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EK381 Homework #4

Problem 4.1 (Video 3.1, 3.2, 3.4, Lecture Problem)

Consider a continuous random variable X with the following PDF:

$$f_X(x) = \begin{cases} c & -1 \le x \le 0 \text{ or } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the PDF of X. Be sure to label your axes.

(b) Determine the value of c that satisfies the normalization property. Set c to this value for the remainder of the problem. This can be done without integration, so your answer should be a number.

(c) What is the expected value of X?

(d) What is the variance of X?

(e) What is $\mathbb{E}[2X^2 - 3X + 1]$?

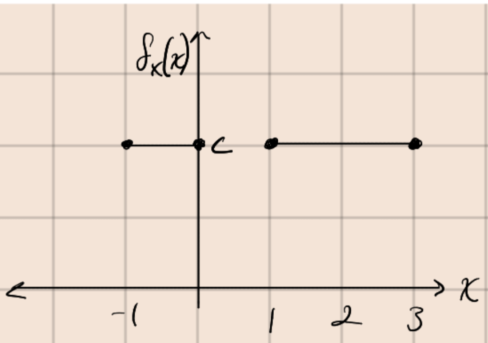
(f) Calculate the probability that $|X|$ is less than 2. (Your answer can be an integral, but you can also take advantage of the simple structure of the PDF.)

(g) Let $B = (-2, 2)$. Determine the conditional PDF $f_{X|B}(x)$ of X given the event $\{X \in B\}$.

(h) Calculate the probability that $X < 0$ given that $|X|$ is less than 2.

(i) Calculate the conditional expectation $\mathbb{E}[X|B]$ of X given that $|X|$ is less than 2.

(a)



(b) $\left.\begin{array}{l}\int_{-1}^{0} c\,dx = c\,[0-(-1)] = c \\ \int_{1}^{3} c\,dx = c\,[3-1] = 2c\end{array}\right\}$ $3c = 1 \to c = 1/3$

(c) $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$= \cancel{\int_{-\infty}^{-1} x f_X(x)\,dx} + \int_{-1}^{0} x f_X(x)\,dx + \cancel{\int_{0}^{1} x f_X(x)\,dx} + \int_{1}^{3} x f_X(x)\,dx + \cancel{\int_{3}^{\infty} x f_X(x)\,dx}$

$= \frac{1}{3}\left[\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{1}^{3}\right] = \frac{1}{3}\left[\left(0-\frac{1}{2}\right) + \left(\frac{9}{2}-\frac{1}{2}\right)\right] = \frac{1}{3}\left(-\frac{1}{2}+\frac{8}{2}\right) = \frac{7}{6}$

(d) $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$= \frac{1}{3}\int_{-1}^{0} x^2\,dx + \frac{1}{3}\int_{1}^{3} x^2\,dx$

$= \frac{1}{3}\left[\frac{x^3}{3}\Big|_{-1}^{0}\right] + \frac{1}{3}\left(\frac{x^3}{3}\Big|_{1}^{3}\right]$

$= \frac{1}{9}(0-(-1)) + \frac{1}{9}(27-1)$

$= \frac{1}{9} + \frac{26}{9} = 3$

$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$= 3 - \left(\frac{7}{6}\right)^2 = 3 - \frac{49}{36}$

$= \frac{108}{36} - \frac{49}{36}$

$= \frac{59}{36}$

(e) $\mathbb{E}[2X^2-3X+1] = 2\mathbb{E}[X^2] - 3\mathbb{E}[X] + 1 = 2\cdot 3 - 3\cdot\frac{7}{6} + 1 = 6 - \frac{21}{6} + 1 = \frac{21}{6}$

(f) $\mathbb{P}[|X|<2] = \int_{-1}^{0} f_X(x)\,dx + \int_{1}^{2} f_X(x)\,dx = \frac{1}{3}[1+1] = 2/3$

(g) $B = (-2,2)$ $\quad \mathbb{P}[X \in B] = \int_{-1}^{0} f_X(x)\,dx + \int_{1}^{2} f_X(x)\,dx = \frac{1}{3}(1) + \frac{1}{3}(1) = 2/3$

$f_{X|B}(x) = \frac{f_X(x)}{\mathbb{P}[X\in B]} = \frac{1/3}{2/3} = 1/2, \quad -1 \le x \le 0 \text{ or } 1 \le x < 2$

(h) $\mathbb{P}[X<0 \mid \overset{B}{|X|<2}] = \int_{-\infty}^{0} f_{X|B}(x)\,dx = \int_{-1}^{0} f_{X|B}(x)\,dx = \frac{1}{2}[1] = 1/2$

(i) $\mathbb{E}[X|B] = \int_{-\infty}^{\infty} x \cdot f_{X|B}(x)\,dx = \int_{-1}^{0} x \cdot \frac{1}{2}\,dx + \int_{1}^{2} x \cdot \frac{1}{2}\,dx = \frac{1}{2}\left[\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{1}^{2}\right] = \frac{1}{2}\left[\left(0 - \frac{1}{2} + \left(2 - \frac{1}{2}\right)\right)\right]$

$= 1/2\left[-1/2 + 3/2\right] = 1/2(1) = 1/2$

Problem 4.2 (**Video 3.3**, **Lecture Problem**) Let X be a Gaussian$(-1, 4)$ random variable.

