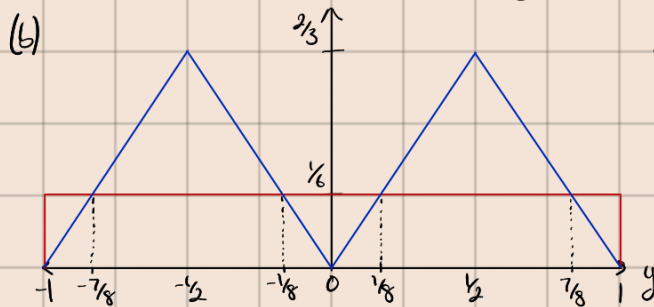


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EK381 Homework #8

Problem 8.1

$$(a) D^{ML}(y) = \begin{cases} 1, & -1 \leq y \leq -\frac{3}{4} \text{ or } -\frac{1}{4} \leq y \leq \frac{1}{4} \text{ or } \frac{3}{4} \leq y \leq 1 \\ 0, & -\frac{3}{4} < y < -\frac{1}{4} \text{ or } \frac{1}{4} < y < \frac{3}{4} \end{cases}$$



$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$1 \cdot \frac{2}{3} = \frac{2}{3}$$

$$f(y) = my + b \quad -\frac{2}{3} \cdot \frac{2}{3} = -\frac{4}{9}$$

$$\frac{1}{6} = -\frac{4}{3}y + \frac{4}{3}$$

$$y = \left(\frac{1}{6} - \frac{4}{3}\right) \cdot \frac{-3}{4} = \frac{-7}{4} \cdot \frac{-3}{4} = \frac{7}{8}$$

$$\frac{1}{6} = \frac{4}{3}y + 0$$

$$y = \frac{1}{6} \cdot \frac{3}{4} = \frac{1}{8}$$

$$D^{MAP}(y) = \begin{cases} 1, & -1 \leq y \leq -\frac{7}{8} \text{ or } -\frac{1}{8} \leq y \leq \frac{1}{8} \text{ or } \frac{7}{8} \leq y \leq 1 \\ 0, & -\frac{7}{8} < y < -\frac{1}{8} \text{ or } \frac{1}{8} < y < \frac{7}{8} \end{cases}$$

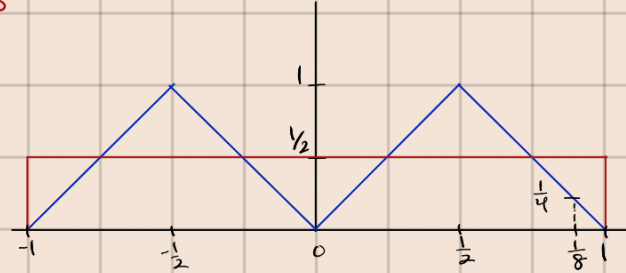
$$(c) P_{FA} = P[Y \in A_0 | H_0] \\ = \frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \cdot 4 \\ = \frac{1}{4}$$

$$P_{MD} = P[Y \in A_0 | H_1] \\ = \left[\frac{1}{2} \left(\frac{1}{2}\right)\right] 2 \\ = \frac{1}{2}$$

$$P_e = P_{FA} P[H_0] + P_{MD} P[H_1] = \frac{1}{4} \left(\frac{2}{3}\right) + \frac{1}{2} \left(\frac{1}{3}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$(d) P_{FA} = P[Y \in A_0 | H_0] \\ = \frac{1}{8} \left(\frac{1}{4}\right) \left(\frac{1}{2}\right) \cdot 4 \\ = \frac{1}{16}$$

$$P_{MD} = P[Y \in A_0 | H_1] \\ = \frac{6}{8} \left(\frac{1}{2}\right) \cdot 2 \\ = \frac{3}{4}$$



$$P_e = P_{FA} P[H_0] + P_{MD} P[H_1] \\ = \frac{1}{16} \cdot \frac{2}{3} + \frac{3}{4} \cdot \frac{1}{3} = \frac{7}{24}$$

