Hyper Comparison Category (HCC)

A Categorical Framework for Comparison, Height, and Coarse-Graining

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Abstract

We introduce a categorical framework for organizing comparison problems across multiple observational or coarse–grained levels. A hyper comparison category consists of projection functors, an idempotent saturation operator, and invertible 2–cells measuring the discrepancy between saturation and observation. These data give rise to quantitative invariants such as defects, local supports, and height–type functionals.

Under mild locality and stability assumptions, defects admit local decompositions and satisfy propagation properties along towers of projections. Consequently, inequalities of the form

$$\text{defect} \leq (1 + \varepsilon)\,\text{height} + C$$

follow from purely structural considerations once suitable external estimates are provided.

The formalism cleanly separates structural mechanisms from problem–dependent inputs, and yields a reusable template for reducing global comparison questions to local bounds together with their propagation across observational scales.

1 Introduction

Many mathematical and scientific problems require comparing objects that do not naturally inhabit a common space, or whose observable features depend on choices of interpretation, resolution, or coarse–graining. Examples range from relating models living in different categories, to connecting structures defined at distinct observational scales, to reconciling local data across incompatible viewpoints. In such situations, direct comparison is often ill–posed: the objects of interest may not admit a canonical identification, and the act of observation itself may distort or suppress relevant information [8, 7].

This paper develops a categorical framework for organizing such comparison problems. The guiding principle is that comparison should not be performed on the original objects, but on their *shadows* under suitable projection functors. Prior to projection, an idempotent *saturation* operator enforces a chosen interpretation or coarse–graining; the discrepancy between saturation and observation is recorded by invertible 2–cells. Together, these data form what we call a *hyper comparison category* (HCC).

The HCC formalism isolates three structural mechanisms that recur across diverse contexts:

- a saturation step that fixes an interpretation or abstraction;
- a projection step that extracts observable information;
- a gauge step measuring the noncommutativity between the two.

From these ingredients one obtains quantitative invariants such as *defects*, *supports*, and *height–type* functionals. Under mild locality and stability assumptions, defects admit local decompositions and satisfy monotonicity along towers of projections. This yields a propagation principle: once suitable local estimates (*external inputs*) are available at a single observational level, global inequalities follow automatically across the entire tower.

A key feature of the framework is the clean separation between *structural* aspects—encoded categorically and valid in complete generality—and *problem–dependent* analytic or arithmetic estimates. The former are universal and reusable; the latter enter only through an external input at a chosen base level. This separation yields a flexible template for reducing global comparison questions to local bounds together with their propagation across scales.

The remainder of the paper develops the theory systematically. Section 2 introduces hyper comparison categories; Section 3 proves the main comparison theorem; Sections 4 and 5 analyze saturation fixed points and horizon phenomena; Sections 6 and 7 treat external inputs and their stability; Section 8 summarizes the resulting protocol; and Section 9 illustrates applications in several settings.

2 Hyper Comparison Categories

This section introduces the structures underlying a *hyper comparison category* (HCC). The purpose is to formalize the interaction between a normalization step (*saturation*), an observation step (*projection*), and the discrepancies that appear when these operations fail to commute. Throughout, $\mathcal{C}$ denotes a base category whose objects represent the systems or models to be compared.

2.1 Basic data

Fix a partially ordered set $(K, \preceq)$ of observational levels. An HCC consists of:

- a category $\mathcal{C}$;
- for each $k \in K$, an *observable category* $\mathcal{O}_k$;
- projection functors $\Pi_k : \mathcal{C} \to \mathcal{O}_k$;
- for each $k \preceq k'$, a *coarse–graining functor* $\sigma_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'}$, subject to

$$\Pi_{k'} = \sigma_{k\to k'} \circ \Pi_k.$$

We additionally require the tower compatibilities $\sigma_{k\to k} = \mathrm{Id}_{\mathcal{O}_k}$ and $\sigma_{k'\to k''} \circ \sigma_{k\to k'} = \sigma_{k\to k''}$ whenever $k \preceq k' \preceq k''$. The order on K is interpreted as a hierarchy of resolutions: larger indices correspond to coarser views.

2.2 Saturation

A central role is played by an idempotent endofunctor

$$S : \mathcal{C} \to \mathcal{C}$$

together with a unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow S$. We refer to S as *saturation*. Intuitively, S enforces a chosen interpretation, abstraction, or coarse–graining before comparison.

Definition 2.1. A *saturation operator* is an idempotent monad (S, η, μ) on $\mathcal{C}$, i.e. the multiplication $\mu : S^2 \Rightarrow S$ is an isomorphism [1, 3]. An object X is *saturated* (a *fixed point*) if η_X is an isomorphism.

Saturation provides a canonical normalization step; later, saturated objects will serve as baselines for “defect–zero” comparisons.

2.3 Observation

For each $k \in K$, the functor Π_k extracts the observable shadow of an object. Comparisons are performed at the level of $\Pi_k(X)$ rather than X itself, and the tower relation $\Pi_{k'} = \sigma_{k\to k'} \circ \Pi_k$ ensures that coarse–graining is compatible with observation.

2.4 Gauge and defect

The interaction between saturation and observation is encoded by invertible 2–cells [4, 6]. For each morphism $f : X \to Y$ in $\mathcal{C}$ and each $k \in K$, we are given an invertible 2–cell in $\mathcal{O}_k$

$$\Phi_f^{(k)} : \Pi_k(Sf) \circ \Pi_k(\eta_X) \;\Rightarrow\; \Pi_k(\eta_Y) \circ \Pi_k(f).$$

We call $\Phi_f^{(k)}$ the *gauge* of f at level k.

To quantify the size of this discrepancy, assume that each $\mathcal{O}_k$ comes equipped with a function assigning to every invertible 2–cell α a number $|\alpha|_k \in \mathbb{R}_{\geq 0}$, invariant under isomorphism of 2–cells. The *defect* of f at level k is

$$C_k(f) := |\Phi_f^{(k)}|_k.$$

2.5 Local decomposition

In many applications, discrepancies decompose into contributions from independent "places" (primes, features, channels, ...). Fix a set $\mathcal{P}$ and, for each (k, p), a localization functor

$$\mathrm{Loc}_{k,p} : \mathcal{O}_k \to \mathcal{O}_{k,p}$$

[5]. Assume the size functional decomposes as

$$|\alpha|_k = \sum_{p \in \mathcal{P}} |\mathrm{Loc}_{k,p}(\alpha)|_{k,p}.$$

Then

$$C_k(f) = \sum_{p \in \mathcal{P}} C_{k,p}(f), \qquad C_{k,p}(f) := |\mathrm{Loc}_{k,p}(\Phi_f^{(k)})|_{k,p}.$$

The set $\{p \in \mathcal{P} : C_{k,p}(f) > 0\}$ is the *support* of f at level k.

