elements by taking some or all at a time out of a given number of distinct element.

Combination of n different object by taking r at a time where $r \leq n$ is denoted by

$C(n, r) / C\binom{n}{r} / {}^{n}C_{r} = \frac{\lfloor n}{\lfloor r \lfloor n-r}$

Relation btw^n^ P(n, r) and C(n, r)

$P(n, r) = r!\ C(n, r)$

Complimentary Combination

2 combinations $C(n, r)$ and $C(n, p)$ are said to be complimentary combination if $r \neq p$ but $C(n, r) = C(n, p)$

NOTE :- If $C(n, r)$ & $C(n, p)$ are complimentary combinations then $n = r + p$

Q Prove that ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

Ans L.H.S

$$\frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \lfloor n \left(\frac{1}{r\,\lfloor r-1\ \lfloor n-r} + \frac{1}{\lfloor r-1\ (n-r+1)\ \lfloor n-r} \right)$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \left(\frac{n-\cancel{r}+1+\cancel{r}}{r(n-r+1)} \right)$$

$$= \frac{n!\,(n+1)}{(r-1)!\cdot r\,(n-r)!\,(n-r+1)}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} = \boxed{{}^{n+1}C_r}$$

Some Basic formula of Combination

1. Combination of n different objects by taking r at a time in which 1 particular thing always occur is $C(n-1, r-1)$

2. Combination of n different objects by taking r at a time in which k particular thing always occur is $\boxed{C(n-k, r-k)}$ (generalized)

3. Combination of n diff. objects by taking r at a time in which one particular thing never occurs is $C(n-1, r)$

4. Combination of n diff. objects by taking r at a time in which k particular thing never occurs is $\boxed{C(n-k, r)}$ (generalized)

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Q How many words can be formed by using the letters of the word LUCKNOW such that
(i) No restriction
(ii) Starting with L
(iii) starting with L & ending with W
(iv) all the vowels are together.

Ans (i) $7!$

(ii) $6!$

(iii) $5!$

(iv) $6! \times 2!$

Q How many bit strings of length 10 ~~contains~~ can be formed containing
(i) exactly 4 no. of 1
(ii) atleast 4 no. of 1
(iii) at most 4 no. of 1
(iv) equall no. of 0 & 1

Ans (i) $\frac{10!}{4!\,6!}$

Ans (ii) Here the no. of 1 is atleast 4 means different cases will arise.

Case I

4 no. of 1 = $\frac{10!}{4!\,6!}$

Case II

5 no. of 1 = $\frac{10!}{5!\,5!}$

Case III

6 no. of 1 = $\frac{10!}{6!\,4!}$

Case IV

7 no. of 1 = $\frac{10!}{7!\,3!}$

Case V

8 no. of 1 = $\frac{10!}{8!\,2!}$

Case VI

9 no. of 1 = $\frac{10!}{9!\,1!}$

Case VII

10 no. of 1 = $\frac{10!}{10!} = 1$

Adding all.

Ans (iii) Here the no. of 1 is atmost 4 means different cases will arise.

Case I

4 no. of 1 = $\frac{10!}{4!\,6!}$

Case II

3 no. of 1 = $\frac{10!}{3!\,7!}$

Case III

2 no. of 1 = $\frac{10!}{2!\,8!}$

Case IV

1 no. of 1 = $\frac{10!}{10!} = 1$

(iv)

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(Q) How many permutations of the letters A, B, C, D, E, F, G. Contain
i) The string BCD
ii) The string CFGA
iii) The string BA & GF
iv) The strings ABC & CDE
v) The strings ABC & FCG

Ans i) 5!

ii) 4!

iii) 5! ~~×2!~~

iv) It is possible only when ABCDE is a string.
3!

v) 0

(9) In a class there are 7 boys 8 girls. In how many diff. ways a comitte of 6 members can be formed such that
i) No restriction.
ii) 2 no. of boys 4 no. of girls.
iii) atleast 4 girls.

Ans i) $^{15}C_6$

ii) $^7C_2 \times {^8C_4}$

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iii) Here different cases will arise

4 girl 2 boy $\rightarrow$ $^7C_2 \times {^8C_4}$

5 girl 1 boy $\rightarrow$ $^7C_1 \times {^8C_5}$

6 girl $\rightarrow$ $^8C_6$

Adding all we get the result.

(Q) How many rectangles ~~can be formed by using the~~ ~~sma~~ are there in the chess board.

Ans In a chess board there are 9 horizontal lines and 9 vertical lines for a rectangle 2 horizontal lines & 2 vertical lines are required.
So the total no. of rectangle is
$^9C_2 \times {^9C_2}$

No. of square $= 8^2 + 7^2 + 6^2 + 5^2 + 4^2 + \cdots + 1^2$

Box size
1×1
2×2
3×3
4×4
5×5 $= 4^2$
6×6 $= 3^2$
7×7 $= 2^2$
8×8 $= 1$

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(9) Suppose that a department contains 10 men & 15 women. How many ways a comite a 6 members can be form. So that the no' of women will be more that the no' of men.

Ans No' of possibility

Men	Women
0	6
1	5
2	4
3	3
4	2
5	1
6	0

} ~~This 3 cases will~~ taken into consideration

3 cases will arise

Case I

0 men 6 women = $^{10}C_0 \times {}^{15}C_6$

1 men 5 women = $^{10}C_1 \times {}^{15}C_5$

2 men 4 women = $^{10}C_2 \times {}^{15}C_4$

(9) Find the dictionary ranking of MOTHER.

Ans

MOTHER

EHMORT → Alphabetically.

