The surprizing complexity of generalized reachability games

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Abstract. Games on graphs with ω-regular objectives provide a natural model for reactive systems. In this talk, we consider generalized reachability games: given subsets $F_1, \ldots, F_k \subseteq V$ of vertices, the objective is to reach one vertex of each F_i. We show that solving generalized reachability games is PSPACE-complete. However, if reachability sets are singletons then we show that the problem is polynomial. If reachability sets have size 2, the problem is still open. We also investigate memory requirements of both players.

Introduction. Graphs games are used to model reactive systems: a finite graph, whose vertices represent the states and edges represent the transitions of the system, models it and its interactions with the environment. If in a given state, the controller can choose the evolution of the system, then the corresponding vertex is controlled by the first player, Eve. Otherwise, the system evolves in an uncertain way: we consider the worst-case scenario, where the second player, Adam, controls those states. A pebble is initially placed on a vertex, representing the initial state of the system. Then Eve and Adam move this pebble along the edges: this constructs an infinite sequence of vertices. Eve tries to ensure that the ensued sequence satisfies some predetermined objective. Hence, in order to synthesize a controller, we are interested in whether Eve can ensure this objective, and what resources are needed to achieve it (see [GTW02] for more details).

Reachability and generalized reachability. One of the simplest objective is reachability: given a subset $F \subseteq V$ of vertices, the objective is to reach a vertex from F. There is an algorithm to solve reachability games in linear time. An extension of reachability is Büchi objective: it requires that a given subset $F \subseteq V$ of vertices is visited infinitely many times. Generalized Büchi objectives are conjunctions of Büchi objectives. To our knowledge, conjunction of reachability objectives have not yet been studied. Generalized reachability objectives require that given subsets $F_1, \ldots, F_k \subseteq V$ of vertices, one vertex of each F_i is visited.

Solving generalized reachability. We study the complexity of solving generalized reachability games: we prove that the problem is PSPACE-complete (see Figure 1 for an idea of the reduction from QBF). Using the same ideas, we also show that the 1-player restriction, where all vertices belong to Eve, is NP-complete (the other 1-player restriction, where all vertices belong to Adam can be solved in polynomial time). Our proof of PSPACE-hardness is correct as long as the size of reachability sets are greater or equal than 3: actually, if for all i, F_i is a singleton, then the problem of solving these games is polynomial. The case where the size of reachability sets are 2 is still open.

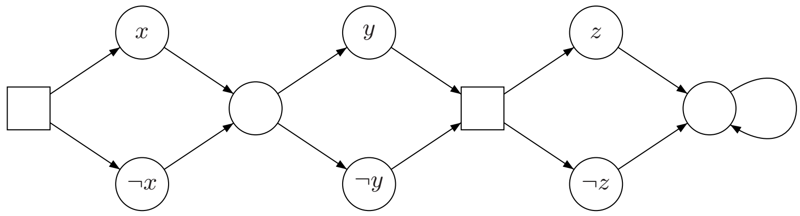

Fig. 1. Reduction from QBF to generalized reachability games, for the formula $\forall x\ \exists y\ \forall z\ (x \vee \neg y) \wedge (\neg y \vee z)$, $F_1 = \{x, \neg y\}$ and $F_2 = \{\neg y, z\}$. Circles are owned by Eve and squares by Adam.

Memory requirements. We study the memory requirements of both players. An upper-bound for both players is $2^k - 1$. We strengthen this result to get an upper-bound of $\binom{\lfloor k/2 \rfloor}{k}$ for Adam. In the case where each reachability set has size at least 2, we show that Eve requires memory $2^{\lfloor k/2 \rfloor}$. If each reachability set has size 1, then Eve needs k bits of memory and Adam only 2 bits.

References

[GTW02] Erich Grädel, Wolfgang Thomas, and Thomas Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001]*, volume 2500 of *LNCS*. Springer-Verlag, 2002.

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