# The Value 1 Problem for Probabilistic Automata

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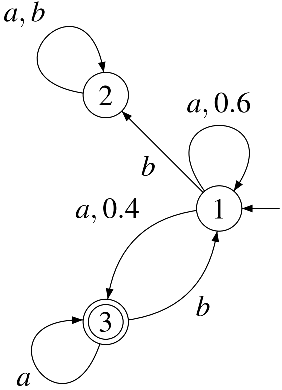
# Probabilistic automata (Rabin, 1963)






$$\mathbb{P}_{\mathcal{A}} : A^* \to [0, 1]$$

$\mathbb{P}_{\mathcal{A}}(w)$ is the probability that a run for $w$ ends up in $F$

# The value 1 problem



This talk is about the value 1 problem:

INPUT: $\mathcal{A}$ a probabilistic automaton
OUTPUT: for all $\varepsilon > 0$, there exists $w \in A^*$, $\mathbb{P}_{\mathcal{A}}(w) \geq 1 - \varepsilon$.

In other words, define $\mathrm{val}(\mathcal{A}) = \sup_{w \in A^*} \mathbb{P}_{\mathcal{A}}(w)$, is $\mathrm{val}(\mathcal{A}) = 1$?

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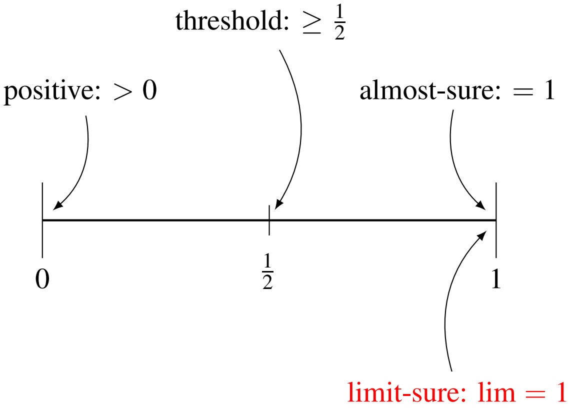
In other words, define $\text{val}(\mathcal{A}) = \sup_{w \in A^*} \mathbb{P}_{\mathcal{A}}(w)$, is $\text{val}(\mathcal{A}) = 1$?

It is undecidable (Gimbert and Oualhadj, 2010).

But *to what extent*?

# Terminology





# A Research Program



Construct an algorithm to decide the value 1 problem,
which is *often* correct.

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Quantify *how often*.

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Construct an algorithm to decide the value 1 problem,
which is *often* correct.

Quantify *how often*.

Argue that you cannot do *more often* than that.

# Outline

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## At first thought



Does the *undecidability* come from the *numerical* values?

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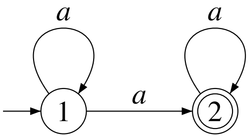
## Does the *undecidability* come from the *numerical* values?

Consider *numberless* probabilistic automata:




Two decision problems:

- for all $\Delta$, $\mathrm{val}(\mathcal{A}[\Delta]) = 1$,
- there exists $\Delta$, such that $\mathrm{val}(\mathcal{A}[\Delta]) = 1$.

# No future (in this direction)



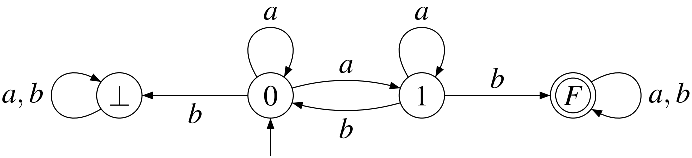
**Theorem (F., Horn, Gimbert and Oualhadj)**

*There is no algorithm such that:*
*On input $\mathcal{A}$ (a numberless probabilistic automaton),*

- *if for all $\Delta$, $val(\mathcal{A}[\Delta]) = 1$ then "YES",*
- *if for all $\Delta$, $val(\mathcal{A}[\Delta]) < 1$ then "NO",*
- *anything in the other cases.*