2.6 Tower monotonicity

Finally, we impose a compatibility between coarse–graining and defect.

Definition 2.2. An HCC satisfies *tower monotonicity* if for all $k \preceq k'$ and all invertible 2–cells α in $\mathcal{O}_k$,

$$|\sigma_{k \to k'}(\alpha)|_{k'} \le |\alpha|_k.$$

Equivalently, $C_{k'}(f) \le C_k(f)$ for all morphisms f.

Tower monotonicity expresses that coarser observations cannot increase the apparent discrepancy between saturation and observation.

The data and axioms above constitute a hyper comparison category. The next section shows that these structures force a canonical three–step factorization of comparison, yielding the main comparison theorem.

3 The Main Comparison Theorem

This section records the main structural consequence of the HCC axioms: comparison at any observational level decomposes canonically into three steps—saturation, observation, and gauge. In particular, the quantitative invariants of the theory (defects, supports, and related height–type functionals) are all induced by this decomposition.

3.1 The three–step comparison diagram

Let $f : X \to Y$ be a morphism in $\mathcal{C}$, and fix an observational level $k \in K$. The unit $\eta_X : X \to SX$ provides the canonical saturation map, and applying S to f gives $Sf : SX \to SY$. After observing at level k, we obtain

$$\Pi_k(X) \xrightarrow{\Pi_k(\eta_X)} \Pi_k(SX) \xrightarrow{\Pi_k(Sf)} \Pi_k(SY) \xleftarrow{\Pi_k(\eta_Y)} \Pi_k(Y).$$

The gauge axiom (Axiom A2) supplies an invertible 2–cell

$$\Phi_f^{(k)} : \Pi_k(Sf) \circ \Pi_k(\eta_X) \;\Rightarrow\; \Pi_k(\eta_Y) \circ \Pi_k(f),$$

which measures the failure of saturation and observation to commute along f.

3.2 Statement of the theorem

Theorem 3.1 (Main Comparison Theorem)**.** *Let $(\mathcal{C}, \mathcal{O}_k, \Pi_k, S, \eta, \Phi_f^{(k)})$ be a hyper comparison category. For every morphism $f : X \to Y$ and every $k \in K$:*

(1) (Three–step factorization) *Comparison at level k factors uniquely through saturation and observation via the gauge 2–cell:*

$$\Pi_k(Sf) \circ \Pi_k(\eta_X) \xRightarrow{\Phi_f^{(k)}} \Pi_k(\eta_Y) \circ \Pi_k(f).$$

(2) (Defect) *The defect of f at level k,*

$$C_k(f) := |\Phi_f^{(k)}|_k,$$

is a well–defined nonnegative quantity, invariant under isomorphism of 2–cells.

(3) (Local decomposition) *Defects decompose over places:*

$$C_k(f) = \sum_{p \in \mathcal{P}} C_{k,p}(f), \qquad C_{k,p}(f) := |\mathrm{Loc}_{k,p}(\Phi_f^{(k)})|_{k,p}.$$

(4) (Subadditivity) *For any composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$,*

$$C_k(g \circ f) \le C_k(f) + C_k(g).$$

(5) (Tower monotonicity) *If $k \preceq k'$, then*

$$C_{k'}(f) \le C_k(f) \qquad \text{and} \qquad C_{k',p}(f) \le C_{k,p}(f) \quad \text{for all } p \in \mathcal{P}.$$

Proof. Item (1) is exactly Axiom A2. Item (2) is the definition of $C_k(f)$ via Axiom A3. Item (3) follows from the local decomposition axiom (Axiom A5). Item (4) is Axiom A4 applied to the gauge 2–cells. Item (5) is tower monotonicity (Axiom A6). ☐

3.3 Interpretation

The theorem shows that every comparison in an HCC reduces to the same canonical three–step process:

1. *saturation* (fix an interpretation),
2. *projection* (extract the observable shadow),
3. *gauge* (measure the residual discrepancy).

All quantitative invariants arise from the gauge data, and the axioms ensure that these invariants behave coherently under composition, localization, and coarse–graining. This rigidity is the mechanism behind the propagation and convergence results developed later.

4 BH Normal Form

Saturation plays a dual role in a hyper comparison category: it both normalizes objects prior to comparison and singles out a distinguished class of objects on which defects vanish. These objects form the *baseline* of the theory; comparisons involving them carry no residual discrepancy. This section makes these ideas precise and records the universal property of saturation.

4.1 Saturated objects and baseline comparison

An object X is *saturated* if the unit $\eta_X : X \to SX$ is an isomorphism. Let

$$\mathcal{C}^{\mathrm{BH}} := \{\, X \in \mathcal{C} : \eta_X \text{ is an isomorphism} \,\}$$

be the full subcategory of saturated ("BH") objects.

Lemma 4.1. *If $X \in \mathcal{C}^{\mathrm{BH}}$, then for every $k \in K$,*

$$C_k(\mathrm{id}_X) = 0.$$

Moreover, if $f : Z \to X$ and X is saturated, then $C_k(f) = 0$ for all $k \in K$.

Proof. If η_X is an isomorphism, then the comparison diagram for id_X collapses to an identity 2–cell. Its size is 0 by Axiom A3. The general case is analogous. □

In this sense, saturated objects are *defect–free targets*.

4.2 Universal property of saturation

The saturation operator is not merely a normalization procedure: it is a reflection of $\mathcal{C}$ onto the baseline subcategory $\mathcal{C}^{\mathrm{BH}}$.

Theorem 4.2 (BH Normal Form)**.** *For every object $X \in \mathcal{C}$, the unit $\eta_X : X \to SX$ exhibits SX as the universal morphism from X to a saturated object. In other words, for any $B \in \mathcal{C}^{\mathrm{BH}}$ and any morphism $u : X \to B$, there exists a unique morphism $\bar{u} : SX \to B$ such that*

$$u = \bar{u} \circ \eta_X.$$

Equivalently, the inclusion $J : \mathcal{C}^{\mathrm{BH}} \hookrightarrow \mathcal{C}$ admits S as a left adjoint with unit η.

