Boundedness Games

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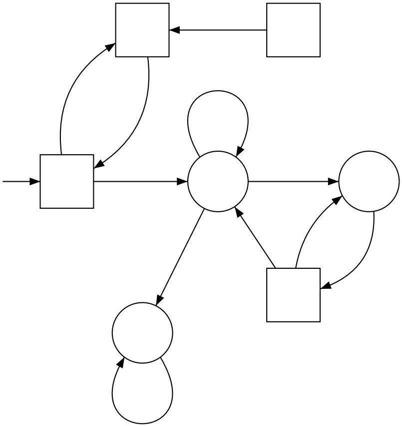
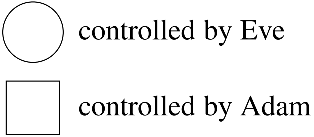
(based on joint works with Krishnendu Chatterjee, Thomas Colcombet, Florian Horn and Martin Zimmermann)

Definition of ωB-games

Two-player turn-based games over **finite** or **infinite** graphs





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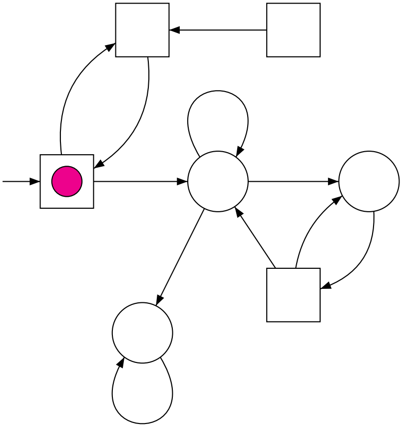

controlled by Eve

controlled by Adam

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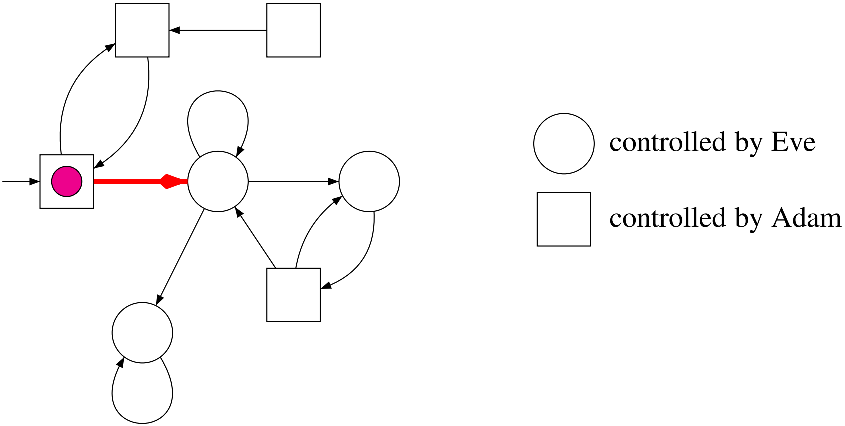

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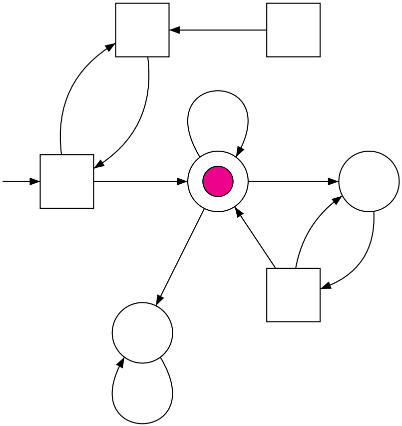

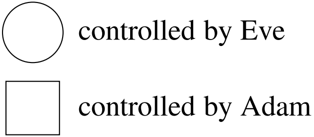

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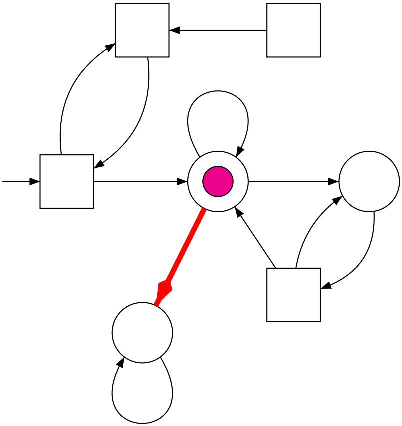