# Outline

# An example



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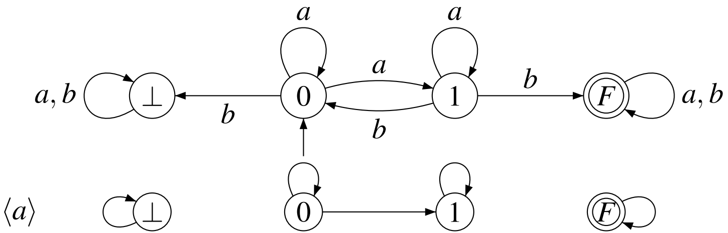

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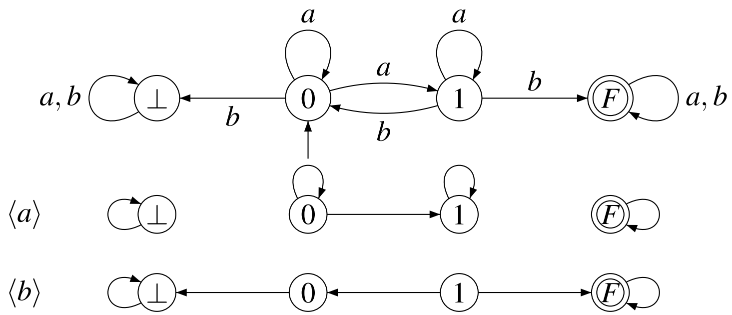

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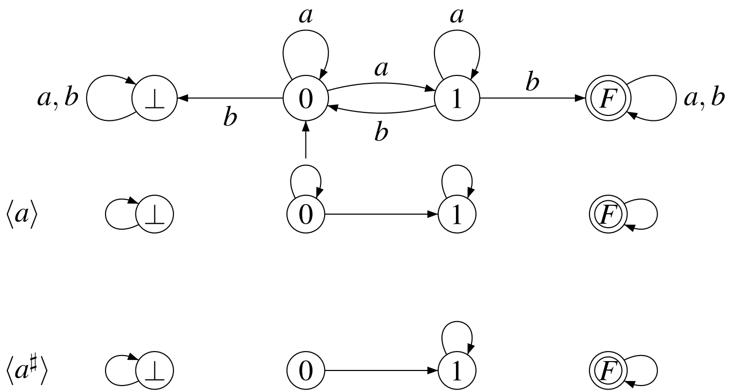

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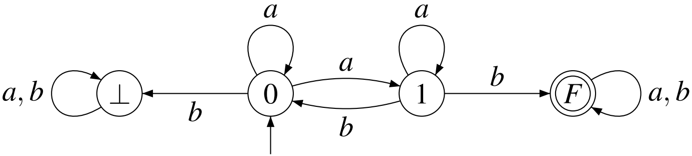
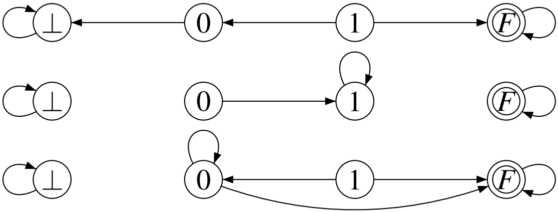



$\langle b \rangle$

$\langle a^\sharp \rangle$

$\langle a^\sharp \cdot b \rangle$



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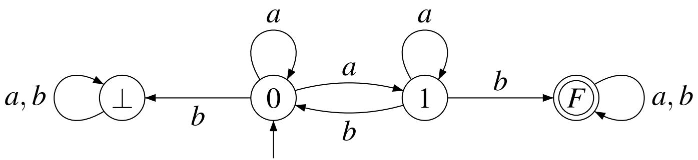
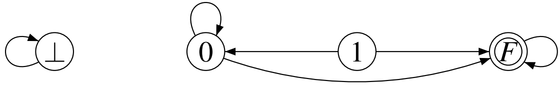


