Logical Formalisms Expressing Boundedness Properties over Infinite Trees

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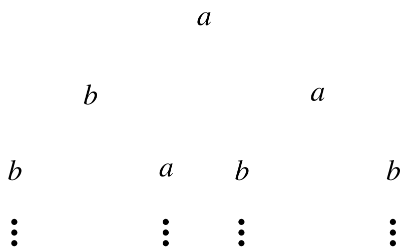
Outline

Logics over infinite (binary) trees

A tree:



A logical property:

"for all nodes a, there are finitely many nodes *below it* that contain a branch with infinitely many b's"

Rabin's theorem: decidability of MSO

The variables $x, y, \ldots$ are interpreted by nodes, $X, Y, \ldots$ by sets of nodes.

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Atomic formulæ:

$$a(x) \quad | \quad x \in X \quad | \quad LeftChild(x, y) \quad | \quad RightChild(x, y)$$

Constructors:

$$\underbrace{\wedge, \vee, \neg}_{\text{boolean connectives}} \quad | \quad \underbrace{\exists x}_{\text{first-order}} \quad | \quad \underbrace{\exists X}_{\text{monadic second-order}}$$

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Theorem (Rabin, 1969)

The following problem (called satisfiability problem) is decidable:

- *Instance: ϕ an MSO formula.*
- *Question: does there exist a tree* $\mathbf{t}$ *satisfying ϕ?*

Can we go further?

i.e. are there *decidable* extensions of MSO over infinite trees?

Can we talk about the *size* of sets?
About their *asymptotic behaviour*?

Some possible extensions

- “X is finite” , $|X| \geq 6$, $|X| \equiv |Y| \bmod 9$

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  $\looparrowright$ undecidable!

- $\mathbb{B}X, \phi$, defined by

$$\exists N \in \mathbb{N},\ \forall X, \quad \phi(X) \Rightarrow |X| \leq N$$

  $\looparrowright$ MSO $+\ \mathbb{B}$ was proposed by Bojańczyk in 2004

- ... ?

Bad news

Theorem (Hummel, Skrzypczak and Toruńczyk, 2010)

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End of the story?

Not quite!

Two directions

Uniform versus non-uniform quantification:

Satisfiability of MSO $+ \mathbb{B}$:

$\exists \mathbf{t}$ (tree),
$\mathbf{t} \models \phi \in \text{MSO} + \mathbb{B}$

non-uniform

Boundedness of cost MSO:
$\exists N \in \mathbb{N}$,
$\forall \mathbf{t}$ (tree),
$t \models \phi(N)$

uniform

First direction: weak MSO + $\mathbb{B}$

Theorem (Bojańczyk and Toruńczyk, 2012)

Weak MSO + $\mathbb{B}$ *is decidable.*

Second direction: cost MSO

Colcombet investigated *uniform* quantifications over bounds:

Add “$|X| \leq N$” to MSO formulæ.

Hope (Colcombet, 2009)

The boundedness problem is decidable:

- *Instance:* $\phi(N)$ *a cost MSO formula.*
- *Question:* $\exists N, \forall \mathbf{t}, \quad \mathbf{t} \models \phi[N]$*?*

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Wide open for infinite trees! It would solve a long-standing open problem (the decidability of the Mostowski index).

Outline

A proof of Rabin's theorem, by Muller and Schupp

Alternating parity automata:

$$\mathcal{A} = (Q, A, q_0, \delta, \text{Parity}),\ \text{where } \delta : Q \times A \to \underbrace{\mathcal{B}^+(Q \times Q)}_{\text{positive boolean combinations}}$$

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A tree **t** induces a two-player game between Eve and Adam:

- Eve chooses disjunctions,
- Adam chooses conjunctions,
- Adam chooses directions.

t is accepted if Eve wins the acceptance game.

q_0

a

b a

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Simulating alternating automata by non-deterministic ones relies on:

- determinization of parity automata over infinite words,
- positional determinacy of parity games.

The three ingredients

1. Determinacy of parity games
2. Determinization of parity automata over infinite words
3. Positional determinacy of parity games

Lifting the proof for cost MSO

1. Define alternating B-parity and S-parity automata
2. Show that the B- and S-variants are equivalent to each other
3. Show that they are equivalent to their non-deterministic variants
4. Show the appropriate closure properties
5. Solve the boundedness problem on non-deterministic S-automata

ND B-automata

ND S-automata

3

3

2

Alternating B-automata

Alternating S-automata

2

Towards cost MSO over infinite trees

We need to generalize the three ingredients to *B*-parity games:

1. Determinacy: ✓ (Borel determinacy takes over)
2. Determinization: ✓ (history-deterministic automata fill in!)
3. Positional determinacy: only partial results...