~~HEMORT → 6!~~

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E _ _ _ _ _ → $\lfloor 5$ = 120
H _ _ _ _ _ → $\lfloor 5$ = 120
M E _ _ _ _ → $\lfloor 4$ = 24
M H _ _ _ _ → $\lfloor 4$ = 24
M O E _ _ _ → $\lfloor 3$ = 6
M O H _ _ _ → $\lfloor 3$ = 6
M O R _ _ _ → $\lfloor 3$ = 6
M O T E _ _ → $\lfloor 2$ = 2
M O T H E R → 1 = 1

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3 4 6 2 1 5
M O T H E R
2 2 3 1 0 0
$\lfloor 5$ $\lfloor 4$ $\lfloor 3$ $\lfloor 2$ $\lfloor 1$ $\lfloor 0$

Towards Right how may small number are less

Rank = $2 \times \lfloor 5 + 2 \times \lfloor 4 + 3 \times \lfloor 3 + 1 \times \lfloor 2 + 0 + 0$
= 309

Binomial Theorm

According the Binomial Th. for positive integral index we know that

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} + {}^nC_2 a^{n-1} x^2 + \dots + {}^nC_n x^n$$

Here, ${}^nC_0, {}^nC_1, {}^nC_2 \dots {}^nC_n$ are binomical coefficients.

Relatⁿ amoung the binomial coefficients

According to Binomial Th we know that

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \dots + {}^nC_n x^n$$

Putting $a=1$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots {}^nC_n x^n \quad \text{----(i)}$$

If we put $x=1$ eqⁿ (i)

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n \quad \text{---(ii)}$$

Putting $x=-1$ in eqⁿ (i)

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots + (-1)^n\,{}^nC_n \quad \text{---(iii)}$$

Adding eqⁿ (ii) & (iii)

$$2^n = 2({}^nC_0 + {}^nC_2 + {}^nC_3 + \dots {}^nC_n) \quad \text{---(IV)}$$

$$\Rightarrow 2^{n-1} = {}^nC_0 + {}^nC_2 + {}^nC_3 \dots$$

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subtracting (iii) from (ii)

$$2^{n-1} = {}^nC_p + {}^nC_3 + {}^nC_5$$

Pascal Triangle

Power of the Binomial	Coefficients
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

Pascal Triangle - It is used for finding the expansion of a binomial when the power of the binomial is any +ve integer.
This expansion is fully dependent on pascal's triangle which is given ~~below~~ above.

Vander mondes Identity -

If m & n are the positive integers then summation k is equal to 0 to n. $\left(\sum_{k=0}^{n}\right)({}^mC_{n-k})({}^nC_k) = {}^{m+n}C_n$

$$\sum_{k=0}^{n} ({}^mC_{n-k})({}^nC_k) = {}^{m+n}C_n$$

According to Binomial Th. we know that

$$(1+x)^m = {}^mC_0 + {}^mC_1 x + {}^mC_2 x^2 + \dots + {}^mC_m x^m \quad \text{--- (I)}$$

Similarly, $$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n \quad \text{--- (II)}$$

Multiplying eqⁿ (I) & (II) we get

$$(1+x)^{m+n} = ({}^mC_0 + {}^mC_1 x + {}^mC_2 x^2 + \dots + {}^mC_{r-1} x^{r-1} + {}^mC_r x^r + \dots + {}^mC_m x^m)$$
$$\cdot ({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{r-1} x^{r-1} + {}^nC_r x^r + \dots + {}^nC_n x^n)$$

$$\Rightarrow {}^{m+n}C_0 + {}^{m+n}C_1 x + {}^{m+n}C_2 x^2 + \dots + {}^{m+n}C_r x^r + \dots + {}^{m+n}C_{m+n} x^{m+n}$$

= same R.H.S as above

Comparing the coeff. of x^r in both sides we get

$${}^{m+n}C_r = {}^nC_0\, {}^mC_r + {}^nC_1\, {}^mC_{r-1} + {}^nC_2\, {}^mC_{r-2} + \dots + {}^nC_r\, {}^mC_0$$

$$\boxed{{}^{m+n}C_r = \sum_{k=0}^{r} {}^mC_{r-k}\, {}^nC_k}$$ (Proved)

Pascal's Identity

$$\boxed{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r}$$

Q. Prove the hexagon identity

$${}^{n-1}C_{k-1} \cdot {}^nC_{k+1} \cdot {}^{n+1}C_k = {}^{n-1}C_k \cdot {}^nC_{k-1} \cdot {}^{n+1}C_{k+1}$$

Ans L.H.S

$${}^{n-1}C_{k-1} \cdot {}^nC_{k+1} \cdot {}^{n+1}C_k$$

$$= \frac{(n-1)!}{(k-1)!\,(n-k)!} \cdot \frac{n!}{(k+1)!\,(n-k-1)!} \cdot \frac{(n+1)!}{k!\,(n+1-k)!}$$

$$= \frac{(n-1)!}{k!\,(n-1-k)!} \cdot \frac{n!}{(k-1)!\,(n-k+1)!} \cdot \frac{(n+1)!}{(n-k)!\,(k+1)!}$$

$$= {}^{n-1}C_k \times {}^nC_{k-1} \times {}^{n+1}C_{k+1} \quad \text{R.H.S}$$

Q. $k\,{}^nC_k = n\,{}^{n-1}C_{k-1}$

L.H.S

$$k \frac{(n)!}{(k)!\,(n-k)!} = \frac{\cancel{k}\; n!\,(n-1)!}{\cancel{k}\,(k-1)!\,(n-k)!}$$

