Exercices in reachability style

GASICS'2010 workshop

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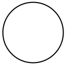
Outline

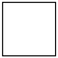
Outline

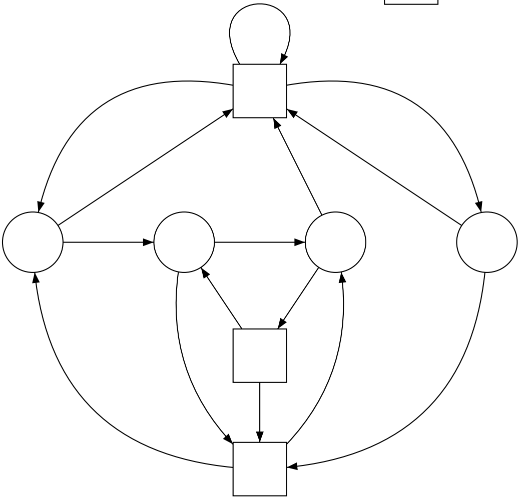
Players

Two players: **Eve** and **Adam**.

Playing

 controlled by Eve

 controlled by Adam



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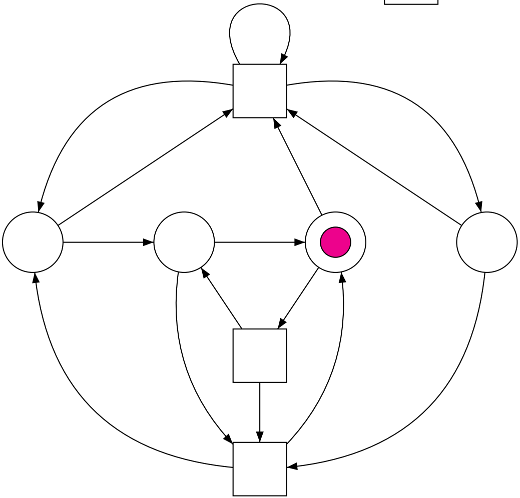

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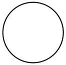
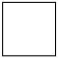
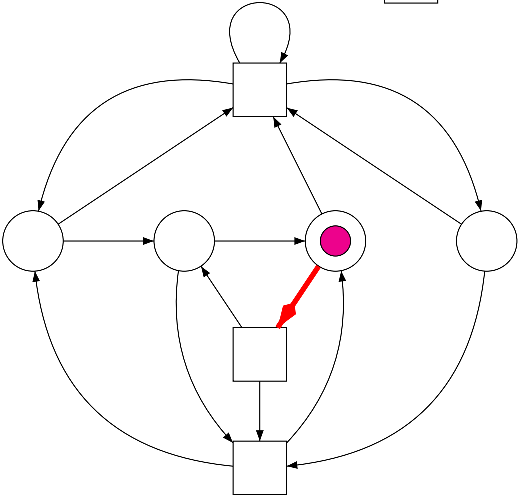
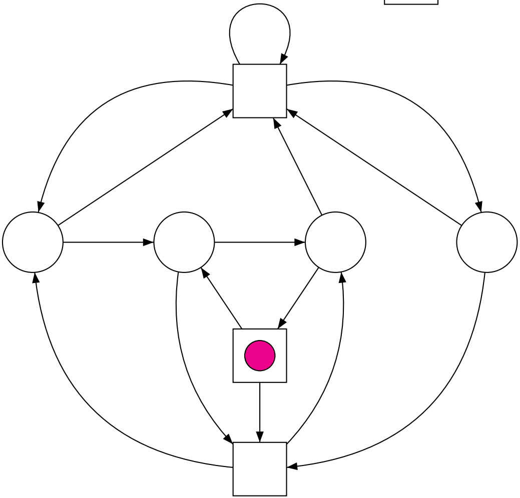
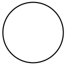
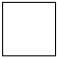
 controlled by Eve