Outline

Definition

controlled by Eve

controlled by Adam

Definition

controlled by Eve

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parity
and
all counters
are bounded

2 0

3 2 3

1

4

parity condition:

the minimal priority seen infinitely often is even

Definition

i, ε

ε, i i, i

i, i

ε, i ε, ε

ε, i

i, r i, ε

r, i

ε, r

$$c_1 = 0$$

$$c_2 = 0$$

ε : nothing

i : increment

r : reset

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r, i

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ε, r

parity
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all counters
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Uniform versus non-uniform quantification

Eve wins means:

$\exists\sigma$ (strategy for Eve),
$\forall\pi$ (paths),
$\exists N \in \mathbb{N}$,

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non-uniform
($\mathrm{MSO} + \mathbb{B}$)

uniform
(cost MSO)

Strategy (for Eve)

General form

$$\sigma : V^+ \to V$$

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$$\sigma : V^+ \to V$$

Positional or memoryless

$$\sigma : V \to V$$

Strategy (for Eve)

General form

$$\sigma : V^+ \to V$$

Positional or memoryless

$$\sigma : V \to V$$

Finite-memory

$$\begin{cases} \sigma : V \times M \to V \\ \mu : M \times E \to M \end{cases}$$

Colcombet's Conjecture

Fix a game G and assume Eve wins $B(N) \cap \text{Parity}$.

Observation

Eve has a strategy with $N+1$ memory states to ensure $B(N) \cap$ Parity.

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Observation

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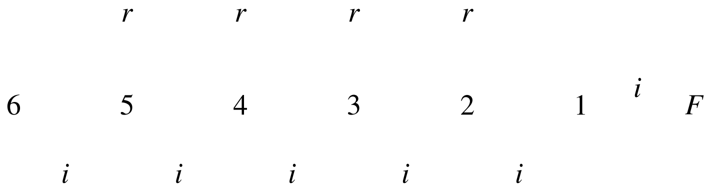
The conjecture involves a trade-off between memory and quality:

Conjecture

There exists a function $\alpha : \mathbb{N} \to \mathbb{N}$ and a constant $m \in \mathbb{N}$ such that for all games:

if Eve wins $B(N) \cap$ Parity,
then she has a strategy with m memory states to ensure
$B(\alpha(N)) \cap$ Parity.

An example



Generalization:

- Eve wins $B(N) \cap \mathit{Reach}(F)$ with $N+1$ memory states,
- Eve wins $B(2 \cdot N) \cap \mathit{Reach}(F)$ with 3 memory states,

Outline

Cost weak MSO

Theorem (Vanden Boom, 2011)

For infinite chronological games:

- *If Eve wins $B(N) \cap$ Büchi, then she has a strategy with 2 memory states to ensure $B(N) \cap$ Büchi.*
- *If Eve wins $\overline{B}(N) \cup$ Büchi, then she has a strategy with 2 memory states to ensure $\overline{B}(N) \cup$ Büchi.*

Corollary

Cost weak MSO is decidable.

Temporal cost MSO

Theorem ("Folklore in the regular cost function community")

For infinite chronological games without ε:

- *If Eve wins $B(N) \cap Parity$, then she has a strategy with 2 memory states to ensure $B(N) \cap Parity$.*
- *If Eve wins $\overline{B}(N) \cup Parity$, then she has a strategy with 2 memory states to ensure $\overline{B}(N) \cup Parity$.*

Corollary

MSO + "$|x - y| \leq N$" (called temporal cost MSO) is decidable.

Cost MSO over thin trees

A tree is thin if it has countably many branches.

Theorem (F.,Horn,Kuperberg,Skrzypczak, unpublished)

Colcombet's Conjecture holds for thin tree games (with non-elementary bounds).

Corollary

Cost MSO is decidable over thin trees.

Conclusion

To extend Rabin's theorem to cost MSO via Muller and Schupp's proof, the following three ingredients are required:

1. Determinacy of B-parity games: ✓
2. Determinization of B-parity automata over infinite words: ✓
3. Finite-memory determinacy for B-parity games: ongoing