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EK381 Homework 2

Problem 2.1 (Video 1.5, Lecture Problem) Consider an experiment with sample space $\Omega = \{1,2,3,4,5,6,7,8\}$. The outcomes have probabilities

$$\mathbb{P}[\{1\}] = \frac{1}{4} \quad \mathbb{P}[\{2\}] = \frac{1}{4} \quad \mathbb{P}[\{3\}] = \frac{1}{8} \quad \mathbb{P}[\{4\}] = \frac{1}{8}$$
$$\mathbb{P}[\{5\}] = \frac{1}{16} \quad \mathbb{P}[\{6\}] = \frac{1}{16} \quad \mathbb{P}[\{7\}] = \frac{1}{16} \quad \mathbb{P}[\{8\}] = \frac{1}{16}.$$

We also define the events

$$A = \{1,3,4\} \quad B = \{2,3,4\} \quad C = \{3,4,5,6,7,8\}$$
$$D = \{2,3,5,6\} \quad E = \{2,4,6,7\} \quad F = \{5,6,7,8\}.$$

3 5 6

$\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{16}$

For each of the following questions, give a "Yes" or "No" answer as well as your reasoning and calculations.

(a) Are the events A, B, and C independent? If not, are they at least pairwise independent?

(b) Are the events A and D independent?

(c) Are the events A and F independent?

(d) Are the events B and Ω independent?

(e) Are the events D, E, and F independent? If not, are they are least pairwise independent?

(f) Are the events A and D conditionally independent given C?

(a) $\mathbb{P}[A] = \frac{1}{4}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$, $\mathbb{P}[B] = \frac{1}{4}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$, $\mathbb{P}[C] = \frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16} = \frac{1}{2}$

$\mathbb{P}[A\cap B] = \mathbb{P}[\{3,4\}] = \frac{1}{4}$, $\mathbb{P}[A]\mathbb{P}[B] = \frac{1}{4} \Rightarrow \mathbb{P}[A\cap B] = \mathbb{P}[A]\mathbb{P}[B]$, A & B are independent

$\mathbb{P}[A\cap C] = \mathbb{P}[\{3,4\}] = \frac{1}{4}$, $\mathbb{P}[A]\mathbb{P}[C] = \frac{1}{4} \Rightarrow \mathbb{P}[A\cap C] = \mathbb{P}[A]\mathbb{P}[C]$, A & C are independent

$\mathbb{P}[B\cap C] = \frac{1}{4}$, $\mathbb{P}[B]\mathbb{P}[C] = \frac{1}{4} \Rightarrow \mathbb{P}[B\cap C] = \mathbb{P}[B]\mathbb{P}[C]$, B & C are independent

$\mathbb{P}[A\cap B\cap C] = \frac{1}{4}$, $\mathbb{P}[A]\mathbb{P}[B]\mathbb{P}[C] = \frac{1}{8} \Rightarrow \mathbb{P}[A\cap B\cap C] \neq \mathbb{P}[A]\mathbb{P}[B]\mathbb{P}[C]$, NO, A, B, and C are NOT independent

YES, A, B, and C are pairwise independent because all the pairs are independent

(b) $\mathbb{P}[A\cap D] = \mathbb{P}[3] = \frac{1}{8}$, $\mathbb{P}[A] = \frac{1}{2}$, $\mathbb{P}[D] = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16} = \frac{1}{2}$

$\mathbb{P}[A\cap D] \neq \mathbb{P}[A]\mathbb{P}[D]$ NO, A and D are NOT independent

(c) $\mathbb{P}[A\cap F] = \emptyset$, $\mathbb{P}[A]\mathbb{P}[F] = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$

$\mathbb{P}[A\cap F] \neq \mathbb{P}[A]\mathbb{P}[F]$ NO, A and F are NOT independent

(d) $\mathbb{P}[B\cap\Omega] = \frac{1}{2}$, $\mathbb{P}[B]\mathbb{P}[\Omega] = \frac{1}{2}$

$\mathbb{P}[B\cap\Omega] = \mathbb{P}[B]\mathbb{P}[\Omega]$ YES, B and Ω are independent

(e) $\mathbb{P}[D] = \frac{1}{2}$, $\mathbb{P}[E] = \frac{1}{2}$, $\mathbb{P}[F] = \frac{1}{4}$

$\mathbb{P}[D\cap E] = \frac{5}{16}$, $\mathbb{P}[D]\mathbb{P}[E] = \frac{1}{4}$

Since $\mathbb{P}[D\cap E] \neq \mathbb{P}[D]\mathbb{P}[E]$, D, E, and F are NOT pairwise independent NO

NOR independent NO

(f) $\mathbb{P}[A\cap D|C] = \frac{\mathbb{P}[A\cap D\cap C]}{\mathbb{P}[C]} = \frac{\mathbb{P}[3]}{\mathbb{P}[C]} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$

$\mathbb{P}[A|C]\mathbb{P}[D|C] = \frac{\mathbb{P}[A\cap C]}{\mathbb{P}[C]}\cdot\frac{\mathbb{P}[D\cap C]}{\mathbb{P}[C]} = \frac{\frac{1}{4}}{\frac{1}{2}}\cdot\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{4}$

YES, A and D are conditionally independent given C

Problem 2.2 (Video 1.5, 1.6, Lecture Problem) Consider the following scenario. You play a simple game with probability of winning 1/4. You play this game repeatedly until your third loss, and then stop playing. Assume all games are independent.

(a) What is the probability of the following specific sequence of game outcomes: Win, Lose, Win, Lose, Lose?

(b) How many different sequences of games are there that end after exactly 5 games? (Hint: you must lose the last game to stop. There aren't that many, so you can enumerate them.)