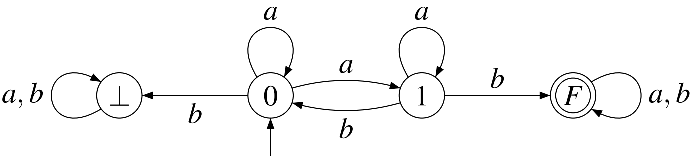

$\langle a^\sharp \cdot b \rangle$



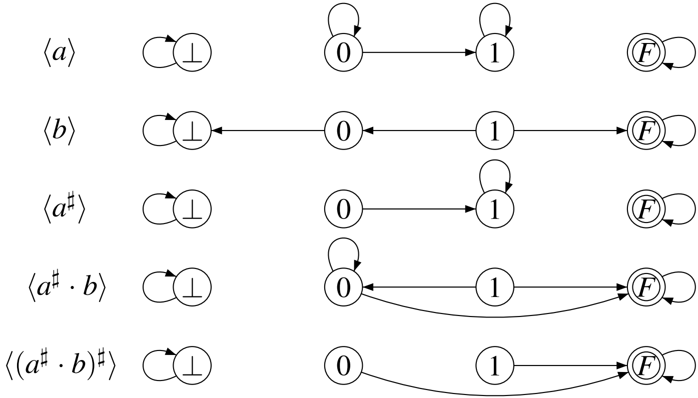

$\langle (a^\sharp \cdot b)^\sharp \rangle$

# An example





$a$
$a$
$a$
$a, b$
$\perp$
$0$
$1$
$F$
$a, b$
$b$
$b$
$b$

$\langle a \rangle$
$\langle b \rangle$
$\langle a^\sharp \rangle$
$\langle a^\sharp \cdot b \rangle$
$\langle (a^\sharp \cdot b)^\sharp \rangle$

# Stabilization monoids (Colcombet)



This is an algebraic structure with two operations:

- binary composition
- stabilization, denoted $\sharp$

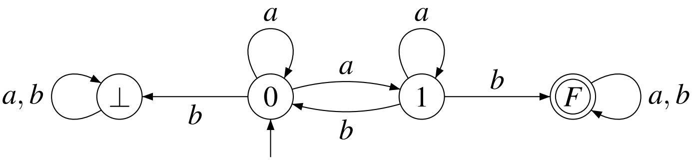
With some natural axioms:

$$(u^\sharp)^\sharp = u^\sharp$$

$$u \cdot (v \cdot u)^\sharp = (u \cdot v)^\sharp \cdot u$$

# Boolean matrices representations



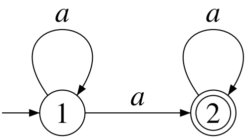



$$\langle a \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \langle b \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$I \cdot \langle u \rangle \cdot F = 1 \quad \text{if and only if} \quad \mathbb{P}_{\mathcal{A}}(u) > 0$$

# Defining stabilization



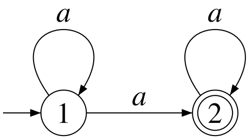



$$\langle a \rangle = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

In $\langle a \rangle$, the state 1 is transient and the state 2 is recurrent.

# Defining stabilization



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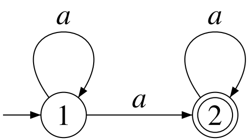


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In $\langle a \rangle$, the state 1 is transient and the state 2 is recurrent.

$$M^{\sharp}(s,t) = \begin{cases} 1 & \text{if } M(s,t) = 1 \text{ and } t \text{ recurrent in } M, \\ 0 & \text{otherwise.} \end{cases}$$

# The Markov Monoid Algorithm



Compute a monoid inside the **finite** monoid $\mathcal{M}_{Q\times Q}(\{0,1\},\vee,\wedge)$.

