How much memory is needed to win regular games?

LMFI Master presentation

Nathanaël Fijalkow, advised by Florian Horn

LIAFA, Paris

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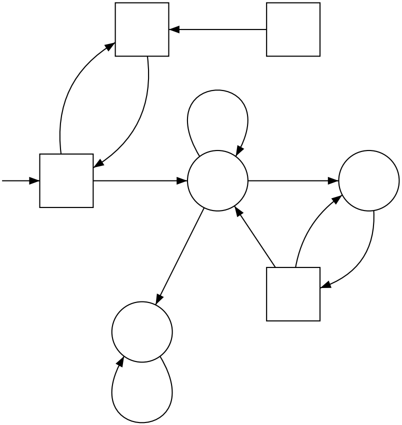
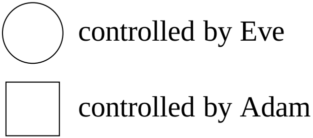
Games





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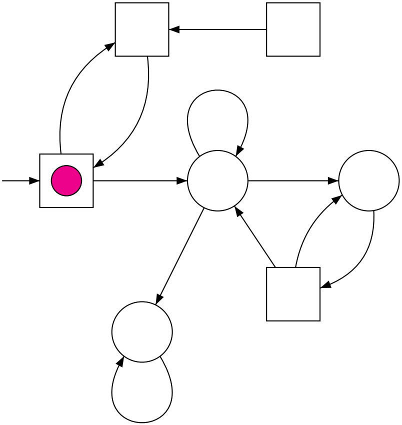

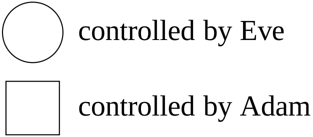

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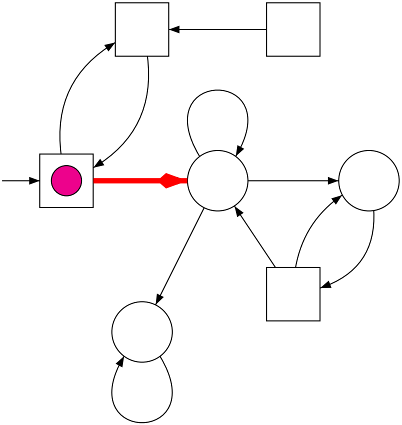

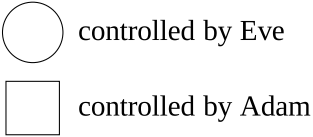

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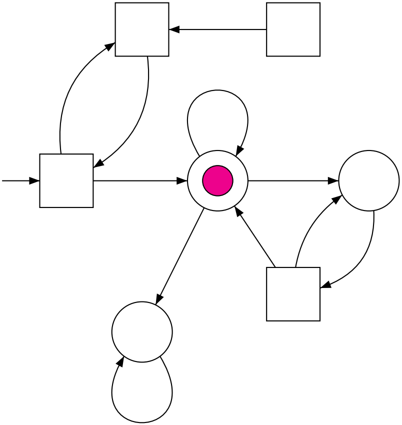

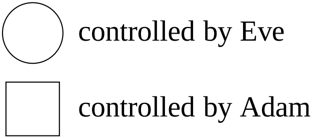

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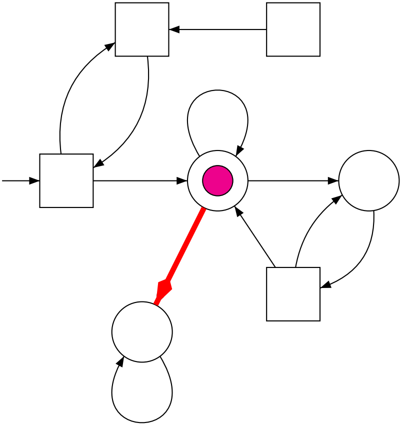

