Boundedness Games

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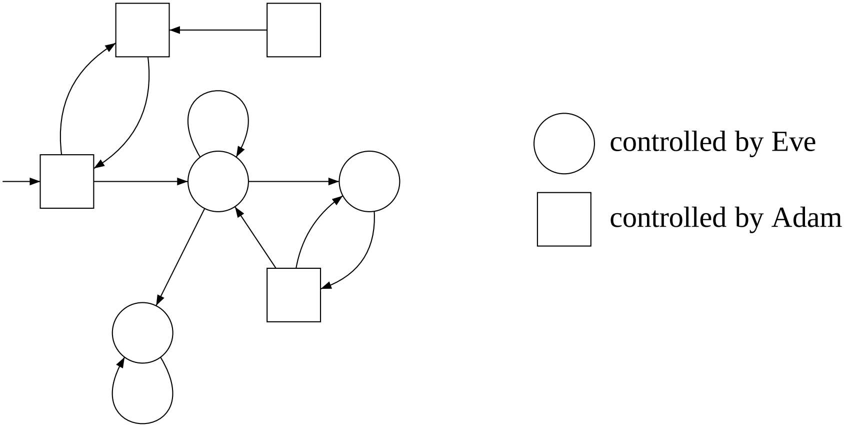
(based on joint works with Krishnendu Chatterjee, Thomas Colcombet, Florian Horn and Martin Zimmermann)

Definition of ωB-games

Two-player turn-based games over **finite** or **infinite** graphs



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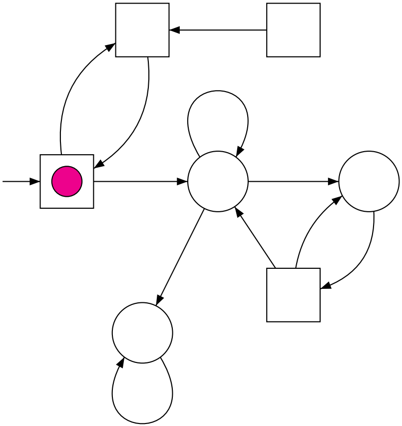

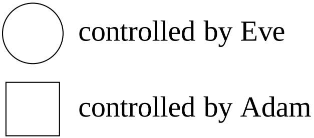

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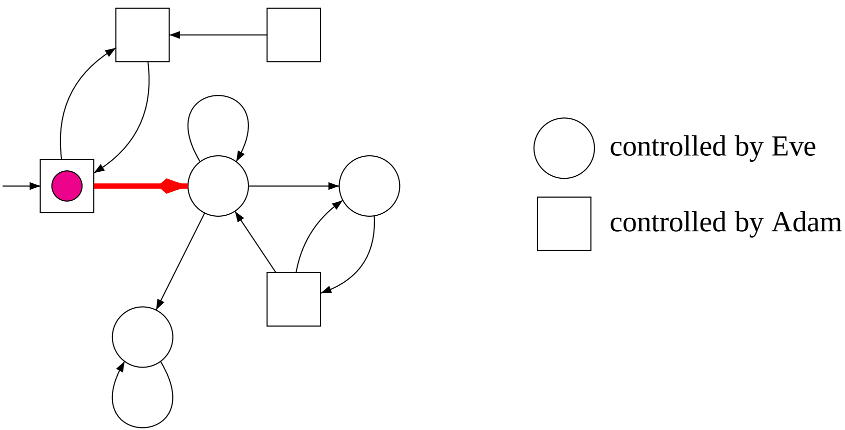

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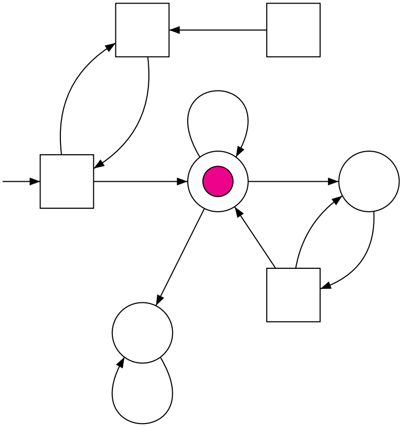