Problem 8.2

$$(a) p^{ML} = \begin{cases} 1, & 0 \leq y < \infty \\ 0, & -\infty < y < 0 \end{cases}$$

$$p^{MAP} = \begin{cases} 1, & -0.347 \leq y < \infty \\ 0, & -\infty < y < -0.347 \end{cases}$$

$$(b) \text{Exponential: } f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\text{Likelihood Ratio} = L(y) = \frac{P_{Y|H_1}(y)}{P_{Y|H_0}(y)} = \frac{2e^{-2y}}{1e^{-y}} = \frac{e^y}{2e^y} = \frac{1}{2e^y}$$

$$p^{ML} = \begin{cases} 1, & L(y) \geq 1 \\ 0, & L(y) < 1 \end{cases} = \begin{cases} 1, & \frac{1}{2e^y} \geq 1 \\ 0, & \frac{1}{2e^y} < 1 \end{cases}$$

$$P_{FA} = P[Y \in A_1 | H_0] = \int_{0.693}^{\infty} 2e^{-2y} dy = \frac{1}{4}$$

$$P_{MD} = P[Y \in A_0 | H_1] = \int_0^{0.693} e^{-y} dy = \frac{1}{2}$$

$$P_e = P_{FA} \left(\frac{1}{2}\right) + P_{MD} \left(\frac{1}{2}\right) = \frac{1}{4} \left(\frac{1}{2}\right) + \frac{1}{2} \left(\frac{1}{2}\right) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

$$(c) \text{Likelihood Ratio} = L(y) = \frac{\binom{3}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{3-y}}{\binom{4}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{4-y}} = \frac{\binom{3}{y} \left(\frac{1}{2}\right)^3}{\binom{4}{y} \left(\frac{1}{2}\right)^4} = \frac{\frac{1}{8} \binom{3}{y}}{\frac{1}{16} \binom{4}{y}} = 2 \frac{\binom{3}{y}}{\binom{4}{y}}$$

$$\frac{P[H_0]}{P[H_1]} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

$$p^{ML} = \begin{cases} 1, & 2 \frac{\binom{3}{y}}{\binom{4}{y}} \geq 1 \\ 0, & 2 \frac{\binom{3}{y}}{\binom{4}{y}} < 1 \end{cases} \quad p^{MAP} = \begin{cases} 1, & 2 \frac{\binom{3}{y}}{\binom{4}{y}} \geq 2 \\ 0, & 2 \frac{\binom{3}{y}}{\binom{4}{y}} < 2 \end{cases}$$

$$P_e = P_{FA} \left(\frac{2}{3}\right) + P_{MD} \left(\frac{1}{3}\right) = 0.3125 \text{ or } ML = \frac{5}{16}$$

$$P_{FA} = P[Y \in A_1 | H_0] = 0.125$$

$$P_{MD} = 0.0625 + 0.25 + 0.375 = 0.6875$$

$$P_e = P_{FA} \left(\frac{2}{3}\right) + P_{MD} \left(\frac{1}{3}\right)$$

$$P_{FA} = 0.042 + 0.125 = \frac{7}{24}$$

$$P_{MD} = 0.042 = \frac{1}{24}$$

$$P_e = \frac{7}{24} \left(\frac{2}{3}\right) + \frac{1}{24} \left(\frac{1}{3}\right)$$

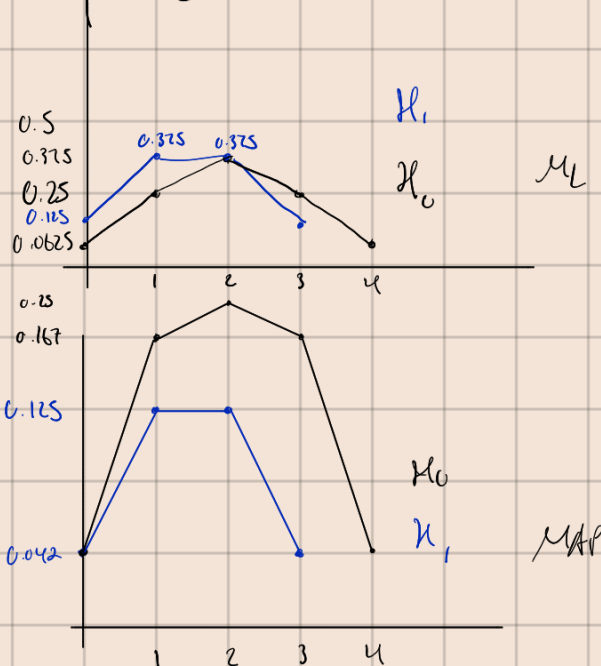
$$= \frac{7}{36} + \frac{1}{72} = \frac{15}{72} = \frac{5}{24}$$