- Compute $\langle a\rangle$ for $a \in A$:

$$\langle a\rangle(s,t) = \begin{cases} 1 & \text{if } \mathbb{P}_{\mathcal{A}}(s \xrightarrow{a} t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Close under product and stabilization.

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- Close under product and stabilization.
- If there exists a matrix $M$ such that

$$\forall t \in Q, \quad M(s_0,t) = 1 \Rightarrow t \in F$$

then "$\mathcal{A}$ has value 1", otherwise "$\mathcal{A}$ does not have value 1".

# Correctness



**Theorem**

*If there exists a matrix $M$ such that*

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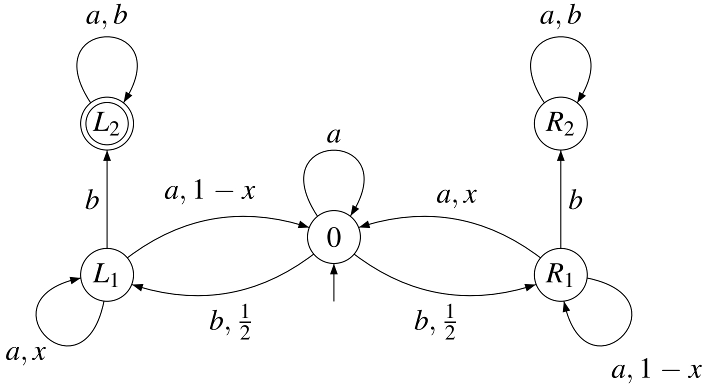
*then $\mathcal{A}$ has value* 1.

But the value 1 problem is undecidable, so…

# No completeness





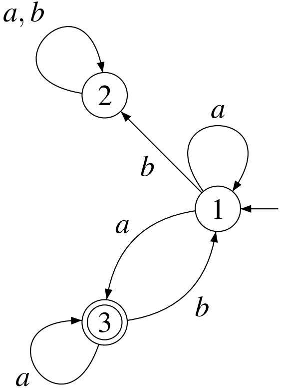

Left and right parts are symmetric, so for all $M$:

$$M(0, L_2) = 1 \iff M(0, R_2) = 1.$$

Yet: it has value 1 if and only if $x > \frac{1}{2}$.

# A leak





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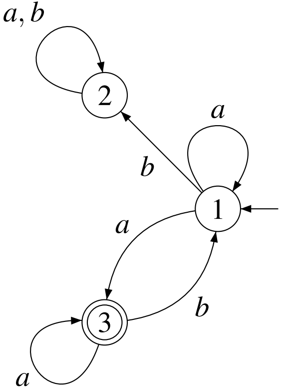
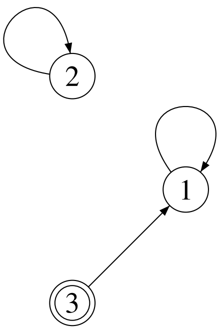
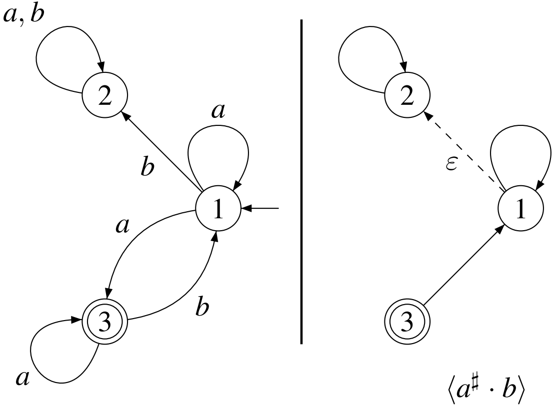