(a) Calculate $\mathbb{P}[X < 2]$. You may leave your answer in terms of the standard normal CDF $\Phi(z)$.

(b) Calculate $\mathbb{P}[X < 0 | X < 2]$. You may leave your answer in terms of the standard normal CDF $\Phi(z)$.

(c) Calculate $\mathbb{E}[2X + 3]$ and $\mathsf{Var}[2X + 3]$.

(d) Let $Y = 2X + 3$. What kind of a random variable is Y?

$\mu = \mathbb{E}[X] = -1$, $\sigma^2 = \mathrm{Var}[X] = 4$, σ = std. deviation of $X = 2$, $\sigma = \sqrt{\mathrm{Var}[X]}$, $\Phi\left(\frac{x-\mu}{\sigma}\right)$

(a) $\mathbb{P}[X < 2] = F_X(2) = \Phi\left(\frac{x+1}{2}\right) = \Phi\left(\frac{3}{2}\right)$

(b) $\mathbb{P}[X < 0 | X < 2] = \frac{\mathbb{P}[X < 0 \cap X < 2]}{\mathbb{P}[X < 2]} = \frac{\mathbb{P}[X < 0]}{\Phi\left(\frac{3}{2}\right)} = \frac{\Phi\left(\frac{1}{2}\right)}{\Phi\left(\frac{3}{2}\right)}$

(c) $\mathbb{E}[2X+3] = 2\mathbb{E}[X] + 3 = 1$

$\mathrm{Var}[2X+3] = 4\,\mathrm{Var}[X] = 16$

(d) $Y = 2X+3$

Y is Gaussian$(1, 16)$

Problem 4.3 (Video 3.1, 3.2, 3.3, 3.4, Quick Calculations)

Calculate each of the requested quantities. **All of these problems are carefully chosen so that they can be completed without integration, so we are expecting exact answers.** For the Gaussian problems, you will sometimes need to lookup values for the standard normal CDF $\Phi(z)$. For example, you could use a lookup table `https://en.wikipedia.org/wiki/Standard_normal_table` or the MATLAB function `normcdf`.

(a) For $f_X(x) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$, sketch the PDF, then calculate $\mathbb{E}[X]$ and $\mathbb{P}[|X| > 1/2]$.

(b) For $f_X(x)$ from part (a), determine and sketch the conditional PDF $f_{X|B}(x)$ given the event $\{|X| > 1/2\}$. Using your sketch, calculate $\mathbb{P}[X > 0 \,|\, |X| > 1/2]$.

(c) Let X be Exponential(2). Calculate $\mathbb{E}[X-1]$ and $\mathbb{E}[(X-1)^2]$.

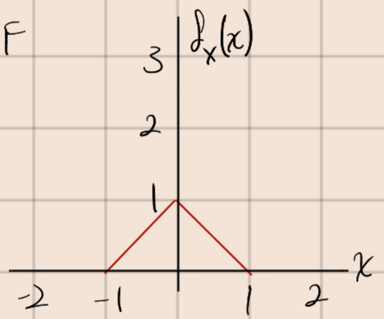
(d) Let X be Uniform$(-2, 3)$. Determine $\mathbb{P}[X - 1 > 0]$ and $\mathbb{P}[X^2 - 1 > 0]$.

(e) Let X be Gaussian$(2, 1)$ and let $Y = 3X - 2$. Determine the mean and variance of Y as well as $\mathbb{P}[Y > 1]$.

(f) Let X be Gaussian$(1, 4)$. Determine (up to two decimal places) the maximum value of a such that $\mathbb{P}[X \le a] \le 0.1$. Determine the minimum value of b such that $\mathbb{P}[X \ge b] \le 0.2$.

(a) $f_X(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$

PDF

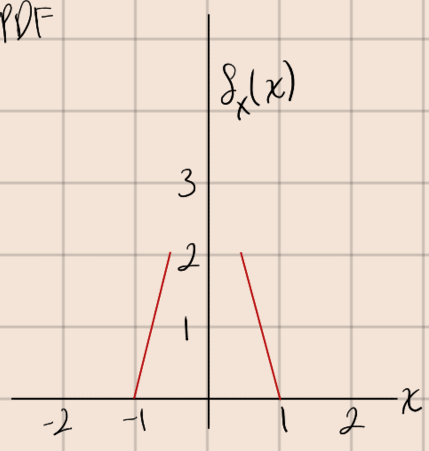


$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_{-1}^{0} x(1+x)\,dx + \int_0^1 x(1-x)\,dx$$
$$= \int_{-1}^{0} x + x^2\,dx + \int_0^1 x - x^2\,dx = \left(\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^3}{3}\Big|_{-1}^{0}\right) + \left(\frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1\right)$$
$$= \left(0 - \frac{1}{2} + 0 + \frac{1}{3}\right) + \left(\frac{1}{2} - 0 - \left(\frac{1}{3} - 0\right)\right)$$
$$= -1/2 + 1/3 + 1/2 - 1/3 = 0$$