$$= \frac{n(n-1)!}{(k-1)!\,(n-1-(k-1))!}$$

$$= n\ {}^{n-1}C_{k-1} \quad \text{R.H.S}$$

Q) $${}^{2n}C_{n+1} + {}^{2n}C_{n} = \frac{1}{2}\ {}^{2n+2}C_{n+1}$$

L.H.S

$${}^{2n}C_{n} + {}^{2n}C_{n+1}$$

$$= {}^{2n+1}C_{n+1}$$

$$= \frac{(2n+1)!}{(n+1)!\,(n)!}$$

$$= \frac{(2n+2)(2n+1)!}{(2n+2)\,(n+1)!\,(n)!}$$

$$= \frac{(2n+2)!}{2\,(n+1)!\,(n+1)!}$$

$$= \frac{1}{2}\ {}^{2n+2}C_{n+1} \quad \text{R.H.S}$$

Generalized Permutation & Combination

In case of Permutation all the objects are distinct; but in case of generalized permutation and combination the objects maybe repeated.

→ **Theorem 1**

If n objects are placed in r places then with repetation then the total number of arrangement is n^r.

Example

Q) How many 4 digited number can be formed using the digits 1, 2, 3, 4, 5.

i) without Repetation

ii) with Repetation.

Ans i) $\boxed{5}\boxed{4}\boxed{3}\boxed{2} = 5\times4\times3\times2$

ii) $\boxed{5}\boxed{5}\boxed{5}\boxed{5} = 5^4$

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Dearrangement

Dearrangement means it is the arrangement of n objects in which all the objects are not arranged in their proper position.

For Ex, suppose in a class there are 5 students. The teacher wants to send the progress card of each of the student. Let the progress of these students are P_1, P_2, P_3, P_4, P_5 and their address are E_1, E_2, E_3, E_4, E_5 respectively. If these progress cards are entered in wrong envelope then it is called dearrangement.

If n objects are dearrangement, then

$$\text{dearrangement} = \lfloor n \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \dots + (-1)^n \frac{1}{\lfloor n}\right]$$

Example: In a school there are 5 classes, in class one, there are 7 students whose attendance are below 50%. The teacher wants to inform it to their parents, through letters. The letters are prepared for individual students and their address are written in the envelopes. In how many different ways these letter can be inserted in the envelope such that none of the letter goes to the right envelope.

Ans: Here $n = 7$,

So, the total no. of dearrangement is

$$D_n = \lfloor 7 \cdot \left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \frac{1}{\lfloor 6} - \frac{1}{\lfloor 7}\right]$$

Chapter Closed

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Relation

If A & B are any 2 non-empty sets, then any subset of $A \times B$ is called a relation from A to B.

Let,
$A = \{a, b, c\}$
$B = \{p, q\}$

$A \times B = \{(a,p), (a,q), (b,p), (b,q), (c,p), (c,q)\}$

Let $R = \{(a,p), (a,q), (b,q)\}$

$R \subset A \times B$
So. R from A to B is a relation

Some Special Relation

(i) Empty Relation

A relation R from A to B is called an empty relⁿ. If R doesnot contain any element from $A \times B$. And it is denoted by ϕ

(ii) Total Relation
A relation R from A to B is called total relⁿ. If R contains all the elements of $A \times B$.
$R = A \times B$

(iii) Identity Relation:
A relation R from A to A is said to be Identity relⁿ. If R contains only and all the identity order pairs of $A \times A$.
$A = \{a, b, c\}$

$A \times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

$R = \{(a,a), (b,b), (c,c)\}$

Inverse of a Relation

If R is a relation from A to B then its inverse relation denoted by R^{-1} is a relation from B to A. which contains the inverse order pair of all the order pairs which are present in R.

$R = \{(a,b), (b,c), (a,a)\}$

$R^{-1} = \{(b,a), (c,b), (a,a)\}$

Compliment of a Relation

The compliment of the relation R from A to B denoted by $\bar{R}$ is defined as it is a relⁿ from A to B and it contains those elements of $A \times B$ which are not in R.

$\bar{R} = A \times B - R$

Let $A = \{a, b, c\}$, $B = \{x, y\}$
$A \times B = \{(a,x), (a,y), (b,x), (b,y), (c,x), (c,y)\}$

$R = \{(a,x), (a,y), (c,x)\}$
$\bar{R} = \{(b,x), (b,y), (c,y)\}$

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NOTE

i) $R \cup \bar{R} = A \times B$

ii) If, $|A| = m$ & $|B| = n$
Then the total no. of relⁿ from A to B is
2^{mn}
Out of which only one is empty relⁿ and the remaining $2^{mn} - 1$ are non-empty relⁿ.

Relation over a single set

A relⁿ R from A to A is called a relation over a single set.

Properties of Relⁿ over a single set

① Reflexive

A relⁿ R from A to A is said to be Reflexive if it contains all the identity order pairs of $A \times A$.

R from A to B
$\forall x \in A \rightarrow (x,x) \in R$

Eg
$A = \{a, b, c\}$
$R = \{(a,a), (b,b), (c,c), (a,b)\}$.

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(ii) Irreflexive

A relⁿ R from A to A is said to be irreflexive relⁿ if R doesnot contain any of the identity order pair from A to A.

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Symmetric Relation

R from $A \rightarrow A$.
$\forall (x,y) \in R \Rightarrow (y,x) \in R$

A relⁿ R from $A \rightarrow A$ is said to be symmetric relⁿ if R must contain the inverse order pair of all the order pairs present in R.

Let $A = \{a, b, c, d\}$
R from $A \rightarrow A$ is a relⁿ
$R = \{(a,a), (a,b), (b,a), (c,c)\}$

A-symmetric Relation

A relⁿ R from $A \rightarrow A$ is said to be Asymmetric relⁿ if R doesn't contain the inverse order pair of any order pair of R.
That means,
$\forall (x,y) \in R \Rightarrow (y,x) \notin R$

In assymmetric relⁿ identity order pair are not present.

