

Jilin Zheng / 1149258796

EK381 Homework #9

Problem 1.

$$(a) E[T] = \sum_{i=1}^n E[T_i] = \sum_{i=1}^{100} E[T_i] = \sum_{i=1}^{100} 3 = 100 \cdot 3 = 300$$

T_i is an uniformly distributed random var. $\rightarrow f_T(t) = \begin{cases} \frac{1}{5-1}, & 1 \leq t \leq 5 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{4}, & 1 \leq t \leq 5 \\ 0, & \text{otherwise} \end{cases}$

$$E[T_i] = \frac{1+5}{2} = 3$$

$$(b) \text{Var}[T] = n \text{Var}[T_i] = 100 \cdot \frac{4}{3} = \frac{400}{3}$$

$$\text{Var}[T_i] = \frac{(5-1)^2}{12} = \frac{16}{12} = \frac{4}{3}$$

$$(c) P[T > 330] = 1 - P[T \leq 330] = 1 - \Phi\left(\frac{330-300}{\sqrt{\frac{400}{3}}}\right) = 1 - \Phi(2.598) \approx 0.0046874$$

$$T \rightarrow \text{Gaussian}(100 \cdot 3, 100 \cdot \frac{4}{3}) = \text{Gaussian}(300, \frac{400}{3})$$

Problem 2.

$$(a) \text{ let } Y_i \text{ be Discrete Uniform } (1, 3) \rightarrow E[Y_i] = \frac{1+3}{2} = 2, \text{Var}[Y_i] = \frac{(3-1)^2 - 1}{12} = \frac{8}{12} = \frac{2}{3}$$

$$\lim_{n \rightarrow \infty} F_{Y_n}(y) = \Phi(y) \quad \text{let } Y_n = \frac{\sum_{i=1}^n (Y_i - 2)}{\sigma \sqrt{n}} \rightarrow \text{Gaussian}(nE[Y_i], n\text{Var}[Y_i]) = \text{Gaussian}(36 \cdot 2, 36 \cdot \frac{2}{3}) = \text{Gaussian}(72, 24)$$

$$P[Y \leq 60] = \Phi\left(\frac{60-72}{\sqrt{24}}\right) \approx 0.0071529$$

$$(b) \lim_{n \rightarrow \infty} P[Y_n = 2n] = 1$$

Yes because as $n \rightarrow \infty$, Y_n becomes Gaussian($nE[Y_i], n\text{Var}[Y_i]$), where $E[Y_n] = nE[Y_i] = 2n$

Problem 3.

$$(a) \hat{\kappa}_{LSE} = E[X] + \frac{\text{Cov}[X, Y]}{\text{Var}[Y]} (y - E[Y]) = 2 + \frac{1/3}{4/3} (y - 3) = 2 + \frac{1}{4} (y - 3) = \frac{1}{4} y + \frac{5}{4}$$

$$E[X] = \frac{1+3}{2} = 2 \quad \text{Var}[X] = \frac{(3-1)^2}{12} = \frac{1}{3}$$

$$E[Y] = E[X+U] = E[X] + E[U] = 2 + 1 = 3$$

$$\text{Var}[Y] = \text{Var}[X+U] = \text{Var}[X] + \text{Var}[U] + 2\text{Cov}[X, U] = \frac{1}{3} + 1 = \frac{4}{3}$$

$\text{Cov}[X, U] = 0$ since X and U are independent

$$\text{Cov}[X, Y] = \text{Cov}[X, X+U] = \text{Var}[X] + 0 + (1+0) \text{Cov}[X, U] = \text{Var}[X] = \frac{1}{3}$$

$$\text{MSE}_{LSE} = \text{Var}[X] - \frac{(\text{Cov}[X, Y])^2}{\text{Var}[Y]} = \frac{1}{3} - \frac{(\frac{1}{3})^2}{\frac{4}{3}} = \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{3} - \frac{1}{12} = \frac{4}{12} - \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$$