$$\mathbb{P}[|X| > \tfrac{1}{2}] = \int_{-1}^{-\frac{1}{2}} (1+x)\,dx + \int_{1/2}^{1} (1-x)\,dx = x\Big|_{-1}^{-1/2} + \frac{x^2}{2}\Big|_{-1}^{-1/2} + x\Big|_{1/2}^{1} - \frac{x^2}{2}\Big|_{1/2}^{1}$$
$$= (-1/2 + 1) + \left(\frac{1}{8} - 1/2\right) + (1 - 1/2) - \left(\frac{1}{2} - \frac{1}{8}\right)$$
$$= 1/2 + (-3/8) + \frac{1}{2} - 3/8 = 2(4/8) + 2(-3/8) = 1/4$$

(b) $f_{X|B}(x) = \dfrac{f_X(x)}{\mathbb{P}[X \in B]} = \begin{cases} 4(1-|x|), & \frac{1}{2} < |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$

PDF



$$\mathbb{P}[X > 0 \,|\, |X| > 1/2] = \frac{\mathbb{P}[X > 0 \cap |X| > 1/2]}{\mathbb{P}[|X| > 1/2]} = \frac{\mathbb{P}[X > 1/2]}{\mathbb{P}[|X| > 1/2]} = 1/2$$

(c) X is Exponential(2)

$\mathbb{E}[X-1] = \mathbb{E}[X] - 1 = 1/2 - 1 = -1/2$

$\mathbb{E}[(X-1)^2] = \mathbb{E}[X^2 - 2X + 1] = \mathbb{E}[X^2] - 2\mathbb{E}[X] + 1 = 1/2 - 2(1/2) + 1 = 1/2 - 1 + 1 = 1/2$

$\mathbb{E}[X^2] = \mathrm{Var}[X] + (\mathbb{E}[X])^2 = 1/4 + (1/2)^2 = 1/2$

(d) X is Uniform$(-2, 3)$ $\quad f_X(x) = \begin{cases} 1/5, & -2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$

$\mathbb{P}[X - 1 > 0] = \mathbb{P}[X > 1] = \int_1^3 1/5\,dx = 1/5\left[x\big|_1^3\right] = 1/5(3-1) = 2/5$

$\mathbb{P}[X^2-1>0]=\mathbb{P}[X^2>1]=\int_{-2}^{-1}\frac{1}{5}dx+\int_{1}^{3}\frac{1}{5}dx=\frac{1}{5}(-1+2)+\frac{1}{5}(3-1)=\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$

(e) X is Gaussian$(2,1)$; $Y=3X-2$

$\mathbb{E}[Y]=\mathbb{E}[3X-2]=3\mathbb{E}[X]-2=3(2)-2=4$ $\quad$ $\mathrm{Var}[Y]=\mathrm{Var}[3X-2]=9\,\mathrm{Var}[X]=9$

$\mathbb{P}[Y>1]=\mathbb{P}[3X-2>1]=\mathbb{P}[3X>3]=\mathbb{P}[X>1]=F_X(1)=\Phi\left(\frac{1-2}{1}\right)=\Phi(-1)=0.1587$

(f) X is Gaussian$(1,4)$

$\mathbb{P}[X\geq a]\leq 0.1 \Rightarrow a=-1.29$

$\mathbb{P}[X\geq b]=1-\mathbb{P}[X<b]\leq 0.2\Rightarrow b=0.85$

Problem 4.4 (Video 3.1, 3.2, 3.4) Consider the following PDF $f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ ce^{x} & x < 0\,. \end{cases}$

(a) Determine the value of c that satisfies the normalization property. Set c to this value for the remainder of the problem.

(b) What is the expected value of X?

(c) What is the variance of X?

(d) Calculate $\mathbb{P}[X < 1]$.

(e) Calculate $\mathbb{P}[X^2 > 4 | X < 1]$.

(f) Determine the CDF of X.

(a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{0} ce^{x}\,dx + \int_{0}^{\infty} ce^{-x}\,dx = c(e^{0} - e^{-\infty}) + c(-e^{-\infty} - (-e^{-0})) = c + c = 2c \Rightarrow c = 1/2$

(b) $\mathbb{E}[X] = \int_{-\infty}^{0} x \cdot f_X(x)\,dx + \int_{0}^{\infty} x \cdot f_X(x)\,dx = \frac{1}{2}\left[\int_{-\infty}^{0} xe^{x}\,dx + \int_{0}^{\infty} xe^{-x}\,dx\right]$

Integration by Parts $\int xe^{x}\,dx$ $\quad u = x \quad du = dx \quad v = e^{x} \quad dv = e^{x}dx$ $\qquad \int xe^{-x}\,dx$ $\quad u = x \quad du = dx \quad v = -e^{-x} \quad dv = e^{-x}dx$

$= xe^{x} - \int e^{x}\,dx = xe^{x} - e^{x} + C \qquad = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C$