$R = \{(a,b), (b,c), (c,a)\}$

Anti-symmetric

A relⁿ R from $A \to A$ is said to be antisymmetric if, it doesnot contain the inverse order pair of any order pair except Identity order pair

$$\forall (x,y)\ (y,x) \in R \Rightarrow x = y.$$

Ex

R_2 from $A \to A$ is a relⁿ

$R_2 = \{(a,a), (b,b), (a,b), (b,c)\}$

Transitive Relation

A relⁿ R from $A \to A$ is said to be transitive relⁿ if

$$\forall (x,y)\ (y,z) \in R \Rightarrow (x,z) \in R.$$

Ex

Let $A = \{a,b,c,d\}$

R from $A \to A$

$R = \{(a,a), (b,b), (a,b), (b,c), (a,c)\}$

Here R is transitive.

$R_1 = \{(b,c), (b,d), (b,a)\}$

Transitive

$R_2 = \{(a,a), (c,c), (a,b), (b,a)\}$

Not-transitive

$(b,a)\ (a,b) \in R_2 \Rightarrow (b,b) \notin R_2$

Equivalence Relⁿ :

A relⁿ R from $A \to A$ is said to equivalence relⁿ when it satisfies reflexive, symmetric & transitive.

Ex

Let $A = \{a,b,c\}$

R from $A \to A$ is a relⁿ ~~where~~ where

$R = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$

Since R is reflexive, symmetric & transitive so, it is an equivalence relⁿ over set A.

NOTE

The Smallest equivalence relⁿ over the set A is Identity relⁿ over A.

& the largest equivalence relⁿ is $A \times A$.

Partial Ordering Relation

A relⁿ R from $A \to A$ is said to be partial order relⁿ if R is reflexive anti-symmetric and transitive.

Ex

Let $A = \{a,b,c\}$

R f $A \to A$

$R = \{(a,a), (b,b), (c,c), (a,b)\}$

Total ordering relⁿ / Linear ordering relⁿ.

A relⁿ R from $A \to A$ is said to be Total ordering relⁿ if R will satisfy the following 2 conditions.

(i) R is a Partial ordering relⁿ

(ii) $\forall (x,y) \in A \Rightarrow (x,y)$ or $(y,x) \in R$

R must contain one order pair from each possible pairing of A.

If $|A| = n$

Ref. $A \to A$

Total Number of :-

i) reflexive $= 2^{n^2-n}$

ii) symmetric $= 2^{(n^2+n)/2}$

iii) reflexive & symmetric $= 2^{(n^2-n)/2}$

Combining of Relation

If R_1 & R_2 are two relⁿ over the same set then their combination can be taken in 4 different ways, that is.

$R_1 \cup R_2$

$R_1 \cap R_2$

$R_1 - R_2$

$R_1 \oplus R_2$

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Q Let $A = \{a, b, c, d\}$

R_1 from $A \to A$

$R_1 = \{(a,b), (a,c), (b,c), (b,d)\}$

R_2 from $A \to A$

$R_2 = \{(a,a), (a,b), (a,d), (b,c), (c,d)\}$

Find $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, $R_1 \oplus R_2$

Ans $R_1 \cup R_2 = \{(a,b), (a,c), (a,a), (a,d), (b,c), (b,d), (c,d)\}$

$R_1 \cap R_2 = \{(a,b), (b,c)\}$

$R_1 - R_2 = \{(a,c), (b,d)\}$

$R_2 - R_1 = \{(a,a), (a,d), (c,d)\}$

$R_1 \oplus R_2 = \{(a,c), (b,d), (a,a), (a,d), (c,d)\}$

$(R_1 - R_2) \cup (R_2 - R_1)$

Composition of ~~Matri~~ Relations

Let R from $A \to B$ ~~is a relⁿ~~ and S from $B \to C$ be 2 relⁿˢ. Since the co-domain of R is equal to the domain of S. So, in this case the composition fxⁿ SOR is defined and it is defined as SOR from $A \to C$, where for
$\forall x \in A$, $SOR(x) = S(R(x))$

Q) Let $A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4\}$
$C = \{7, 8, 9\}$

R from $A \to B$
$R = \{(1,1), (1,4), (2,3), (3,1)\}$

S from $B \to C$
$S = \{(1,7), (4,8), (3,8)\}$

Find SOR and ROS?

Ans SOR
Since the co-domain of R is equal to domain of S. So, SOR is defined; so the domain of SOR is $\{1,2,3\}$

$SOR(1) = S(R(1)) = S(1,4) = (7,8)$

$SOR(2) = S(R(2)) = S(3) = 8$

$SOR(3) = S(R(3)) = S(1) = 7$

$SOR = \{(1,7), (1,8), (2,8), (3,7)\}$

In this case ROS is not defined because the codomain of S is not equal to domain of R.

NOTE

If R is a relation from $A \to A$ and all the elements of A are taking part in the relⁿ. Then in this case we can find out the composition of R with itself.

$ROR = R^2$
$R^3 = R^2 OR = ROR^2$
$R^4 = R^3 OR$
$R^5 = R^4 OR$
$\vdots$
$R^n = R^{n-1} OR$

Q Let R from $A \to A$ be a relⁿ
where $A = \{a, b, c\}$
$R = \{(a,b), (b,c), (c,a)\}$
Find R^2 & R^3.