or MAP

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, x=0, \dots, n$$

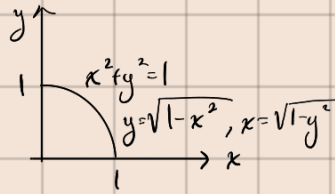
$$P_{Y|H_0}(y) = \text{Binomial}(4, \frac{1}{2}) = \binom{4}{y} \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{4-y} = \binom{4}{y} \frac{1}{16}$$

$$P_{Y|H_1}(y) = \text{Binomial}(3, \frac{1}{2})$$



Problem 8.3

$$f_{X,Y}(x,y) = \begin{cases} \frac{4}{\pi}, & x \geq 0, y \geq 0, x^2 + y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$



$$(a) \hat{x}_{\text{MMSE}}(y) = E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

$$f_Y(y) = \int_0^{\sqrt{1-y^2}} \frac{4}{\pi} dx = \frac{4}{\pi} \sqrt{1-y^2}, \quad 0 \leq y \leq 1$$

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{\sqrt{1-y^2}}, & 0 \leq x \leq \sqrt{1-y^2}, \quad 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\rightarrow \int_0^{\sqrt{1-y^2}} x \frac{4}{\pi \sqrt{1-y^2}} dx = \frac{1}{\sqrt{1-y^2}} \left[\frac{x^2}{2} \right]_0^{\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}} (1-y^2) = \frac{\sqrt{1-y^2}}{2}$$

$$(b) E[(X - \hat{x}_{\text{MMSE}}(Y))^2] = E[X^2] - E[\hat{x}_{\text{MMSE}}^2(Y)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \hat{x}_{\text{MMSE}}(y))^2 f_{X,Y}(x,y) dx dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \left(x - \frac{\sqrt{1-y^2}}{2}\right)^2 \frac{4}{\pi} dx dy = \frac{1}{16}$$

$$(c) \hat{x}_{\text{LSE}}(y) = E[X|Y] + \frac{\text{Cov}[X,Y]}{\text{Var}[Y]} (y - E[Y])$$

$$f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{4}{\pi} dy = \frac{4}{\pi} \sqrt{1-x^2}, \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_0^{\sqrt{1-y^2}} \frac{4}{\pi} dx = \frac{4}{\pi} \sqrt{1-y^2}, \quad 0 \leq y \leq 1$$

$$E[X] = \frac{4}{3\pi}$$

$$E[Y] = \int_0^1 y \left(\frac{4}{\pi} \sqrt{1-y^2}\right) dy = \frac{4}{3\pi}$$

$$E[XY] = \int_0^1 \int_0^{\sqrt{1-y^2}} xy \left(\frac{4}{\pi}\right) dx dy = \frac{4}{3\pi} \cdot \frac{1}{2}$$

$$E[Y^2] = \int_0^1 y^2 \left(\frac{4}{\pi} \sqrt{1-y^2}\right) dy = \frac{1}{4}$$

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{4}{3\pi} \cdot \frac{1}{2} - \left(\frac{4}{3\pi}\right)^2$$