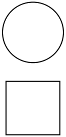

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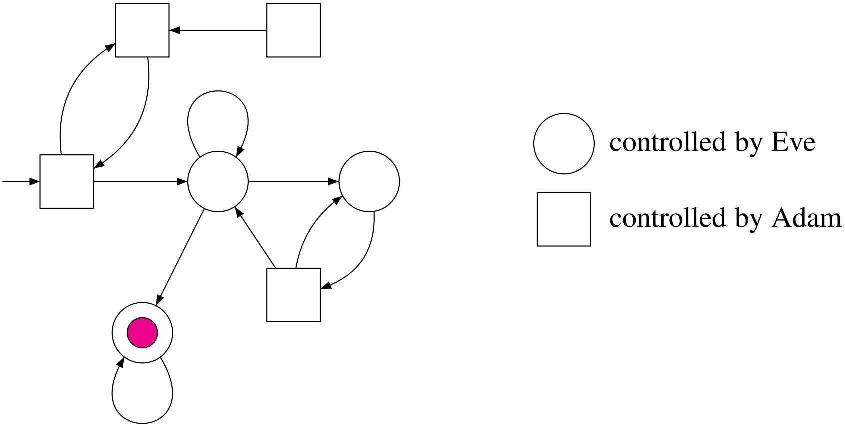

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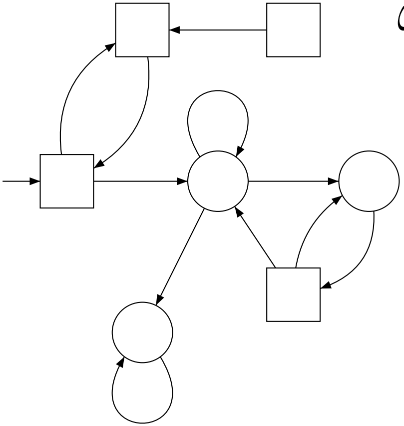

ωB winning condition:

parity
and
all counters
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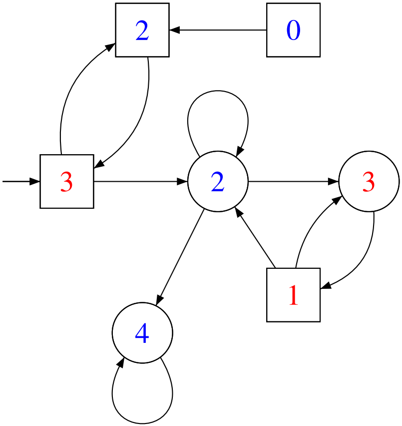

parity condition:

the minimal priority seen infinitely often is even

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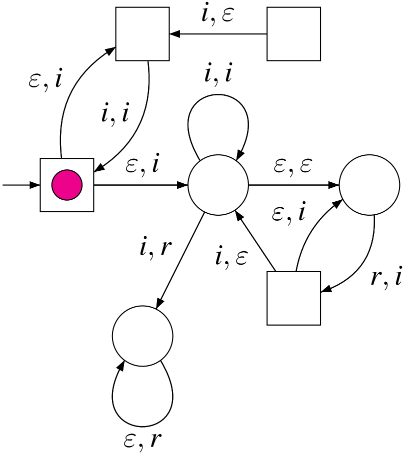

$$c_1 = 0$$
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ε : nothing

i : increment

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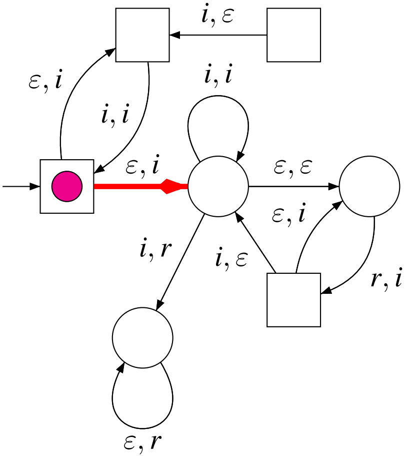