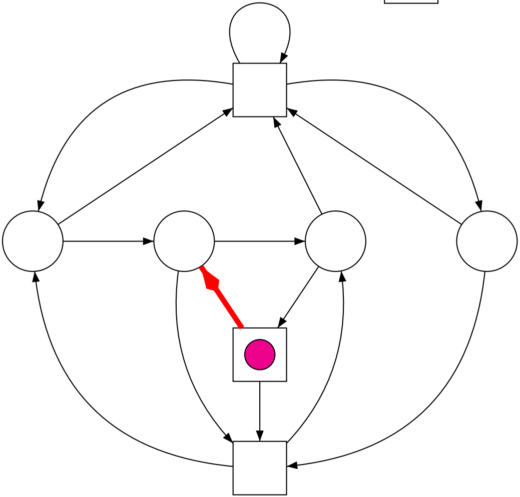
 controlled by Adam



Playing ○ controlled by Eve □ controlled by Adam



Playing  controlled by Eve  controlled by Adam



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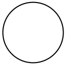
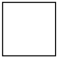
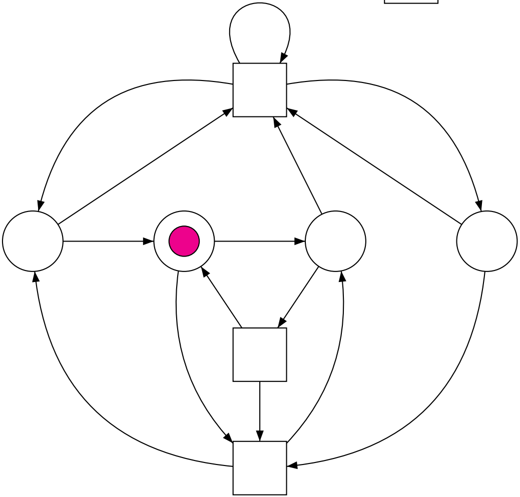
 controlled by Eve  controlled by Adam



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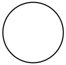
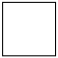
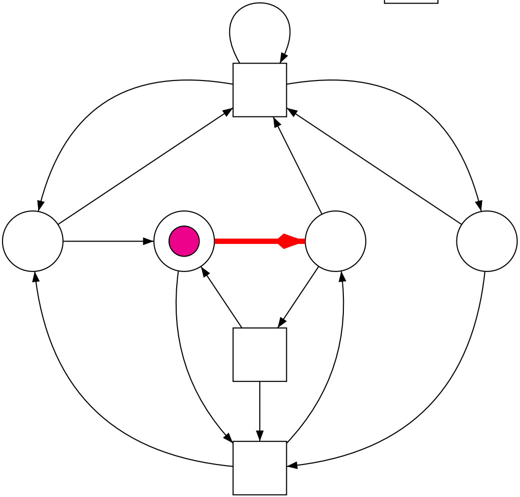
 controlled by Eve

 controlled by Adam



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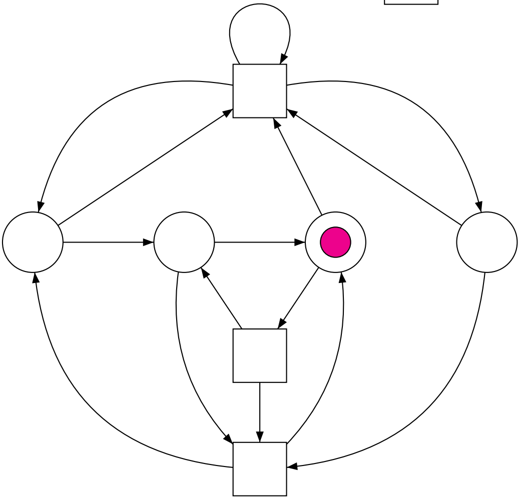
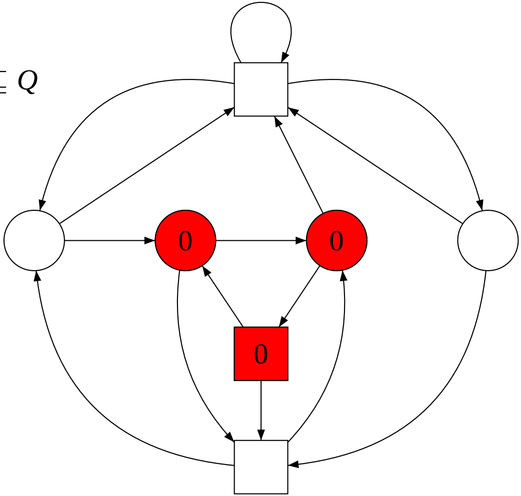