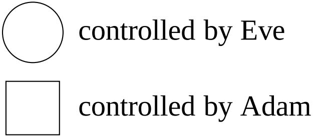

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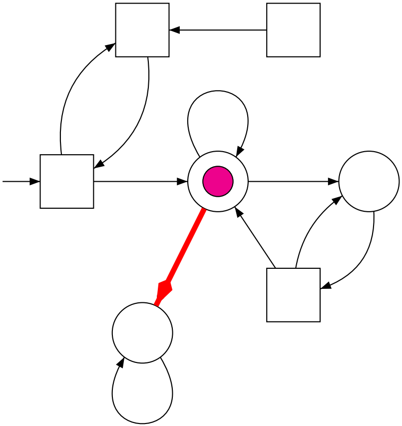

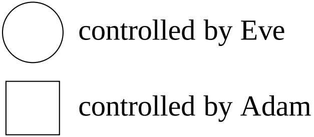

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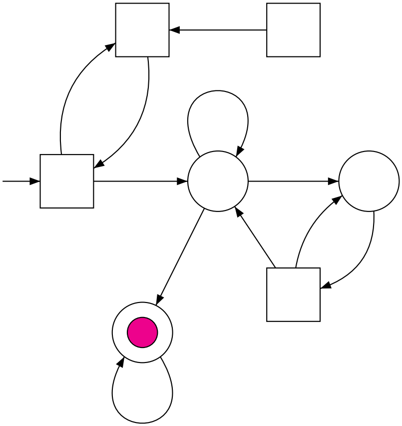

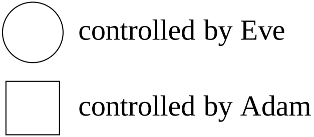

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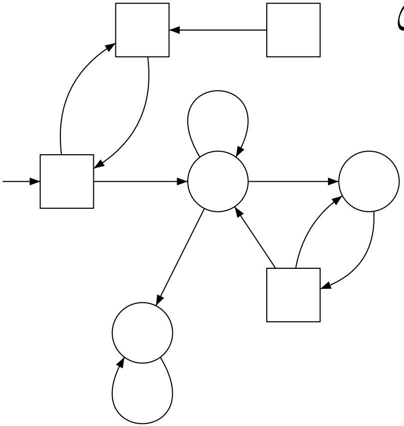

ωB winning condition:

parity
and
all counters
are bounded

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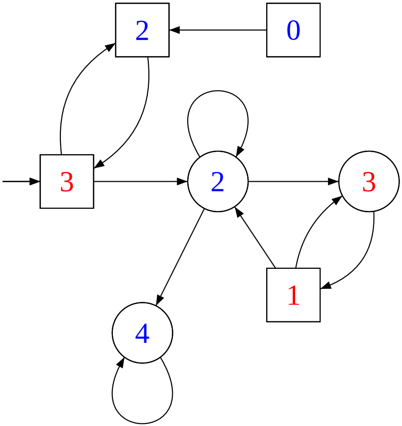

parity condition:

the minimal priority seen infinitely often is even

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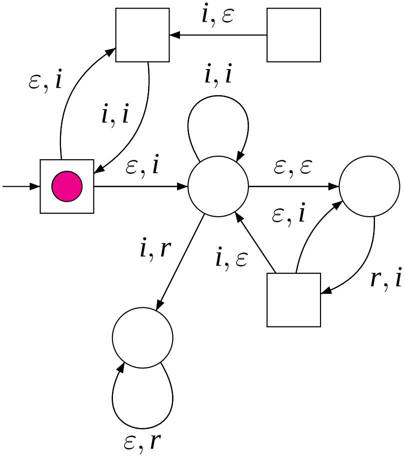

$$c_1 = 0$$
$$c_2 = 0$$

ε : nothing

i : increment

r : reset

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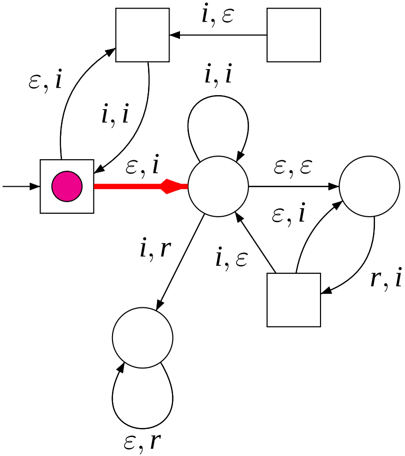

