Jilin Zheng // U49258796

EK381 Homework #7

Problem 7.1

$X = U + V$, $Y = U - 2V$, U and V are independent, standard Gaussians (Gaussian $(0,1)$)

(a) Yes, X and Y are linear functions of independent, standard Gaussian random variables, so they are jointly Gaussian

(b) $\mathbb{E}[X] = \mathbb{E}[U+V] = \mathbb{E}[U] + \mathbb{E}[V] = \emptyset$

$\mathbb{E}[Y] = \mathbb{E}[U - 2V] = \emptyset - 2(\emptyset) = \emptyset$

(c) $\text{Cov}[X,Y] = \text{Cov}[U+V, U-2V] = 1 \cdot \text{Var}[X] + (-2)\text{Var}[Y] + (-2+1)\text{Cov}[U,V]$

$= 1 + (-2) + (-1)(\emptyset) = -1$

(d) $\text{Var}[X] = \text{Var}[U+V] = \text{Var}[U] + \text{Var}[V] + 2\text{Cov}[U,V] = 1 + 1 + 2(\emptyset) = 2$

$\text{Var}[Y] = \text{Var}[U - 2V] = \text{Var}[U] + (-2)^2\text{Var}[V] + 2(1)(-2)\text{Cov}[U,V] = 1 + 4 = 5$

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = \frac{-1}{\sqrt{2 \cdot 5}} = \frac{-1}{\sqrt{10}}$$

(e) $\mathbb{E}[X | Y = y] = \mu_X + \frac{\text{Cov}[X,Y]}{\text{Var}[Y]}(y - \mu_Y) = \emptyset + \frac{-1}{5}(y - \emptyset) = \frac{-y}{5}$

(f) $\text{Var}[X|Y] = (1 - \rho_{X,Y}^2)\sigma_X^2 = \left(1 - \left(\frac{-1}{\sqrt{10}}\right)^2\right)2 = \left(1 - \left(\frac{1}{10}\right)\right)2 = \frac{9}{10} = \frac{9}{5}$

Problem 7.2

(a) $\underline{\mu_Y} = \begin{bmatrix} \mathbb{E}[Y_1] \\ \mathbb{E}[Y_2] \end{bmatrix} = \begin{bmatrix} \mathbb{E}[2X_1+1] \\ \mathbb{E}[X_1 - X_2] \end{bmatrix} = \begin{bmatrix} 2(\emptyset)+1 \\ \emptyset - \emptyset \end{bmatrix} = \begin{bmatrix} 1 \\ \emptyset \end{bmatrix}$

(b) $\underline{\Sigma_Y} = \begin{bmatrix} Cov[Y_1, Y_1] & Cov[Y_1, Y_2] \\ Cov[Y_2, Y_1] & Cov[Y_2, Y_2] \end{bmatrix} = \begin{bmatrix} 4 & 4/3 \\ 4/3 & 4/3 \end{bmatrix}$

$Var[Y_1] = Var[2X_1 + 1] = 4Var[X_1] = 4$

$Var[Y_2] = Var[X_1 - X_2] = Var[X_1] + Var[X_2] + 2(1)(-1)Cov[X_1, X_2] = 1 + 1 + (-2)(1/3) = 2 - 2/3 = 6/3 - 2/3 = 4/3$

$Cov[Y_1, Y_2] = Cov[2X_1 + 1, X_1 - X_2] = 2Var[X_1] + \emptyset + (2(-1) + \emptyset)Cov[X_1, X_2] = 2 + (-2)\frac{1}{3} = 6/3 + (-2/3) = 4/3$

$= Cov[Y_2, Y_1]$

(c) Y_1 and Y_2 are linear functions of jointly Gaussian X_1 and X_2, so Y_1 and Y_2 are also jointly Gaussian

$\rho_{Y_1, Y_2} = \frac{Cov[Y_1, Y_2]}{\sqrt{Var[Y_1]Var[Y_2]}} = \frac{4/3}{\sqrt{4 \cdot 4/3}} = \frac{4/3}{\sqrt{16/3}} = \frac{4}{3} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}$