$\Rightarrow \frac{1}{2}\left[xe^{x} - e^{x}\Big|_{-\infty}^{0} + \left(-xe^{-x} - e^{-x}\Big|_{0}^{\infty}\right)\right] = \frac{1}{2}\left[(0 - 1 - (0 - 0)) + (0 - 0 - (0 - 1))\right] = \frac{1}{2}(-1 + 1) = 0$

(c) $\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-\infty}^{0} x^2 \cdot \frac{1}{2}e^{x}\,dx + \int_{0}^{\infty} x^2 \cdot \frac{1}{2}e^{-x}\,dx$

$= 1 + 1 = 2$

(d) $\mathbb{P}[X < 1] = \int_{-\infty}^{0} f_X(x)\,dx + \int_{0}^{1} f_X(x)\,dx = \int_{-\infty}^{0} \frac{1}{2}e^{x}\,dx + \int_{0}^{1} \frac{1}{2}e^{-x}\,dx = \frac{1}{2} + \frac{1 - e^{-1}}{2} = \frac{2 - e^{-1}}{2} = 1 - \frac{1}{2e} \approx 0.316$

(e) $\mathbb{P}[X^2 > 4 | X < 1] = \dfrac{\mathbb{P}[X^2 > 4 \cap X < 1]}{\mathbb{P}[X < 1]} = \dfrac{\int_{-\infty}^{-2} f_X(x)\,dx}{\frac{2 - e^{-1}}{2}} = \dfrac{\frac{1}{2}\left[e^{x}\Big|_{-\infty}^{-2}\right]}{\frac{2 - e^{-1}}{2}} = \dfrac{\frac{1}{2}e^{-2}}{\frac{2 - e^{-1}}{2}} = \dfrac{e^{-2}}{2 - e^{-1}} \approx 0.0829$

(f) CDF of X

$\int_{-\infty}^{x} \frac{1}{2}e^{x}\,dx = \frac{1}{2}\left[e^{x}\Big|_{-\infty}^{x}\right] = \frac{1}{2}(e^{x} - 0) = \frac{1}{2}e^{x}$

$\int_{0}^{x} \frac{1}{2}e^{-x}\,dx = \frac{1}{2}\left[-e^{-x}\Big|_{0}^{x}\right] = \frac{1}{2}\left[-(e^{-x} - 1)\right] = -1/2\, e^{-x}$

$F_X(x) = \begin{cases} \frac{1}{2}e^{x} & , x < 0 \\ \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} & , x \geq 0 \end{cases}$

Problem 4.5 (Video 3.3, 3.4)

Let X be a continuous random variable representing the (exact) lifetime of your TV set, measured in years. A simple model for X is that is an Exponential(λ) random variable. You may assume that your brand of TV has an average lifetime of 20 years. $\mathbb{E}[X] = 20 = 1/\lambda \to \lambda = 1/20$

(a) What is the probability that the TV fails in the first year?

(b) What is the probability that it lasts more than 5 years?

(c) Consider the event $B = \{X \geq 10\}$ that your TV has already lasted 10 years. What is the conditional PDF $f_{X|B}(x)$?

(d) Let $Y = X - 10$. What is the conditional probability of Y given $B = \{X \geq 10\}$? You can get this by simply transforming $f_{X|B}(x)$ as $f_{Y|B}(y) = f_{X|B}(y + 10)$.

(e) Assume your TV has already lasted 10 years. What is the probability that it fails during the next year?

X is Exponential($1/20$)

(a) $\mathbb{P}[X \leq 1] = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{20} e^{-\frac{1}{20}x}\,dx$ let $u = -\frac{1}{20}x$, $du = -\frac{1}{20}dx$ $\Rightarrow \int_0^1 \frac{1}{20} e^{u} \cdot -20\,du = -1\left[e^{-\frac{1}{20}x}\Big|_0^1\right]$

$= -\left[e^{-1/20} - 1\right] = 1 - e^{-1/20}$

(b) $\mathbb{P}[X > 5] = 1 - \mathbb{P}[X \leq 5] = 1 - \left(-\left[e^{-5/20} - e^{0}\right]\right) = 1 - \left(1 - e^{-1/4}\right) = e^{-1/4}$

(c) $B = \{X \geq 10\}$ find PDF $f_{X|B}(x)$

$\mathbb{P}[X \in B] = 1 - \mathbb{P}[X < 10] = 1 - \left(-\left[e^{-10/20} - 1\right]\right) = 1 - \left(1 - e^{-1/2}\right) = e^{-1/2}$

$$f_{X|B}(x) = \frac{f_X(x)}{\mathbb{P}[X \in B]} = \begin{cases} \frac{\frac{1}{20} e^{-\frac{1}{20}x}}{e^{-1/2}}, & x \in B \\ 0, & x \notin B \end{cases} = \begin{cases} \frac{1}{20} e^{-\frac{1}{20}x + \frac{1}{2}}, & x \in B \\ 0, & x \notin B \end{cases}$$

(d) $$f_{Y|B}(y) = f_{X|B}(y+10) = \begin{cases} \frac{1}{20} e^{-\frac{1}{20}(y+10) + \frac{1}{2}}, & y+10 \in B \\ 0, & y+10 \notin B \end{cases} = \begin{cases} \frac{1}{20} e^{-\frac{1}{20}y}, & y+10 \in B \\ 0, & y+10 \notin B \end{cases}$$

