Playing Safe

FSTTCS'2014

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Delhi, India

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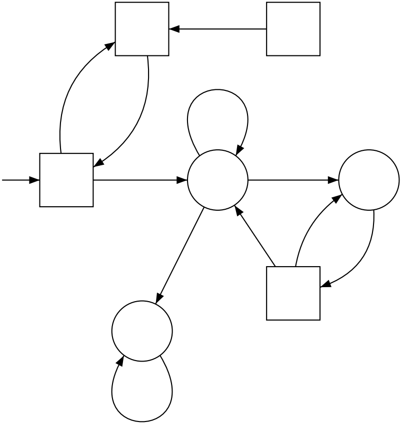
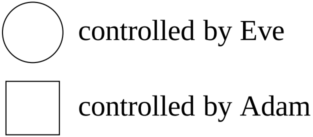
Games





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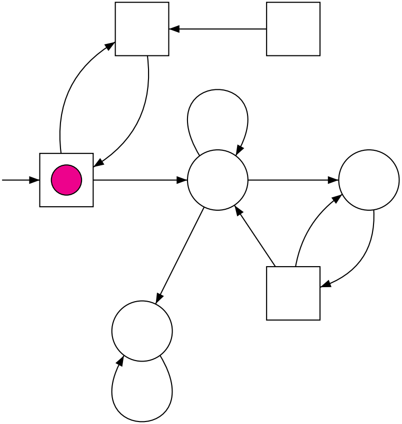

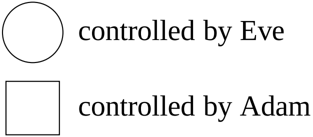

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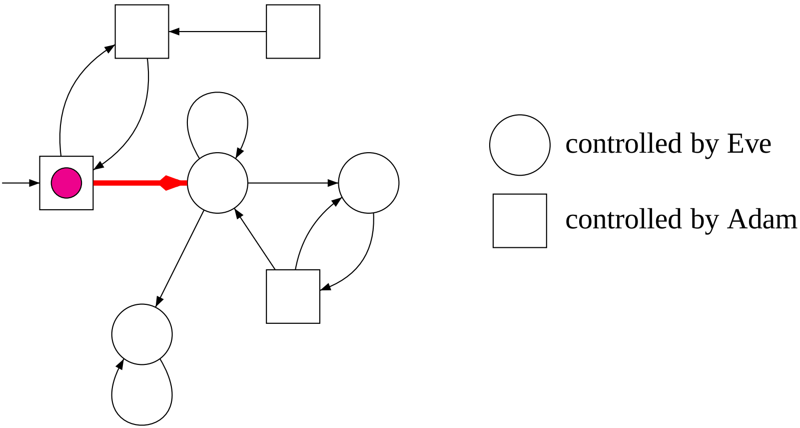

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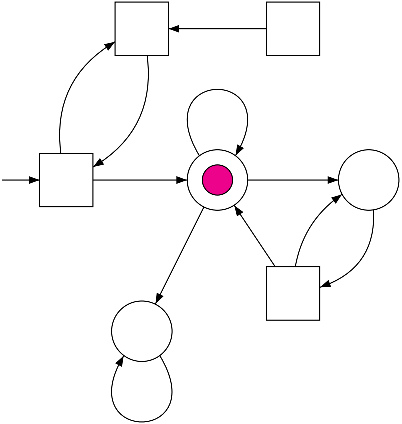

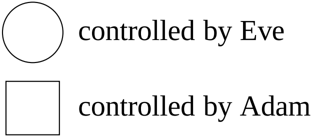

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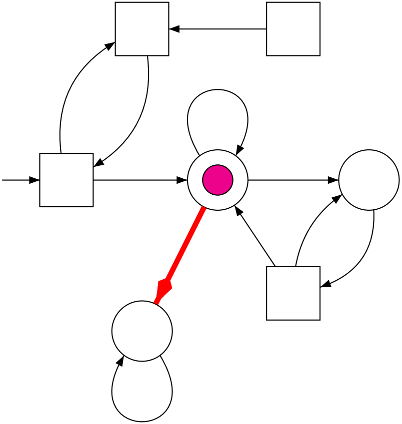