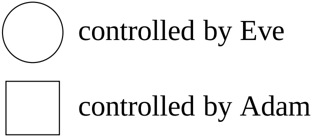

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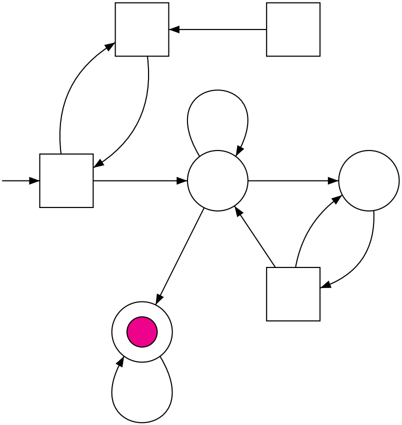
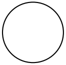
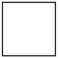


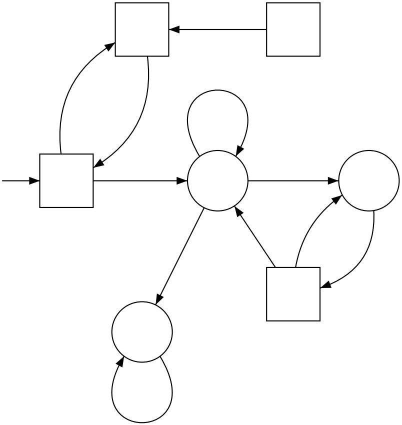


controlled by Eve



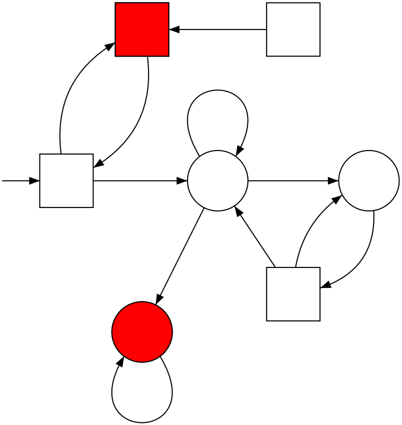
controlled by Adam

Introduction: reachability games



Given $F \subseteq Q$

Introduction: reachability games



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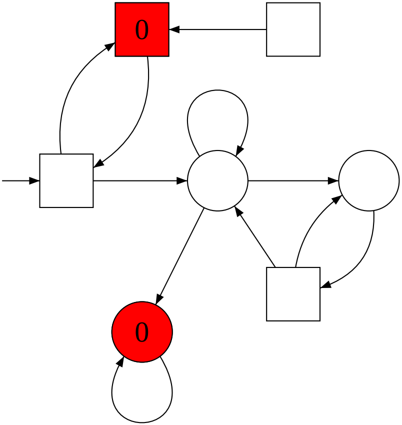

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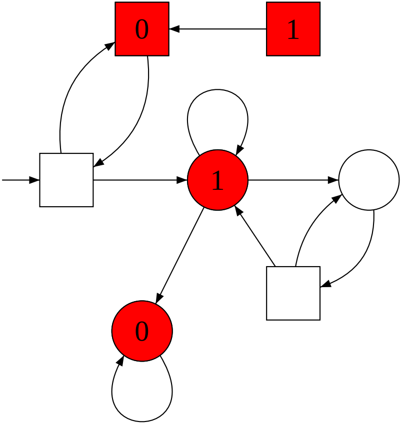

Given $\color{red}{F} \subseteq Q$

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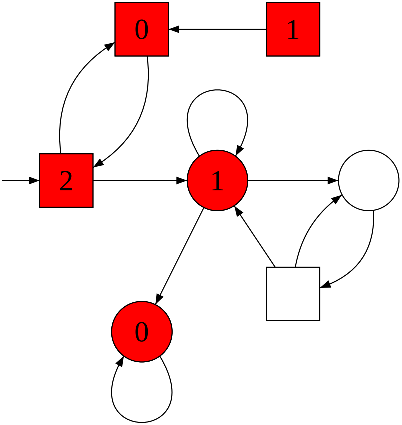