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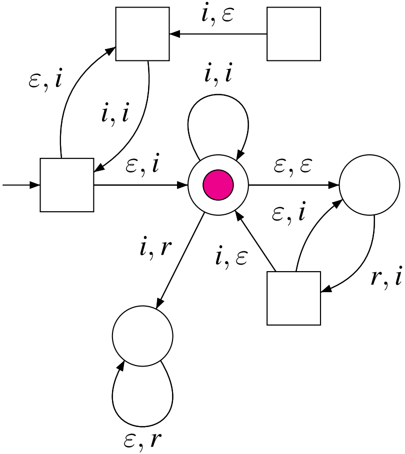

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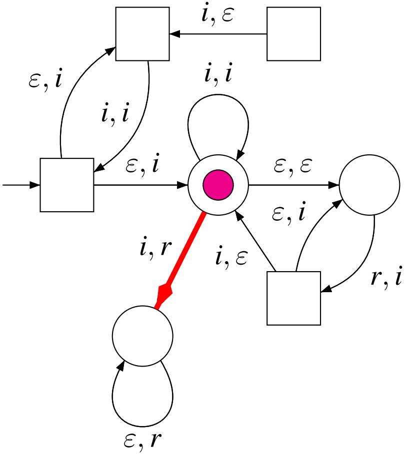

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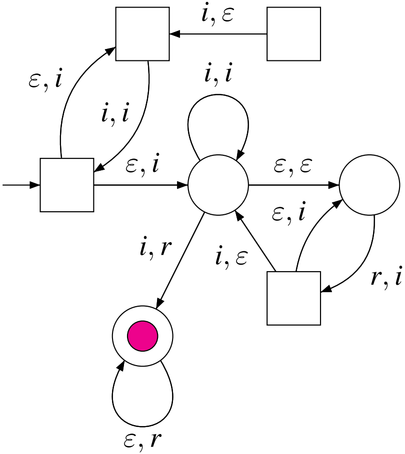

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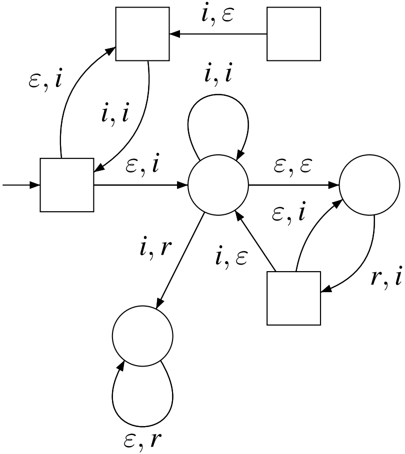

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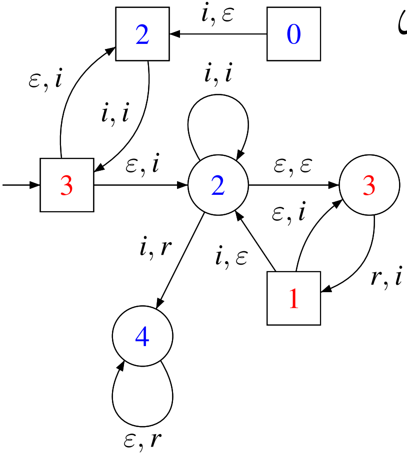

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General form

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What about ωB games?

Finite-memory

$$\begin{cases} \sigma : V \times M \to V \\ \mu : M \times E \to M \end{cases}$$

Why finite-memory strategies?

Thomas Colcombet's habilitation:

> le fait 2.22 et en déduire que la domination entre formules de la logique monadique de coût est décidable sur les arbres infinis. Ainsi, la conjecture 9.2 implique la conjecture 9.1.
>
> En fait, il est possible de pointer avec encore plus de précision où se trouve la difficulté. Si l'on cherche à démontrer la conjecture 9.2, tout comme dans le cas des arbres finis, le point crucial est l'existence de stratégies gagnantes à mémoire finie. Il suffirait d'établir la conjecture suivante.
>
> **Conjecture 9.3.** *Les objectifs* $\mathtt{hB} \wedge \mathtt{parit\acute{e}}$ *et* $\neg\mathtt{B} \wedge \mathtt{parit\acute{e}}$ *sont à* $\approx$*-mémoire finie, sur toutes les arènes/sur les arènes chronologiques/sur les arènes «arborescentes».*

Existence of finite-memory strategies in (some) boundedness games
$\Longrightarrow$ Decidability of cost MSO over infinite trees
$\Longrightarrow$ Decidability of the index of the non-deterministic Mostowski's hierarchy (open for 40 years)!