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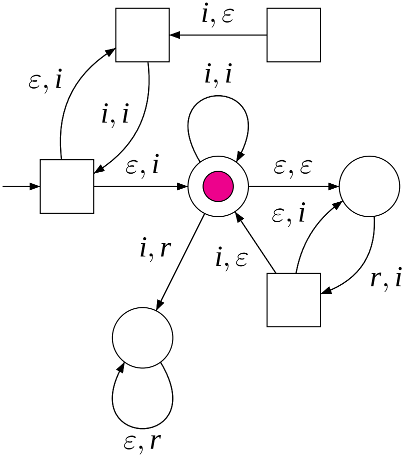

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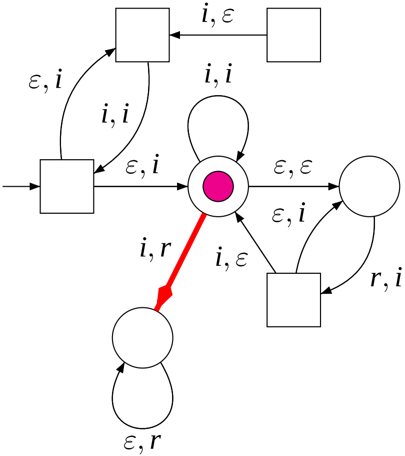

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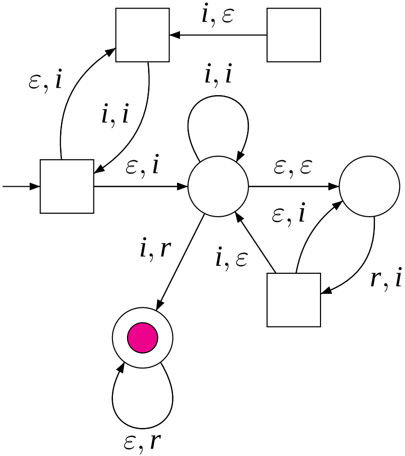

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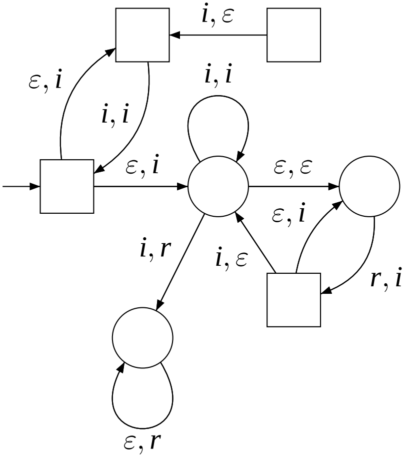

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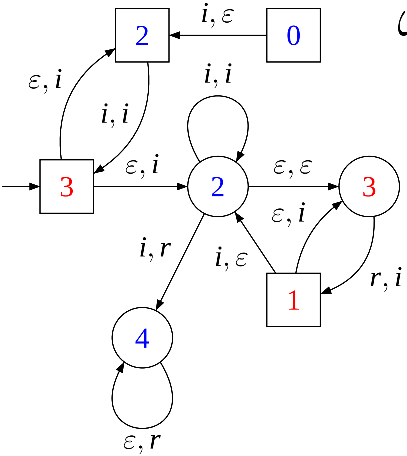

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Strategy (for Eve)

General form

$$\sigma : V^+ \to V$$

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What about ωB games?

Finite-memory

$$\begin{cases} \sigma : V \times M \to V \\ \mu : M \times E \to M \end{cases}$$

Why proving the existence of finite-memory strategies?