Proof. This is the standard universal property of an idempotent monad; see Axiom A1. □

In particular, every object admits a canonical "baseline approximation" SX, and comparisons involving X factor through this baseline. The comparison diagram of Section 3.1 always takes the form

$$X \xrightarrow{\eta_X} SX \xrightarrow{Sf} SY \xleftarrow{\eta_Y} Y,$$

with all ambiguity concentrated in the gauge.

4.3 Defect collapse on the baseline

Proposition 4.3. *If X is saturated, then for every $f : Z \to X$ and every $k \in K$,*

$$C_k(f) = 0.$$

More generally, for any morphism $f : X \to Y$,

$$C_k(f) = C_k(Sf).$$

Proof. If X is saturated, then η_X is an isomorphism, so the gauge $\Phi_f^{(k)}$ is isomorphic to an identity 2–cell; hence $C_k(f) = 0$ by Axiom A3. The second statement follows from functoriality of S and Axiom A2. □

Thus saturation behaves as an absorbing state for defects: once an object is in $\mathcal{C}^{\mathrm{BH}}$, further saturation does not change its comparison behavior.

4.4 Interpretation

The BH normal form provides a conceptual baseline for comparison:

- SX is the canonical interpretation of X;
- comparisons factor through SX and SY;
- saturated objects form the defect–free core of the theory;
- later results (Sections 5–7) can be viewed as flows toward this baseline.

5 Support and Horizon

Defects in a hyper comparison category decompose canonically over "places." This section develops the resulting geometric intuition: the set of places that contribute nontrivially to a comparison forms a support, and its behavior under coarse–graining leads to a natural *horizon* phenomenon. As the observational level becomes coarser, visible support can only shrink, and defects decrease monotonically. These facts provide the structural backbone for the propagation and convergence results of later sections.

5.1 Support

For a morphism $f : X \to Y$ and a level $k \in K$, recall the local decomposition

$$C_k(f) = \sum_{p\in\mathcal{P}} C_{k,p}(f), \qquad C_{k,p}(f) := |\mathrm{Loc}_{k,p}(\Phi_f^{(k)})|_{k,p}.$$

The *support* of f at level k is

$$\mathrm{supp}_k(f) := \{\, p \in \mathcal{P} : C_{k,p}(f) > 0 \,\}.$$

Intuitively, $\mathrm{supp}_k(f)$ records the loci at which the comparison between X and Y fails to be defect–free (e.g. primes, features, or independent information channels).

5.2 Monotonicity of support

Tower monotonicity (Axiom A6) implies that coarse–graining does not increase defect; the same statement holds placewise.

Proposition 5.1 (Support monotonicity)**.** *If $k \preceq k'$, then for all $p \in \mathcal{P}$,*

$$C_{k',p}(f) \le C_{k,p}(f),$$

and consequently

$$\mathrm{supp}_{k'}(f) \subseteq \mathrm{supp}_k(f).$$

Proof. Apply Axiom A6 to the localized invertible 2–cells $\mathrm{Loc}_{k,p}(\Phi_f^{(k)})$. □

Thus the visible support can only shrink as observations become coarser; categorically, this expresses the principle that coarse–graining suppresses fine distinctions.

5.3 Horizon events

The inclusion in Proposition 5.1 may be strict.

Definition 5.1. A *horizon event* for f between levels $k \preceq k'$ is the strict containment

$$\mathrm{supp}_{k'}(f) \subsetneq \mathrm{supp}_k(f).$$

At a horizon event, certain places stop contributing to the defect once the observational resolution is lowered; such events mark qualitative changes in the comparison landscape.

Proposition 5.2 (Defect drop at horizon events)**.** *If $k \preceq k'$ and a horizon event occurs, then*

$$C_{k'}(f) < C_k(f)$$

provided there exists $p \in \mathrm{supp}_k(f) \setminus \mathrm{supp}_{k'}(f)$ with $C_{k,p}(f) > 0$.

Proof. By local decomposition,

$$C_k(f) = \sum_{p \in \mathrm{supp}_k(f)} C_{k,p}(f), \qquad C_{k'}(f) = \sum_{p \in \mathrm{supp}_{k'}(f)} C_{k',p}(f).$$

If a place p disappears from the support and satisfies $C_{k,p}(f) > 0$, then its contribution is absent from the sum for $C_{k'}(f)$, yielding strict inequality. □

5.4 Horizon as a coarse–graining boundary

The terminology is justified by the picture that the horizon is the boundary beyond which certain distinctions become invisible. As coarse–graining proceeds, the support contracts and the defect decreases monotonically. In the extreme case where the support becomes empty, the defect must vanish.

Corollary 5.3 (Support collapse implies defect collapse)**.** *If $\mathrm{supp}_k(f) = \varnothing$ for some k, then $C_k(f) = 0$.*

Proof. Immediate from the local decomposition. □

5.5 Interpretation

Support and horizon give a geometric intuition for comparison:

- support identifies the loci of nontrivial discrepancy;
- coarse–graining shrinks support and reduces defect;

- horizon events mark qualitative transitions in visibility;
- support collapse corresponds to convergence toward the baseline (BH) regime.

This perspective will be used in Sections 6 and 7 to analyze external inputs and their propagation.

6 External Input

Sections 2–5 describe how defects behave under saturation, observation, localization, and coarse–graining. By themselves, however, these structural properties do not yield quantitative bounds relating defects to height–type functionals. Such inequalities necessarily depend on problem–specific analytic or arithmetic estimates.

This section formalizes the notion of an *external input* (EI): a collection of local bounds at a chosen base level which, once supplied, serves as the unique problem–dependent ingredient of the theory. The propagation results of Section 7 then transport this input throughout the entire tower.

6.1 Height–type functionals

For each level $k \in K$, fix a nonnegative functional

$$\mathrm{Rad}_k : \mathrm{Mor}(\mathcal{C}) \to \mathbb{R}_{\geq 0},$$

called the *height* at level k. In applications, Rad_k often arises from a system of local weights $w_k(p) \geq 0$ via

$$\mathrm{Rad}_k(f) = \sum_{p \in \mathcal{P}} w_k(p)\, \mathbf{1}_{\{p \in \mathrm{supp}_k(f)\}},$$

although no particular form will be used here.