Outline

Solving reachability games

Given $F \subseteq Q$



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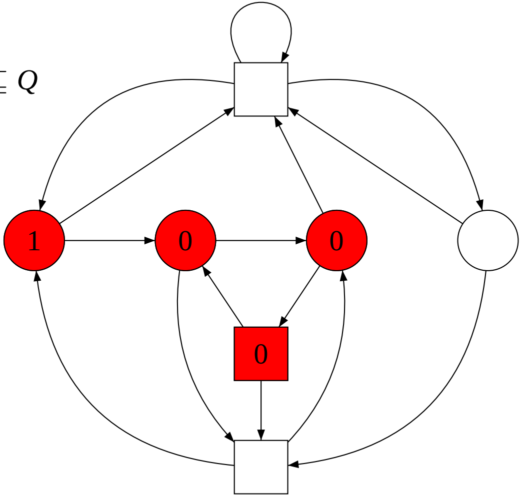

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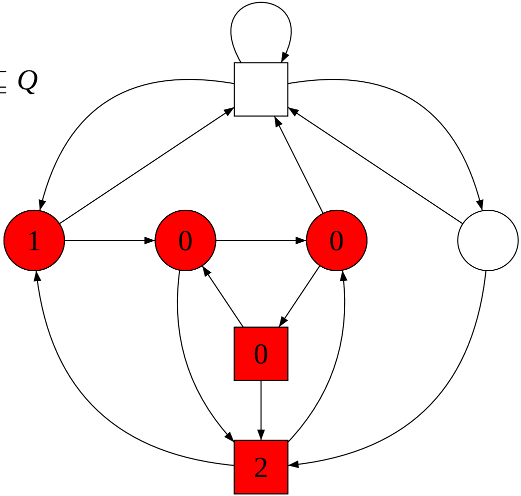

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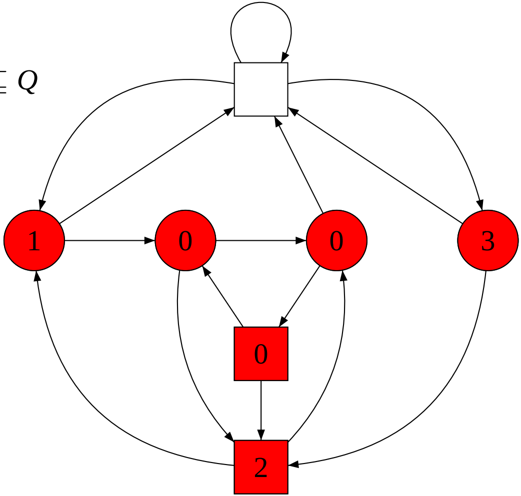


Outline

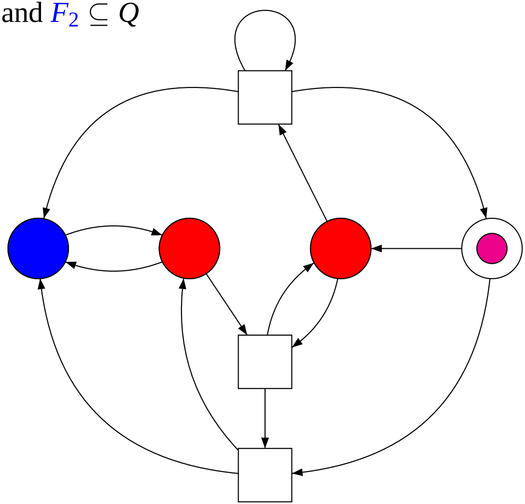
1. Generalized reachability games
   - Games
   - Reachability games
   - Generalized reachability games
   - The surprizing complexity
2. Weakening weak Müller games
   - Weak Müller games
   - Weakening
   - Universal reachability games
   - A possible complexity gap
   - Memory requirements