Thomas Colcombet's habilitation:

le fait 2.22 et en déduire que la domination entre formules de la logique monadique de coût est décidable sur les arbres infinis. Ainsi, la conjecture 9.2 implique la conjecture 9.1.

En fait, il est possible de pointer avec encore plus de précision où se trouve la difficulté. Si l'on cherche à démontrer la conjecture 9.2, tout comme dans le cas des arbres finis, le point crucial est l'existence de stratégies gagnantes à mémoire finie. Il suffirait d'établir la conjecture suivante.

Conjecture 9.3. *Les objectifs* `hB` $\wedge$ `parité` *et* $\neg$`B` $\wedge$ `parité` *sont à* $\approx$*-mémoire finie, sur toutes les arènes/sur les arènes chronologiques/sur les arènes «arborescentes».*

Existence of finite-memory strategies in (some) boundedness games
$\Longrightarrow$ Decidability of cost MSO over infinite trees
$\Longrightarrow$ Decidability of the index of the non-deterministic Mostowski's hierarchy!

Quantification

Eve wins means:

$\exists\sigma$ (strategy for Eve),
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Is this:

1. Equivalent? Sometimes ...
2. Decidable? Not always ...

The questions I am interested in

Boundedness games:

1. Over finite graphs: decide the winner efficiently and construct small finite-memory strategies.

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   $\looparrowright$ Cost-parity games [F. and Zimmermann, 2012].
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2. Over pushdown graphs: decide the winner and construct finite-memory strategies.
   ↬ Pushdown finitary games [Chatterjee and F., 2013].
3. Over infinite graphs: prove the existence of finite-memory winning strategies.
   ↬ Ongoing work with Thomas Colcombet and Florian Horn.

Outline

1. Finite-memory strategies
   - Some examples
   - Worst-case strategies
   - Finitary conditions

2. Equivalence for pushdown games
   - The case of finite graphs
   - The case of pushdown graphs
   - Application: ωB-games with max

Outline

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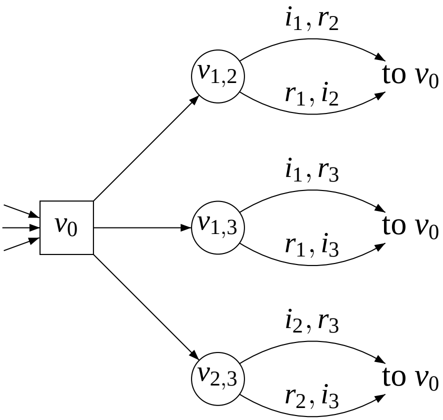
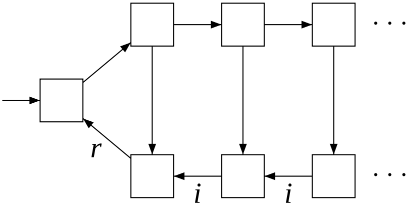
Eve needs some memory



Adam needs infinite memory



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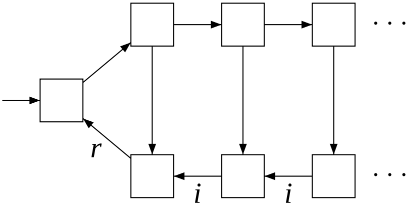

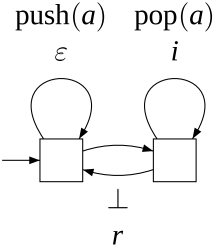

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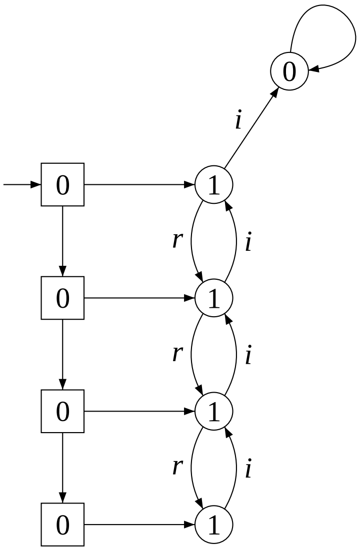