(c) What is the probability of playing exactly 5 games?

(d) Given that you play exactly 5 games, what is the probability that your first game ended in a loss?

(e) Now, let's generalize this a bit. Say the probability of winning an individual game is p and that you play until your m^{th} loss. What is the probability of playing exactly k games?

(a) $\mathbb{P}[W_1 \cap L_2 \cap W_3 \cap L_4 \cap L_5] = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{1024}$

(b) $\underbrace{\_\ \_\ \_\ \_}\ L$

2 losses in the 1st 4 trials → How many ways to put 2 losses into 4 places? $\binom{4}{2}$

L L W W

L W L W

L W W L

W L L W

W L W L

W W L L

$\Big\}$ 6 ways $= \binom{4}{2}$

∴ there are 6 different sequences of games that end after exactly 5 games

(c) there are 6 ways to play exactly 5 games, with each way having a probability of $\frac{27}{1024}$

∴ the probability of playing exactly 5 games $= 6 \cdot \frac{27}{1024} = \frac{81}{512} = \left(\frac{1}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^3 \cdot 6$

(d) $\mathbb{P}[\text{first match loss} \mid \text{play 5 games}] = \dfrac{\mathbb{P}(\text{first match loss} \cap \text{play 5 games})}{\mathbb{P}(\text{play 5 games})} = \frac{3}{6} = \frac{1}{2}$

(e) $\mathbb{P}[\text{play exactly } k \text{ games}] = \binom{k-1}{m-1} \cdot p^{(k-m)} \cdot (1-p)^m$

Problem 2.3 (Video 1.6) You would like to evaluate the probability of success for testing a batch of n widgets. To start out, let's assume that if there is a problem with the batch, exactly 1 out of the n widgets are defective. You are willing to test only k of the widgets (due to budget or times constraints).

(a) How many ways are there of testing k out of n widgets?

(b) How many ways are there of testing k widgets with the defective widget included?

(c) Use your answers from parts (a) and (b) to determine the probability of catching a defective batch.

(d) Evaluate your answer from part (c) for $n = 20$ and $k = 5$.

(e) Now, say that a defective batch contains exactly 2 defective widgets. How many ways are there of testing k widgets with *at least one* defective widget included? (You may assume that $k > 2$.)

(f) Use your answer from part (e) to determine the probability of catching a defective batch.

(g) Evaluate your answer from part (f) for $n = 20$ and $k = 5$.

(a) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

(b) $\binom{1}{1}\binom{n-1}{k-1} = \binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}$

(c) $\mathbb{P}[\text{catch defective batch}] = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}}$

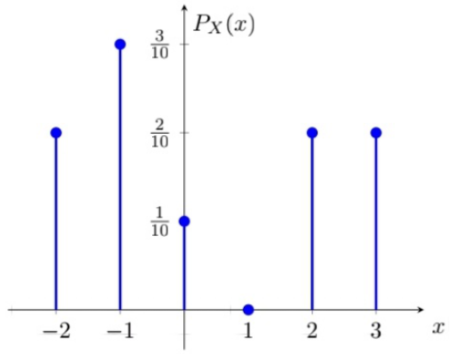
(d) $n=20, k=5$ $\quad \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{\frac{19!}{4!\,15!}}{\frac{20!}{5!\,15!}} = \frac{\frac{19\cdot18\cdot17\cdot16}{4\cdot3\cdot2\cdot1}}{\frac{20\cdot19\cdot18\cdot17\cdot16}{5\cdot4\cdot3\cdot2\cdot1}} = \frac{\cancel{19\cdot18\cdot17\cdot16}}{\cancel{4\cdot3\cdot2\cdot1}}\cdot\frac{5\cdot\cancel{4\cdot3\cdot2\cdot1}}{20\cdot\cancel{19\cdot18\cdot17\cdot16}} = 1/4$

(e) $\binom{2}{1}\binom{n-2}{k-1} + \binom{2}{2}\binom{n-2}{k-2} = 2\frac{(n-2)!}{(k-1)!(n-k-1)!} + \frac{(n-2)!}{(k-2)!(n-k)!}$

(f) $\mathbb{P}[\text{catch defective batch}] = \frac{2\frac{(n-2)!}{(k-1)!(n-k-1)!} + \frac{(n-2)!}{(k-2)!(n-k)!}}{\frac{n!}{k!(n-k)!}}$

(g) $n=20, k=5$ $\quad \frac{2\frac{18!}{4!\,14!} + \frac{18!}{3!\,15!}}{\frac{20!}{5!\,15!}} = \frac{2\frac{18\cdot17\cdot16\cdot15}{4\cdot3\cdot2\cdot1} + \frac{18\cdot17\cdot16}{3\cdot2\cdot1}}{\frac{20\cdot19\cdot18\cdot17\cdot16}{5\cdot4\cdot3\cdot2\cdot1}} = \left[\frac{18\cdot17\cdot16\cdot15}{4\cdot3} + \frac{18\cdot17\cdot16}{3\cdot2}\right]\frac{5\cdot4\cdot3\cdot2}{20\cdot19\cdot18\cdot17\cdot16} = 17/38$

Problem 2.4 (Video 2.1, 2.2, Lecture Problem)



Consider the PMF above and let $A = \{-2, -1, 3\}$.

(a) Calculate the probability that X falls into A, $\mathbb{P}[X \in A]$.

(b) Calculate the probability that X^2 exceeds 1, $\mathbb{P}[X^2 > 1]$.

(c) Given that $\{X \in A\}$ occurs, what is the conditional probability that X^2 exceeds 1, $\mathbb{P}[X^2 > 1 | X \in A]$?

(d) Determine the CDF $F_X(x)$.