(e) $\mathbb{P}[X \geq 11 \mid X \geq 10] = \mathbb{P}[10 \leq X \leq 11] = \mathbb{P}[Y \leq 1]$

$= \int_0^1 f_{Y|B}(y)\,dy = \int_0^1 f_{X|B}(y+10)\,dy = \int_0^1 \frac{1}{20} e^{-\frac{1}{20}y}\,dy$ let $u = -\frac{1}{20}y$, $du = -\frac{1}{20}dy$ $\Rightarrow \int_0^1 \frac{1}{20} e^{u} \cdot -20\,du$

$= -\left[e^{-\frac{1}{20}y}\Big|_0^1\right] = -\left[e^{-1/20} - e^{0}\right] = 1 - e^{-1/20}$

Problem 4.6

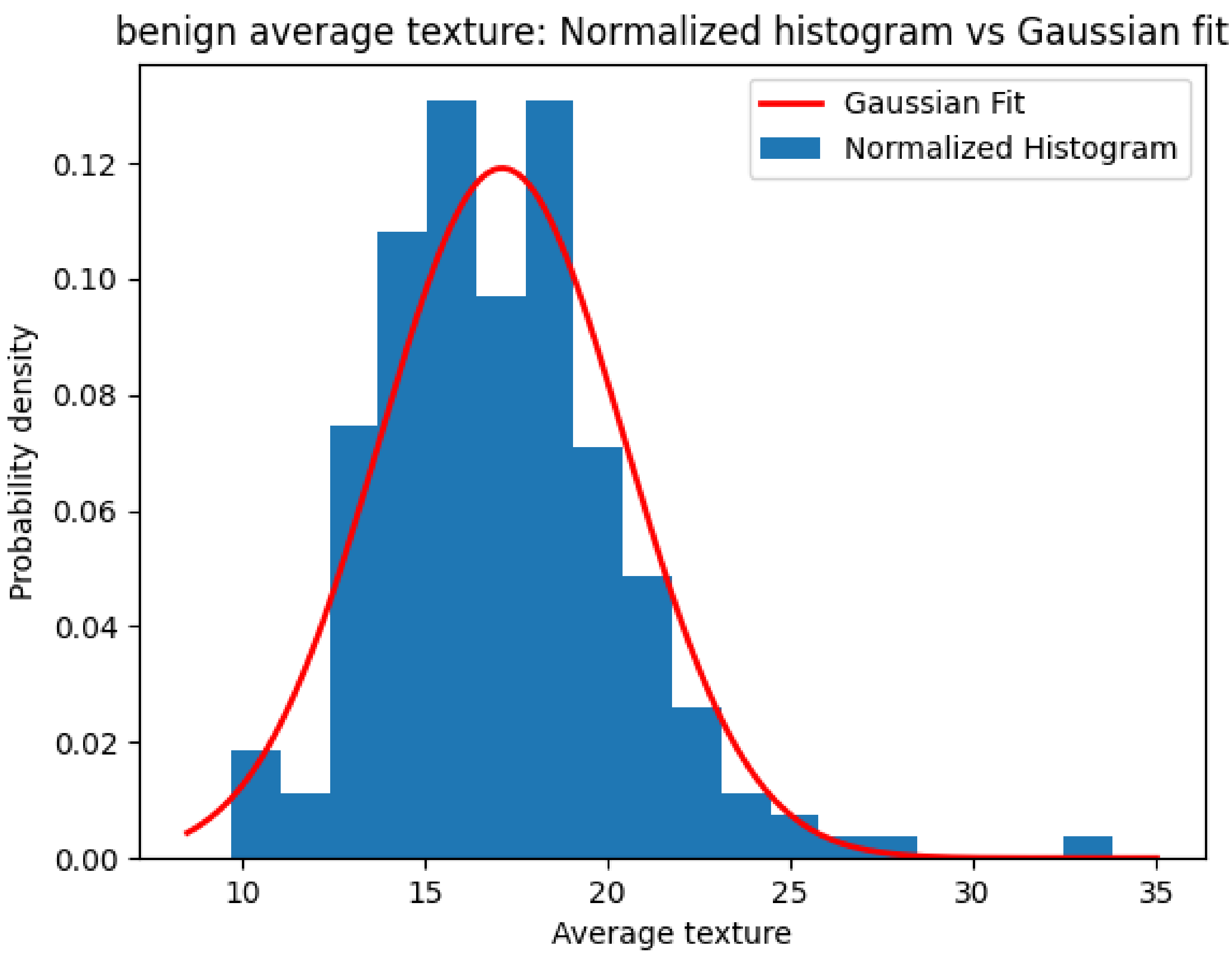
Part (a). Means

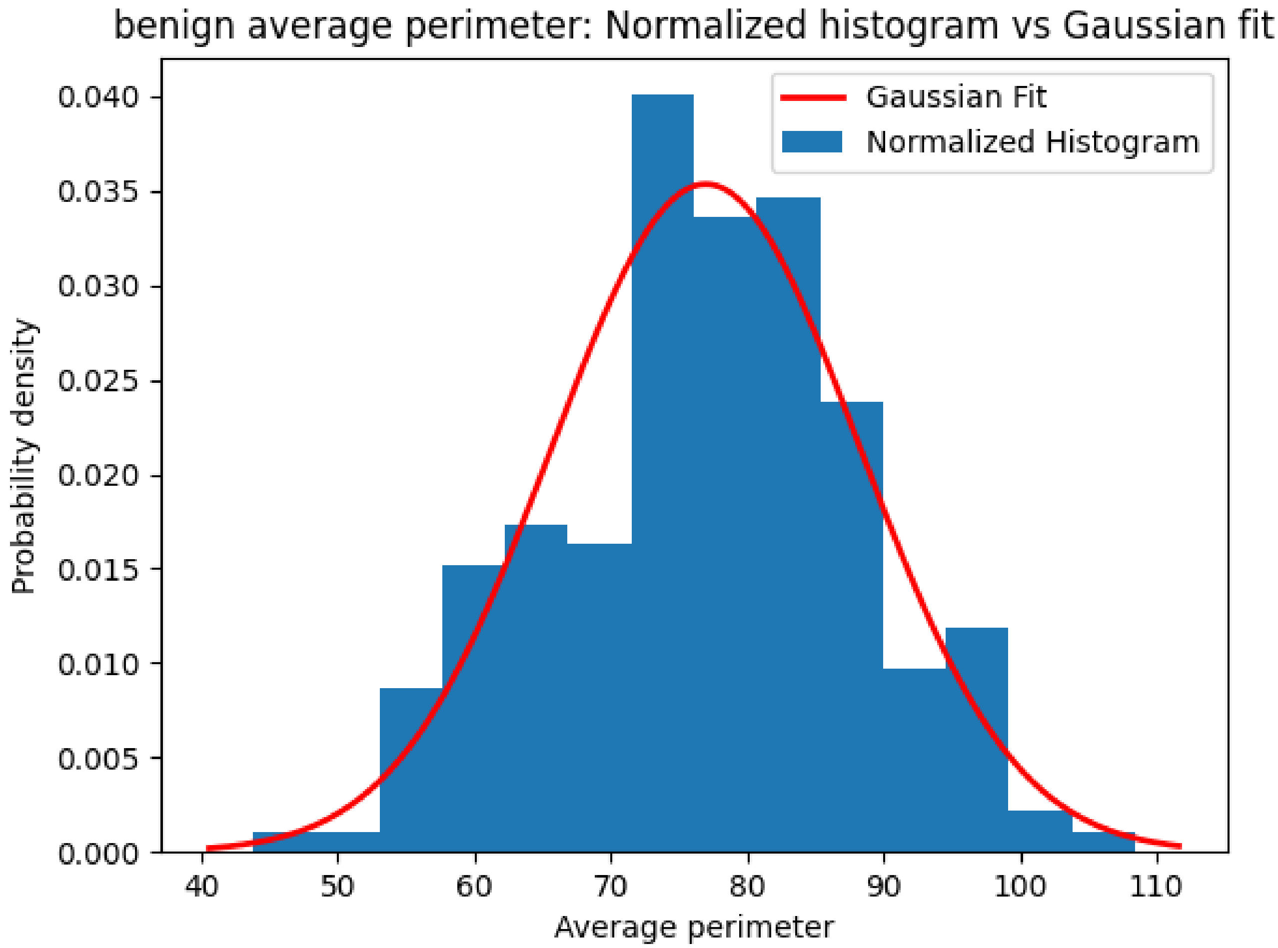
	Texture	Perimeter
Benign	[17.1157	76.96375]
Malignant	[21.4498	114.53195]