Generalized reachability objectives

- Reachability objectives: given $F \subseteq Q$, reach at least one vertex in F;
- Generalized reachability objectives: given $F_1, F_2, \ldots, F_p \subseteq Q$, reach at least one vertex in each F_i.

Example

Given F_1 and $F_2 \subseteq Q$



Generalized reachability games

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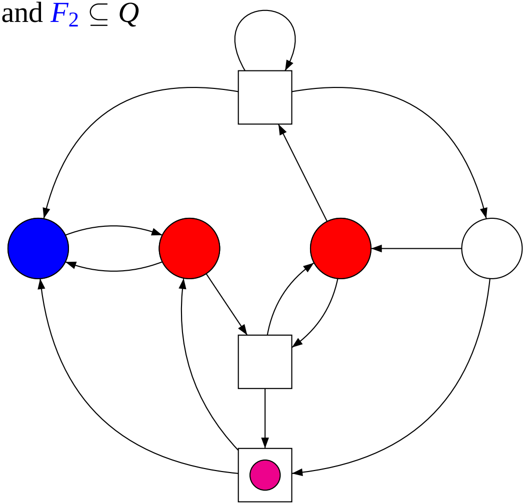

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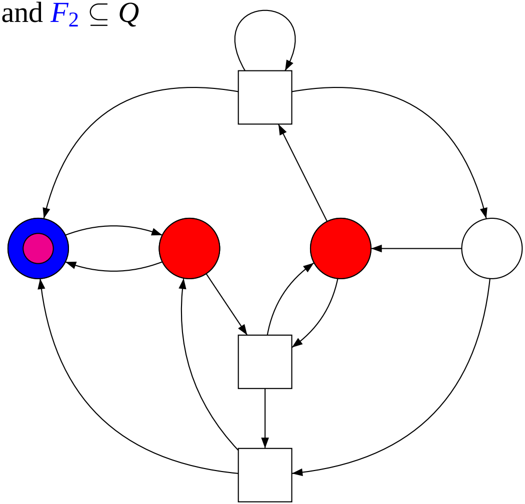

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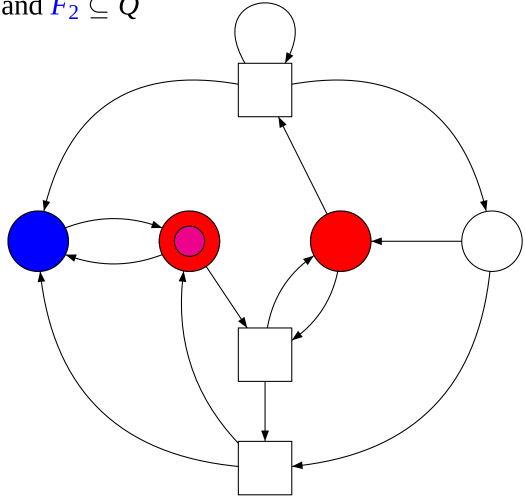

Outline

Reduction from QBF to generalized reachability games

ϕ quantified boolean formula in conjunctive normal form:

$$\phi = \forall x \, \exists y \, \forall z \;\; (x \vee \neg y) \wedge (\neg y \vee z)$$

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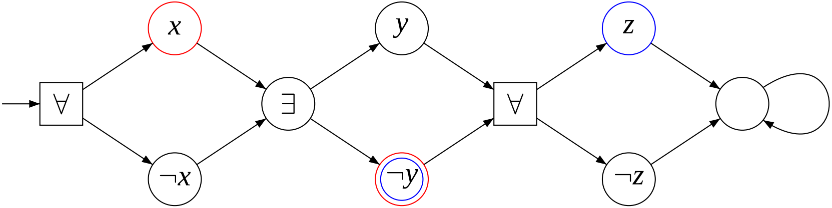

Eve wins if and only if ϕ is true.