Ans $R^2(a) = ROR(a)$
$= R(R(a))$
$= R(b)$
$= c$

$R^2(b) = R(R(b))$
$= R(c)$
$= a$

$R^2(c) = R(R(c))$
$= R(a)$
$= b$

$R^2 = \{(a,c), (b,a), (c,b)\}$

$R^3(a) = R^2 \circ R^2(a)$
$= R(R^2(a))$
$= R(c)$
$= a$

$R^3(b) = R(R^2(b))$
$= R(a)$
$= b$

$R^3(c) = R(R^2(c))$
$= R(b)$
$= c$

$R^3 = \{(a,a), (b,b), (c,c)\}$

NOTE
A relⁿ R will be transitive if R_n is a subset of R.

Representation of a relⁿ using Matrix

If A & B are 2 sets where $|A| = m$ & $|B| = n$ & R is a relⁿ from $A \to B$. Then its corresponding matrix representation M_R

$\boxed{M_R \text{ is of order } m \times n}$

whose elements are either 1 or 0.

$$a_{ij} = \begin{cases} 1, & (a_i, a_j) \in R \\ 0 & (a_i, a_j) \notin R \end{cases}$$

Ex Let, $A = \{a, b, c\}$
$B = \{p, q, r, s\}$
R is relⁿ from $A \to B$.
$R = \{(a,p), (b,q), (b,r), (b,s)\}$

Represent this relⁿ in ~~Mat~~ terms of Matrix.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Q Let $A = \{x, y, z\}$
R from $A \to A$ is a relⁿ where
$R = \{(x,x), (y,y), (z,z), (x,y), (y,x)\}$

$$M_R = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

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NOTE

If R is a rel^n from $A \to A$ and its matrix representation is M_R then from M_R we can easily identify Reflexive, Symmetric & anti-Symmetric properties.

Reflexive
It is reflexive if all the main diagonal elements are 1.

Symmetric
It is symmetric if $a_{ij} = a_{ji}$

Anti-Symmetric
It is anti-symmetric if $a_{ij} \neq a_{ji}$

Matrix Representation of combination of 2 relations

If R_1 & R_2 be 2 rel^{ns} whose matrix representat^n are M_{R_1} & M_{R_2} then the matrix representat^n of $R_1 \cup R_2$ is

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$
$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$
$$ROS = M_S \odot M_R$$

Q) Let $A = \{a, b, c\}$. R_1 from $A \to A$ be a rel^n, where
$R_1 = \{(a,a), (a,c), (b,a), (c,b)\}$
and R_2 is another rel^n from $A \to A$
$R_2 = \{(a,a), (a,c), (b,b), (b,c), (c,a)\}$
Find :-
M_{R_1}, M_{R_2}, $M_{R_1 \cup R_2}$, $M_{R_1 \cap R_2}$, M_{ROS} & M_{SOR}

Ans

$$M_{R_1} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 1 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \end{array}$$

$$M_{R_2} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 0 \end{array}$$

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \vee \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_{R_2 \circ R_1} = M_{R_1} \odot M_{R_2}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$M_{R_1 \circ R_2} = M_{R_2} \odot M_{R_1}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Representation of rel^n using ~~digraphs~~ digraph

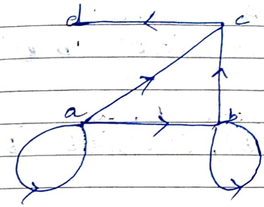
$A = \{a, b, c, d\}$

Every rel^n ~~can be~~ over a single set can be represented through a directed graph which is called the digraph of the relations.

Ex $A = \{a, b, c, d\}$

R from $A \to A$

$R = \{(a,a), (b,b), (a,b), (b,c), (a,c), (c,d)\}$



Closure of a rel^n

Closure

Let R from A to A be a rel^n. Then, R_1 from $A \to A$ is called a closure of rel^n ~~the~~ R. If R_1 is a super set of R.

(i) Reflexive Closure of a rel^n

Let R from $A \to A$ be a rel^n R_1 from $A \to A$ is called the reflexive closure of R. If R_1 will satisfy the following 2 conditions.

(a) R_1 is a superset of R $(R_1 \supset R)$

(b) R_1 is itself reflexive

NOTE:-

→ R_1 must be obtained by adding some identity order pairs of R

→ If R is itself reflexive then the reflexive closure of R is R.

Ex

Let $A = \{a, b, c\}$
R from $A \to A$, where
$R = \{(a,a), (b,b), (a,b), (b,c)\}$

Its reflexive closure R_1 is
$\{(a,a), (b,b), (c,c), (a,b), (b,c)\}$

(ii) Symmetric Closure

The symmetric closure of the relⁿ R from $A \to A$ will be the relⁿ R_1 from $A \to A$ which must satisfy the following 2 conditions.

(a) $R_1 \supset R$
(b) R_1 is itself symmetric
$(\forall (a,b) \in R \Rightarrow (b,a) \in R_1)$

Ex

Let $A = \{a, b, c\}$
$R = \{(a,b), (b,c), (a,a)\}$
$R_1 = \{(a,b), (b,c), (a,a), (b,a), (c,b)\}$

NOTE

→ In symmetric close only non identity order pairs are added.
→ If R is itself symmetric then its symmetric closure is itself R.

(iii) Transitive ~~Closure~~ Extension

Let R from $A \to A$ is a relⁿ.
R_1 from $A \to A$ is called the transitive Extension of R if :-
(A) It will satisfy the following 2 conditⁿˢ.
(a) $R_1 \supset R$.
(b) $\forall (a,b), (b,c) \in R \not\Rightarrow (a,c) \in R_1$

NOTE
The transitive extension of a relⁿ, may be or may ~~or~~ not be ~~or~~ transitive.