Quantification

Eve wins means:

$\exists \sigma$ (strategy for Eve),
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Is this:

1. Equivalent? Sometimes ...
2. Decidable? Not always ...

The questions I am interested in

Boundedness games:

1. Over finite graphs: decide the winner efficiently and construct small finite-memory strategies.

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   ↬ Pushdown ωB games [Chatterjee and F., 2013].
3. Over infinite graphs: prove the existence of finite-memory winning strategies.
   ↬ Ongoing work with Thomas Colcombet and Florian Horn.

Outline

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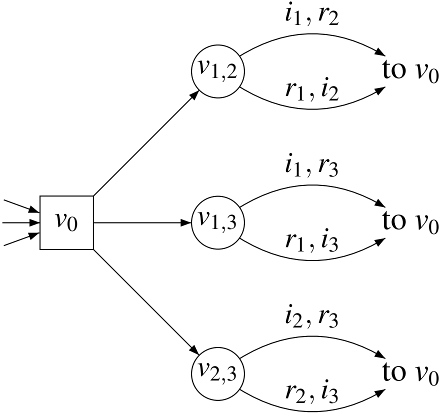
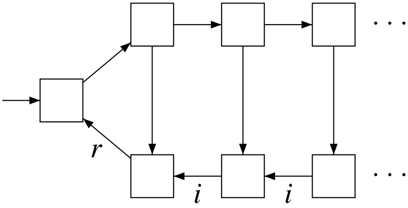
Eve needs some memory



Adam needs infinite memory



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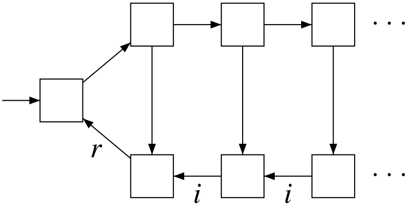

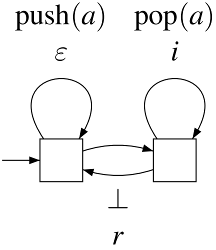

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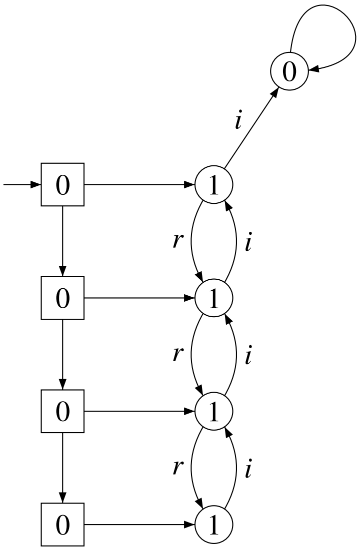

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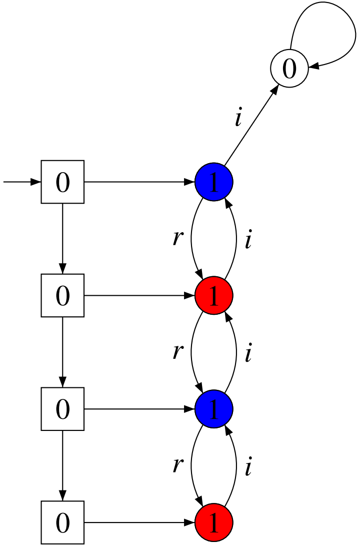