6.2 Local budgets and exceptions

Fix a base level $k_0 \in K$ and a parameter $\varepsilon > 0$. The aim is to bound $C_{k_0}(f)$ in terms of $\mathrm{Rad}_{k_0}(f)$ by controlling local contributions $C_{k_0,p}(f)$.

Definition 6.1 (Local budget)**.** An (ε, k_0)–*budget* is a family of inequalities

$$C_{k_0,p}(f) \leq (1+\varepsilon)\, w_{k_0}(p) \qquad \text{for all but finitely many } p \in \mathcal{P}.$$

Equivalently, the exceptional set

$$E_{\varepsilon,k_0}(f) := \{\, p \in \mathcal{P} : C_{k_0,p}(f) > (1+\varepsilon)\, w_{k_0}(p) \,\}$$

is required to be finite.

Exceptional places are those at which the defect exceeds the prescribed budget; their contribution must be handled separately.

6.3 Excess control

Definition 6.2 (Excess bound)**.** An *excess bound* at (ε, k_0) is a constant $K_{\varepsilon,k_0} \geq 0$ such that, for all morphisms f,

$$\sum_{p \in E_{\varepsilon,k_0}(f)} C_{k_0,p}(f) \leq K_{\varepsilon,k_0}.$$

Combining the budgeted and exceptional contributions yields the seed inequality.

Proposition 6.1 (Seed inequality)**.** *If an (ε, k_0)–budget and an excess bound K_{ε,k_0} are available, then for all morphisms f,*

$$C_{k_0}(f) \leq (1+\varepsilon)\,\mathrm{Rad}_{k_0}(f) + K_{\varepsilon,k_0}.$$

Proof. Write $C_{k_0}(f) = \sum_{p \notin E_{\varepsilon,k_0}(f)} C_{k_0,p}(f) + \sum_{p \in E_{\varepsilon,k_0}(f)} C_{k_0,p}(f)$. Use the budget on the first sum and the excess bound on the second. $\square$

The inequality of Proposition 6.1 is the *external input* required for propagation.

6.4 Failure modes

The EI may fail for several distinct reasons, all of which are structural and independent of the specific application:

- (F1) *Infinite exceptions:* the set $E_{\varepsilon,k_0}(f)$ is infinite.
- (F2) *Unbounded excess:* the exceptional sum cannot be bounded uniformly.
- (F3) *Weight mismatch:* the chosen weights $w_{k_0}(p)$ do not reflect the scale of $C_{k_0,p}(f)$.
- (F4) *Tower incompatibility:* the height functionals fail to satisfy the monotonicity $\mathrm{Rad}_{k'}(f) \leq \mathrm{Rad}_k(f)$ for $k \preceq k'$.

These are the only obstructions to establishing the seed inequality. Once EI holds at the base level, Section 7 guarantees its propagation throughout the tower.

6.5 Interpretation

The external input isolates the analytic or arithmetic core of a comparison problem:

- the HCC structure determines the formal behavior of defects;

- EI supplies quantitative bounds at a single level k_0;
- once supplied, those bounds propagate automatically.

In this way, EI is the unique interface between the categorical framework and problem–specific estimates; the remaining arguments are purely structural.

7 Stability and Propagation

Once an external input (EI) is established at a single observational level, the structural axioms of a hyper comparison category ensure that the resulting inequality is stable under composition, localization, and coarse–graining. This section records these stability properties and proves the propagation theorem: a one–shot estimate at a base level extends automatically to all higher levels in the tower.

7.1 Stability under composition

Defects are subadditive under composition (Theorem 3.1(4)):

$$C_k(g \circ f) \le C_k(f) + C_k(g).$$

Assume likewise that the height–type functionals satisfy

$$\mathrm{Rad}_k(g \circ f) \le \mathrm{Rad}_k(f) + \mathrm{Rad}_k(g),$$

for instance when the weights $w_k(p)$ are additive over supports.

Proposition 7.1 (Compositional stability)**.** *Fix $\varepsilon > 0$ and $K \ge 0$. If*

$$C_k(h) \le (1 + \varepsilon)\,\mathrm{Rad}_k(h) + K$$

holds for all h in a class $\mathcal{G} \subseteq \mathrm{Mor}(\mathcal{C})$, then the same inequality holds for any finite composite of morphisms in $\mathcal{G}$.

Proof. Iterate subadditivity for C_k and the corresponding subadditivity for Rad_k. □

In applications, this reduces verification of EI to a convenient generating set.

7.2 Stability under localization

Suppose defects and heights admit compatible local decompositions at level k.

Proposition 7.2 (Local stability)**.** *Assume the seed inequality at level k,*

$$C_k(f) \le (1+\varepsilon)\,\mathrm{Rad}_k(f) + K,$$

holds for all morphisms f. Then for each $p \in \mathcal{P}$ there exist constants $K_{k,p} \ge 0$ with $\sum_{p\in\mathcal{P}} K_{k,p} \le K$ such that

$$C_{k,p}(f) \le (1+\varepsilon)\,w_k(p) + K_{k,p}$$

for all f.

Proof. Decompose both sides into sums over $p \in \mathcal{P}$ and absorb any exceptional contribution into $K_{k,p}$. □

7.3 Stability under coarse–graining

Tower monotonicity (Theorem 3.1(5)) yields:

Proposition 7.3 (Tower stability)**.** *If $k \preceq k'$ and the seed inequality holds at level k, then*

$$C_{k'}(f) \le C_k(f) \qquad \textit{and} \qquad \mathrm{Rad}_{k'}(f) \le \mathrm{Rad}_k(f)$$

for all morphisms f.

Coarse–graining therefore cannot destroy an inequality; it can only improve it.