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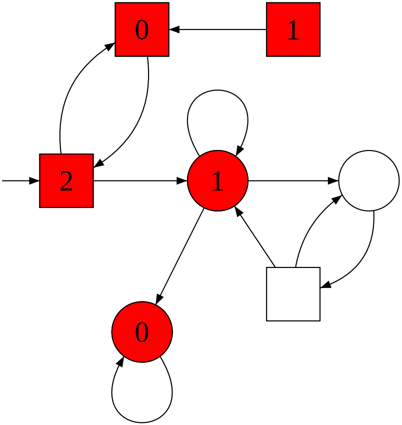
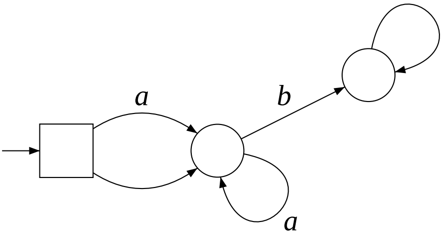


Given $F \subseteq Q$

Memoryless winning strategies
$\sigma : V \to E$ and $\tau : V \to E$

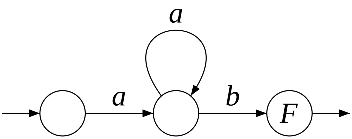
Introduction: regular games

arena



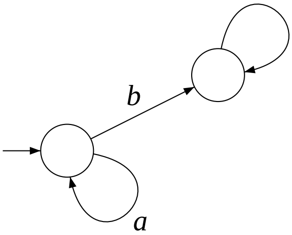
winning condition

$$L = a^+ \cdot b$$



Regular games need memory

arena



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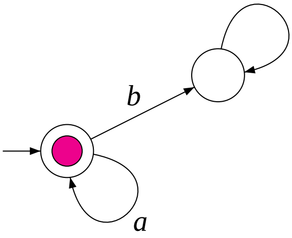

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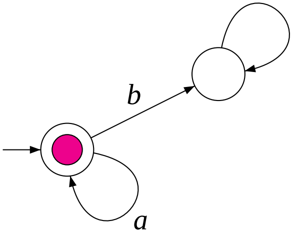

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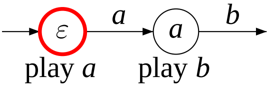
A winning strategy for Eve uses two memory states.



a memory structure

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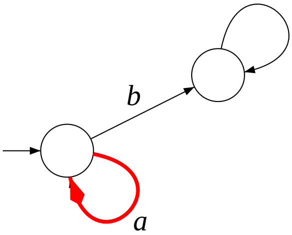

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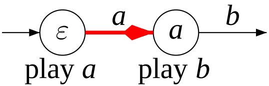

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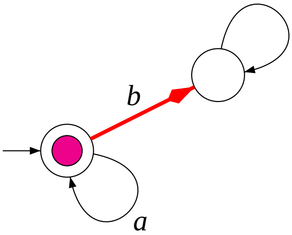

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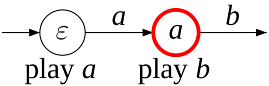

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Introduction: how much memory is needed to win?

Question: given a regular language L, what is the memory required by winning strategies?

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In other words, compute $m_L \in \mathbb{N}^*$ such that:

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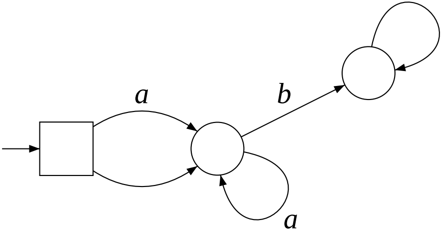
Question: given a regular language L, what is the memory required by winning strategies?
In other words, compute $m_L \in \mathbb{N}^*$ such that:

- in any arena, if Eve wins the regular game for L, then she has a winning strategy with m_L memory states,
- there is an arena where Eve wins but there are no winning strategies with less than m_L memory states.