(a) $\mathbb{P}[X \in A] = 2/10 + 3/10 + 2/10 = 7/10$

(b) $\mathbb{P}[X^2 > 1] = P_X(-2) + P_X(2) + P_X(3) = 2/10 + 2/10 + 2/10 = 6/10 = 3/5$

$R_X = \{-2, -1, 0, 2, 3\}$; $R_{X^2} = \{4, 1, 0, 4, 9\}$

(c) $\mathbb{P}[X^2 > 1 | X \in A] = \dfrac{\mathbb{P}[X^2 > 1 \cap X \in A]}{\mathbb{P}[X \in A]} = \dfrac{P_X(-2) + P_X(3)}{P_X(-2) + P_X(-1) + P_X(3)} = \dfrac{2/10 + 2/10}{2/10 + 3/10 + 2/10} = \dfrac{4}{10} \cdot \dfrac{10}{7} = 4/7$

$A = \{-2, -1, 3\}$; $A^2 = \{4, 1, 9\}$

(d)

$$\text{CDF } F_X(x) = \begin{cases} 0, & x < -2 \\ 2/10, & -2 \le x < -1 \\ 5/10, & -1 \le x < 0 \\ 6/10, & 0 \le x < 2 \\ 8/10, & 2 \le x < 3 \\ 10/10, & 3 \le x \end{cases}$$

Problem 2.5 (Video 1.5, 1.6, 2.1, 2.2, Quick Calculations) Calculate each of the requested quantities.

(a) Let A and B be independent events with $\mathbb{P}[A] = 1/5$ and $\mathbb{P}[B] = 1/4$. Calculate $\mathbb{P}[A \cap B]$ and $\mathbb{P}[A \cup B]$.

(b) Let A_1, A_2, A_3 be events that are conditionally independent given B. *Additionally, assume that A_1, A_2, A_3 are conditionally independent given B^c.* Assume that $\mathbb{P}[A_i|B] = 1/4$ and $\mathbb{P}[A_i|B^c] = 1/2$ for $i = 1, 2, 3$ and $\mathbb{P}[B] = 1/3$. Calculate $\mathbb{P}[A_1 \cap A_2^c \cap A_3|B]$ and $\mathbb{P}[A_1 \cap A_2^c \cap A_3]$.

(c) Consider a packet of jellybeans that contains 9 jellybeans, of which 4 are lemon and the remaining 5 are raspberry. You reach in and pull out 3 jellybeans. What is the probability that they are all lemon? What is the probability that they are all raspberry?

(d) Let X be a random variable with PMF $P_X(x) = \begin{cases} 1/6 & x = -1, +1 \\ 2/3 & x = 0 \end{cases}$. Calculate $\mathbb{P}[X \neq 0]$ and $\mathbb{P}[X > 0|X \neq 0]$.

(e) If the random variable Y has CDF $F_Y(y) = \begin{cases} 0 & y < 1 \\ 1/4 & 1 \le y < 5 \\ 1 & 5 \le y \end{cases}$, what is the PMF of Y?

(a) $\mathbb{P}[A] = 1/5$, $\mathbb{P}[B] = 1/4$

$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B] = 1/20$

$\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = 1/5 + 1/4 - 1/20 = 4/20 + 5/20 - 1/20 = 8/20 = 2/5$

(b) $\mathbb{P}[A_1 \cap A_2^c \cap A_3 | B] = \mathbb{P}[A_1|B]\,\mathbb{P}[A_2^c|B]\,\mathbb{P}[A_3|B] = 1/4 \cdot 3/4 \cdot 1/4 = 3/64$

$\mathbb{P}[A_1 \cap A_2^c \cap A_3] = \mathbb{P}[A_1 \cap A_2^c \cap A_3|B] + \mathbb{P}[A_1 \cap A_2^c \cap A_3|B^c]$

$= 3/64 + 1/2 \cdot 1/2 \cdot 1/2 = 3/64 + 8/64 = 11/64$

(c) $$\mathbb{P}[\text{all lemon}] = \frac{\binom{4}{3}\binom{5}{0}}{\binom{9}{3}} = \frac{\frac{4!}{3!1!} \cdot \frac{5!}{0!5!}}{\frac{9!}{3!6!}} = \frac{4}{\frac{9\cdot8\cdot7}{3\cdot2}} = \frac{24}{504} = 1/21$$

$$\mathbb{P}[\text{all raspberry}] = \frac{\binom{4}{0}\binom{5}{3}}{\binom{9}{3}} = \frac{5\cdot4}{\frac{9!}{3!6!}} = \frac{20}{\frac{9\cdot8\cdot7}{3\cdot2}} = \frac{120}{504} = 5/21$$