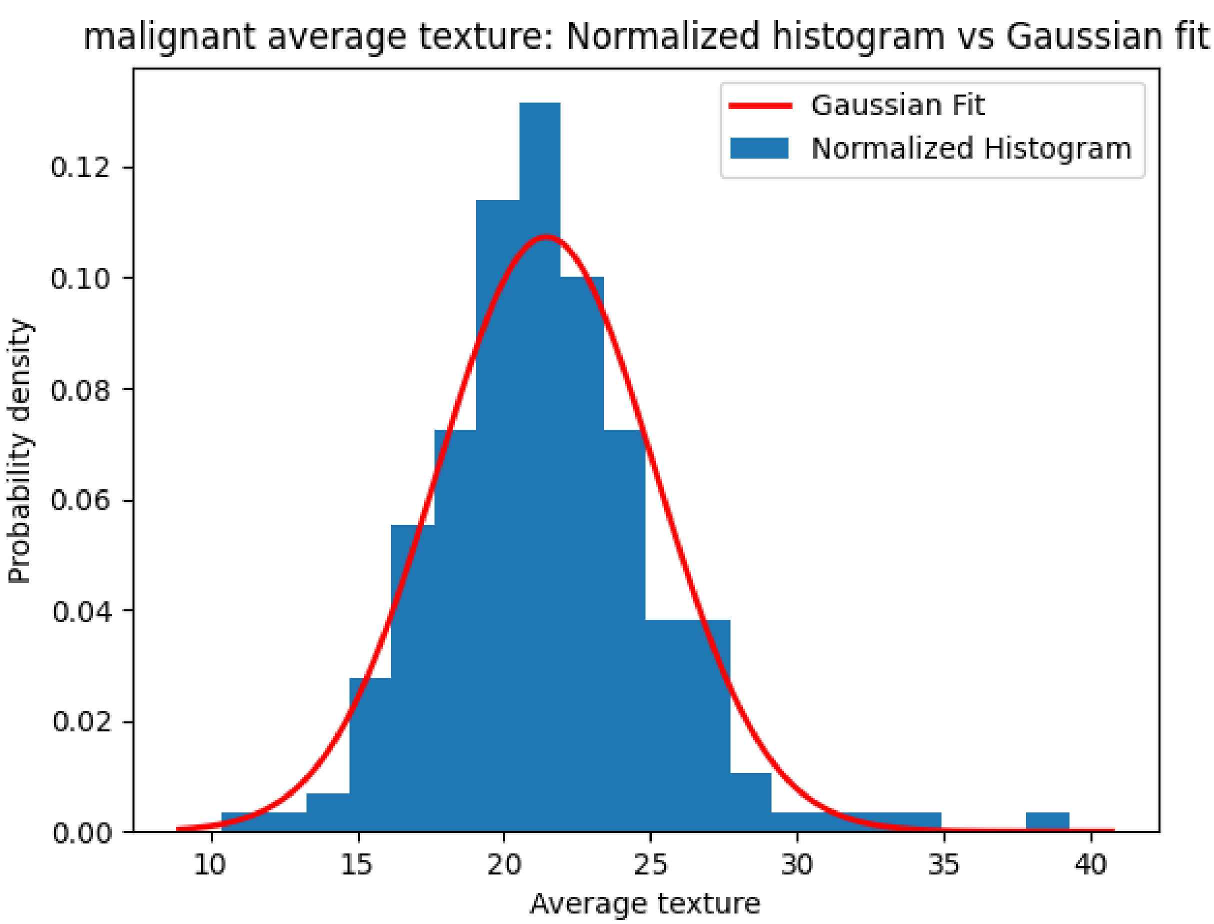
Part (b). Variance

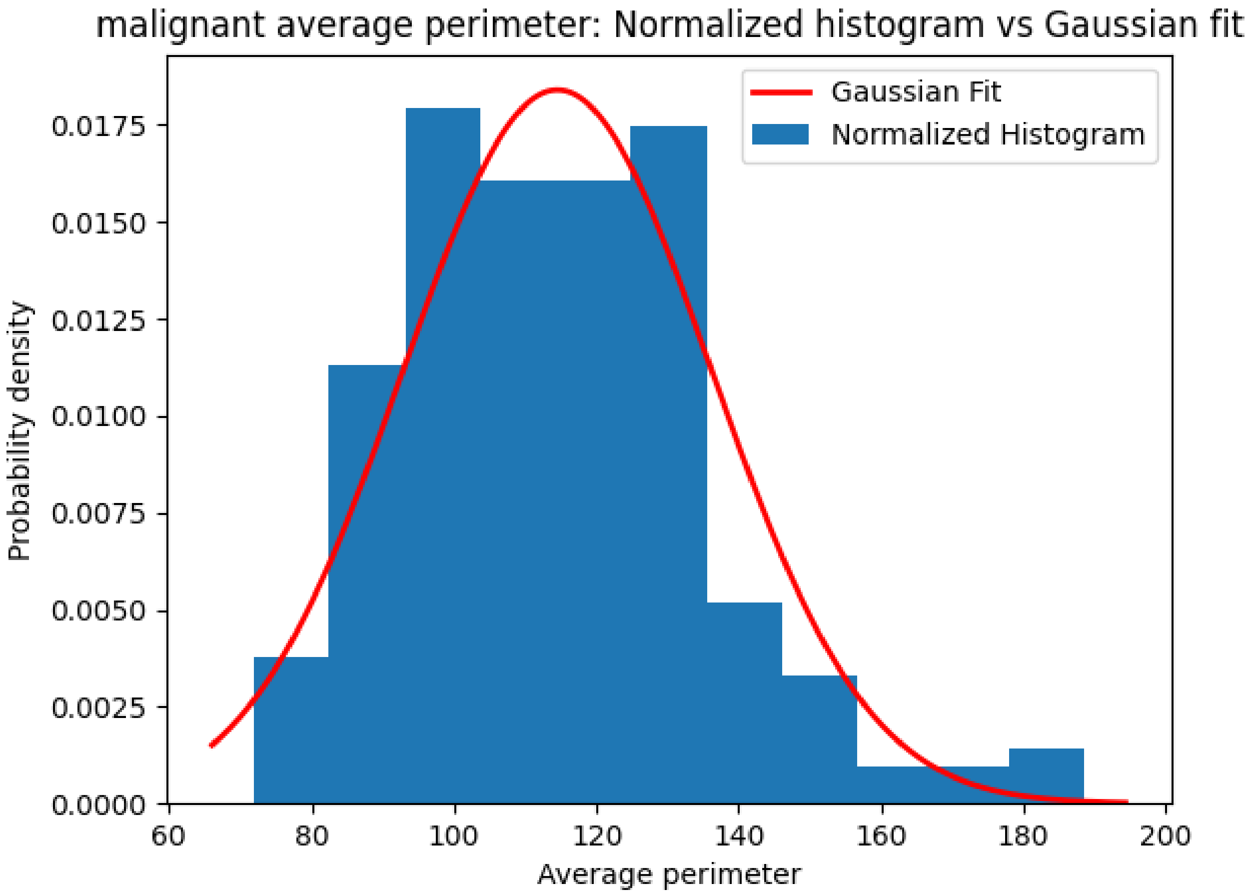
	Texture	Perimeter
Benign	[11.27652514	127.96211401]
Malignant	[13.89054569	472.34102382]

