Jilin Zheng

EK381 Homework #11

Problem 11.1

(a) $P = \begin{bmatrix} 1/3 & 2/3 & \emptyset \\ 1/2 & \emptyset & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

(b) $P(2) = P^2 = \begin{bmatrix} 1/3 & 2/3 & \emptyset \\ 1/2 & \emptyset & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & \emptyset \\ 1/2 & \emptyset & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 4/9 & 2/9 & 1/3 \\ 7/24 & 11/24 & 1/4 \\ 1/3 & 7/24 & 3/8 \end{bmatrix}$

(c) $\rho_0 = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \end{bmatrix}$

$\rho_1 = P^T \rho_0 = \begin{bmatrix} 11/30 \\ 7/30 \\ 2/5 \end{bmatrix}$ $\qquad \rho_2 = P^T \rho_1 = \begin{bmatrix} 61/180 \\ 31/90 \\ 19/60 \end{bmatrix}$

(d) $\mathbb{P}[X_0 = 1 \cap X_1 = 2 \cap X_2 = 2]$

$= \mathbb{P}[X_0 = 1]\,\mathbb{P}[X_1 = 2 \mid X_0 = 1]\,\mathbb{P}[X_2 = 2 \mid X_1 = 2 \cap X_0 = 1]$

$= \frac{1}{5} \cdot \frac{2}{3} \cdot \emptyset = \emptyset$

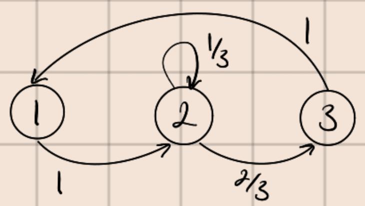
$\mathbb{P}[X_0 = 3, X_1 = 1, X_2 = 2]$

$= \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{2}{3}$

$= \frac{4}{60} = \frac{1}{15}$

Problem 11.2

(a)



(b) Period of State 1 = 3

(c) $\mathbb{P}[X_0 = 2, X_1 = 2, X_2 = 3]$

$= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$

$= \frac{2}{27}$

(d) An unique limiting state probability vector $\underline{\pi}$ does exist because the Markov chain is aperiodic

$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1/3 & 0 \\ 0 & 2/3 & 0 \end{bmatrix}$

$P^T \underline{\pi} = \underline{\pi} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1/3 & 0 \\ 0 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} \rightarrow$

$\pi_3 = \pi_1$
$\pi_1 + \frac{1}{3}\pi_2 = \pi_2$
$\frac{2}{3}\pi_2 = \pi_3 = \pi_1$

$\pi_1 + \pi_2 + \pi_3 = 1$
$\frac{2}{3}\pi_2 + \pi_2 + \frac{2}{3}\pi_2 = 1$
$\frac{7}{3}\pi_2 = 1 \rightarrow \pi_2 = \frac{3}{7}$

$\pi_1 = \pi_3 = \frac{2}{3}\left(\frac{3}{7}\right) = \frac{2}{7}$

$\underline{\pi} = \begin{bmatrix} 2/7 \\ 3/7 \\ 2/7 \end{bmatrix}$

Problem 11.3

(a) $C_1 = \{1, 2\}$, $C_2 = \{3, 4, 5\}$

(b) Period of $C_1 = 1$, Recurrent

Period of $C_2 = 2$, Transient

(c) $$P = \begin{bmatrix} \emptyset & 1 & \emptyset & \emptyset & \emptyset \\ 1/2 & 1/2 & \emptyset & \emptyset & \emptyset \\ \emptyset & 1/2 & \emptyset & 1/2 & \emptyset \\ \emptyset & \emptyset & 1/2 & \emptyset & 1/2 \\ \emptyset & \emptyset & \emptyset & 1 & \emptyset \end{bmatrix}$$

(d) An unique limiting state probability vector $\underline{\pi}$ does exist because the Markov chain has one recurrent communicating class and is aperiodic

$$\underline{\pi} = P^T \underline{\pi}$$

$$= \begin{bmatrix} \emptyset & 1/2 & \emptyset & \emptyset & \emptyset \\ 1 & 1/2 & 1/2 & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & 1/2 & \emptyset \\ \emptyset & \emptyset & 1/2 & \emptyset & 1 \\ \emptyset & \emptyset & \emptyset & 1/2 & \emptyset \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \emptyset \\ \emptyset \\ \emptyset \end{bmatrix} = \begin{cases} \frac{1}{2}\pi_2 = \pi_1 \\ \pi_1 + \frac{1}{2}\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases} \quad \frac{1}{2}\pi_2 + \pi_2 = 1 \quad \pi_2 = \frac{2}{3}$$

$$\pi_1 = \frac{1}{3}$$

$$\underline{\pi} = \begin{bmatrix} 1/3 \\ 2/3 \\ \emptyset \\ \emptyset \\ \emptyset \end{bmatrix}$$

Problem 11.4

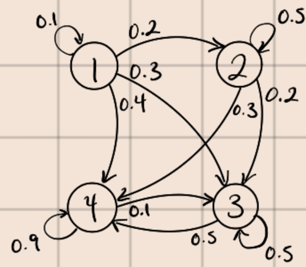
(a) $C_1 = \{1, 2\}$, $C_2 = \{3, 4\}$, $C_3 = \{5, 6, 7, 8\}$

(b) Period of $C_1 = 2$, Period of $C_2 = 1$, Period of $C_3 = 3$

(c) C_1 is Recurrent, C_2 is Transient, C_3 is Recurrent

Problem 11.5

(a)



(b) Transient States: 1 and 2 ; Recurrent States: 3 and 4

(c) The Markov chain is NOT irreducible because there are 3 communicating classes

(d) The Markov chain is aperiodic because all the states are aperiodic

(e) $$\underline{\pi} = P^T \underline{\pi}$$

$$= \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0 \\ 0.3 & 0.2 & 0.5 & 0.1 \\ 0.4 & 0.3 & 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{cases} 0.5\pi_3 + 0.1\pi_4 = \pi_3 \\ 0.5\pi_3 + 0.9\pi_4 = \pi_4 \\ \pi_3 + \pi_4 = 1 \end{cases}$$

$$0.8\pi_4 = \pi_4 - \pi_3 \qquad 1.2\pi_4 = 1$$

$$\pi_3 = 0.2\pi_4 \qquad \pi_4 = 1/1.2 = 0.83$$

$$0.2\pi_4 + \pi_4 = 1 \qquad \pi_3 = 0.17$$

$$= \begin{bmatrix} \emptyset \\ \emptyset \\ 1/6 \\ 5/6 \end{bmatrix}$$