Outline

A first remark

arena



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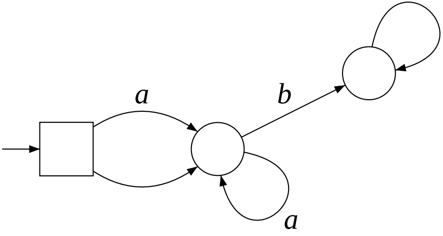

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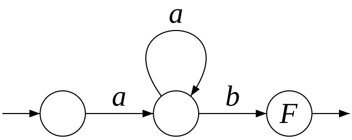

Any deterministic automaton that recognizes L is a good memory structure.

Proof: the synchronized product is a reachability game.

An upper bound for both players

We describe a good memory structure for L using left quotients: for $u \in \Sigma^*$,

$$u^{-1}L = \{v \mid u \cdot v \in L\}.$$

- the initial memory state is $\varepsilon^{-1}L = L$,
- each time a letter a is read from $u^{-1}L$, the memory is updated to $(u \cdot a)^{-1}L$.

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Lemma (An upper bound for both players)

For all regular games $\mathcal{G} = (\mathcal{A}, \mathrm{Reach}(L))$, both players have winning strategies using this memory structure (denoted $\mathcal{M}_L$).

Another example

“read at most ten consecutive b’s, and then an a”.

$$L = a + b \cdot a + bb \cdot a + \ldots + b^{10} \cdot a.$$

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In every regular game for L, Eve wins without memory.
This shows that the memory structure $\mathcal{M}_L$ is not optimal.

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Order left quotients inclusion-wise

If Adam wins in $\mathcal{G} \times \mathcal{M}_L$ from $(q, u^{-1}L)$ and $v^{-1}L \subseteq u^{-1}L$, then he wins from $(q, v^{-1}L)$

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Lemma (A tighter upper bound for Adam)

For all regular games $\mathcal{G} = (\mathcal{A}, \mathrm{Reach}(L))$, Adam has a winning strategy from his winning set that uses k memory states.

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Idea: whenever in $(q, v^{-1}L)$, play as from $(q, u^{-1}L)$, where $u^{-1}L$ is maximal winning from q.

Optimality

Lemma (Matching lower bound for Adam)

For all regular languages L, there exists an arena $\mathcal{A}$ such that Adam needs k memory states to win in $\mathcal{G} = (\mathcal{A}, \mathrm{Reach}(L))$.

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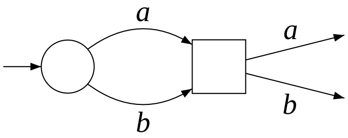
Exemplified: $L = (a+b)^* \cdot (aa + bb)$.

inclusion

$$\begin{tikzcd}
 & \color{red}{a^{-1}L} \arrow[dr] & \\
\varepsilon^{-1}L \arrow[ur] \arrow[dr] & & (aa)^{-1}L \\
 & \color{red}{b^{-1}L} \arrow[ur] &
\end{tikzcd}$$

arena



Outline

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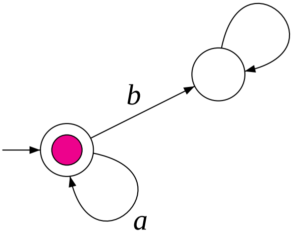
If Eve wins in $\mathcal{G} \times \mathcal{M}_L$ from $(q, u^{-1}L)$ and $u^{-1}L \subseteq v^{-1}L$, then she wins from $(q, v^{-1}L)$

A wrong intuition

If Eve wins in $\mathcal{G} \times \mathcal{M}_L$ from $(q, u^{-1}L)$ and $u^{-1}L \subseteq v^{-1}L$, then she wins from $(q, v^{-1}L)$ ... however the *same strategy* might fail!