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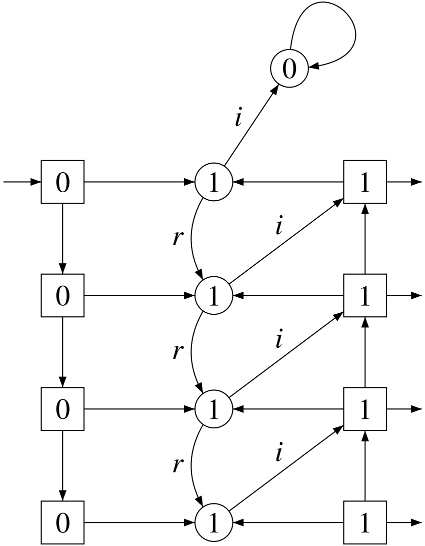

Conjecture's status

Conjecture (Thomas Colcombet's conjecture)

Over arbitrary graphs / chronological graphs / tree-like graphs,

if Eve has a strategy ensuring $\mathrm{Parity} \cap \mathrm{Bound}(N)$,
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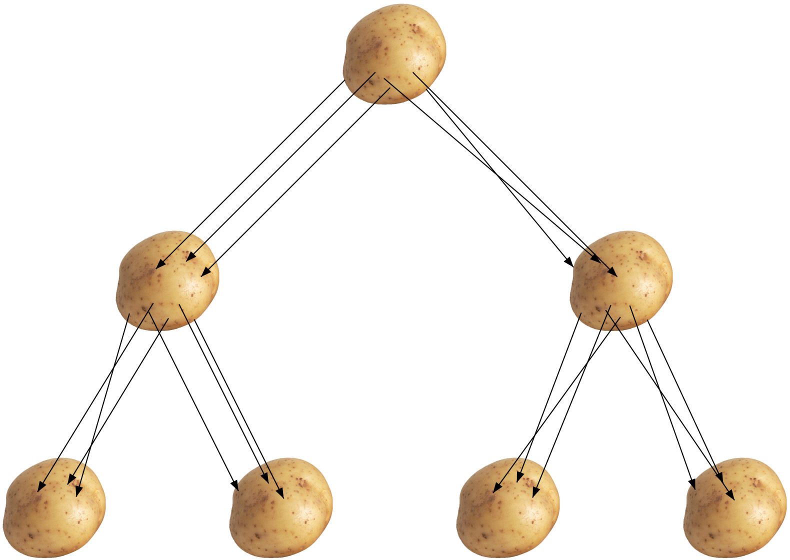
With some more effort:

Theorem

The conjecture does not hold over chronological graphs.

Potato-trees



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Counting conditions

A subclass of ωB conditions where the counters and parity are not independent.

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Over general graphs:

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- *Eve has finite-memory winning strategies in finitary parity games.*

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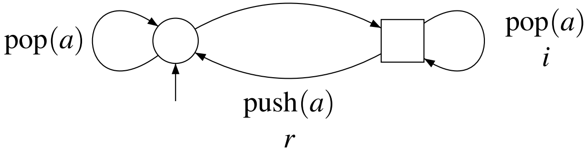
In some sense, the class of counting conditions is the maximal subclass of ωB conditions for such results.

Outline

Some more examples (1)



Eve should maintain a low stack.

Objective

Theorem

For all pushdown games, the following are equivalent:

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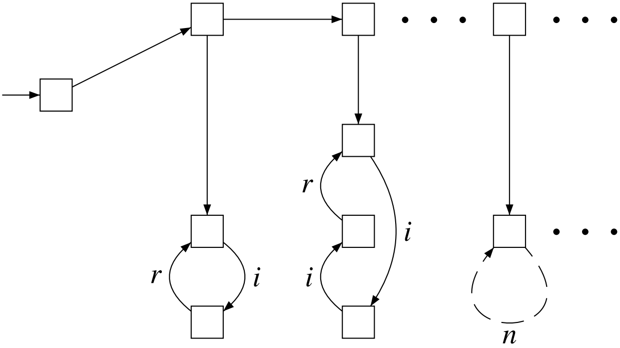
Counter-example for the general case



Eve wins but she does not know the bound!

Outline

A simple proof for the case of finite graphs

Condition: parity and all counters are bounded.