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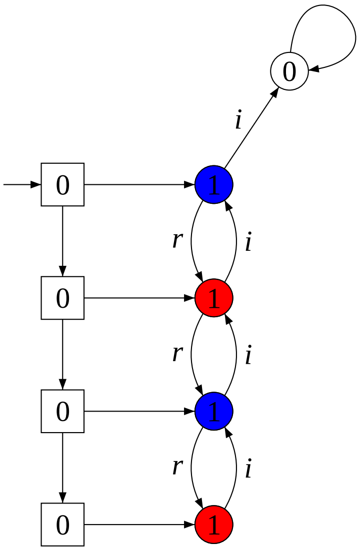

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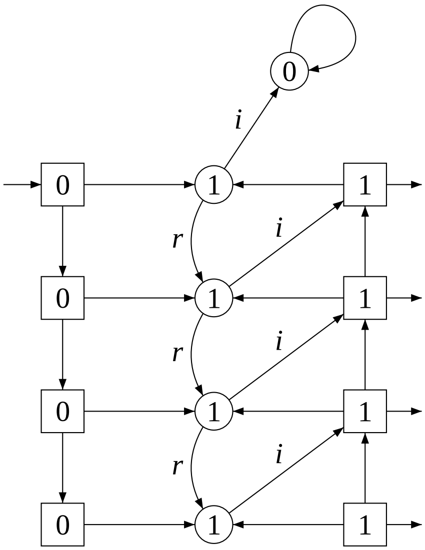

Outline

Playing for the worst, hoping for the best

Condition: the only counter is bounded by N.

Observation

For every vertex v, there is a maximal $N(v) \leq N$ such that Eve wins.

Idea: play from v as you would with counter value $N(v)$.

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Bottom-line: possible scenarios are *linearly ordered*.

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Lemma

Eve has positional winning strategies for the condition "the counter is bounded by N*".*

Outline

Finitary Büchi

Condition: there exists N such that Büchi vertices are visited every N steps.

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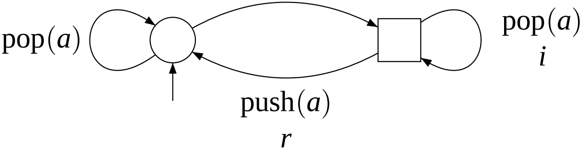
Theorem

- *Eve has positional winning strategies in finitary Büchi games.*
- *Eve has finite-memory winning strategies in finitary parity games.*

Outline

Some more examples (1)



Eve should maintain a low stack.

Objective

Theorem

For all pushdown games, the following are equivalent:

- $\exists\sigma$ *(strategy for Eve)*, $\forall\pi$ *(paths)*, $\exists N \in \mathbb{N}$,
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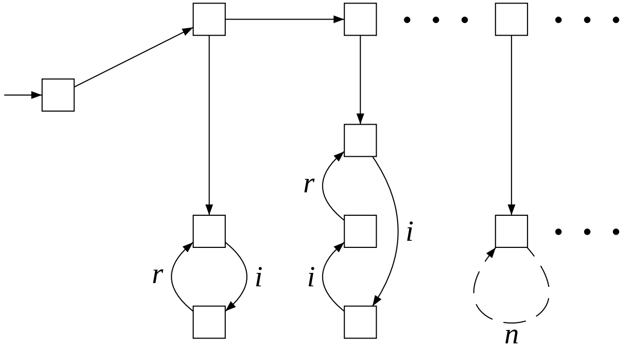
Counter-example for the general case



Eve wins but she does not know the bound!

Outline

A simple proof for the case of finite graphs

Condition: parity and all counters are bounded.

Define:

- $\mathcal{W}_E(N)$ the set of vertices where Eve wins for the bound N.
- $\mathcal{W}_E$ the set of vertices where Eve wins for some (non-uniform) bound.