Part (c). Four Plots

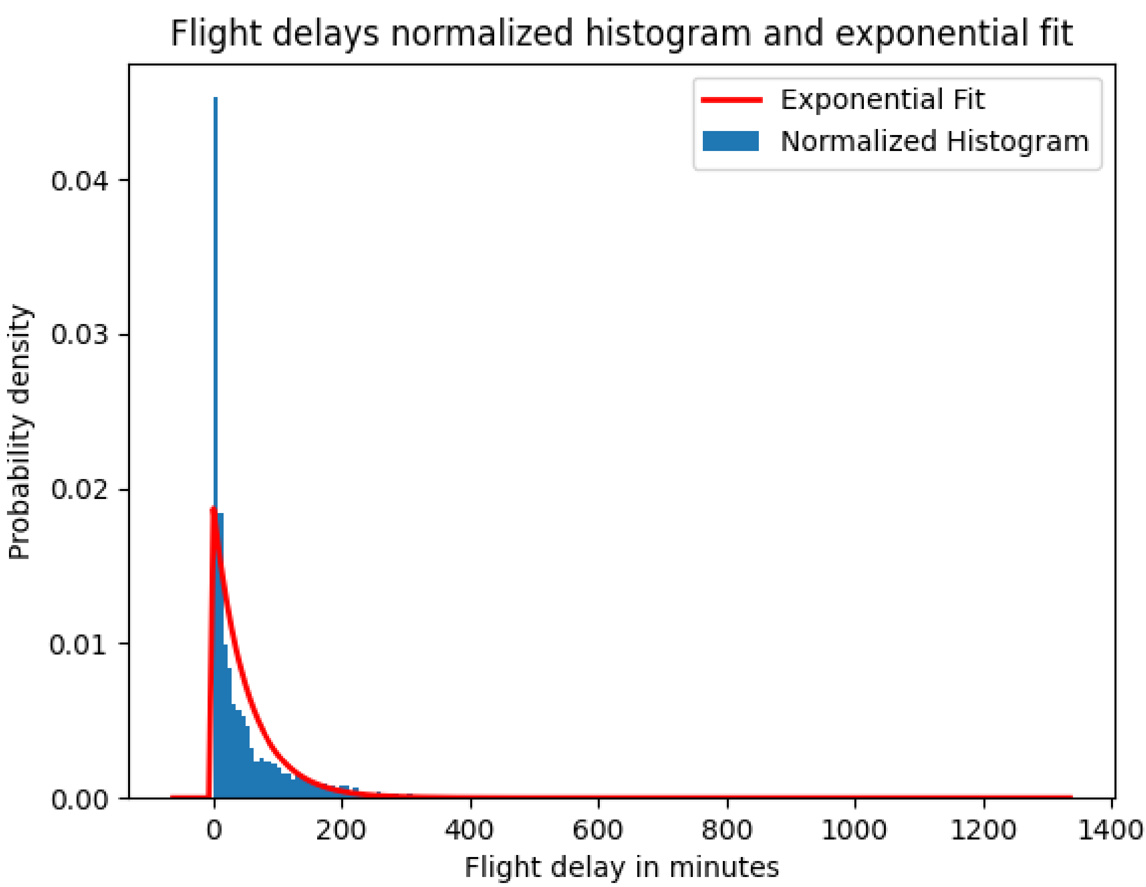



benign average perimeter: Normalized histogram vs Gaussian fit
Gaussian Fit
Normalized Histogram
Probability density
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
40
50
60
70
80
90
100
110
Average perimeter


malignant average texture: Normalized histogram vs Gaussian fit
Gaussian Fit
Normalized Histogram
Probability density
0.00
0.02
0.04
0.06
0.08
0.10
0.12
10
15
20
25
30
35
40
Average texture



Part (d). Exponential histogram



Part (e). Probability, fraction, comment

Probability[Z > 60] = 0.32109077601786185

Fraction of flights delayed by more than 60 mins = 1205/4611 ~ 0.261

Comment: This is an ok approximation – the percent error is about 22.8%, which is not very ideal, but I suppose it's not outrageous. The plot also generally fits the data, so it is not a terrible approximation.