Define:

- $\mathcal{W}_E(N)$ the set of vertices where Eve wins for the bound N.
- $\mathcal{W}_E$ the set of vertices where Eve wins for some (non-uniform) bound.

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2. *There exists N such that $\mathcal{W}_E(N) = \mathcal{W}_E(N+1) = \cdots$.*
3. *For such N, Adam wins from $V \setminus \mathcal{W}_E(N)$, hence $\mathcal{W}_E = \mathcal{W}_E(N)$.*

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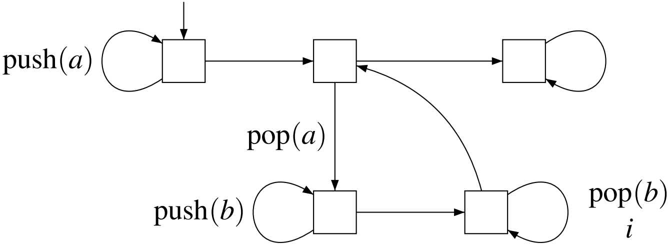

Adam may use the stack as “credit”.

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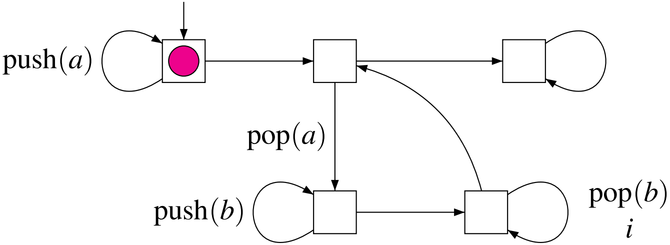

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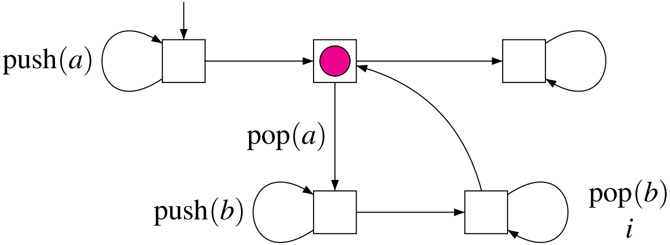

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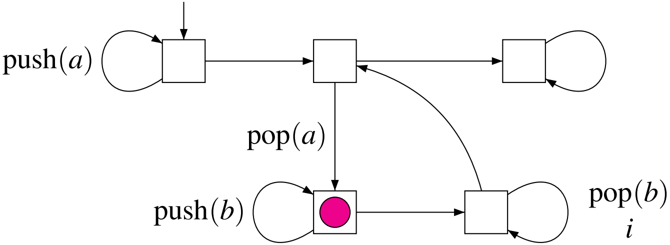

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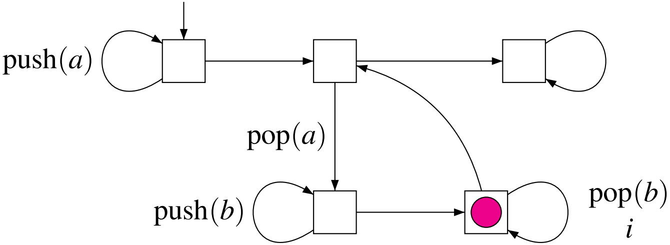

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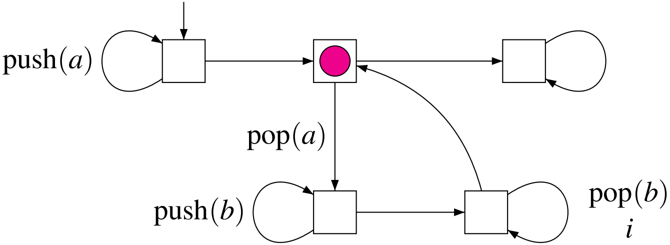

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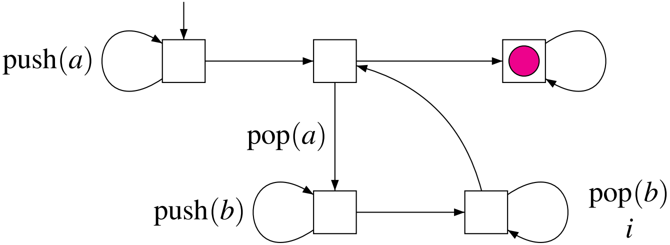

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Proof sketch

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3. *For such N, Adam wins from $V \setminus \mathcal{W}_E(N)$, hence $\mathcal{W}_E = \mathcal{W}_E(N)$.*

Why is 2. true?

