

How much memory is needed to win regular games?

ENS Cachan

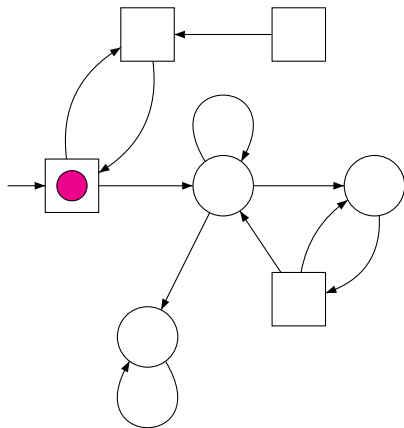
Thomas Colcombet, Nathanaël Fijalkow and Florian Horn

LIAFA, Paris

September 8th, 2011

controlled by Adam

Games

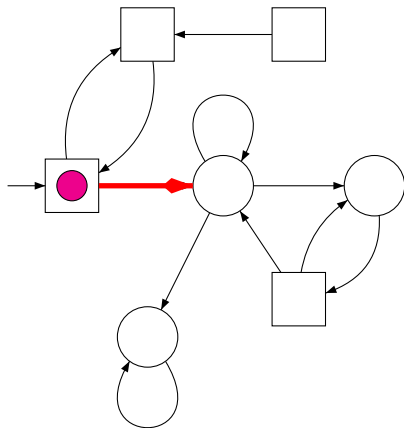


controlled by Eve



controlled by Adam

Games

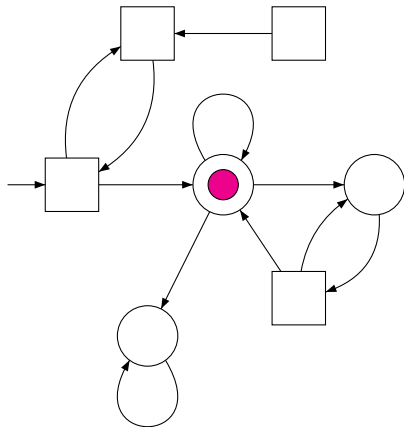


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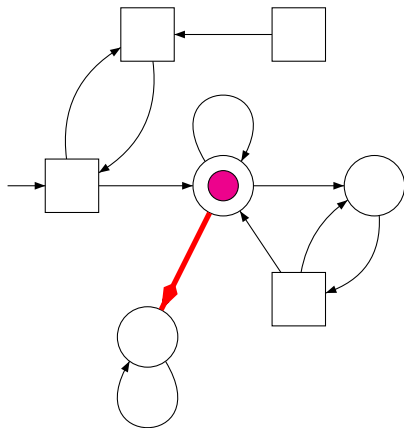


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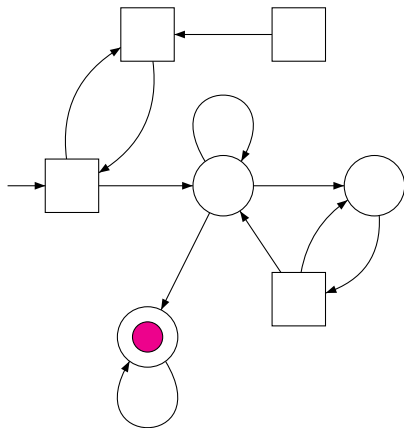
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controlled by Adam

Games



Games

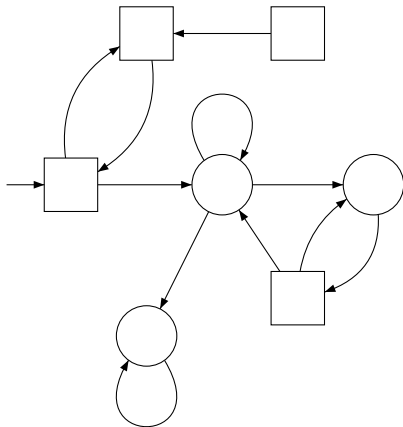


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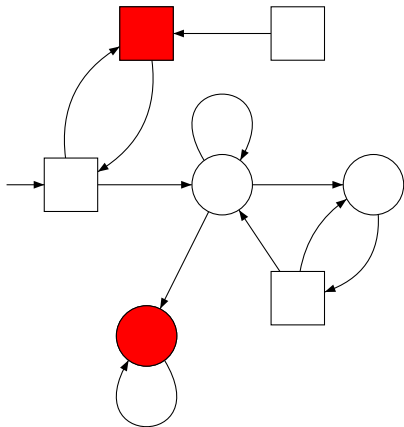
controlled by Adam