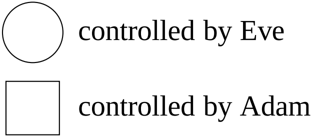

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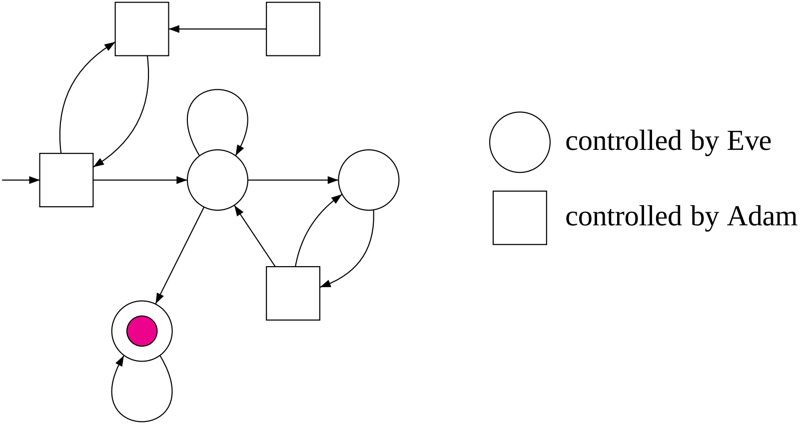

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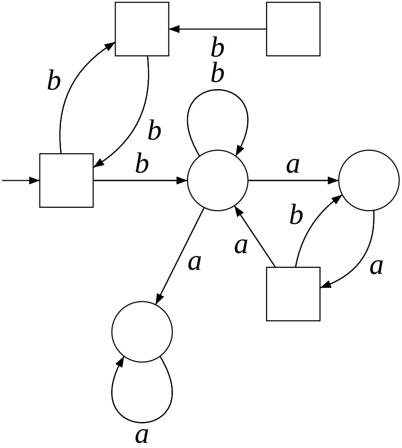

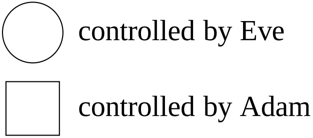

$$W \subseteq \{a, b\}^{\omega}$$

Strategy (for Eve)

General form

$$\sigma : V^{+} \to V$$

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Positional or memoryless

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Positional or memoryless

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Finite-memory

$$\begin{cases} \sigma : V \times M \to V \\ \mu : M \times E \to M \end{cases}$$

How much memory is needed to win?

Let $W \subseteq A^\omega$, compute:

$$\mathrm{mem}(W) \doteq \sup_{\mathcal{G}=(\mathcal{A},W) \text{ game}} \ \inf_{\substack{\sigma \text{ winning} \\ \text{strategy}}} \mathrm{mem}(\sigma) \ .$$

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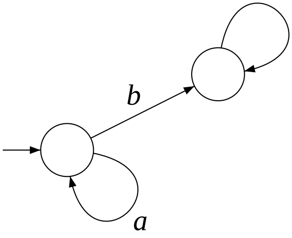
Equivalently:

- *upper bound:* for all games $\mathcal{G} = (\mathcal{A}, W)$, if Eve has a winning strategy, then she has a winning strategy using at most $\mathrm{mem}(W)$ memory states,
- *lower bound:* there exists a game $\mathcal{G} = (\mathcal{A}, W)$ where Eve has a winning strategy, but no winning strategy using less than $\mathrm{mem}(W)$ memory states.

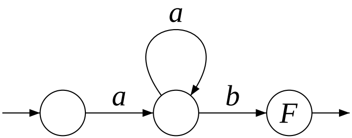
A Simple Example

arena



winning condition

$$W = a^+ \cdot b$$



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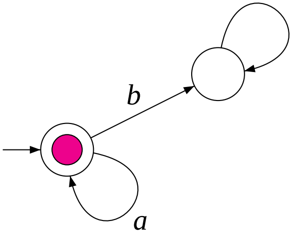

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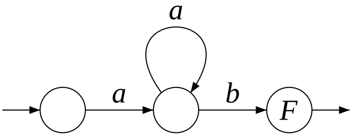

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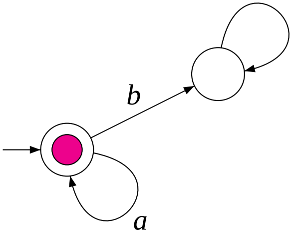