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2. *There exists N such that $\mathcal{W}_E(N) = \mathcal{W}_E(N+1) = \cdots$.*

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2. *There exists N such that $\mathcal{W}_E(N) = \mathcal{W}_E(N+1) = \cdots$.*
3. *For such N, Adam wins from $V \setminus \mathcal{W}_E(N)$, hence $\mathcal{W}_E = \mathcal{W}_E(N)$.*

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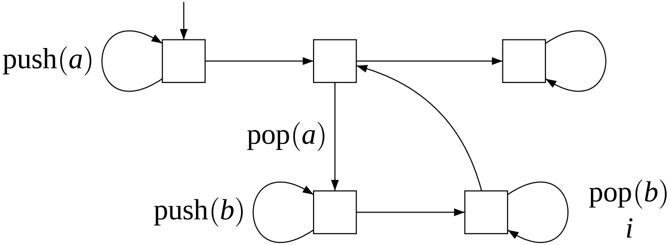
Some more examples (2)



Adam may use the stack as “credit”.

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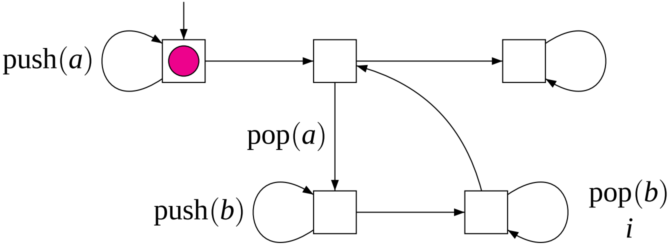

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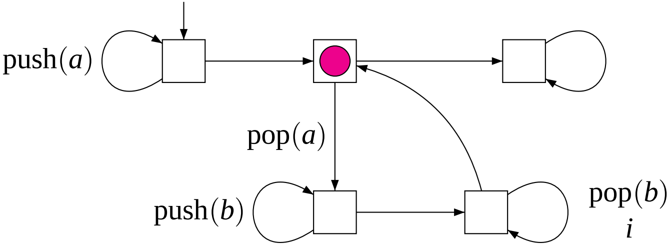

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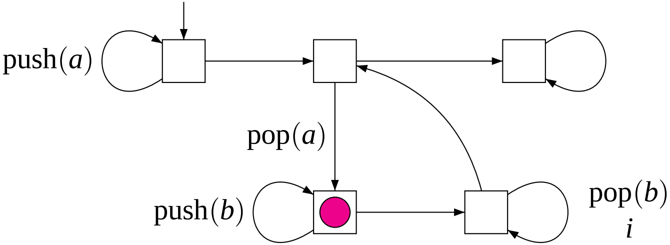

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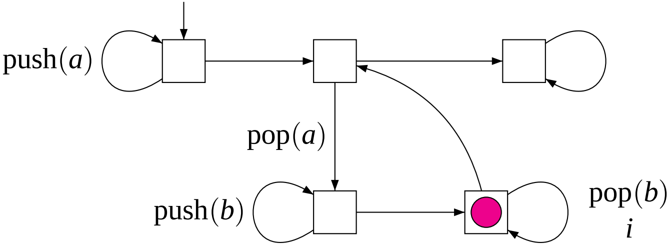

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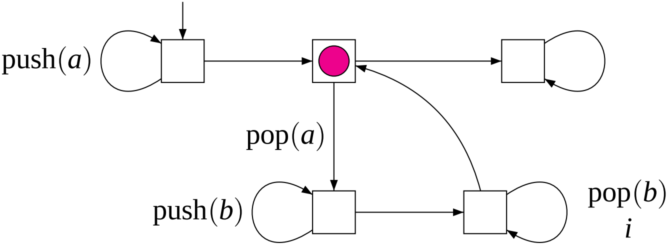

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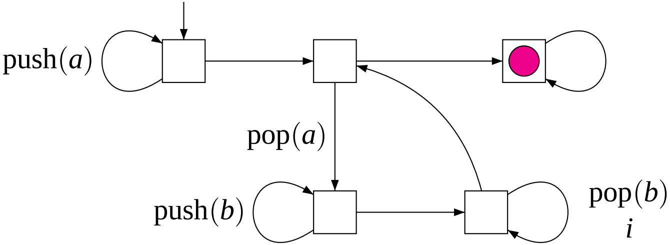

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Proof sketch

Condition: parity and all counters are bounded.