7.4 Propagation

Theorem 7.4 (Propagation of external input)**.** *Fix $\varepsilon > 0$ and a base level $k_0 \in K$. Suppose the seed inequality*

$$C_{k_0}(f) \le (1+\varepsilon)\,\mathrm{Rad}_{k_0}(f) + K_{\varepsilon,k_0}$$

holds for all morphisms f. Then for every $k \succeq k_0$,

$$C_k(f) \le (1+\varepsilon)\,\mathrm{Rad}_k(f) + K_{\varepsilon,k_0}.$$

Proof. By tower monotonicity, $C_k(f) \le C_{k_0}(f)$. By monotonicity of the height, $\mathrm{Rad}_{k_0}(f) \ge \mathrm{Rad}_k(f)$. Substitute the seed inequality at level k_0. □

7.5 Propagation from generators

Corollary 7.5. *If the seed inequality holds at level k_0 for all morphisms in a class $\mathcal{G}$ whose elements generate* $\mathrm{Mor}(\mathcal{C})$ *under finite composition, then it holds for all morphisms at every level $k \succeq k_0$.*

Proof. Combine Proposition 7.1 with Theorem 7.4. □

7.6 Interpretation

The tower acts as a stabilizing mechanism:

- EI is required only once, at a single base level;
- all higher levels inherit the inequality automatically;
- composition and localization preserve admissible bounds;
- coarse–graining can only improve them.

This principle underlies the protocol of Section 8.

8 Protocol

The preceding sections separate comparison problems into two components: (1) a universal structural part encoded by the HCC axioms, and (2) a problem–dependent analytic/arithmetic part supplied as an external input (EI). The purpose of this section is to assemble these ingredients into a practical procedure for deriving global comparison inequalities.

The protocol has five steps (Steps 0–4): structural verification, BH normalization, EI at a seed level, propagation, and (when available) convergence.

8.1 Step 0: Structural verification

Before applying any quantitative estimate, check that the ambient data indeed form a hyper comparison category and that the quantitative functionals used in EI are compatible with the tower. Concretely, one typically verifies:

- defect monotonicity under coarse–graining;
- local decomposition of defects and heights;
- subadditivity under composition;
- compatibility of projection towers;
- finiteness of support at each level.

If any of these fail, the propagation mechanism of Section 7 cannot be invoked.

8.2 Step 1: BH normalization

Given $f : X \to Y$, the BH normal form (Theorem 4.2) provides the canonical factorization

$$X \xrightarrow{\eta_X} SX \xrightarrow{Sf} SY \xleftarrow{\eta_Y} Y.$$

All ambiguity in comparing X and Y is concentrated in the gauge 2–cells attached to Sf. Accordingly, the defect $C_k(f)$ is a structural invariant determined by the HCC data alone.

8.3 Step 2: External input at a seed level

Choose a base observational level $k_0 \in K$. The only problem–dependent task is to prove an EI inequality at level k_0:

$$C_{k_0}(f) \le (1+\varepsilon)\,\mathrm{Rad}_{k_0}(f) + K_{\varepsilon,k_0}.$$

As in Section 6, this typically consists of:

- a local budget for $C_{k_0,p}(f)$ outside a finite exceptional set;
- an excess bound controlling the total contribution of exceptional places.

Once these are established, the seed inequality follows (Proposition 6.1).

8.4 Step 3: Propagation

The propagation theorem (Theorem 7.4) upgrades the seed estimate at k_0 to every higher level in the tower:

$$C_k(f) \le (1+\varepsilon)\,\mathrm{Rad}_k(f) + K_{\varepsilon,k_0} \qquad \text{for all } k \succeq k_0.$$

No further analytic input is required: tower monotonicity forces both C_k and Rad_k to decrease under coarse–graining.

If a generating class $\mathcal{G}$ is available, it suffices to verify EI on $\mathcal{G}$; the resulting inequality then extends to all morphisms (Corollary 7.5).

8.5 Step 4: Convergence

If the tower admits eventual support collapse—i.e. if for each f there exists $k_\infty \succeq k_0$ with $\mathrm{supp}_{k_\infty}(f) = \varnothing$— then Corollary 5.3 implies

$$C_{k_\infty}(f) = 0.$$

Thus the comparison converges to the BH regime at sufficiently coarse levels.

8.6 Summary

1. Verify the HCC axioms and quantitative compatibility.
2. Normalize via the BH factorization.
3. Establish EI at a single seed level k_0.
4. Propagate the inequality throughout the tower.
5. When support collapses, conclude convergence to defect–zero.

9 Applications

The hyper comparison category (HCC) formalism is designed to isolate the structural mechanisms underlying comparison across multiple observational levels. Since the axioms are minimal and purely categorical, the framework applies broadly. The examples below illustrate how the protocol of Section 8 organizes global comparison problems into local estimates together with their propagation.

9.1 General applications

Many systems naturally come with multiple observational or coarse–grained levels and projection maps between them; in such situations, HCC–type data often arise essentially for free.

(1) Multi–resolution analysis. Hierarchies of function spaces (e.g. wavelet towers, Sobolev scales) provide natural projection functors. Saturation corresponds to a canonical normalization (e.g. orthogonal projection), while defects measure discrepancies between representations at different resolutions. Propagation then yields stability of bounds across scales.

(2) Coarse–graining in physics. Renormalization–group flows produce towers of effective theories. Saturation corresponds to fixing a renormalization scheme, projection to integrating out degrees of freedom, and defects quantify the mismatch between coarse–grained and saturated dynamics. External inputs arise from local energy or coupling bounds.

(3) Model comparison in machine learning. Different architectures or feature maps define observational levels. Saturation corresponds to canonical preprocessing, projection to feature extraction, and defects measure the noncommutativity of these operations. Local budgets correspond to per–feature error bounds.

(4) Interpretation selection in cognitive systems. Competing interpretations of a stimulus can be modeled by different saturation operators. Projection extracts observable behavior, and defects measure interpretation–dependent discrepancies. Horizon events correspond to the loss of discriminability under coarse observation.

These examples share the same pattern: comparison becomes well–behaved only after saturation, and defects propagate predictably along towers of observations.

9.2 Arithmetic applications

Arithmetic settings often come equipped with natural local decompositions (e.g. primes, valuations) and height–type functionals. The HCC framework provides a uniform language for organizing such data.

(1) Height inequalities. Local contributions at each prime define $C_{k,p}(f)$, while the height $\mathrm{Rad}_k(f)$ aggregates prescribed weights. An external input at a single level then yields global height bounds by propagation.

(2) Local–global principles. Local budgets control contributions at almost all primes, while excess bounds control finitely many exceptional ones. The seed inequality then produces global estimates.

(3) abc–type inequalities. In many contexts, the defect measures the complexity of a morphism and the height measures the size of its support. The HCC protocol reduces global inequalities of the form

$$\text{defect} \leq (1+\varepsilon)\,\text{height} + C$$

to verifying local bounds at a single level.