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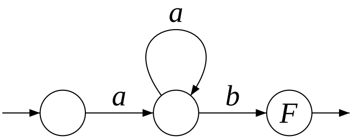
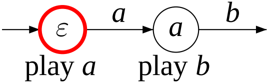


A winning strategy for Eve uses two memory states.



a memory structure

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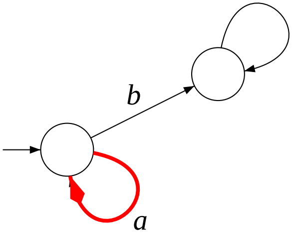

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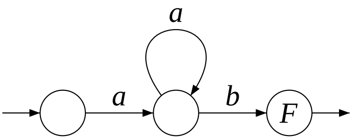

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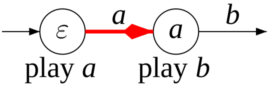

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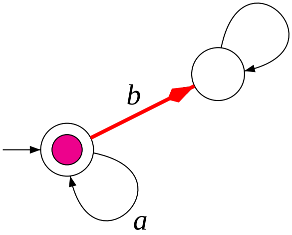

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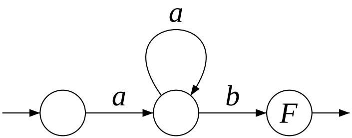

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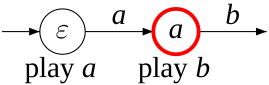

a memory structure

What is Known about Computing $\mathrm{mem}(W)$

Theorem (Dziembowski, Jurdziński, Walukiewicz, 1997)

For W a boolean combination of "infinitely many $a \in A$", $\mathrm{mem}(W)$ is computable (and characterized through the Zielonka tree).

Theorem (Kopczyński, 2007)

For W which is ω-regular, $\mathrm{mem}_{\text{chromatic}}(W)$ is computable.

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Conjecture (Kopczyński, 2008)

For W which is ω-regular, $\mathrm{mem}_{\mathrm{chromatic}}(W) = \mathrm{mem}(W)$.

Our Results

$\mathrm{Res}(W)$ is the set of residuals of W: for $u \in \Sigma^*$,

$$u^{-1}W = \{v \mid u \cdot v \in W\} \ .$$

Theorem (Colcombet, F., Horn)

For all safety conditions W,
$\mathrm{mem}(W)$ *is the width of* $(\mathrm{Res}(W), \subseteq)$,
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$\mathrm{mem}(W)$ *is the width of* $(\mathrm{Res}(W), \subseteq)$,
i.e. the size of the largest antichain.

- We make no regularity assumption!
- This holds for infinite arenas of finite degree.

An upper bound

The memory structure $\mathcal{M}_W$ uses $\mathrm{Res}(W)$ as set of memory states, and:

- the initial memory state is $\varepsilon^{-1}W = W$,
- each time a letter a is read from $u^{-1}W$, the memory is updated to $(u \cdot a)^{-1}W$.

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- each time a letter a is read from $u^{-1}W$, the memory is updated to $(u \cdot a)^{-1}W$.

Lemma (An upper bound)

For all games $\mathcal{G} = (\mathcal{A}, W)$, Eve has a winning strategy using $\mathcal{M}_W$.

Another example

“read at most ten consecutive a’s, and then an b”.

$$W = a + b \cdot a + bb \cdot a + \ldots + b^{10} \cdot a.$$

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$\looparrowright$ This shows that the memory structure $\mathcal{M}_W$ is not optimal.

Order left quotients inclusion-wise

If Eve wins from $(q, u^{-1}L)$ and $u^{-1}L \subseteq v^{-1}L$, then she wins from $(q, v^{-1}L)$

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Intuition: whenever in $(q, v^{-1}L)$, play as from $(q, u^{-1}L)$, where $u^{-1}L$ is minimally winning from q.

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Problems:

- there may not exist minimally winning left quotients!
- winning or losing depends on the current position, which makes updating the memory state not trivial.

Our results

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- Fails for infinite arenas of infinite degree.
- Unifies several results from the literature: boundedness condition, energy condition, generalized reachability.

The end

Thanks!