Define:

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Proposition

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3. *For such N, Adam wins from $V \setminus \mathcal{W}_E(N)$, hence $\mathcal{W}_E = \mathcal{W}_E(N)$.*

Why is 2. true?

Regularity of the winning regions

Theorem (derived from Serre)

*For all N, $\mathcal{W}_E(N)$ is a regular set of configurations, recognized by an alternating automaton of size **independent of** N.*

Decidability

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Corollary

Determining the winner in a pushdown ωB*-game is decidable.*

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Theorem

For all pushdown games, the following are equivalent:

- $\exists \sigma$ *(strategy for Eve),* $\forall \pi$ *(paths),* $\exists N \in \mathbb{N}$*,*
  π *satisfies parity and each counter is bounded by* N*.*
- $\exists \sigma$ *(strategy for Eve),* $\exists N \in \mathbb{N}$*,* $\forall \pi$ *(paths),*
  π *satisfies parity and* ***eventually*** *each counter is bounded by* N*.*

Corollary

Determining the winner in a pushdown ωB*-game is decidable.*

Remark: one can show that the collapse bound is doubly-exponential!

Outline

The max operator

We add a new feature for counters: $\gamma \leftarrow \max(\gamma_1, \gamma_2)$.

Theorem (derived from Bojańczyk and Toruńczyk)

Deterministic max-automata are equivalent to Weak MSO + $\mathbb{U}$.

We consider ωB-games with max.

Co-determinisation

$$
\begin{tikzcd}
0 \arrow[r, "i"] & 1 \arrow[r, "r"] & 0 \arrow[r] & 0 \arrow[r, "\max(\gamma_1, \gamma_2)"] & 2 \arrow[r, "i"] & 3 \\
0 \arrow[r] & 0 \arrow[r, "i"] & 1 \arrow[r, "\max(\gamma_2, \gamma_3)"] & 2 \arrow[r, "i"] \arrow[ur, dashed] & 3 \arrow[r, "r"] \arrow[dr, dashed] & 0 \\
0 \arrow[r, "i"] & 1 \arrow[r, "i"] & 2 \arrow[r, "i"] \arrow[ur, dashed] & 3 \arrow[r, "r"] & 0 \arrow[r, "\max(\gamma_1, \gamma_2)"'] & 3
\end{tikzcd}
$$

Co-determinisation

$$
\begin{tikzcd}
0 \arrow[r, "i"] & 1 \arrow[r, "r"] & 0 \arrow[r] & 0 \arrow[r, "\max(\gamma_1,\gamma_2)"] & 2 \arrow[r, "i"] & 3 \\
0 \arrow[r] & 0 \arrow[r, "i"] & 1 \arrow[r, "\max(\gamma_2,\gamma_3)"] & 2 \arrow[r, "i", red] \arrow[ur, dashed] & 3 \arrow[r, "r"] \arrow[dr, red] & 0 \\
0 \arrow[r, "i", red] & 1 \arrow[r, "i", red] & 2 \arrow[r, "i"] \arrow[ur, red] & 3 \arrow[r, "r"] & 0 \arrow[r, "\max(\gamma_1,\gamma_2)"'] & 3
\end{tikzcd}
$$

To prove that a counter value is high, one can count backwards!

Reduction

We reduce ωB-games with max to pushdown ωB-games (without max).

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Idea: simulate the game and store the play in the stack.

Whenever he wants, Adam can declare “this counter has a very large value”: from there, play backwards using the stack until a reset is met.

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We reduce ωB-games with max to pushdown ωB-games (without max).

Idea: simulate the game and store the play in the stack.

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Theorem

Determining the winner in an ωB-game with max is decidable.

The end.

Thank you!

Outline

1. Finite-memory strategies
   - Some examples
   - Worst-case strategies
   - Finitary conditions

2. Equivalence for pushdown games
   - The case of finite graphs
   - The case of pushdown graphs
   - Application: ωB-games with max