Logical Formalisms Expressing Boundedness Properties over Infinite Trees

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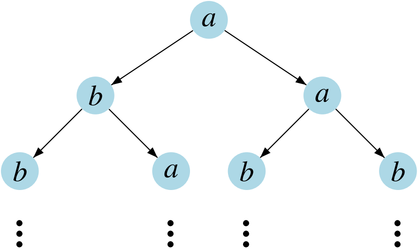
Outline

Logics over infinite (binary) trees

A tree:



A logical property:

"for all nodes a, there are finitely many nodes *below it* that contain a branch with infinitely many b's"

Rabin's theorem

The variables $x, y, \ldots$ are interpreted by nodes, $X, Y, \ldots$ by sets of nodes.

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Atomic formulæ:

$$a(x) \quad | \quad x \in X \quad | \quad \mathit{LeftChild}(x, y) \quad | \quad \mathit{RightChild}(x, y)$$

Constructors:

$$\underbrace{\wedge, \vee, \neg}_{\text{boolean connectives}} \quad | \quad \underbrace{\exists x}_{\text{first-order}} \quad | \quad \underbrace{\exists X}_{\text{monadic second-order}}$$

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Theorem (Rabin, 1969)

The following problem (called satisfiability problem) is decidable:

- *Instance: ϕ an MSO formula.*
- *Question: does there exist a tree* $\mathbf{t}$ *satisfying ϕ?*

Can we go further?

i.e. are there *decidable* extensions of MSO over infinite trees?

Can we talk about the *size* of sets?
About their *asymptotic behaviour*?

Some possible extensions

- "X is finite" , $|X| \geq 6$, $|X| \equiv |Y| \bmod 9$

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  $\looparrowright$ undecidable!

- $\mathbb{B}X, \phi$, defined by

$$\exists N \in \mathbb{N},\ \forall X, \quad \phi(X) \Rightarrow |X| \leq N$$

  $\looparrowright$ MSO $+\ \mathbb{B}$ was proposed by Bojańczyk in 2004

- ... ?

Bad news

Theorem (Hummel, Skrzypczak and Toruńczyk, 2010)

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The decidability of MSO $+ \mathbb{B}$ *over infinite trees cannot be proved in* ZFC.

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End of the story?

Not quite!

Regular cost functions

Colcombet investigated *uniform* quantifications over bounds:

Add "$|X| \leq N$" to MSO formulæ.

Hope (Colcombet, 2009)

The following problem (called boundedness problem) is decidable:

- *Instance:* $\phi(N)$ *a cost MSO formula.*
- *Question:* $\exists N, \forall \mathbf{t}, \quad \mathbf{t} \models \phi[N]$*?*

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Wide open for infinite trees! It would solve a long-standing open problem (the decidability of the Mostowski index).

Outline

Alternating parity automata:

$$\mathcal{A} = (Q, A, q_0, \delta, \text{Parity}), \text{ where } \delta : Q \times A \to \underbrace{\mathcal{B}^+(Q \times Q)}_{\text{positive boolean combinations}}$$

A proof of Rabin's theorem, by Muller and Schupp

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- Eve chooses disjunctions,
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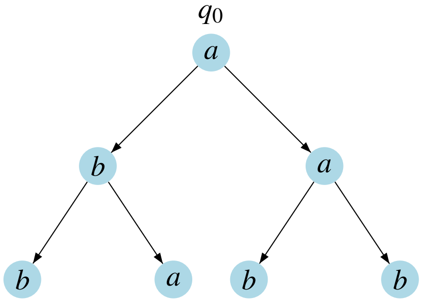
t is accepted if Eve wins the acceptance game.



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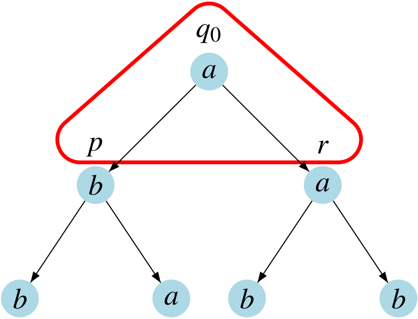

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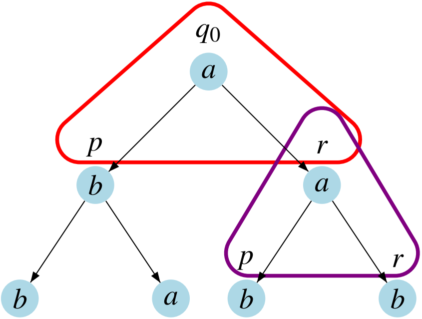

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Simulating alternating automata by non-deterministic ones relies on:

- determinization of parity automata over infinite words,
- positional determinacy of parity games.

The three ingredients

1. Determinacy of parity games
2. Determinization of parity automata over infinite words
3. Positional determinacy of parity games

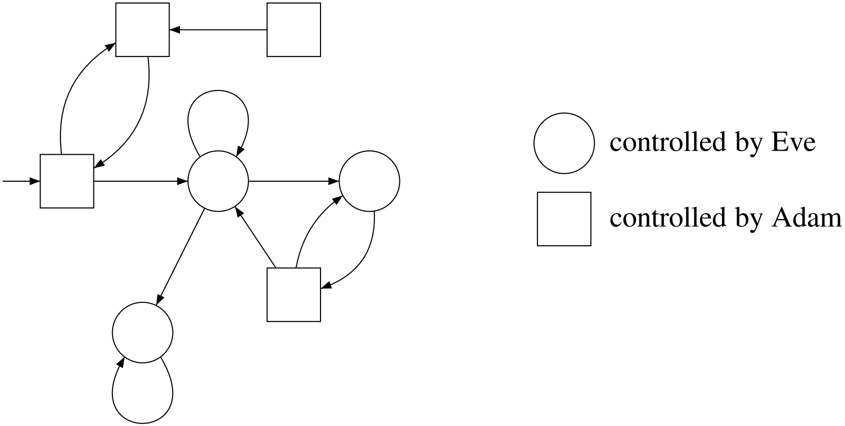
Towards cost MSO over infinite trees

We need to generalize these three ingredients to cost-parity games:

1. Determinacy: ✓ (Borel determinacy takes over)
2. Determinization: ✓ (history-deterministic automata fill in!)
3. Positional determinacy: only partial results...