Introduction: reachability games



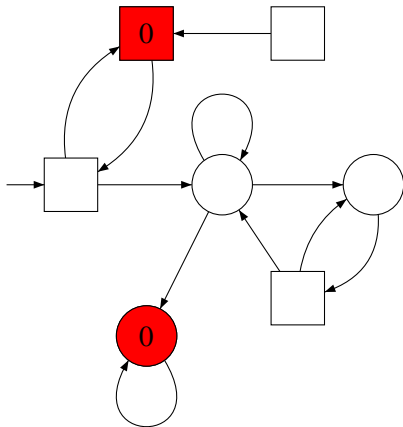
Given $F \subseteq Q$

Introduction: reachability games



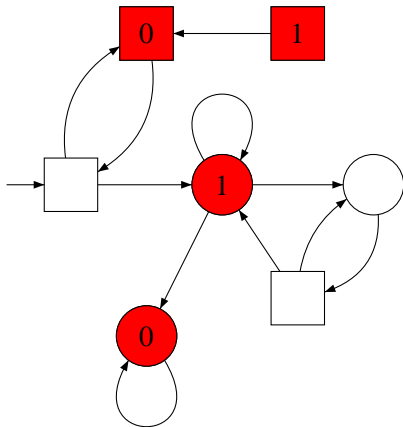
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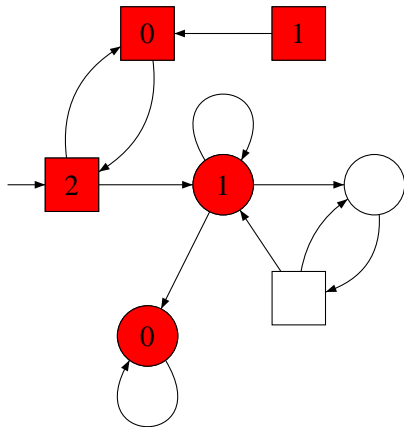
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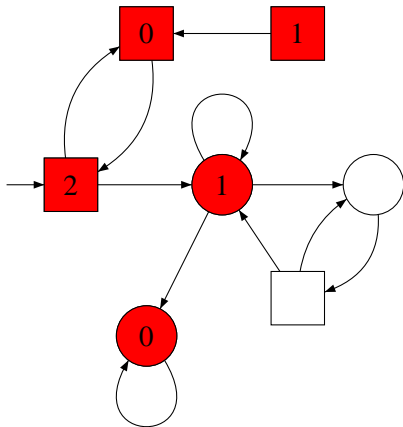
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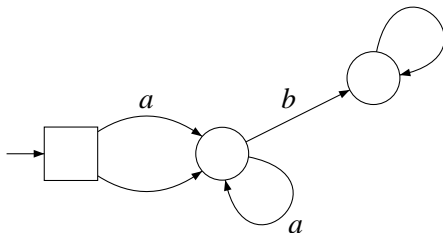
Given $F \subseteq Q$

Memoryless winning strategies

$\sigma : V \rightarrow E$ and $\tau : V \rightarrow E$

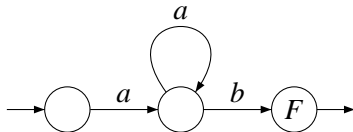
Introduction: regular games

arena



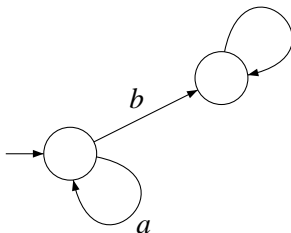
winning condition

$$L = a^+ \cdot b$$



Regular games need memory

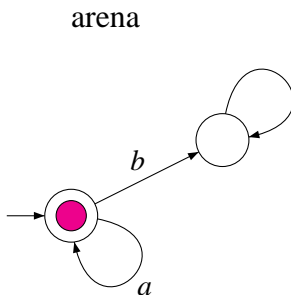
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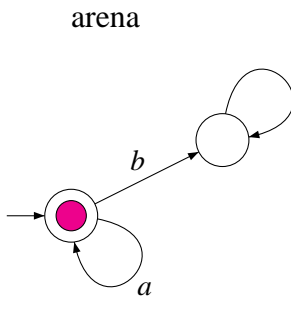
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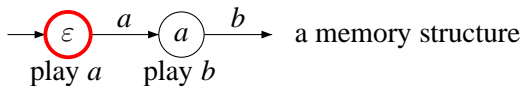
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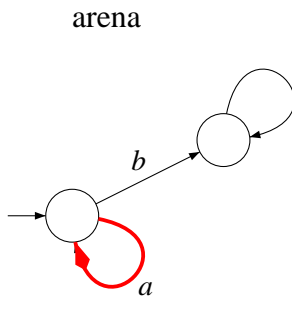
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A winning strategy for Eve uses two memory states.