Ex

$A = \{a, b, c, d\}$
R fr $A \to A$
$R = \{(a,a), (b,b), (a,b), (b,c)\}$
Let R_1 be the transitive extenⁿ of R.
$R_1 = \{(a,a), (b,b), (a,b), (b,c), (a,c)\}$

$A = \{a, b, c, d, e\}$
$R = \{(a,b), (b,d), (d,a)\}$

$R_1 = \{(a,b), (b,d), (d,a), (a,d), (b,a), (d,b)\}$

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Transitive Closure

Let R from A → A be a relⁿ suppose R_1 is the transitive extension of R. R_2 is the transitive extension of R_1. R_3 is the transitive extension of R_2. If we continue this process we get R_4, R_5, R_6 ... so on so for. If 2 consiquite extension say R_K & R_{K+1} are equal. Then the process is terminated and the last extension R_K is called the Transitive closure of Relⁿ R.

Ex Let $A = \{a,b,c,d\}$
R from A → A
$R = \{(a,a), (a,b), (b,c), (b,d)\}$

$R_1 = \{(a,a), (a,b), (b,c), (b,d), (a,c), (a,d)\}$

$R_2 = \{(a,a), (a,b), (b,c), (b,d), (a,c), (a,d)\}$

Since $R_1 = R_2$
Hence, R_1 is the transitive closure of the relⁿ R.

NOTE
Let R from A → A be a relⁿ suppose R^2 is the composition of R with itself. that means
$R^2 = R o R$
Similarly $R^3 = R^2 o R$
$R^4 = R^3 o R$

Continuing in similar manner we get,
R^5, R^6, R^7, etc.

So the transitive closure of a Relⁿ R is denoted by

$$\boxed{R^* = \bigcup_{i=1}^{\infty} R^i} = R \cup R^2 \cup R^3 \cup \dots \cup R^\infty$$

Let $A = \{a,b,c,d\}$
R from A → A.
$R = \{(a,a), (a,b), (b,c), (b,d)\}$
$R^2 = R o R =$ $R^2(a) = R(R(a))$
$= R(a,b)$
$= \{a,b,c,d\}$

$R^2(b) = R(R(b))$
$= R(c,d)$
Not defined

Similarly, $R^2(c)$ & $R^2(d)$ is not defined.

$R^2 = \{(a,a), (a,b), (b,c) (a,d)\}$

$R^3 = R^2 o R =$ $R^3(a) = R^2(R(a))$
$= R^2(a,b)$
$= \{a,b,c,d\}$

$R^3(b) = R^2(R(b))$
$= R^2(c,d)$
= Not defined

similarly $R^3(c)$ & $R^3(d)$ are not defined

$R^3 = \{(a,a), (a,b), (a,c), (a,d)\}$

$R^4 = R^3 o R = R^3 o R(a) = R^3(R(a))$
$= R^3(a, b)$
$= \{a, b, c, d\}$

$R^4(b) = R^3 o R(b) = R^3(R(b))$
$= R^3(c, d)$
$=$ Not defined.

$R^4(c)$ & $R^4(d)$ is not defined.

$R^4 = \{(a, a), (a, b), (a, c), (a, d)\}$

$R^* = \underbrace{R' \cup R^2}_{R^2} \cup R^3 \cup R^4$

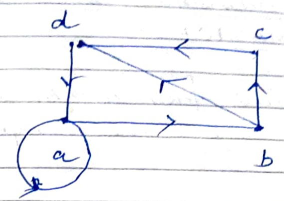
$= R \cup R^2$
$= \{(a, a), (a, b), (b, c), (b, d), (a, c), (a, d)\}$

Path is a directed graph (di-graph)

It is a sequence of edges, such that the end vertex of one edge will be the starting point of another edge.

In other words, it is a sequence of edges in a particular direction.

Ex
Let $A = \{a, b, c, d\}$
R from $A \to A$ is a rel^n where
$R = \{(a, a), (a, b), (b, c), (c, d), (b, d), (d, a)\}$



Here, The paths from the vertex a to d are,
a b c d
a b d

Length of the Path
It is the no. of edges covered in that path.
Ex
In the path a b c d the length is 3
but in the path a b d the length is 2.

Circuit
A path is said to be circuit if its ending & starting vertex are same.
For example a b c d a is a circuit.

Connectivity Relation

A rel^n R from $A \to A$ is said to be a connectivity rel^n if ~~By for every~~ $\forall (a, b) \in R \Rightarrow$ there is a path betw^n a & b whose length is ≥ 1.

Determination of Transitive Closure of a Relⁿ using MATICES.

Let R be any relⁿ from $A \to A$, suppose $R^2, R^3, R^4, \ldots$ are the composition relⁿ of R.

We know that,

The transitive closure of R

$R^* = R \cup R^2 \cup R^3 \ldots$

$$\boxed{M_R^* = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}}$$

$M_{R \circ S} = M_S \odot M_R$

Q) Find the transitive closure of relⁿ R whose matrix representation

$$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$M_{R^2} = M_R \odot M_R$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_{R^2} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$M_{R^3} = M_R \odot M_{R^2}$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_R^* = M_R \vee \underbrace{M_{R^2} \vee M_{R^3}}_{\text{same}}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

5/1/22

Warshall's Algorithm for determinatⁿ of transitive closure of a relⁿ.

Let R be a relⁿ from $A \to A$ where cardinality of $|A| = n$

Step ①

Let W be the matrix form of representation of the relⁿ R.

It is clear that the order of the matrix W is $n \times n$

Step (2)

Let ω_1 be the extension of ω, By using the first vertex as interior vertex. that mean if a is the first vertex and there is a path from ba & ac, then their implication order pair be ~~must~~ will be introduced in ~~the~~ ω_1.