$$(b) E[Y] = 0 \quad \text{Var}[Y] = \frac{\text{Var}[X]}{400} = \frac{(\frac{1}{3})^2}{400} = \frac{1}{400} = 0.0025$$

$$(c) P[Y \leq 1.9] \approx \Phi\left(\frac{1.9-2}{\sqrt{\frac{1}{400}}}\right) = \Phi(-0.05) \approx 0.480$$

$$E[\text{Exponential}(\frac{1}{2})] = \frac{1}{1/2} = 2 \quad \text{Var}[\text{Exponential}(\frac{1}{2})] = (\frac{1}{1/2})^2 = \frac{1}{1/4} = 4$$

Problem 4.

$$(a) E[S_{300}] = 300 E[X] = 300 \cdot (2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2}) = 300(1+2) = 900$$

$$E[X^2] = 2^2 \cdot \frac{1}{2} + 4^2 \cdot \frac{1}{2} = 2 + 8 = 10$$

$$\text{Var}[X] = E[X^2] - (E[X])^2 = 10 - 9 = 1$$

$$\text{Var}[S_{300}] = 300 \text{Var}[X] = 300(1) = 300$$

$$(b) P[|S_{300} - E[S_{300}]| \geq 40] = 1 - P[|S_{300} - E[S_{300}]| < 40]$$

$$= 1 - P[-40 < S_{300} - E[S_{300}] < 40]$$

$$= 1 - P[860 < S_{300} < 940]$$

$$= 1 - \left[\Phi\left(\frac{940-900}{\sqrt{300}}\right) - \Phi\left(\frac{860-900}{\sqrt{300}}\right) \right]$$

$$= 1 - \left[\Phi\left(\frac{4}{\sqrt{3}}\right) - \Phi\left(-\frac{4}{\sqrt{3}}\right) \right]$$

$$(c) P[\mu - \epsilon \leq \mu_n \leq \mu + \epsilon] = 1 - \alpha = 0.95 \rightarrow \alpha = 0.05$$

$$\epsilon = \frac{\sigma Q^{-1}\left(\frac{\alpha}{2}\right)}{\sqrt{n}} = \frac{1 \cdot Q^{-1}(0.025)}{\sqrt{300}} = \frac{1.96}{\sqrt{300}} = 0.113$$

$$[\mu_{300} \pm 0.113]$$

$$(d) V_{300} = 1.21$$

$$\epsilon = \frac{-\sqrt{V_{300}} F_{T_{299}}^{-1}(0.025)}{\sqrt{300}} = \frac{-\sqrt{1.21} (-1.9679)}{\sqrt{300}} = 0.125$$

$$[\mu_{300} \pm 0.125]$$

Problem 5.

$$(a) X \text{ is Gaussian}(5.02 \mu\text{m}, 0.25 \mu\text{m}^2)$$

$$\text{Var}[\mu_{100}] = \frac{\text{Var}[X]}{n} = \frac{0.25}{100} = 0.0025 \mu\text{m}^2$$

$$(b) \mu_{100} = 5.10 \mu\text{m}, \alpha = 0.05$$

$$\epsilon = \frac{\sigma Q^{-1}\left(\frac{\alpha}{2}\right)}{\sqrt{n}} = \frac{0.50 \cdot Q^{-1}(0.025)}{\sqrt{100}} = \frac{0.5 \cdot 1.96}{10} = 0.098$$

$$[5.1 \pm 0.098] = [5.002, 5.198]$$

No the sample is NOT significantly different @ a significance level of 0.05 because the baseline is within the confidence interval.

$$(c) 1 - \alpha = 0.9 \rightarrow \alpha = 0.1$$

$$\epsilon = \frac{\sigma Q^{-1}\left(\frac{\alpha}{2}\right)}{\sqrt{n}} = \frac{0.5 Q^{-1}(0.05)}{10} = 0.082$$

$$[5.10 \mu\text{m} \pm 0.082]$$