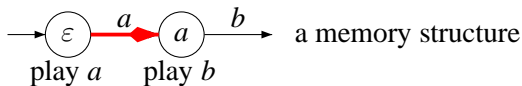
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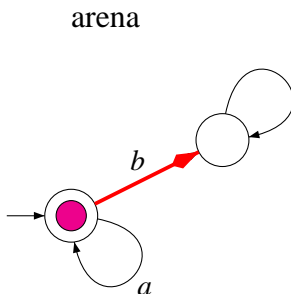
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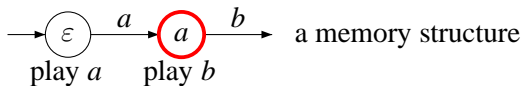
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Introduction: how much memory is needed to win?

Question: given a regular language L , what is the memory required by winning strategies?

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In other words, compute $m_L \in \mathbb{N}^*$ such that:

- in any arena, if Eve wins the regular game for L , then she has a winning strategy with m_L memory states,

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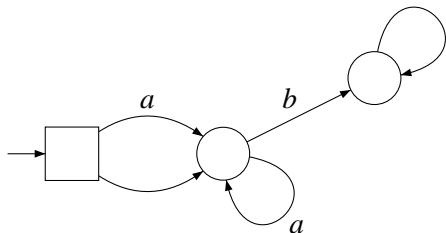
- in any arena, if Eve wins the regular game for L , then she has a winning strategy with m_L memory states,
- there is an arena where Eve wins but there are no winning strategies with less than m_L memory states.

Outline

- 1 Examples
- 2 Playing safety
- 3 Playing reachability
- 4 Playing optimally in the stochastic case

A first remark

arena

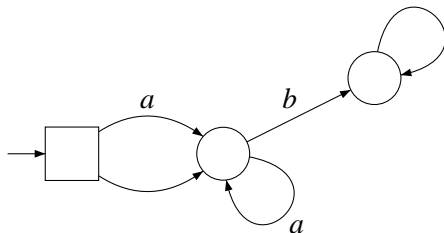


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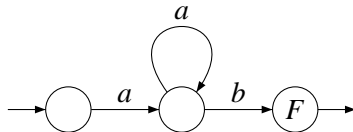
A first remark

arena



winning condition

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Any deterministic automaton that recognizes L is a good memory structure.

Proof: the synchronized product is a reachability game.

An upper bound for both players

We describe a good memory structure for L using left quotients: for $u \in \Sigma^*$,

$$u^{-1}L = \{v \mid u \cdot v \in L\}.$$

- the initial memory state is $\varepsilon^{-1}L = L$,
- each time a letter a is read from $u^{-1}L$, the memory is updated to $(u \cdot a)^{-1}L$.

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Lemma (An upper bound for both players)

For all regular games $\mathcal{G} = (\mathcal{A}, \text{Reach}(L))$, both players have winning strategies using this memory structure (denoted \mathcal{M}_L).

Another example

“read at most ten consecutive b ’s, and then an a ”.

$$L = a + b \cdot a + bb \cdot a + \dots + b^{10} \cdot a.$$

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In every regular game for L , Eve wins without memory.

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This shows that the memory structure \mathcal{M}_L is not optimal.

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Order left quotients inclusion-wise

If Adam wins in $\mathcal{G} \times \mathcal{M}_L$ from $(q, u^{-1}L)$ and $v^{-1}L \subseteq u^{-1}L$, then he wins from $(q, v^{-1}L)$

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Let k the maximal number of incomparable (with respect to inclusion) left quotients of L .

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Lemma (A tighter upper bound for Adam)

For all regular games $\mathcal{G} = (\mathcal{A}, \text{Reach}(L))$, Adam has a winning strategy from his winning set that uses k memory states.

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Idea: whenever in $(q, v^{-1}L)$, play as from $(q, u^{-1}L)$, where $u^{-1}L$ is maximal winning from q .

Optimality

Lemma (Matching lower bound for Adam)

For all regular languages L , there exists an arena \mathcal{A} such that Adam needs k memory states to win in $\mathcal{G} = (\mathcal{A}, \text{Reach}(L))$.