These applications indicate that HCC captures the structural essence of local–global comparison in arithmetic.

9.3 IUT–type comparison architectures (brief)

Certain comparison architectures in arithmetic geometry exhibit the same structural pattern encoded in the HCC axioms. Without assuming any specific theory, one can view the following correspondences as a neutral structural dictionary:

- saturation models fixing an interpretation or abstraction before comparison;
- projection functors encode observable data at various levels;

- gauge 2–cells record the noncommutativity between saturation and observation;
- local decomposition reflects contributions from primes (or other independent places);
- towers encode hierarchies of coarse–grained or partially interpreted structures.

In such settings, the HCC protocol provides a reusable template for organizing long comparison chains: normalize via saturation, establish local estimates at a seed level, and propagate across the tower.

9.4 Interpretation

Across these examples, HCC serves as a unifying language for comparison problems with multiple observational levels. The protocol of Section 8 reduces global inequalities to local bounds, while propagation ensures that once an estimate holds at a single level, it holds uniformly across the hierarchy.

10 Conclusion

We introduced a categorical framework for comparison across multiple observational (or coarse–grained) levels. The guiding principle is that comparison becomes well behaved once three operations are disentangled: *saturation* (fixing an interpretation), *projection* (extracting observable data), and *gauge* (recording their noncommutativity). From these data one obtains quantitative invariants such as defects, supports, and height–type functionals.

The HCC axioms force robust structural behavior: defects admit local decompositions, decrease under coarse–graining, and interact predictably with composition and localization. These mechanisms culminate in the propagation theorem: once a suitable external input is available at a single seed level, the corresponding inequality automatically holds throughout the tower. The protocol of Section 8 packages this logic into an actionable template that reduces global comparison problems to local bounds together with their propagation.

A central feature is the separation between universal categorical structure and problem–dependent analytic or arithmetic estimates. The former are encoded entirely in the HCC axioms; the latter enter only via external input at a chosen base level. This division allows the same structural machinery to be reused across diverse settings, from multi–resolution analysis and coarse–graining in physics to arithmetic height inequalities and other local–global phenomena.

More broadly, HCC provides a unifying language for comparison across scales, clarifying how local information controls global behavior once stability and monotonicity are built into the ambient structure. We expect further applications in contexts where interpretation, observation, and coarse–graining interact in nontrivial ways.

Appendix A. Examples of Saturation (Sel)

This appendix collects several examples of saturation operators. The purpose is not to exhaust all possibilities, but to illustrate how the abstract notion of an idempotent monad arises naturally in contexts where comparison requires fixing an interpretation, normalization, or coarse–graining before observable data can be meaningfully extracted.

A.1 Reflective subcategories

Let $\mathcal{C}$ be a category and $\mathcal{D} \subseteq \mathcal{C}$ a reflective subcategory. The reflector

$$S : \mathcal{C} \to \mathcal{D} \hookrightarrow \mathcal{C}$$

is an idempotent monad, with unit $\eta_X : X \to SX$ exhibiting SX as the universal approximation of X inside $\mathcal{D}$. Typical examples include:

- abelianization of groups;
- sheafification of presheaves;
- completion of metric spaces;
- normalization of schemes.

In each case, SX represents the canonical form of X with respect to a chosen interpretation.

A.2 Normalization procedures

Many constructions in geometry and analysis produce canonical normalizations that are idempotent by design. Examples include:

- orthogonal projection onto a closed subspace of a Hilbert space;
- taking the reduced subscheme of a scheme;
- passing to the closure of a subset in a topological space;
- projecting a matrix onto a fixed subalgebra (e.g. diagonal part).

These operations enforce a structural constraint before comparison.

A.3 Coarse–graining and abstraction

In settings with multiple levels of description, saturation may represent the act of fixing an interpretation or abstraction. Examples include:

- selecting a coordinate system or gauge in physics;
- canonical preprocessing of data (e.g. centering, whitening);
- abstraction maps in logic or type theory;
- quotienting by symmetries or equivalence relations.

Here SX is the canonical representative of X under the chosen abstraction.

A.4 Algebraic and arithmetic examples

In arithmetic geometry, saturation often corresponds to imposing a canonical structure before comparison. Examples include:

- passing from a ring to its integral closure;
- taking the Néron model of an abelian variety;
- forming the maximal unramified extension of a local field;
- normalizing divisors or line bundles.

These operations ensure that subsequent comparisons reflect intrinsic rather than incidental features.

A.5 Information–theoretic examples

In information processing, saturation corresponds to canonical preprocessing steps that remove irrelevant variation:

- projecting data onto a feature subspace;
- enforcing invariances (e.g. translation or scale invariance);
- canonical encoding of signals;
- normalization of probability distributions.

Such operations are idempotent and serve as interpretation–fixing preparations for comparison.

Across all these examples, saturation plays the same conceptual role: it produces a canonical form of an object, eliminating interpretational ambiguity and enabling meaningful comparison at the observable level.

Appendix B. Examples of Projection Towers

This appendix provides examples of projection functors and tower structures. The purpose is to illustrate how hierarchies of observational levels arise naturally in many mathematical and scientific contexts, and how coarse–graining manifests as a functorial passage to coarser categories.

B.1 Multi–resolution analysis

Let $\mathcal{C}$ be a space of functions (e.g. $L^2(\mathbb{R})$) and let $\mathcal{O}_k$ denote the subspace spanned by wavelets at scales $\geq k$. The projection

$$\Pi_k : \mathcal{C} \to \mathcal{O}_k$$

is the orthogonal projection onto the coarse resolution k. The tower maps

$$\sigma_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'} \qquad (k \preceq k')$$

are the natural inclusions of coarser scales. Coarse–graining removes fine detail, and defects measure the mismatch between normalized and observed representations.

B.2 Coarse–graining in physics

In renormalization group (RG) theory, one considers a hierarchy of effective theories obtained by integrating out high–energy degrees of freedom. Let $\mathcal{C}$ be the category of microscopic models, and $\mathcal{O}_k$ the category of effective theories at scale k. The projection

$$\Pi_k : \mathcal{C} \to \mathcal{O}_k$$

is the RG flow map, and the tower maps

$$\sigma_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'}$$

represent further coarse–graining. The HCC axioms reflect the physical principle that coarse–graining suppresses fine distinctions.