Complexity

Theorem (Complexity of generalized reachability games)

- *Solving two players generalized reachability games is* PSPACE-*complete;*
- *Solving one player (Eve) generalized reachability games is* NP-*complete.*

Outline

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Müller games and weak Müller games

- Müller objectives: given $\mathcal{F} \subseteq 2^Q$, the set of vertices visited infinitely often is in $\mathcal{F}$;
- weak Müller objectives: given $\mathcal{F} \subseteq 2^Q$, the set of visited vertices is in $\mathcal{F}$.

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Solving Müller games, as well as weak Müller games, is PSPACE-*complete.*

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How to weaken weak Müller games to get lower complexity?

Outline

Downward-closed objectives

- A condition $\mathcal{F} \subseteq 2^Q$ is downward-closed if $Y \in \mathcal{F}, X \subseteq Y \Rightarrow X \in \mathcal{F}$;
- weak Müller downward-closed objectives are generalized (existential) reachability objectives;
- weak Müller upward-closed objectives are generalized universal reachability objectives.

Cutting on formulas

$$
\begin{tikzcd}
\bigvee_i \mathrm{Reach}(v_i) \arrow[d] \arrow[dr] & & \wedge_i \mathrm{Reach}(v_i) \arrow[d] \arrow[dll] \\
\vee_i \wedge_j \mathrm{Reach}(v_{i,j}) & \vdots & \wedge_i \vee_j \mathrm{Reach}(v_{i,j}) \\
\vdots & & \vdots
\end{tikzcd}
$$

Cutting on formulas

(existential) reachability

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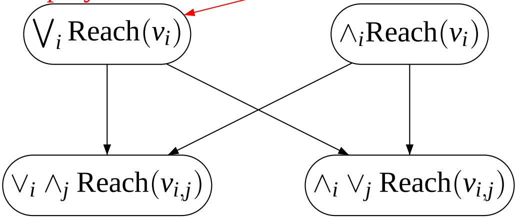
Weakening

Cutting on formulas

(existential) reachability

polynomial



Cutting on formulas

universal reachability

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???

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generalized (existential) reachability

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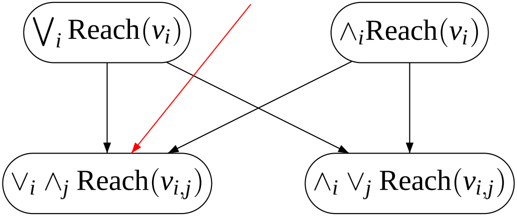
PSPACE-complete

Cutting on formulas

generalized universal reachability



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???

Outline

Solving

$\boxed{\wedge_i \mathrm{Reach}(v_i)}$

Solving

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Theorem (Complexity of universal reachability games)

Solving universal generalized reachability games is in P*. Furthermore, Eve requires at most k memory states, and Adam at most* 2*.*

Sketch of a proof

We consider two cases:

- If there exists a permutation f over $\{1, \ldots, k\}$ such that for all $1 \leq i \leq k-1$, we have $v_{f(i)} \in \mathrm{Attr}\big(v_{f(i+1)}\big)$.

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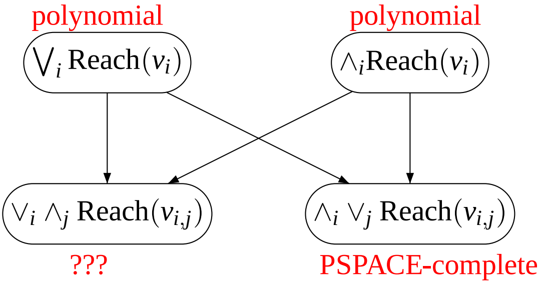
  Then Eve wins from $\cap_i \mathrm{Attr}(v_i)$.
- Otherwise, there exists v_i and v_j such that $v_i \notin \mathrm{Attr}(v_j)$ and $v_j \notin \mathrm{Attr}(v_i)$.
  A winning strategy for Adam is: ”if v_i or v_j has been reached, then avoid the other”.