(d) $\mathbb{P}[X \neq 0] = 1/6 + 1/6 = 2/6 = 1/3$

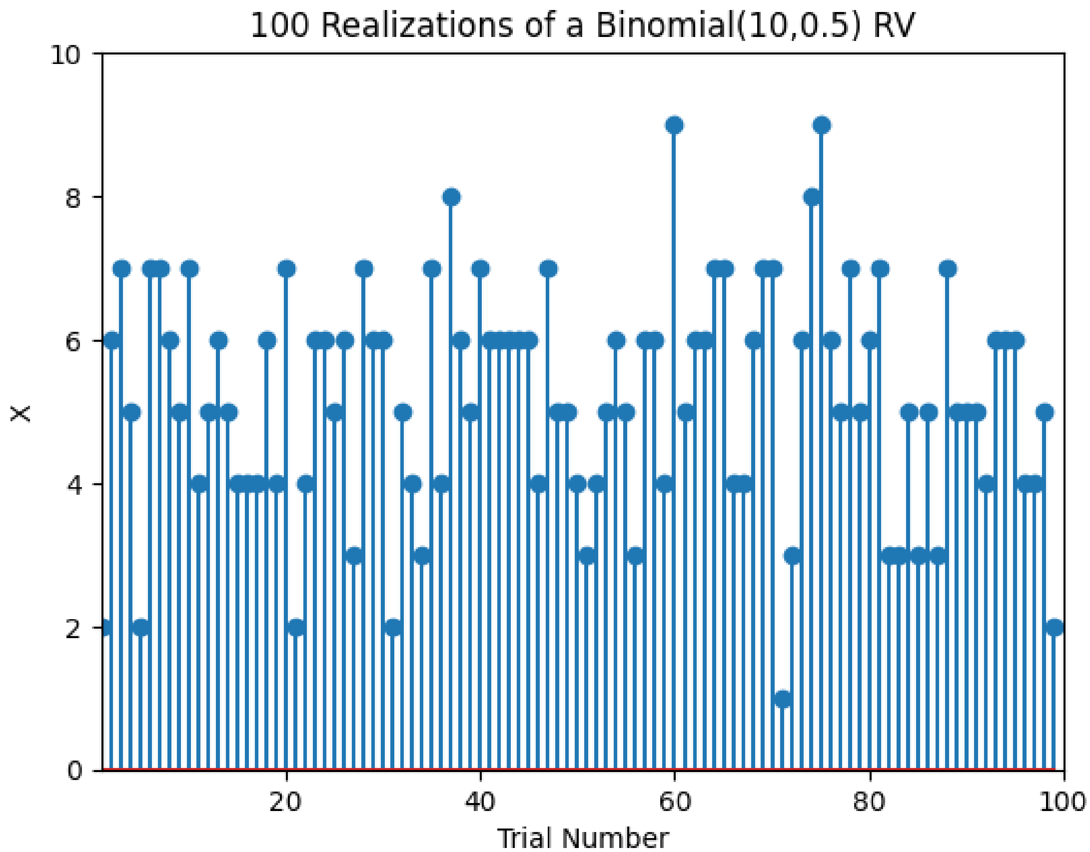
$$\mathbb{P}[X>0 | X \neq 0] = \frac{\mathbb{P}[X>0 \cap X \neq 0]}{\mathbb{P}[X \neq 0]} = \frac{1/6}{2/6} = 1/2$$

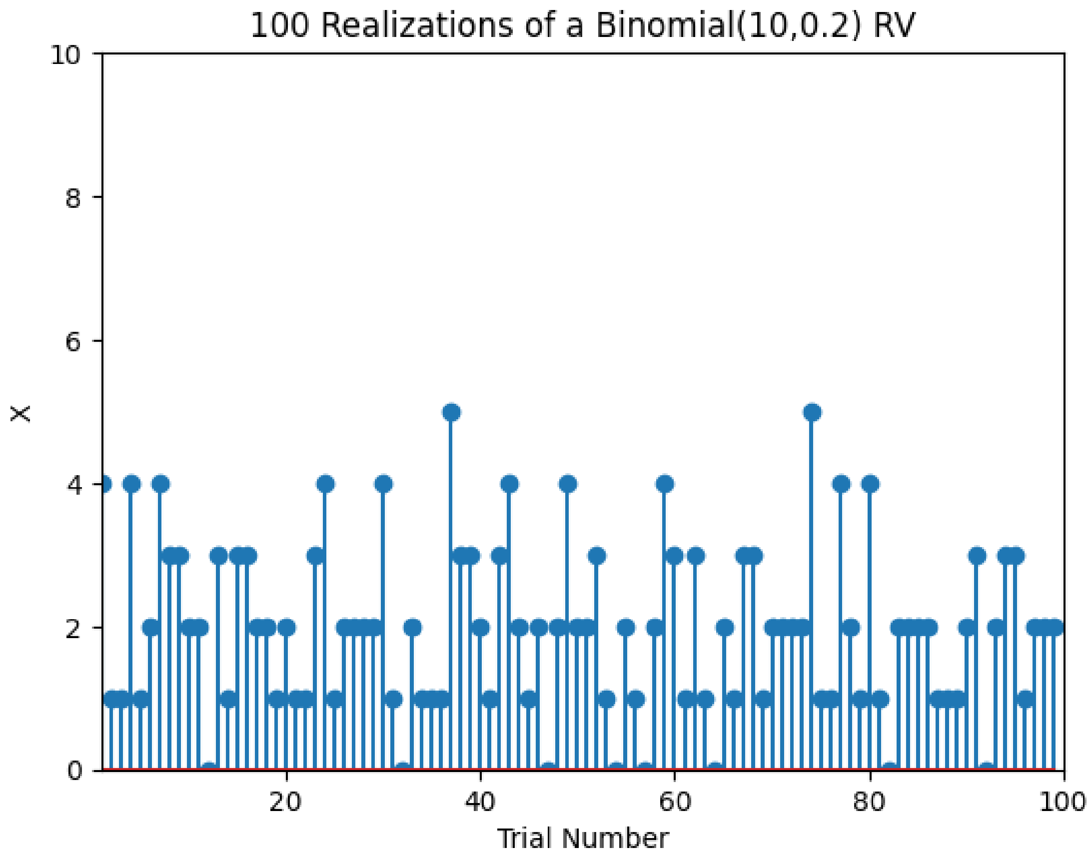
(e) $$\text{PMF } P_Y(y) = \begin{cases} 1/4, & 1 \le y < 5 \\ 3/4, & 5 \le y \end{cases}$$

```python
 #part(a)
import matplotlib.pyplot as plt
import numpy as np
import math
```