B.3 Feature hierarchies in machine learning

Let $\mathcal{C}$ be a space of data objects (e.g. images). A feature extractor at level k produces a representation

$$\Pi_k : \mathcal{C} \to \mathcal{O}_k,$$

where $\mathcal{O}_k$ is the feature space at depth k of a neural network. Deeper layers correspond to coarser, more abstract features. The tower maps

$$\sigma_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'}$$

are induced by the network architecture. Defects measure the noncommutativity between preprocessing (saturation) and feature extraction.

B.4 Valuation towers

Let $\mathcal{C}$ be a category of arithmetic objects (e.g. number fields, schemes). For each k, let $\mathcal{O}_k$ encode data visible at a set of valuations of bounded complexity (e.g. primes of norm $\leq k$). The projection

$$\Pi_k : \mathcal{C} \to \mathcal{O}_k$$

forgets contributions from valuations of complexity $> k$. The tower maps

$$\sigma_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'}$$

add back valuations of intermediate complexity. This produces a natural local–global hierarchy.

B.5 Abstract poset–indexed towers

More generally, let $(K, \preceq)$ be any poset and let $\{\mathcal{O}_k\}_{k\in K}$ be a diagram of categories with functors

$$\sigma_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'} \qquad (k \preceq k')$$

satisfying the coherence conditions of a functor from K to **Cat**. Any family of projections

$$\Pi_k : \mathcal{C} \to \mathcal{O}_k$$

compatible with the tower maps defines a projection tower in the sense of the HCC axioms.

These examples demonstrate that projection towers arise naturally in contexts where information is organized hierarchically. The HCC framework abstracts the common structural features of such hierarchies and provides a unified language for comparison across observational levels.

Appendix C. Defect, Support, and Height: Additional Examples

This appendix provides concrete illustrations of the quantitative invariants introduced in Sections 3–6. The goal is to clarify how defects arise from gauge 2–cells, how support identifies the loci of nontrivial discrepancy, and how height–type functionals aggregate local information.

C.1 A simple defect computation

Let $\mathcal{O}_k$ be a category enriched in groupoids, and let $\alpha : A \Rightarrow B$ be an invertible 2–cell. Suppose the size functional $|\cdot|_k$ is given by

$$|\alpha|_k = \ell(\alpha),$$

where ℓ is a length function on the automorphism group of A. Then for a morphism $f : X \to Y$ in $\mathcal{C}$, the defect

$$C_k(f) = |\Phi_f^{(k)}|_k$$

measures the minimal "cost" of adjusting the comparison diagram so that saturation and observation commute.

Even in this simple setting, defects encode the essential obstruction to direct comparison.

C.2 Support: finite and infinite cases

Let $\mathcal{P}$ be a set of places (e.g. primes, features, channels). For a morphism f, the localized defects

$$C_{k,p}(f) = |\mathrm{Loc}_{k,p}(\Phi_f^{(k)})|_{k,p}$$

identify the contributions from each place.

Finite support. If $C_{k,p}(f) = 0$ for all but finitely many p, then

$$\mathrm{supp}_k(f) = \{\, p : C_{k,p}(f) > 0 \,\}$$

is finite. This is typical in arithmetic settings where only finitely many primes contribute to a given comparison.

Infinite support. In analytic or geometric settings, support may be infinite. For instance, if $C_{k,p}(f)$ decays rapidly with p, then

$$C_k(f) = \sum_{p \in \mathcal{P}} C_{k,p}(f)$$

converges even though $\mathrm{supp}_k(f)$ is infinite. The HCC axioms accommodate both behaviors.

C.3 Horizon events

Consider a tower $k \preceq k' \preceq k''$. Suppose

$$\mathrm{supp}_k(f) = \{p_1, p_2, p_3\}, \qquad \mathrm{supp}_{k'}(f) = \{p_1, p_2\}, \qquad \mathrm{supp}_{k''}(f) = \{p_1\}.$$

Then the transitions

$$\mathrm{supp}_k(f) \supsetneq \mathrm{supp}_{k'}(f) \quad \text{and} \quad \mathrm{supp}_{k'}(f) \supsetneq \mathrm{supp}_{k''}(f)$$

are horizon events. At each horizon, the defect drops by the contribution of the disappearing places:

$$C_{k'}(f) = C_k(f) - C_{k,p_3}(f), \qquad C_{k''}(f) = C_{k'}(f) - C_{k',p_2}(f).$$

This illustrates how coarse–graining suppresses fine distinctions.

C.4 Height functionals

Height–type functionals aggregate local weights. A typical example is

$$\mathrm{Rad}_k(f) = \sum_{p \in \mathrm{supp}_k(f)} w_k(p),$$

where $w_k(p) \geq 0$ is a prescribed weight.

Uniform weights. If $w_k(p) = 1$ for all p, then $\mathrm{Rad}_k(f)$ counts the size of the support.

Complexity weights. If $w_k(p)$ increases with the complexity of p, then $\mathrm{Rad}_k(f)$ measures the "weighted size" of the support.

Decay weights. If $w_k(p)$ decreases with p, then $\mathrm{Rad}_k(f)$ emphasizes low–complexity contributions.

The HCC framework does not impose any specific choice of weights; only monotonicity under coarse–graining is required for propagation.

C.5 Local decomposition in practice

Let $f : X \to Y$ be a morphism with defect

$$C_k(f) = 7.3.$$

Suppose the localized contributions are

$$C_{k,p_1}(f) = 3.1, \quad C_{k,p_2}(f) = 2.0, \quad C_{k,p_3}(f) = 2.2,$$

and $C_{k,p}(f) = 0$ for all other p. Then

$$\mathrm{supp}_k(f) = \{p_1, p_2, p_3\},$$

and the height

$$\mathrm{Rad}_k(f) = w_k(p_1) + w_k(p_2) + w_k(p_3)$$

aggregates the weights of these places.

This decomposition is the basis for the external input: local budgets control each $C_{k,p}(f)$, and excess bounds control the exceptional places.

These examples illustrate how defects, supports, and height–type functionals behave in concrete settings. They provide the intuition behind the structural results of Sections 5–7 and the protocol of Section 8.

Appendix D. Categorical Background

This appendix summarizes the categorical notions used in the definition of a hyper comparison category. The material is standard, and the presentation is tailored to the needs of the main text; see [1, 4, 3, 2] for background.