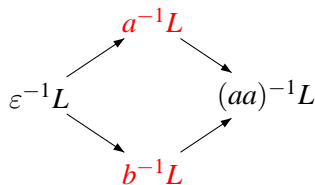
Optimality

Lemma (Matching lower bound for Adam)

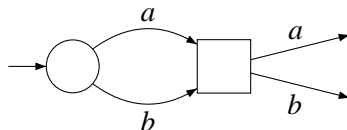
For all regular languages L , there exists an arena \mathcal{A} such that Adam needs k memory states to win in $\mathcal{G} = (\mathcal{A}, \text{Reach}(L))$.

Exemplified: $L = (a + b)^* \cdot (aa + bb)$.

inclusion



arena



Outline

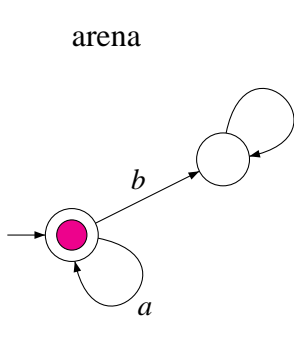
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A wrong intuition

If Eve wins in $\mathcal{G} \times \mathcal{M}_L$ from $(q, u^{-1}L)$ and $u^{-1}L \subseteq v^{-1}L$, then she wins from $(q, v^{-1}L)$

A wrong intuition

If Eve wins in $\mathcal{G} \times \mathcal{M}_L$ from $(q, u^{-1}L)$ and $u^{-1}L \subseteq v^{-1}L$, then she wins from $(q, v^{-1}L)$... however the *same strategy* might fail!



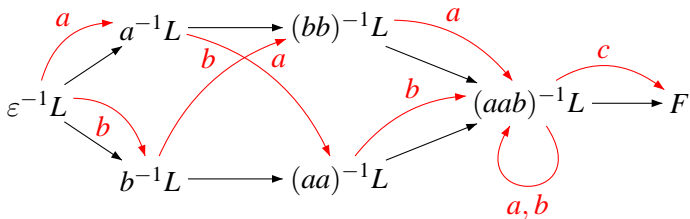
winning condition

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$$\varepsilon^{-1}L \subset a^{-1}L$$

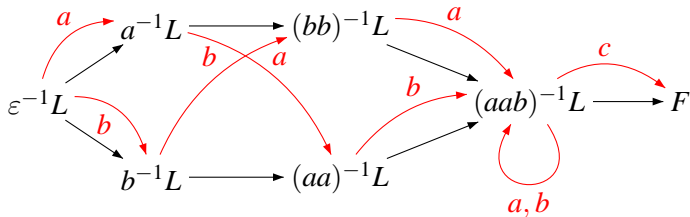
A global merging policy

$$L = (aab + baa) \cdot (a + b)^* \cdot c$$



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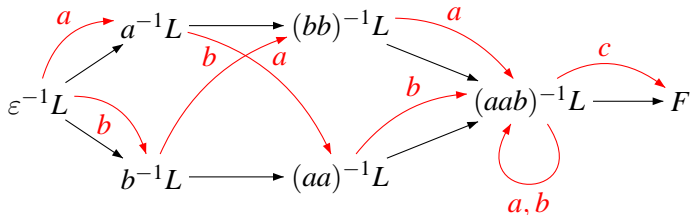
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Eve can play from $(bb)^{-1}L$ as from $a^{-1}L$, hence merge the two memory states.

