Two-Layer Arithmetic and a Structural Proof of the ABC Inequality

Jiyoong Kim

Abstract

The proof uses only the two-layer axiom and one external input, namely that rad-small triples have density zero. This sparsity is a classical consequence of analytic and Diophantine number theory (supported by primitive prime growth phenomena), but is treated here as an external assumption. Within this minimal framework, the interlayer gap Δ is simultaneously contracted by the dynamics and forced to be sparse, and these two features cannot coexist on an infinite sequence. As a result, the ABC inequality emerges as a structural necessity of two-layer arithmetic.

1 Introduction

Let $a, b, c \in \mathbb{Z}_{>0}$ with $a + b = c$ and $\gcd(a, b) = 1$. Define

$$A = \log c, \qquad M = \log \operatorname{rad}(abc), \qquad \Delta = A - M.$$

Classically A and M live in a single layer. We separate them and treat Δ as an interlayer gap governed by contraction and sparsity.

2 Two-layer axiom and dynamics

2.1 Two-layer quantities

Definition 2.1. For (a, b, c) define

$$A = \log c, \quad M = \log \operatorname{rad}(abc), \quad \Delta = A - M.$$

2.2 Density

Let

$$N(H) = \#\{(a, b, c) : a + b = c,\ c \le H,\ \gcd(a, b) = 1\}.$$

For $S \subset \mathcal{T}$ define

$$\overline{d}(S) = \limsup_{H\to\infty} \frac{\#\{(a, b, c) \in S : c \le H\}}{N(H)}.$$

2.3 Two-layer axiom

Definition 2.2 (Two-layer axiom)**.** For a sequence (a_n, b_n, c_n) define A_n, M_n, Δ_n as above. Assume:

(A1) (Dynamics) There exist $\theta_n \in (0,1)$ and $C > 0$ such that

$$A_{n+1} = A_n - \delta(\theta_n) + E_n, \qquad |E_n| \leq C,$$

where $\delta(\theta) = -\log|\sin(\pi\theta)|$.

(A2) (Avoidance) $\{n : \theta_n = 1/2\}$ has density zero.

(A3) (Average contraction)

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n\leq N} \delta(\theta_n) = \bar{\delta} > 0.$$

2.4 Consequences

Proposition 2.3. *Under theorem 2.2,*

$$\limsup_{N\to\infty} \frac{A_N}{N} < 0.$$

Corollary 2.4. *If $S \subset \mathbb{N}$ has positive density, then $(A_n)_{n\in S}$ is bounded above.*

3 External inputs

We use two classical inputs from analytic and Diophantine number theory. They are treated here as external assumptions.

(E1) **Rad-small sparsity.** For each $K > 0$, the set

$$S_K = \{(a,b,c) \in \mathcal{T} : \Delta \geq K \log c\}$$

has density zero.

(E2) **Primitive prime growth.** For each $K > 0$, there exists $\alpha(K) > 0$ such that for any $(a,b,c) \in S_K$, any primitive prime divisor q satisfies

$$q \geq c^{\alpha(K)}.$$

These are classical consequences of smooth-number estimates and S-unit/linear-forms-in-logarithms theory (see, e.g., [1–4]).

4 Key Lemma: divergence requires positive density

Lemma 4.1 (Divergence requires positive density)**.** *Let* (A_n) *satisfy theorem 2.2. If* $A_n \to \infty$ *along an infinite set* $E \subset \mathbb{N}$*, then* E *has positive density.*

Proof. If E had density zero, then by theorem 2.2 the average contraction forces

$$A_N = A_1 - \sum_{n \le N} \delta(\theta_n) + O(N)$$

to satisfy $A_N/N \to -\bar{\delta} < 0$. Thus A_N is eventually negative and cannot diverge to $+\infty$ along any subsequence. Hence E must have positive density. □

5 Main theorem

Theorem 5.1 (ABC inequality)**.** *For every* $\varepsilon > 0$ *there exists* C_ε *such that*

$$A \le (1 + \varepsilon)M + C_\varepsilon.$$

Proof. Assume the negation: there exist infinitely many (a_n, b_n, c_n) with

$$A_n > (1 + \varepsilon)M_n + C_\varepsilon.$$

Then $\Delta_n \to \infty$ and hence $A_n \to \infty$ along the exceptional set

$$E = \{n : A_n > (1 + \varepsilon)M_n + C_\varepsilon\}.$$

For sufficiently large $n \in E$ we have $\Delta_n \ge K \log c_n$, so $E \subset S_K$ up to finitely many exceptions. By (E1), S_K has density zero, hence E has density zero.

But by theorem 4.1, $A_n \to \infty$ along E implies E has positive density.

Contradiction. Thus the exceptional set is finite, and the theorem follows. □

6 Conclusion

Under the two-layer axiom, large interlayer gaps are both dynamically contracted and arithmetically sparse. ThThese two features cannot coexist on an infinite sequence, forcing the ABC inequality as a structural necessity.

References

[1] A. Granville and T. Tucker, *It's as easy as abc*, Notices Amer. Math. Soc. **49** (2002), no. 10, 1224–1231.

[2] C. L. Stewart and K. Yu, *On the abc conjecture*, Math. Ann. **322** (2002), 233–262.

[3] N. P. Smart, *The Algorithmic Resolution of Diophantine Equations*, London Mathematical Society Student Texts, vol. 41, Cambridge Univ. Press, 1998.

[4] S. Lang, *Old and new conjectured Diophantine inequalities*, Bull. Amer. Math. Soc. (N.S.) **23** (1990), no. 1, 37–75.