Topological, Automata-Theoretic and Logical Characterizations of Finitary Languages

Presentation for YR-CONCUR 2010

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Introduction: system specification

- non-terminating (e.g web server);
- discrete time;
- non-deterministic.

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  (e.g ”available”, ”waiting”, ”critical error”)

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- runs are infinite words: $w = w_0 \cdot w_1 \ldots w_n \ldots \in \Sigma^\omega$;
- specification given as a language $L \subseteq \Sigma^\omega$;
- ω-regular language: safety + liveness;
- liveness properties: ”something good happens eventually”.

Outline

Motivations

Characterizations

Expressions

Classical liveness properties

A first example, Büchi:

- a given set of propositions appears infinitely often; (e.g ”job done”)

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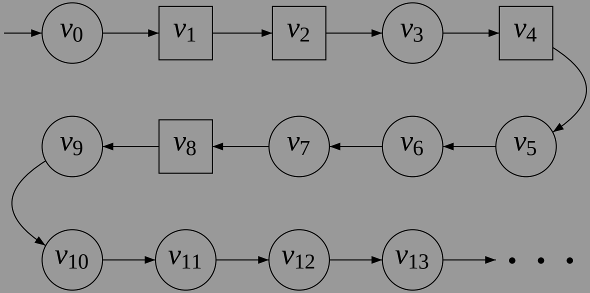
A second example, parity:

- integers are assigned to propositions, representing a priority;
- along an execution, some integers appear infinitely often;
- parity specifies that the least priority appearing infinitely often is even.

A drawback of classical ω-regular specifications



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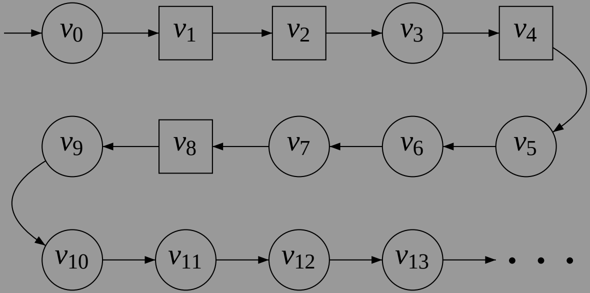

Specification: Büchi with $F = \{v_{2^k} \mid k \in \mathbb{N}\}$.

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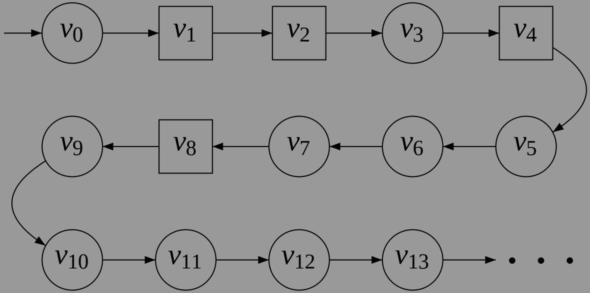

Specification: Büchi with $F = \{v_{2^k} \mid k \in \mathbb{N}\}$.

Satisfied, but the time until something good happens may grow unbounded!

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unknown: retain independence from granularity.

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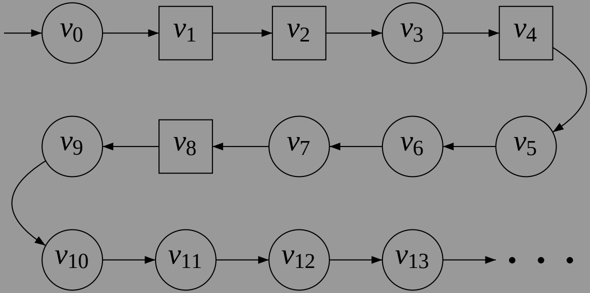
- closed: involves Cantor topology;
- ω-regular: involves ω-regularity;
- restriction operator: $\mathrm{fin}(L) \subseteq L$.

Back to the example

Finitary Büchi: $F = \{v_{2^k} \mid k \in \mathbb{N}\}$



Not satisfied!

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Describing classical finitary objectives: Büchi

Let $F \subseteq \Sigma$,

$$\text{Büchi}(F) = \{w \mid \text{Inf}(w) \cap F \neq \emptyset\}$$

$\text{Inf}(w)$ is the set of propositions that appear infinitely often in w.