arena



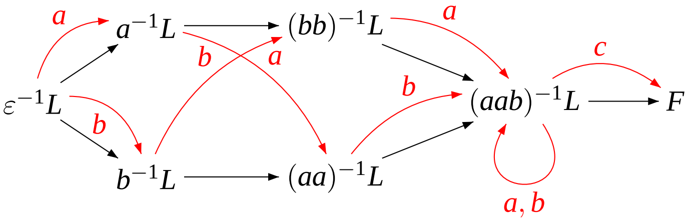
winning condition

$$L = a^+ \cdot b$$

$$\varepsilon^{-1}L \subset a^{-1}L$$

A global merging policy

$$L = (aab + baa) \cdot (a + b)^* \cdot c$$



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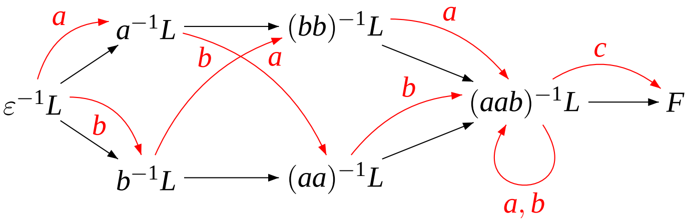

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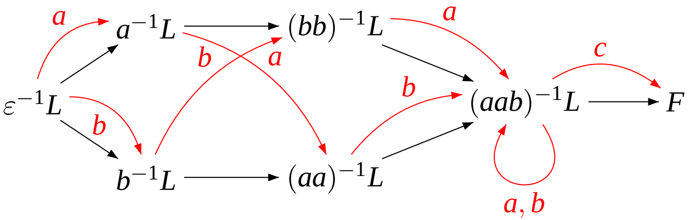

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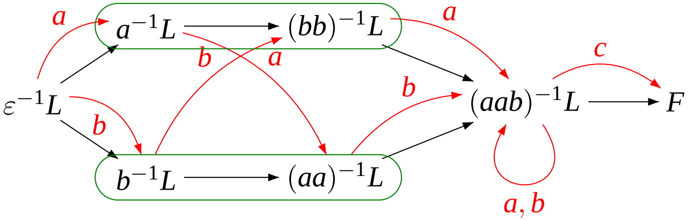

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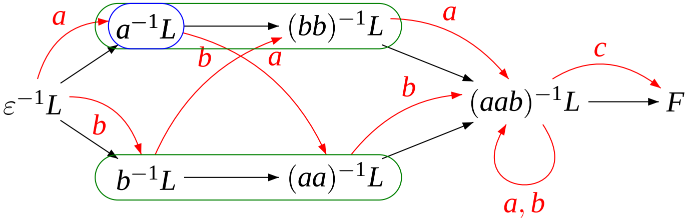

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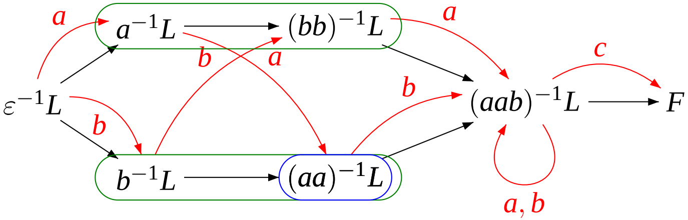

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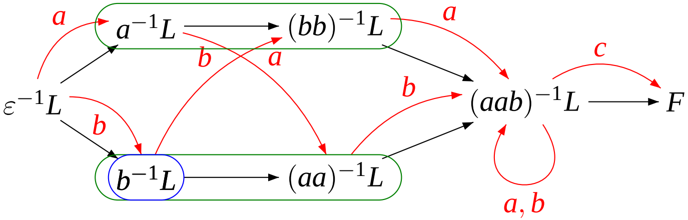