Part a. Generate numtrials realizations of a Binomial(n, p) random variable and plots them in sequence. Generate and turn in a plot for 100 realizations of a Binomial(10,1/2) random variable. Generate and turn in a plot for 100 realizations of a Binomial(10,0.2)

```python
#Parameters
n = 10
p = 0.5
numtrials = 100


#Generate Binomial(n,p) random variables
X = np.random.binomial(n,p,numtrials)
#Plot
fig = plt.figure()
plt.stem(X)
plt.axis([1, numtrials, 0, n])
plt.xlabel('Trial Number')
plt.ylabel('X')
plt.title(f"{numtrials} Realizations of a Binomial({n},{p}) RV")
plt.show()
fig.savefig('hw2a1python.png')

#Parameters
n = 10
p = 0.2
numtrials = 100


#Generate Binomial(n,p) random variables
X = np.random.binomial(n,p,numtrials)
#Plot
fig = plt.figure()
plt.stem(X)
plt.axis([1, numtrials, 0, n])
plt.xlabel('Trial Number')
plt.ylabel('X')

plt.title(f"{numtrials} Realizations of a Binomial({n},{p}) RV")
plt.show()
fig.savefig('hw2a2python.png')
```

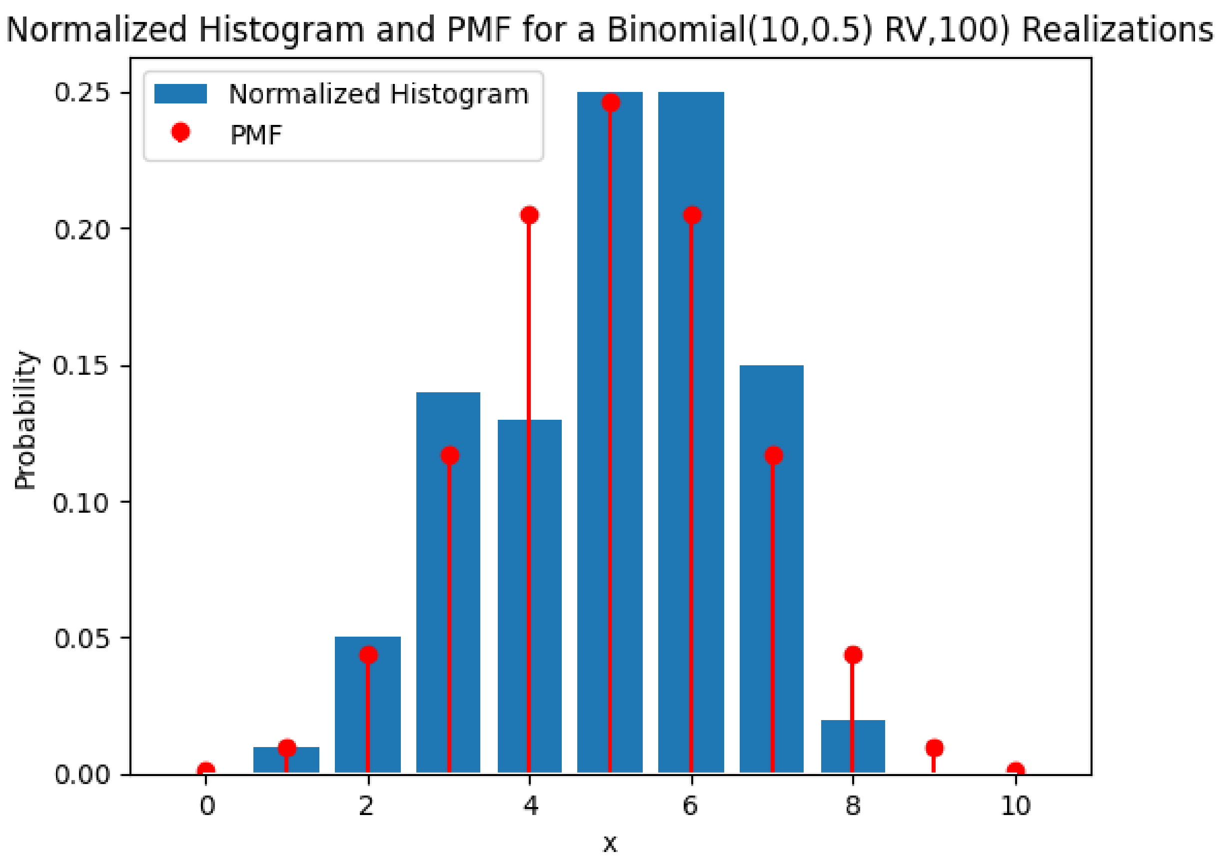

100 Realizations of a Binomial(10,0.5) RV
X
Trial Number
0
2
4
6
8
10
20
40
60
80
100

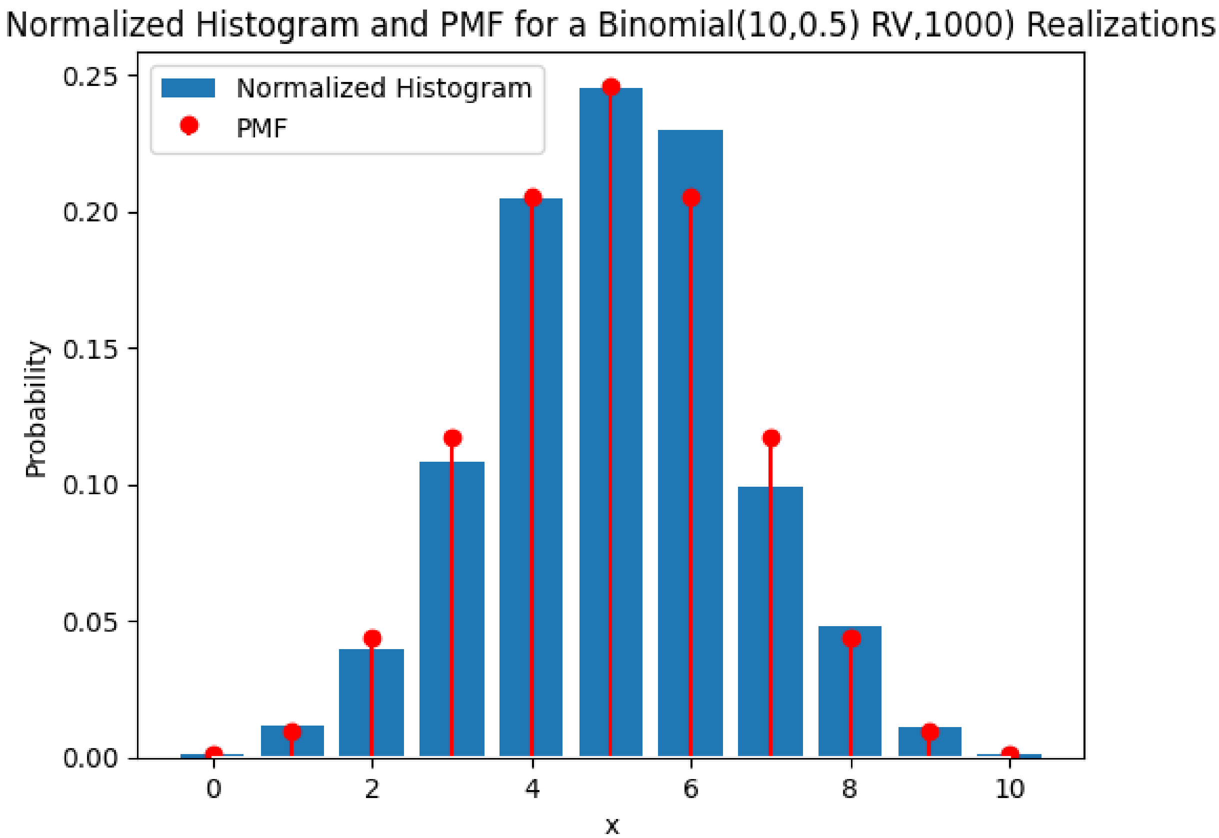


Part b. The code framework below generates a bar plot of the number of occurrences of each value from 0 to n, normalized by the number of trials. Your job is to fill in the lines of code needed to count the number of occurrences of the values i in the sequence X and also calculate the PMF of a Binomial(n, p) random variable for X = i. Then, for n = 10 and p = 1/2, generate and turn in plots for 100, 1000, and 10000 trials.