(a)

~~ab~~ ba ac $\Rightarrow$ (b,c)

Step (3)

Let ω_2 be the extension of ω_1 by using 2^{nd} vertex as interior vertex.

Step (4)

Continuing in similar manner we get $\omega_3, \omega_4, \omega_5, \dots, \omega_n$

This ω_n is the transitive closure of the relⁿ R.

Qn Using warshall's algorithm find the transitive closure of the relⁿ R from A→A where R = {(a,d), (b,a), (b,c), (c,a), (c,d), (d,c)}

where A = {a,b,c,d}

Ans Let ω be the matrix form of the relⁿ R. We know that, ~~i~~ it has 4 element, so 4×4 matrix.

$$\omega = \begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

~~ω_1 =~~ If we take a as intermediate vertex, then we have 2 paths (ba) & (ca). But starting with a there is only pair (a d), so, comparing ba and (ca) with ad we get (b d) & (cd).

Introducing these two in ω we get ω_1

$$\omega_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since in 2^{nd} column of ω_1 all the elements are 0. So, we can't take the 2^{nd} vertex as the interior vertex. ~~Since~~ Hence, $\boxed{\omega_2 = \omega_1}$

In 3^{rd} column there are 2 vertices (b c) & (d c) but starting with c there are 2 order pairs (ca) & (cd).

Now,

(b c), (c a) $\Rightarrow$ (b a)

(b c), (c d) $\Rightarrow$ (b d)

(d c), (c a) $\Rightarrow$ (d a)

(d c), (c d) $\Rightarrow$ (d d)

Introducing these 4 in ω_2 we get ω_3.

$$W_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

(a d) (b d) (c d) (d d)

(d a) (d c) (d d)

ad, da ⇒ aa
ad dc ⇒ ac
ad dd ⇒ ad

bd da ⇒ ba
bd dc ⇒ bc
bd dd ⇒ bd

cd da ⇒ ca
cd dc ⇒ cc
cd dd ⇒ cd

dd da ⇒ da
dd dc ⇒ dc
dd dd ⇒ dd

$$W_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

So the transitive closure of the relⁿ is W_4

Hence, the transitive closure is,

{(a,a), (a,c), (a,d), (b,a), (b,c), (b,d), (c,a), (c,c), (c,d), (d,a), (d,c), (d,d)}

Q Let $A = \{a, b, c\}$
R from $A \to A$
$R = \{(a,a), (a,b), (b,a), (b,c)\}$

Ans

$$W_0 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(a,a) (b,a) — (a,a) (a,b) = (a,a) (a,b) (b,a) (b,b)

$$W_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

~~(b,a)~~ (a,b) (b,b) — (b,a) (b,b) (b,c) = (a,a) (a,b) (a,c) (b,a) (b,b) (b,c)

$$W_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

c a
c b

$W_3 = W_2$

Hence, the transitive closure is
{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c)}

NOTE
Whenever we find out W_i by taking i^{th} vertex as the interior vertex and if in its previous matrix W_{i-1}, either all the elements of i^{th} row or all the elements of i^{th} column are 0. Then that i^{th} vertex cannot be taken as interior vertex.

So, in this case,
$$\boxed{\omega_{i-1} = \omega_i}$$

Equivalence Relation

Congruent modulo n

If n be any natural number and a & b are any 2 integers then a is congruent to b modulo n if n is a divisor of a − b

$a \equiv b \pmod n \Leftrightarrow n \mid (a-b)$

$30 \equiv 3 \pmod 9 \Leftrightarrow 9 \mid (30-3)$

$37 \equiv 2 \pmod 7 \Leftrightarrow 7 \mid (37-2)$

5

Q) Show that the relⁿ parallelism is an equivalence relⁿ over the set of lines L.

Ans Here, L is the set of lines.
R = Parallel
R from $L \to L$ is a relⁿ where,
$(L_1, L_2) \in R \Leftrightarrow L_1 \parallel L_2$

We have to show that R is an equivalence relⁿ.

That means, R is reflexive, symmetric & transitive.

Reflexive

We have to show that
$\forall L_1 \in L \Rightarrow (L_1, L_1) \in R$.
~~Since R is reflexive~~
We know that if 2 lines are parallel then angle betwⁿ them is 0°.
But it is clear that angle betwⁿ L_1 & L_1 is 0°.
It ⇒ order pair (L_1, L_1) is parallel.
So, it is reflexive.

Symmetric

$\forall (L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$
$L_1 \parallel L_2$
$\Rightarrow L_2 \parallel L_1$
$\Rightarrow (L_2, L_1) \in R$

So, it is symmetric

Transitive

$\forall (L_1, L_2) \in R, (L_2, L_3) \in R \Rightarrow (L_1, L_3) \in R$

$L_1 \parallel L_2 \qquad L_2 \parallel L_3$
$\Rightarrow L_1 \parallel L_2 \parallel L_3$
$\Rightarrow L_1 \parallel L_3$
$\Rightarrow (L_1, L_3) \in R$
So, it is transitive.
Hence, R is an equivalence relⁿ.

Q. Show that the rel^n 'congruent modulo n' ~~over the~~ is an equivalence rel^n over the set

Ans → Here,
R from $Z \to Z$ is a rel^n which is defined as
$(a,b) \in R \Leftrightarrow a \equiv b \pmod{n}$

Reflexive
we have to show, $\forall x \in Z \Rightarrow (x,x) \in R$

$n \mid 0$
$\Rightarrow n \mid x-x$
$\Rightarrow x \equiv x \pmod{n}$
$\Rightarrow (x,x) \in R$

Symmetric

we have to show, $\forall x\ (x,y) \in R \Rightarrow (y,x) \in R$.