Since Y_1 and Y_2 are NOT uncorrelated ($\rho_{Y_1, Y_2} \neq \emptyset$), they are NOT independent

(d) $\mathbb{E}[Y_1 \mid Y_2 = y_2] = \mu_{Y_1} + \frac{Cov[Y_1, Y_2]}{Var[Y_2]}(y_2 - \mu_{Y_2})$

$= 1 + \frac{4/3}{4/3}(y_2 - \emptyset)$

$= 1 + y_2$

Problem 7.3

(a) $\mathbb{E}[\underline{X}] = \mu_{\underline{X}} = \begin{bmatrix} \mathbb{E}[X_1] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix} = \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix}$

$$\Sigma_{\underline{X}} = \begin{bmatrix} \mathrm{Cov}[X_1,X_1] & \cdots & \mathrm{Cov}[X_1,X_n] \\ \vdots & & \vdots \\ \mathrm{Cov}[X_n,X_1] & \cdots & \mathrm{Cov}[X_n,X_n] \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^2 \end{bmatrix}$$

(b) $\underset{1\times1}{Y} = \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right) = \underset{1\times n}{A}\,\underset{n\times 1}{\underline{X}} = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

(c) $\mathbb{E}[Y] = \mathbb{E}[A\underline{X}] = A\,\mathbb{E}[\underline{X}] = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}\begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} = \sum_{i=1}^{n} \frac{\mu}{n} = \frac{\mu}{n}(n) = \mu$

(d) $\Sigma_{\underline{Y}} = A\Sigma_{\underline{X}}A^T = \underset{1\times n}{\begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}}\underset{n\times n}{\begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^2 \end{bmatrix}}\underset{n\times 1}{\begin{bmatrix} 1/n \\ \vdots \\ 1/n \end{bmatrix}}$

$$= \underset{1\times n}{\begin{bmatrix} \frac{\sigma^2}{n} & \cdots & \frac{\sigma^2}{n} \end{bmatrix}}\underset{n\times 1}{\begin{bmatrix} 1/n \\ \vdots \\ 1/n \end{bmatrix}} = \frac{\sigma^2}{n}\left(\frac{1}{n}\right)n = \underset{1\times 1}{\frac{\sigma^2}{n}}$$

$\mathrm{Var}[Y] = \frac{\sigma^2}{n}$

(e) $\mathbb{P}[|Y-\mathbb{E}[Y]| > \delta] = 1-\mathbb{P}[|Y-\mathbb{E}[Y]| < \delta]$

$= 1-\mathbb{P}[-\delta < Y-\mathbb{E}[Y] < \delta]$

$= 1-\mathbb{P}[-\delta+\mu < Y < \delta+\mu]$ $\qquad \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$

$= 1-\left[\Phi\left(\frac{\delta+\mu-\mu}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{-\delta+\mu-\mu}{\sigma/\sqrt{n}}\right)\right]$

$= 1-\left[\Phi\left(\frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(\frac{-\delta\sqrt{n}}{\sigma}\right)\right]$

$= Q\left(\frac{\delta\sqrt{n}}{\sigma}\right) + \Phi\left(\frac{-\delta\sqrt{n}}{\sigma}\right) = Q\left(\frac{\delta}{\frac{\sigma}{\sqrt{n}}}\right) + \Phi\left(\frac{-\delta}{\frac{\sigma}{\sqrt{n}}}\right) = 2Q\left(\frac{\delta}{\frac{\sigma}{\sqrt{n}}}\right)$

(f) $\delta = \frac{1}{10}$

$\frac{1}{1000} = 2Q\left(\frac{\frac{1}{10}}{\frac{\sigma}{\sqrt{n}}}\right) = 2Q\left(\frac{\sqrt{n}}{10\sigma}\right)$

$Q\left(\frac{\sqrt{n}}{10\sigma}\right) = \frac{1}{2000}$

$\frac{\sqrt{n}}{10\sigma} = 3.29$

$\sqrt{n} = 32.9\sigma$

$n = 1082.41\sigma^2$