Regularity of the winning regions

Theorem (derived from Serre)

*For all N, $\mathcal{W}_E(N)$ is a regular set of configurations, recognized by an alternating automaton of size $|Q|$ (**independent of** N).*

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Corollary

Determining the winner in a pushdown ωB*-game is decidable.*

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Theorem

For all pushdown games, the following are equivalent:

- $\exists\sigma$ *(strategy for Eve),* $\forall\pi$ *(paths),* $\exists N \in \mathbb{N}$,
  π *satisfies parity and each counter is bounded by* N.
- $\exists\sigma$ *(strategy for Eve),* $\exists N \in \mathbb{N}$, $\forall\pi$ *(paths),*
  π *satisfies parity and* ***eventually*** *each counter is bounded by* N.

Corollary

Determining the winner in a pushdown ωB*-game is decidable.*

Remark: one can show that the collapse bound is doubly-exponential!

Outline

The max operator

We add a new feature for counters: $\gamma \leftarrow \max(\gamma_1, \gamma_2)$.

Theorem (derived from Bojańczyk and Toruńczyk)

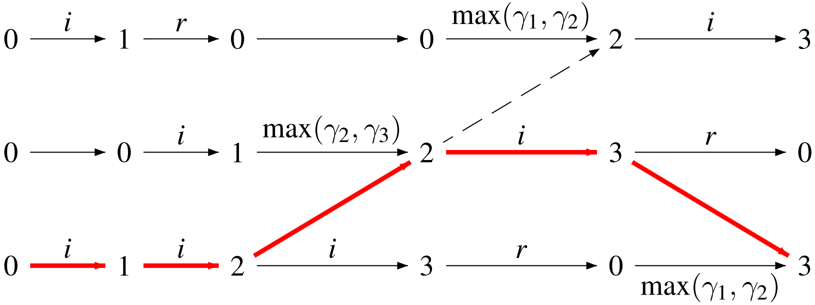
Deterministic max-automata are equivalent to Weak MSO + $\mathbb{U}$.

We consider ωB-games with max.

Co-determinisation

$$
\begin{tikzcd}
0 \arrow[r, "i"] & 1 \arrow[r, "r"] & 0 \arrow[r] & 0 \arrow[r, "\max(\gamma_1,\gamma_2)"] & 2 \arrow[r, "i"] & 3 \\
0 \arrow[r] & 0 \arrow[r, "i"] & 1 \arrow[r, "\max(\gamma_2,\gamma_3)"] & 2 \arrow[r, "i"] \arrow[ur, dashed] & 3 \arrow[r, "r"] \arrow[dr, dashed] & 0 \\
0 \arrow[r, "i"] & 1 \arrow[r, "i"] & 2 \arrow[r, "i"] \arrow[ur, dashed] & 3 \arrow[r, "r"] & 0 \arrow[r, "\max(\gamma_1,\gamma_2)"'] & 3
\end{tikzcd}
$$

Co-determinisation



To prove that a counter value is high, one can count backwards!

Reduction

We reduce ωB-games with max to pushdown ωB-games (without max).

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Idea: simulate the game and store the play in the stack.

Whenever he wants, Adam can declare “this counter has a very large value”: from there, play backwards using the stack until a reset is met.

Reduction

We reduce ωB-games with max to pushdown ωB-games (without max).

Idea: simulate the game and store the play in the stack.

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Theorem

Determining the winner in an ωB-game with max is decidable.

The end.

Thank you!

Outline

1. Finite-memory strategies
   - Some examples
   - Counting conditions
2. Equivalence for pushdown ωB games
   - The case of finite graphs
   - The case of pushdown graphs
   - Application: ωB-games with max