Let, $(x,y) \in R$
$\Rightarrow x \equiv y \pmod{n}$
$\Rightarrow n \mid x-y$
$\Rightarrow n \mid -(x-y)$
$\Rightarrow n \mid (y-x)$
$\Rightarrow y \equiv x \pmod{n}$
$\Rightarrow (y,x) \in R$.

Transitive
we have to show that,
$\forall (x,y), (y,z) \in R \Rightarrow (x,z) \in R$
$(x,y) \in R$ & $(y,z) \in R$

$x \equiv y \pmod{n}$ & $y \equiv z \pmod{n}$
$\Rightarrow n \mid x-y$ & $n \mid y-x$
$\Rightarrow n \mid x-y+y-z$
$\Rightarrow n \mid x-z$
$\Rightarrow x \equiv z \pmod{n}$
$\Rightarrow (x,z) \in R$

Q. R from $Z \to Z$ be a rel^n where $(a,b) \in R \Leftrightarrow a+b$ is even. Verify R is a equivalence rel^n or not.

Ans → Reflexive
we have to show that,
$\forall a \in Z \Rightarrow (a,a) \in R$
Here, $a+a = 2a$ is also even
So, $(a,a) \in R$

Symmetric
we have to show that,
$\forall (a,b) \in R \Rightarrow (b,a) \in R$

$\Rightarrow$ $a+b$ is even
$\Rightarrow$ $b+a$ is also even.
So,
$(b,a) \in R$

Transitive

$\forall (a,b), (b,c) \in R \Rightarrow (a,c) \in R$

Let, $(a,b) \in R$. $(b,c) \in R$

$a+b$ is even $b+c$ is even.

$\Rightarrow a+b = 2k$ — (I) $\Rightarrow b+c = 2m$ — (II)

Adding (I) & (II)

$a+b+b+c = 2k+2m$

$\Rightarrow a+c+2b = \cancel{4k}\ 2k+2m$

$\Rightarrow a+c = \cancel{4k-2k}\ 2k+2m-2b$

$\Rightarrow a+c = 2(k+m-b)$

$\Rightarrow a+c$ is even

$(a,c) \in R$

(9) Let R for $Z_1 \to Z$

$(a,b) \in R \Leftrightarrow a|b$

$Z_1 = Z - [0]$

Verify R is a equivalence rel^n or not.

Ans → Ref: $\forall x \in Z_1 \Rightarrow (x,x) \in R$

It is clear that

$x | x$

$\Rightarrow (x,x) \in R$

Symmetric

$\forall (x,y) \in R \Rightarrow (y,x) \in R$.

$\Rightarrow x|y \not\Rightarrow y|x$

$\Rightarrow (y,x) \notin R$

Not, Symmetric

Transitive

$\forall (x,y) (y,z) \in R \Rightarrow (x,z) \in R$

$(x,y) \in R$ $(y,z) \in R$

x/y & y/z

$x/y/z$

$\Rightarrow x/z$

$(x,z) \in R$.

Not a equivalence becoz it is not symmetric.

Partition of a Set

Let A be any non-empty set $(A_1, A_2, A_3 \dots, A_n)$ are subsets of A.

These subsets is called ~~the~~ a partition of A. If this ~~partiti~~ subsets will satisfy the following 3 conditions:-

(i) All the subsets are non-empty.
That mean, $A_i \neq \phi$ for every i $(\forall i)$

(ii) $A_1 \cup A_2 \cup A_3 \dots \cup A_n = A$.

(iii) These subsets are mutually disjoints.
$A_i \cap A_j = \phi$ $i \neq j$

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Qn $A = \{a, b, c, d, e, f, g, h\}$

$A_1 = \{a, b\}$
$A_2 = \{c\}$
$A_3 = \{d, g\}$
$A_4 = \{e, f, h\}$

Here A_1, A_2, A_3, A_4 is a partition of set A. And this partition is written blockwise as given below.

$A = \{\overline{ab}\ \bar{c}\ \overline{dg}\ \overline{efh}\}$

Equivalence Classes on Congruence Class

R from $A \to A$.

Equivalence class → If R from $A \to A$ is an equivalence rel^n then all the elements of the set A. Must have equivalence classes. If $x \in A$ Then its equivalence class, denoted by $[x]_R = \{y \mid y \in A, (x, y) \in R\}$

That means,

equivalence class of x contains those element of A; which are related with x. in the equivalence rel^n R.

Qn

Let $A = \{a, b, c, d\}$
R from $A \to A$
$R = \{(a,a), (b,b), (c,c), (d,d), (a,b)(b,a)\}$

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$[a] = \{a, b\}$
$[b] = \{b, a\}$
$[c] = \{c\}$
$[d] = \{d\}$

Properties of equivalence classes.

1. Equivalence classes always depends upon the equivalence rel^n.
That means over the same set, if we take diff.^{nt} equivalence rel^n then their equivalence classes will be diff.^{nt}.

2. If $x \in [y] \Leftrightarrow [x] = [y]$

3. ~~Over the same set~~
If we take the smallest equivalence rel^n over any set A. Then it gives, the lastest number of distinct equivalent classes which is equal to $|A|$.

4. If we take the largest equivalence rel^n over any set A. Then we get the smallest number of distinct equivalence classes which is equal to 1.

Partition of a set induced by an Equivalence rel^n.

Let A be any non-empty set and R from $A \to A$ is an equivalence rel^n. The distinct equivalence classes of the equivalence rel^n R must break the set A as a partition. This partition is called partition induced by an Equivalence rel^n.