$\langle a^{\sharp} \cdot b \rangle$

# A leak

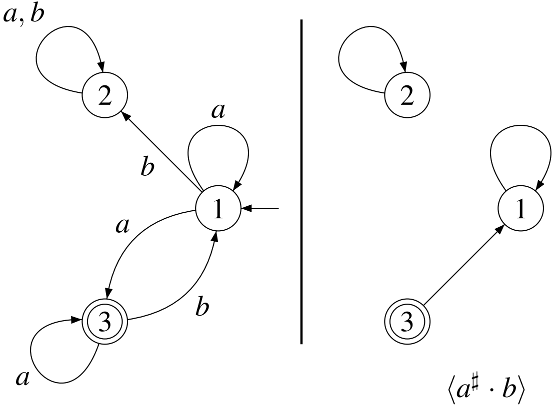





There is a leak from 1 to 2.

# A leak






There is a leak from 1 to 2.

**Definition**

An automaton $\mathcal{A}$ is leaktight if it has no leak.

# Leaktight automata



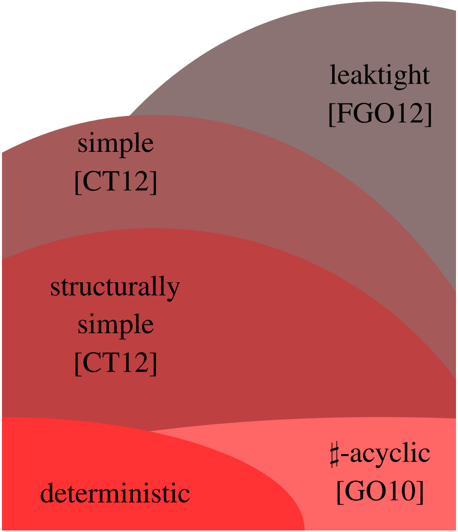
Theorem (F.,Gimbert and Oualhadj 2012)

*The algorithm is complete for leaktight automata.*
*Hence, the value* 1 *problem is decidable for leaktight automata.*

The proof relies on Simon's factorization forest theorem.

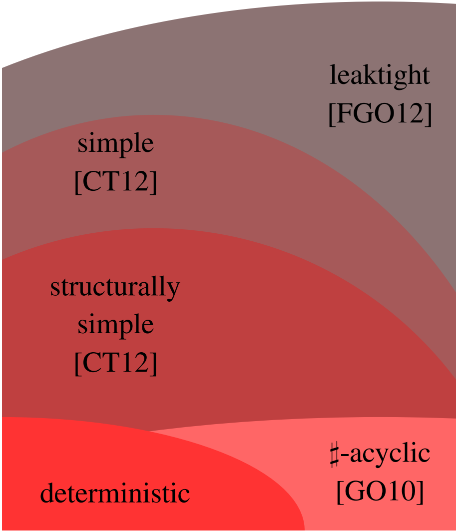
# Other decidable subclasses: in 2012





# Other decidable subclasses: today





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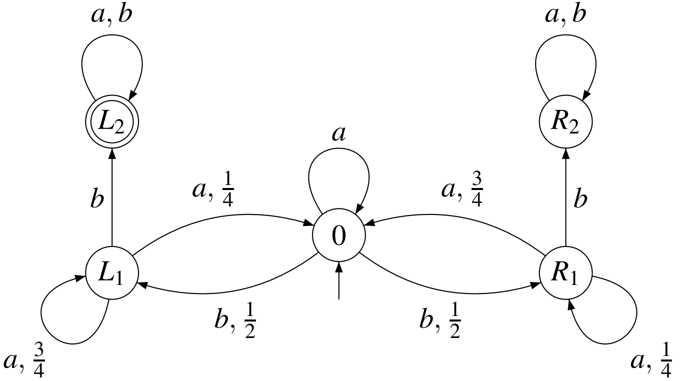
But for *how long*?