D.1 Idempotent monads and reflective subcategories

A monad on a category $\mathcal{C}$ consists of an endofunctor $S : \mathcal{C} \to \mathcal{C}$ together with natural transformations

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow S, \qquad \mu : S^2 \Rightarrow S,$$

satisfying the usual associativity and unit axioms. The monad is *idempotent* if μ is an isomorphism.

Idempotent monads correspond precisely to reflective subcategories: if $\mathcal{D} \subseteq \mathcal{C}$ is reflective with reflector $S : \mathcal{C} \to \mathcal{D}$, then S extends to an idempotent monad on $\mathcal{C}$. Conversely, the fixed points of an idempotent monad form a reflective subcategory.

In the HCC framework, saturation is modeled by an idempotent monad.

D.2 2-categories and invertible 2-cells

A 2-category consists of:

- objects;
- 1-morphisms between objects;
- 2-morphisms between 1-morphisms.

Composition of 1-morphisms is associative up to coherent 2-isomorphism, and 2-morphisms compose both vertically and horizontally.

In a hyper comparison category, the observable categories $\mathcal{O}_k$ are treated as 2-categories (or categories enriched in groupoids), and the gauge $\Phi_f^{(k)}$ is an invertible 2-cell measuring the failure of saturation and observation to commute.

D.3 Localization functors

Given a category $\mathcal{O}_k$ and a set of "places" $\mathcal{P}$, a localization functor

$$\mathrm{Loc}_{k,p} : \mathcal{O}_k \to \mathcal{O}_{k,p}$$

extracts the contribution of the place p. These functors are required to be compatible with the size functionals:

$$|\alpha|_k = \sum_{p \in \mathcal{P}} |\mathrm{Loc}_{k,p}(\alpha)|_{k,p}.$$

Localization is used to define support and to decompose defects into local contributions.

D.4 Size functionals on 2-cells

For each level k, the size functional

$$| \cdot |_k : \{\text{invertible 2-cells in } \mathcal{O}_k\} \to \mathbb{R}_{\geq 0}$$

assigns a nonnegative real number to each invertible 2-cell. The functional is required to satisfy:

- invariance under isomorphism of 2-cells;
- subadditivity under vertical and horizontal composition;
- compatibility with localization;
- monotonicity under coarse–graining.

These conditions ensure that defects behave predictably under the operations of the HCC.

D.5 Poset-indexed diagrams and coherence

Let $(K, \preceq)$ be a poset. A K-indexed diagram of categories consists of:

- a category $\mathcal{O}_k$ for each $k \in K$;

- functors $\sigma_{k\to k'} : \mathcal{O}_k \to \mathcal{O}_{k'}$ for each $k \preceq k'$;
- coherence conditions

$$\sigma_{k\to k} = \mathrm{Id}, \qquad \sigma_{k\to k''} = \sigma_{k'\to k''} \circ \sigma_{k\to k'} \quad (k \preceq k' \preceq k'').$$

Such diagrams model hierarchies of observational levels. The projection functors $\Pi_k : \mathcal{C} \to \mathcal{O}_k$ are required to satisfy

$$\Pi_{k'} = \sigma_{k\to k'} \circ \Pi_k,$$

ensuring that observation commutes with coarse–graining.

These categorical notions provide the structural foundation for the HCC framework. They ensure that saturation, observation, and gauge interact coherently, and that defects behave predictably under localization, composition, and coarse–graining.

Appendix E. Structural Correspondence with IUT-type Architectures (Optional)

This appendix outlines a purely structural correspondence between the components of a hyper comparison category (HCC) and certain comparison architectures that arise in arithmetic geometry. The purpose is not to assert any mathematical claims about specific theories, but simply to illustrate how the abstract mechanisms of saturation, projection, gauge, local decomposition, and propagation may be recognized in settings with similar structural features.

The correspondence is schematic and depends only on the formal pattern of comparison across multiple observational levels.

E.1 Saturation and interpretation fixing

In HCC, saturation is an idempotent monad

$$S : \mathcal{C} \to \mathcal{C}$$

that fixes an interpretation before comparison. In IUT-type architectures, one encounters analogous procedures in which objects are first placed into a canonical or interpretation-stable form before observable data are extracted. The role of SX is played by the interpretation-fixed version of X.

E.2 Projection and observable data

Projection functors

$$\Pi_k : \mathcal{C} \to \mathcal{O}_k$$

extract observable shadows at level k. In IUT-type settings, various "visible"invariants or partially interpreted structures serve a similar role, providing the data on which comparisons are performed.

E.3 Gauge and comparison discrepancy

The gauge 2-cells

$$\Phi_f^{(k)} : \Pi_k(Sf) \circ \Pi_k(\eta_X) \Rightarrow \Pi_k(\eta_Y) \circ \Pi_k(f)$$

measure the failure of saturation and observation to commute. In IUT-type architectures, comparison often involves reconciling data obtained after different interpretation or abstraction steps. The resulting discrepancies play the same structural role as the gauge.

E.4 Local decomposition

In HCC, defects decompose over places:

$$C_k(f) = \sum_{p \in \mathcal{P}} C_{k,p}(f).$$

IUT-type settings frequently involve contributions indexed by primes, valuations, or other local data. The structural similarity lies in the existence of independent local channels whose contributions aggregate to a global invariant.

E.5 Towers and hierarchical comparison

The projection tower

$$\Pi_{k'} = \sigma_{k \to k'} \circ \Pi_k$$

models comparison across multiple observational levels. IUT-type architectures also organize comparison across hierarchies of partially interpreted structures. The structural pattern—comparison at one level feeding into comparison at another—is formally similar.

E.6 External input

In HCC, the only analytic or arithmetic input enters through a seed inequality at a single level:

$$C_{k_0}(f) \leq (1 + \varepsilon)\,\mathrm{Rad}_{k_0}(f) + K.$$

IUT-type settings likewise require external estimates at specific stages of a comparison chain. The structural role of these estimates matches the role of the EI in the HCC protocol.

E.7 Propagation along the tower

The propagation theorem shows that once the seed inequality holds at k_0, it holds for all $k \succeq k_0$. In IUT-type architectures, comparison results often propagate along a hierarchy of structures in a similar manner. The HCC framework abstracts this propagation mechanism.

This appendix is intended only as a structural guide. The HCC formalism is independent of any specific arithmetic theory, and the correspondences above are schematic analogies rather than mathematical identifications.

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