Completing the picture



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Theorem (Complexity of generalized universal reachability games)

Solving generalized universal games is PSPACE*-complete.*

The proof is the same, using QBF in disjunctive normal form.

but...

If we look carefully at our reduction, it does not imply that solving generalized reachability games where reachability sets have size 2 is PSPACE-hard.

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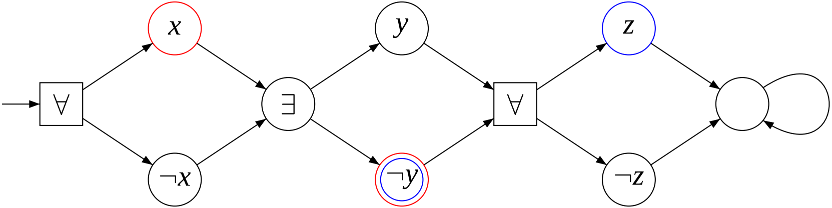

Generalized universal reachability games

Theorem (Complexity of restricted generalized universal reachability games)

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Theorem (Complexity of restricted generalized universal reachability games)

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However, the problem is still open for generalized (existential) reachability games, as well as for the dual version: generalized universal reachability games where there are two reachability sets.

Completing the picture

polynomial | polynomial

$$\begin{tikzcd}
\bigvee_i \mathrm{Reach}(v_i) \arrow[d] \arrow[dr] & \bigwedge_i \mathrm{Reach}(v_i) \arrow[d] \arrow[dl] \\
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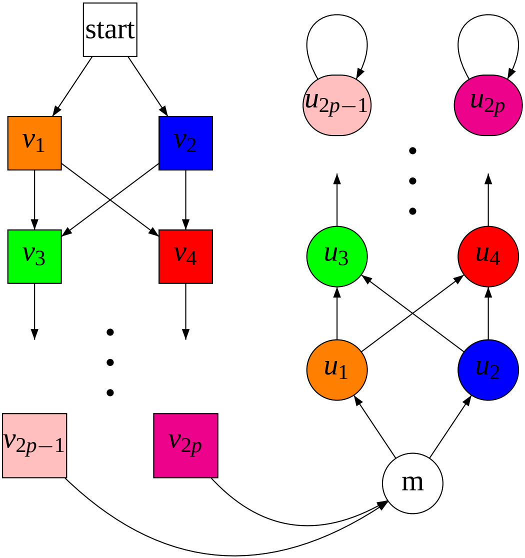
\end{tikzcd}$$

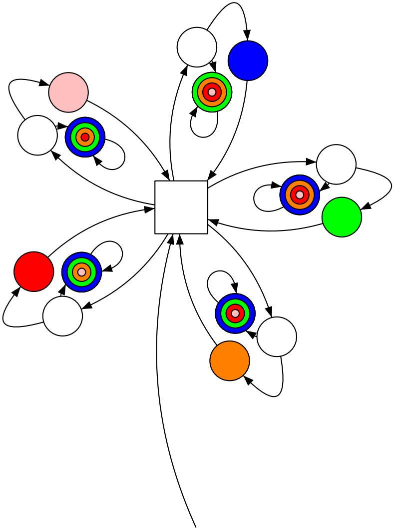
PSPACE-complete | PSPACE-complete

Outline

Exponential lower bound for Eve, reachability sets of size 2



Florian’s piece of art; exponential lower bound for Eve



Conclusion and further work

- Many restrictions over weak Müller games are still PSPACE-complete;
- Open case: generalized reachability games where reachability sets have size 2.

The end.

Thank for your attention!