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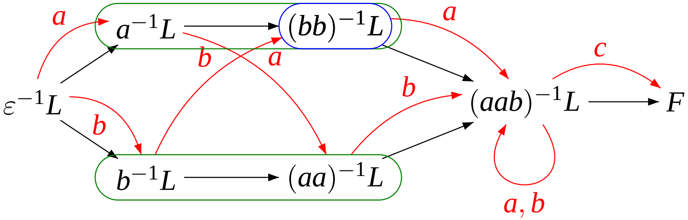

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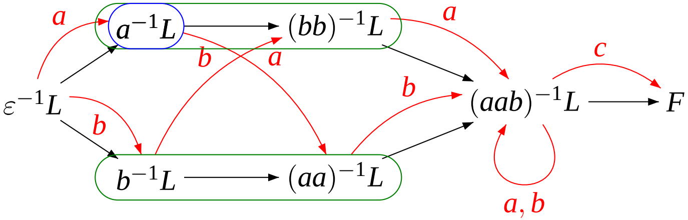

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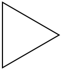
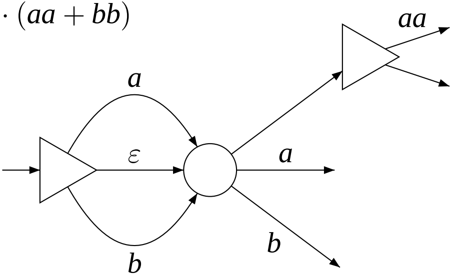
Outline

Playing optimally in stochastic arenas


Stochastic vertex

$L = (a + b)^* \cdot (aa + bb)$



Upper bound for both players

Since stochastic reachability games enjoy memoryless determinacy:

Lemma (Upper bound for both players)

For all stochastic regular games $\mathcal{G} = (\mathcal{A}, \mathrm{Reach}(L))$, both players have winning strategies using $\mathcal{M}_L$ as memory structure.

Lower bound for Eve

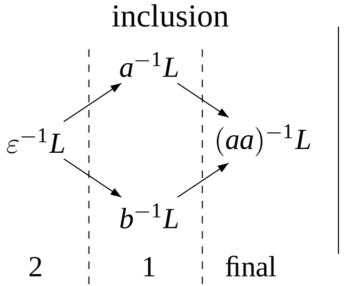
Lemma (Memory lower bound in stochastic games for Eve)

For all regular languages L, let n be the number of non-final left quotients of L, there exists an arena $\mathcal{A}$ such that Eve needs n memory states to play optimally in $\mathcal{G} = (\mathcal{A}, \mathrm{Reach}(L))$.

Again, we order left quotients inclusion-wise.

The construction exemplified

We construct an arena for the condition $L = (a+b)^* \cdot (aa+bb)$.



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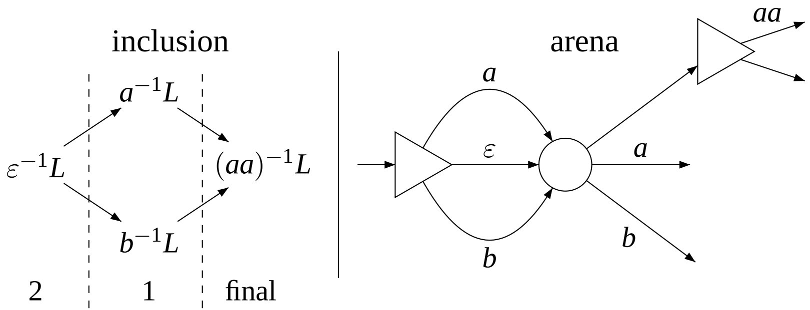

Adam case

The same applies to Adam, using a very similar construction.

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As opposed to the deterministic case, memory requirements are **symmetric** in stochastic regular games!

The end

Thank you for your attention!