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = \frac{1}{4} - \left(\frac{4}{3\pi}\right)^2 = \frac{1}{4} - \frac{16}{9\pi^2} \approx 0.0699$$

$$\rightarrow \frac{4}{3\pi} + \frac{y \cdot \frac{4}{3\pi} - \frac{16}{9\pi^2}}{\frac{1}{4} - \frac{16}{9\pi^2}} \left(y - \frac{4}{3\pi}\right)$$

$$(d) \text{MSE}_{\text{LSE}} = \text{Var}[X] - \frac{(\text{Cov}[X,Y])^2}{\text{Var}[Y]}$$

$$\left(E[X^2] - E[Y^2] = \frac{1}{4}, \text{Var}[X] - \text{Var}[Y] = \frac{1}{4} - \frac{16}{9\pi^2} \right)$$

$$\rightarrow \frac{1}{4} - \frac{16}{9\pi^2} - \frac{\left(y \cdot \frac{4}{3\pi} - \left(\frac{4}{3\pi}\right)^2\right)^2}{\left(\frac{1}{4} - \frac{16}{9\pi^2}\right)}$$

Problem 8.4

$$(a) \hat{X}_{MSE}(y) = E[X|Y=y]$$

$$= \mu_X + \frac{\text{Cov}[X, Y]}{\text{Var}[Y]} (y - \mu_Y) = 0 + \frac{1}{2} (y - 0) = \frac{1}{2} y$$

$$(b) \hat{Z}_{MSE}(u) = E[Z] + \frac{\text{Cov}[u, Z]}{\text{Var}[Z]} (u - E[u]) = \frac{1}{5} \left(\frac{21}{10} \right) u = \frac{21}{50} u$$

$$E[Z] = E[U^3 + V] = E[U^3] + E[V] = 0$$

$$\text{Var}[Z] = E[Z^2] - (E[Z])^2 = \frac{10}{21} - 0 = \frac{10}{21}$$

$$E[U^3] = \int_{-1}^1 u^3 \cdot \frac{1}{2} du = \frac{1}{2} \left[\frac{u^4}{4} \right]_{-1}^1 = \frac{1}{8} (1 - 1) = 0$$

$$E[Z^2] = E[U^6 + 2U^3V + V^2] = \frac{1}{7} + \frac{1}{3} = \frac{2}{7} + \frac{2}{21} = \frac{10}{21}$$

$$\text{Cov}[u, Z] = \text{Cov}[u, U^3 + V] = \frac{1}{5}$$

$$E[U^6] = \int_{-1}^1 u^6 \cdot \frac{1}{2} du = \frac{1}{2} \left[\frac{u^7}{7} \right]_{-1}^1 = \frac{1}{14} (1 - (-1)) = \frac{1}{7}$$

$$= E[U^6 + UV] - E[U]E[U^3 + V]$$

$$2E[U^3V] = \int_{-1}^1 \int_{-1}^1 u^3 v \cdot \frac{1}{2} \cdot \frac{1}{2} du dv = 0$$

$$E[U^4] = \int_{-1}^1 u^4 \cdot \frac{1}{2} du = \frac{1}{2} \left[\frac{u^5}{5} \right]_{-1}^1 = \frac{1}{10} (2) = \frac{1}{5}$$

$$\int_{-1}^1 u^3 v \cdot \frac{1}{4} du = \frac{1}{4} v \left[\frac{u^4}{4} \right]_{-1}^1 = \frac{1}{16} v (1 - 1) = 0$$

$$E[V] = \int_{-1}^1 v^2 \cdot \frac{1}{2} dv = \frac{1}{2} \left[\frac{v^3}{3} \right]_{-1}^1 = \frac{1}{6} (1 - (-1)) = \frac{1}{3}$$

$$(c) MSE_{MSE} = \text{Var}[Z] - \frac{(\text{Cov}[Z, U])^2}{\text{Var}[U]}$$

$$\text{Var}[U] = E[U^2] - (E[U])^2 = \frac{1}{3} - 0 = \frac{1}{3}$$

$$E[U^2] = \int_{-1}^1 u^2 \cdot \frac{1}{2} du = \frac{1}{2} \left[\frac{u^3}{3} \right]_{-1}^1 = \frac{1}{6} (2) = \frac{1}{3}$$

$$= \frac{10}{21} - \frac{\frac{1}{25}}{\frac{1}{3}} = \frac{187}{525} \approx 0.356 = MSE_{MSE}$$

Problem 8.5

(a) 0.068

(b) 0.05

(c) 0.012

(d) 0.385

(e) 0.085

(f) 0.065

(g) 0.015