# Outline

# What it misses: different convergence speeds






$$\lim_n \mathbb{P}_{\mathcal{A}} \left( b(a^n b)^{f(n)} \right) = 1 \text{ if and only if } \lim_n f(n) \cdot \left( \frac{3}{4} \right)^n = \infty \, ,$$

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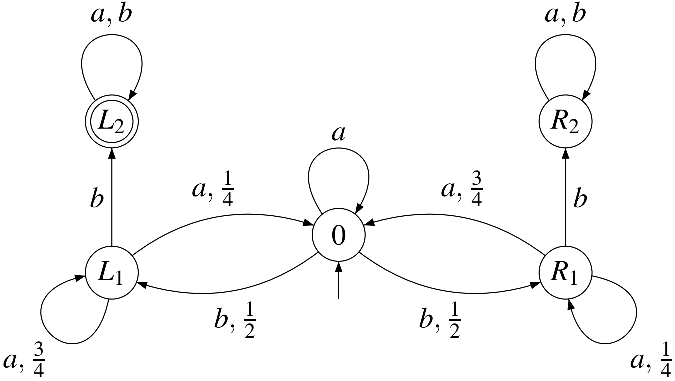


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so $f(n) = 2^n$ works but $f(n) = n$ does not.

# A characterization



$\widetilde{A^*}$ is the space of prostochastic words.

$$A^* = \widetilde{A^*}[0] \subsetneq \widetilde{A^*}[1] \subsetneq \widetilde{A^*}[2] \subsetneq \cdots \subsetneq \widetilde{A^*}\,.$$

**Lemma**

*The following are equivalent:*

- *The value* 1 *problem over finite words,*
- *The emptiness problem over prostochastic words.*

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**Lemma**

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**Theorem**

1. *The Markov Monoid Algorithm answers "YES" if and only if there exists $x \in \widetilde{A^*}[1]$ accepted by $\mathcal{A}$,*
2. *The following problem is undecidable: determine whether there exists $x \in \widetilde{A^*}[2]$ accepted by $\mathcal{A}$.*

# Prostochastic words



**Definition**

$(u_n)_{n \in \mathbb{N}}$ converges if for every $\mathcal{A}$, the limit $\lim_n \mathbb{P}_{\mathcal{A}}(u_n)$ exists.

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Two (converging) sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are equivalent if for every $\mathcal{A}$,

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**Definition**

A prostochastic word is an equivalence class of converging sequences.

# The $\omega$ operators



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Let $u$ be a converging sequence.
$u^{\omega_1}$ is the converging sequence $(u_n^{n!})_{n\in\mathbb{N}}$.

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**Example**

The prostochastic words $(a^{\omega_1}b)^{\omega_1}$ and $(a^{\omega_1}b)^{\omega_2}$ are not equal.

# An equivalent characterization



**Theorem**

*The Markov Monoid Algorithm answers "YES" if and only if there exists a regular sequence $(u_n)_{n\in\mathbb{N}}$ of finite words such that $\lim_n \mathbb{P}_{\mathcal{A}}(u_n) = 1$.*

The regular sequences are described by the following grammar:

$$u \quad = \quad a \quad | \quad u \cdot u \quad | \quad (u_n^n)_{n\in\mathbb{N}} \, .$$

## Corollary



In *some* sense,

the Markov Monoid Algorithm is
the *most correct* algorithm
to solve the value 1 problem.

# Outline



1. Theory
   - A first attempt: get rid of numerical values
   - A second attempt: the Markov Monoid Algorithm
   - On the optimality of the Markov Monoid Algorithm
2. Practice: ACMÉ

*The tool ACME (Automata with Counters, Monoids and Equivalence) has been written in OCaml by Nathanaël Fijalkow and Denis Kuperberg.*

*************************************

*Statistics on the Markov Monoid Algorithm:*

*************************************

*Over* 1000 *Automata of size* 7 *with* 15 *transitions:*

540 *are leaktight and do not have value 1*

133 *are leaktight and have value 1*

17 *are not leaktight and may have value 1*

310 *are not leaktight and have value 1*