Describing classical finitary objectives: Büchi

Let $F \subseteq \Sigma$,

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$$\text{next}_k(w, F) = \inf\{k' - k \mid k' \geq k, w_{k'} \in F\}$$

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$$w = v_0 \ldots v_k \underbrace{v_{k+1} \ldots v_{k'-1}}_{\notin F} \underbrace{v_{k'}}_{\in F}$$

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$$\text{fin}(\text{Büchi}(F)) = \{w \mid \limsup_k \text{next}_k(w, F) < \infty\}$$

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Lemma

$$\text{fin}(\text{Büchi}(F)) = \\ \{w \mid \exists B \in \mathbb{N}, \exists n \in \mathbb{N}, \forall k \geq n, \text{next}_k(w, F) \leq B\}$$

Describing classical finitary objectives: parity

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$$\text{Parity}(p) = \{w \mid \min(p(\text{Inf}(w))) \text{ is even}\}$$

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$$\mathrm{dist}_k(w, p) = \inf\{k' - k \mid \begin{array}{l} k' \geq k, p(w_{k'}) \text{ even and} \\ p(w_{k'}) \leq p(w_k)\} \end{array}$$

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Topological characterization

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Parity lies in the boolean closure of Σ_2 *and* Π_2.

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Theorem

1. *For all* $p : \Sigma \to \mathbb{N}$, *we have* $\mathrm{fin}(\mathrm{Parity}(p)) \in \Sigma_2$.
2. *For all* $\emptyset \subset F \subset \Sigma$, *we have that* $\mathrm{fin}(\text{Büchi}(F))$ *is* Σ_2*-complete.*
3. *There exists* $p : \Sigma \to \mathbb{N}$ *such that* $\mathrm{fin}(\mathrm{Parity}(p))$ *is* Σ_2*-complete.*

Automata-theoretic characterization

We consider automata on infinite words, whose acceptance conditions are finitary Büchi and finitary parity.

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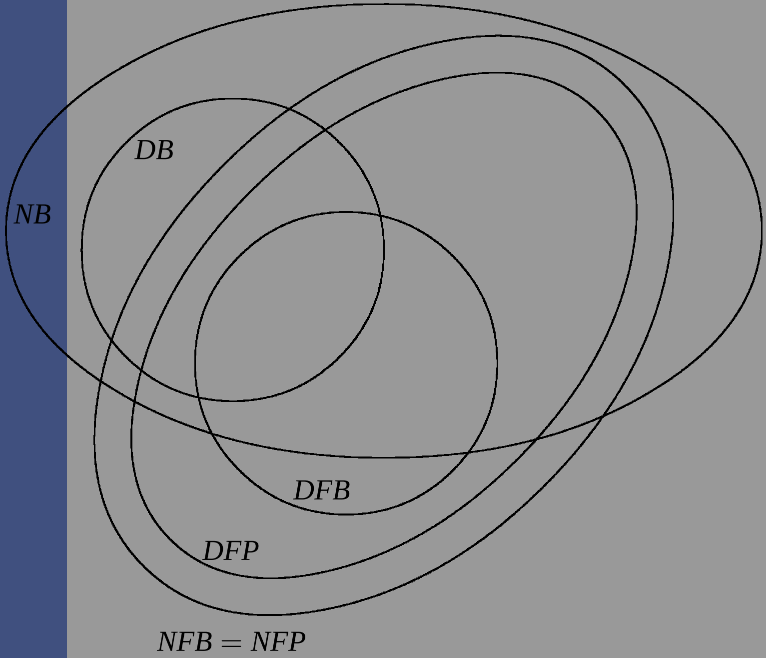
$$\left\{ \begin{array}{c} D \\ N \end{array} \right\} \cdot \left\{ \begin{array}{c} \varepsilon \\ F(\mathit{finitary}) \end{array} \right\} \cdot \left\{ \begin{array}{c} B(\mathit{B\ddot{u}chi}) \\ P(\mathit{parity}) \end{array} \right\}$$



DB
NB
DFB
DFP
NFB = NFP

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ω-regular expressions

Regular expressions defines regular languages over finite words:

$$L := \emptyset \mid \varepsilon \mid \sigma \mid L \cdot L \mid L^* \mid L + L; \quad \sigma \in \Sigma$$

ω-regular languages are finite union of $L \cdot L'^{\omega}$, where L and L' are regular languages of finite words.

The bound operator B [BC06]

$$L^\omega = \{u_0 \cdot u_1 \cdot \ldots \cdot u_k \ldots \mid u_0, u_1, \ldots, u_k, \ldots \in L\}$$

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(complete definitions require the use of infinite sequences of finite words)

Star-free ωB-regular expressions

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Theorem *NFP (non-deterministic finitary parity) has exactly the same expressive power as star-free ωB-regular expressions.*

Conclusion

- finitary objectives is a refinment for specification purposes;
- for ω-regular languages, topological, logical and automata-theoretic characterizations were known;
- for finitary languages, all were missing; we established:
  - topological characterization;
  - automata-theoretic characterization, comparison to ω-regular languages, closure properties;
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Future work:

- algorithmic issues: equivalence with ωB-regular expressions, emptiness problem of finitary automata;
- a tool for finitary objectives...

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The end

Thank you for your attention!