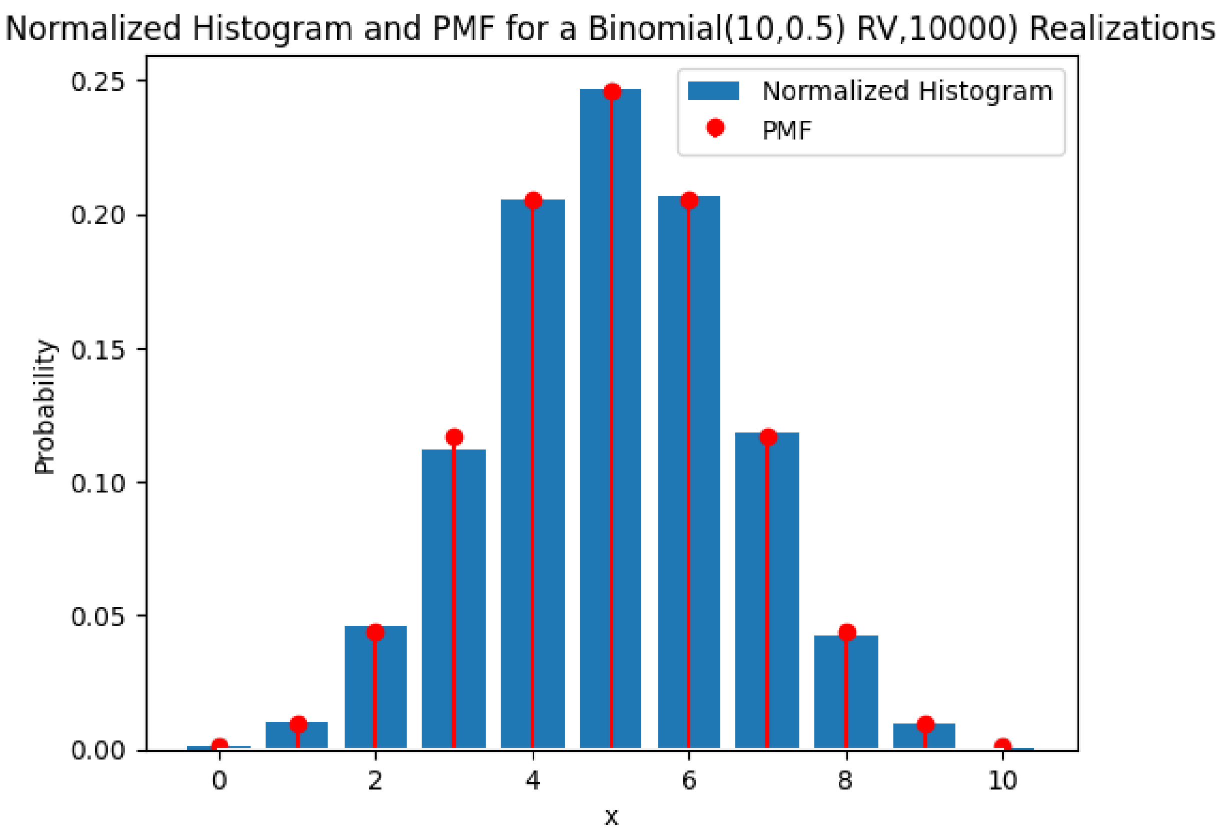
```
#part(b)
#Parameters
n = 10
p = 0.5
trials = [100,1000,10000]
for numtrials in trials:
    #Generate Binomial(n,p) random variables
    X = np.random.binomial(n,p,numtrials)
    #Initialize arrays
    counts = np.zeros(n+1)
    pmf = np.zeros(n+1)
    #Count number of times each value occurs
    for value in X:
        counts[value] += 1

    #Divide counts by number of trials to get a normalized histogram
    normhist = counts/numtrials
```

```
    #Calculate probability mass function
    for k in range(0,n+1):
        # Your code for the PMF of value i; this is pmf[k] =
(combinations of k out of n)*(p**k) * ((1-p)**(n-k))
        #  see if you can use the function math.comb(n,k) which gives
combinations of k out of n elements
        pmf[k] = math.comb(n, k) * pow(p, k) * pow(1-p, n-k)

    #Plot
    xvals = np.arange(len(counts)) #Array of possible values
    fig = plt.figure()
    plt.bar(xvals,normhist)
    plt.xlabel('x')
    plt.ylabel('Probability')
    plt.stem(xvals,pmf,linefmt='r',markerfmt='ro',basefmt='w')
    plt.title(f"Normalized Histogram and PMF for a Binomial({n},{p})
RV,{numtrials}) Realizations")
    plt.legend(['Normalized Histogram','PMF'])
    plt.show()
    fig.savefig("hw3bpython_"+str(numtrials)
+"trials.png",bbox_inches='tight')
```