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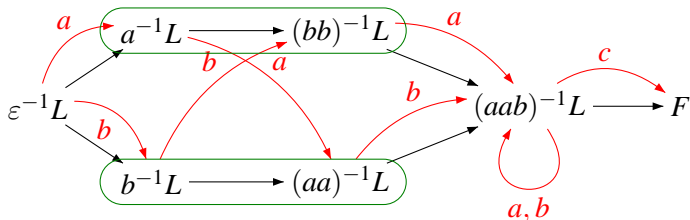
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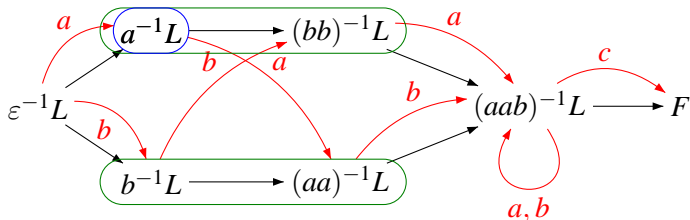
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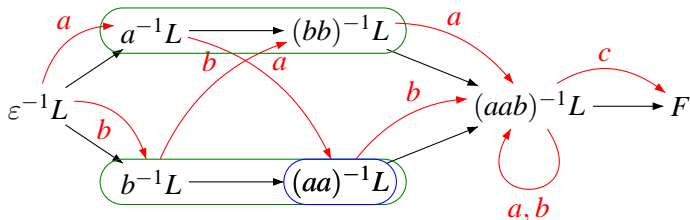
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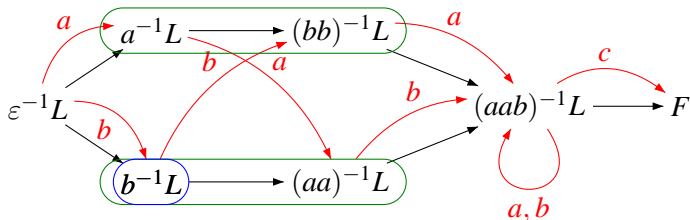
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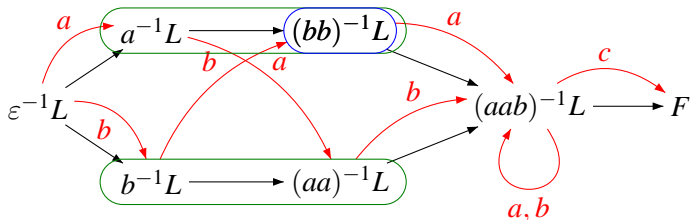
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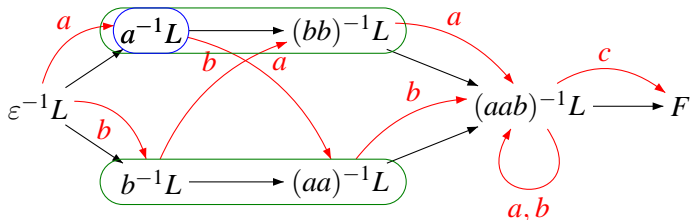
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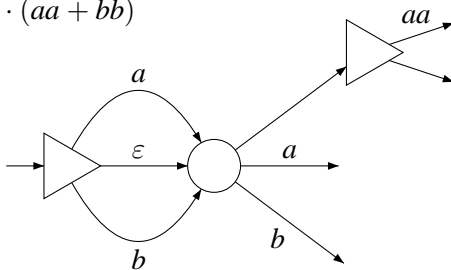
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Playing optimally in stochastic arenas



$$L = (a + b)^* \cdot (aa + bb)$$



Upper bound for both players

Since stochastic reachability games enjoy memoryless determinacy:

Lemma (Upper bound for both players)

For all stochastic regular games $\mathcal{G} = (\mathcal{A}, \text{Reach}(L))$, both players have winning strategies using \mathcal{M}_L as memory structure.

Lower bound for Eve

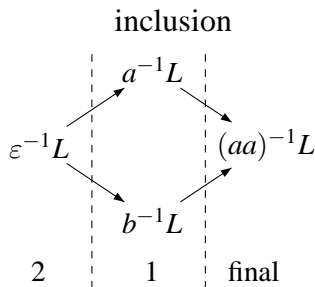
Lemma (Memory lower bound in stochastic games for Eve)

For all regular languages L , let n be the number of non-final left quotients of L , there exists an arena \mathcal{A} such that Eve needs n memory states to play optimally in $\mathcal{G} = (\mathcal{A}, \text{Reach}(L))$.

Again, we order left quotients inclusion-wise.

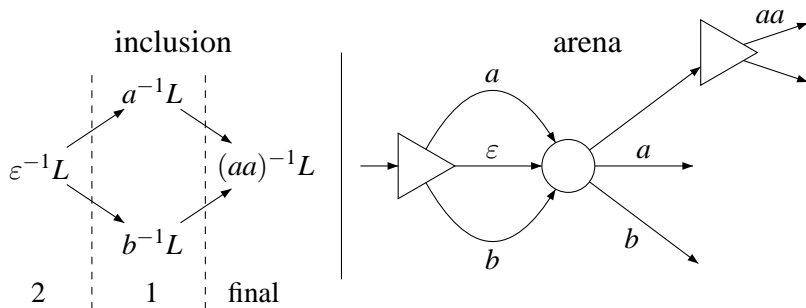
The construction exemplified

We construct an arena for the condition $L = (a + b)^* \cdot (aa + bb)$.



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Adam case

The same applies to Adam, using a very similar construction.

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As opposed to the deterministic case, memory requirements are **symmetric** in stochastic regular games!

The end

Thank you for your attention!