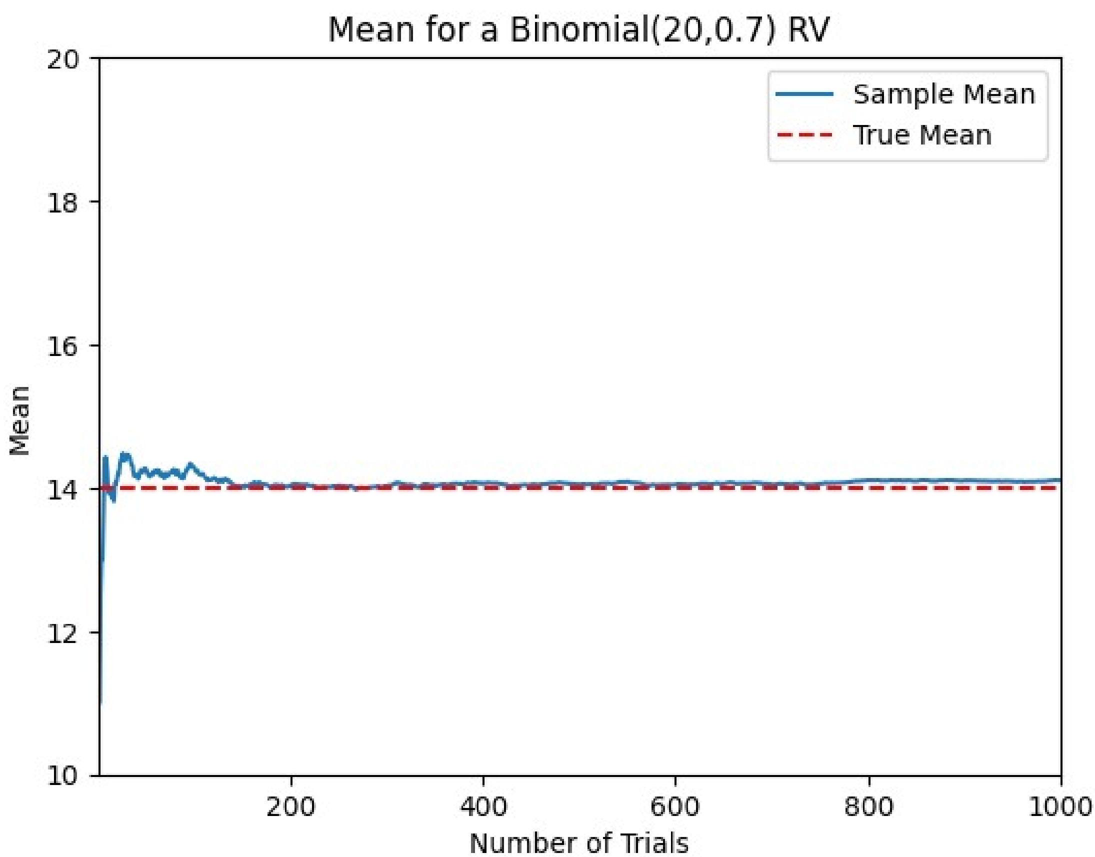
Normalized Histogram and PMF for a Binomial(10,0.5) RV,1000) Realizations
Normalized Histogram
PMF
Probability
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
10
x



Part c. Generate numtrials realizations of a Binomial(n,p) random variable, calculate the sample mean up to the mth realization for m from 1 to numtrials, and plots the resulting sequence of sample means. For n = 20, p = 0.7, and numtrials= 1000, generate and turn in a plot of the sample and true mean.

```
#Parameters
n = 20;
p = 0.7;
numtrials = 1000;

#Generate random samples
X = np.random.binomial(n,p,numtrials)

#Determine sample means
samplemeanX = np.zeros(numtrials)
for m in range(0,numtrials):
    #Your code to calculate the average of X[0],...,X[m]
    subtrials = X[0:m]
    if len(subtrials) == 0:
        samplemeanX[m] = X[m]
    else:
        samplemeanX[m] = sum(subtrials)/len(subtrials)

#Calculate true mean
```

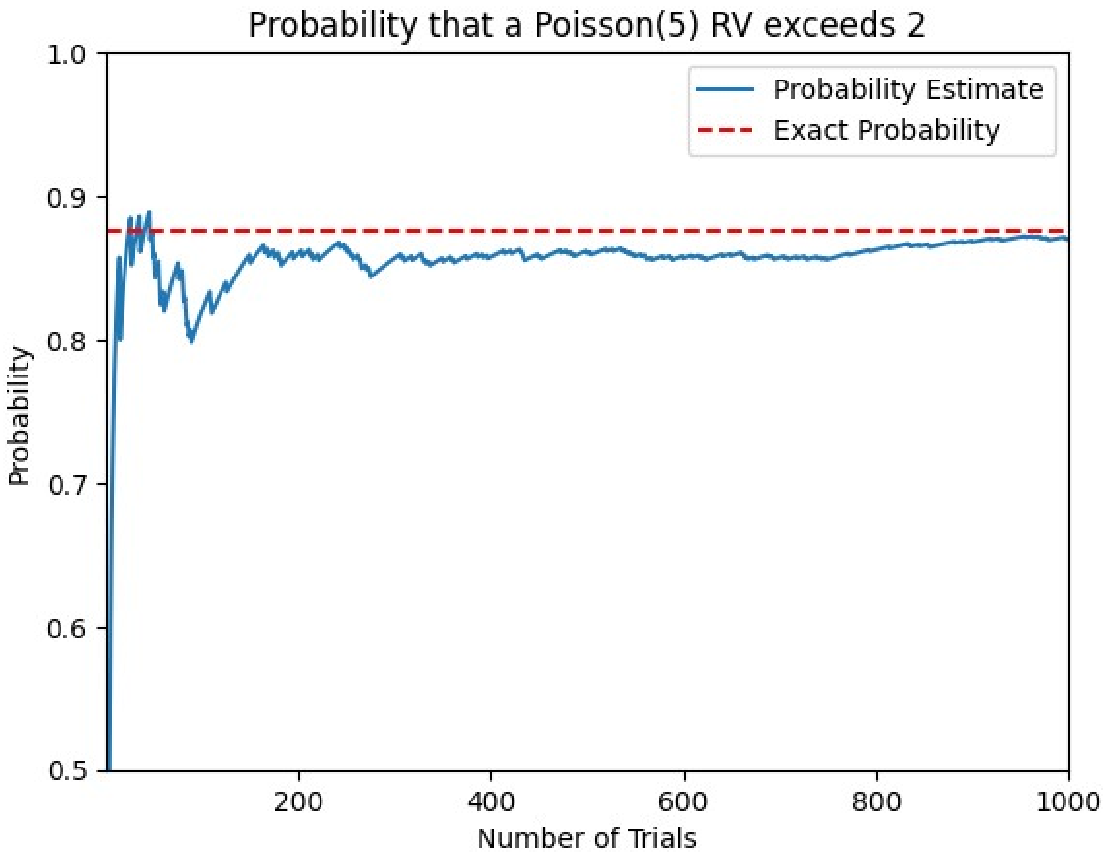
```
truemeanX = n*p

#Plot
fig = plt.figure()
plt.plot(np.arange(1,numtrials+1),samplemeanX)
plt.plot(np.arange(1,numtrials+1),truemeanX*np.ones(numtrials),'r--')
plt.axis([1, numtrials, 10, 20])
plt.xlabel('Number of Trials')
plt.ylabel('Mean')
plt.legend(['Sample Mean','True Mean'])
plt.title(f"Mean for a Binomial({n},{p}) RV")
plt.show()
fig.savefig('hw3_s23_cpython.png')
```



Part d. Generate 1000 realizations of a Poisson(5) random variable, estimates the probability P[X > b] using up to the mth realization for m from 1 to realization for m from 1 to numtrials, and plot the resulting sequence of probability estimates as well as the exact probability. Your job is to fill in the code needed to generate the estimate the probability. Then, generate and turn in a plot of the estimated and exact probability.

```
#Parameters
alpha = 5
```

```python
b = 2
numtrials = 1000

#Generate Poisson(alpha) random variables
X = np.random.poisson(alpha,numtrials)

#Determine probability estimate
probestimate = np.zeros(numtrials)
for m in range(0,numtrials):
    #Your code for (number of times X[0],...,X[m] exceeds b)/(m+1)
    count = 0
    for k in range(0, m):
        if X[k] > b:
            count += 1
    probestimate[m] = count/(m+1)

#Determine exact probability (using complement)
complementprob = 0
for i in range(0,b+1):
    complementprob = complementprob +
float(alpha**i)/(math.factorial(i)) * np.exp(-alpha)

exactprob = 1 - complementprob

#Plot
fig = plt.figure()
plt.plot(np.arange(1,numtrials+1),probestimate)
plt.plot(np.arange(1,numtrials+1),exactprob*np.ones(numtrials),'r--')
plt.axis([1, numtrials, 0.5, 1])
plt.xlabel('Number of Trials')
plt.ylabel('Probability')
plt.legend(['Probability Estimate','Exact Probability'])
plt.title(f"Probability that a Poisson({alpha}) RV exceeds {b}")
plt.show()
fig.savefig('hw3d_s23_python.png')
```


Probability that a Poisson(5) RV exceeds 2
Probability Estimate
Exact Probability
Probability
Number of Trials
1.0
0.9
0.8
0.7
0.6
0.5
200
400
600
800
1000