Outline

Definition



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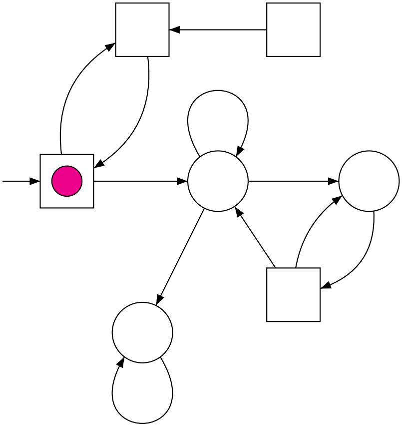

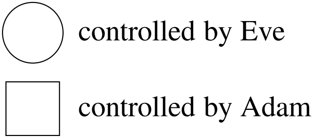

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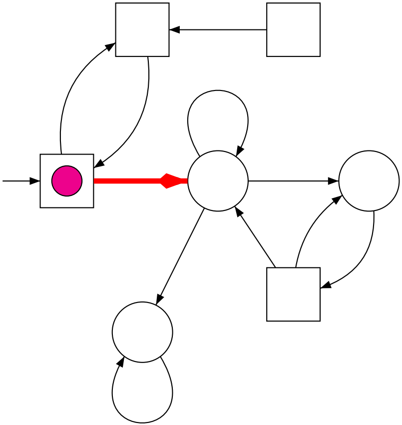

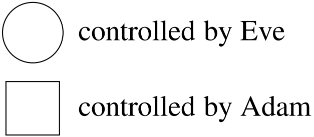

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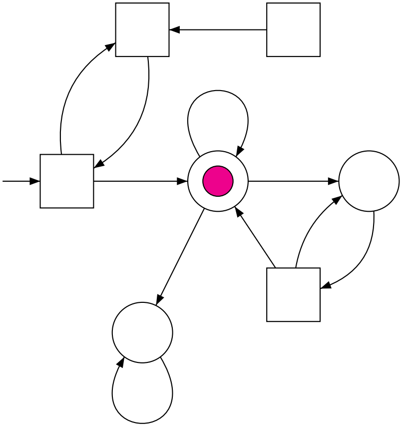

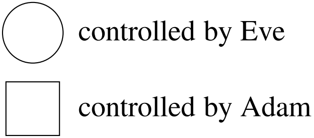

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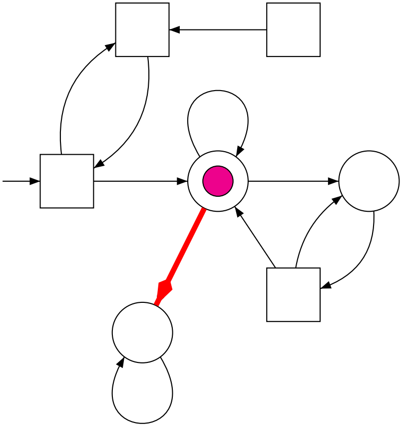

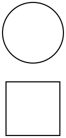

controlled by Eve

controlled by Adam

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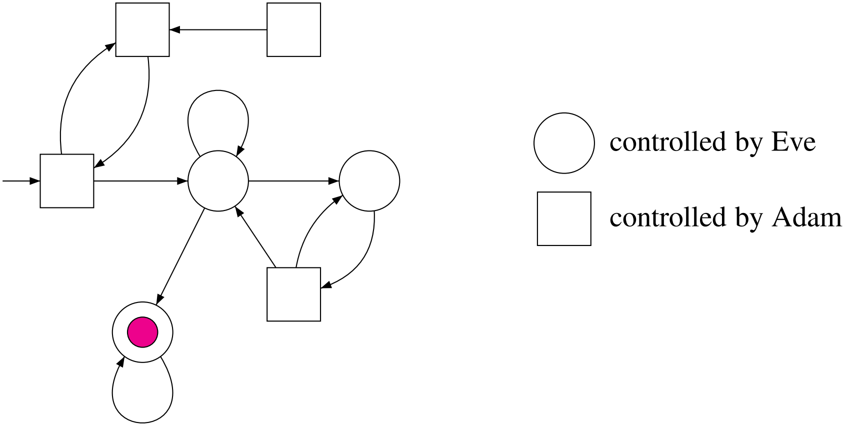

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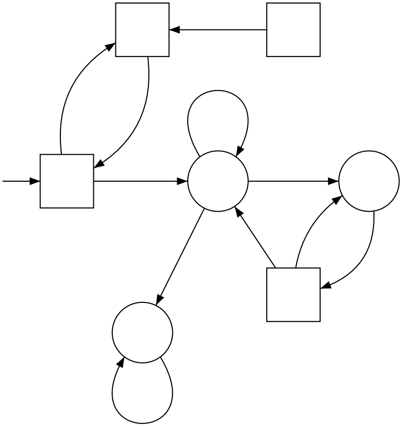

parity
and
all counters
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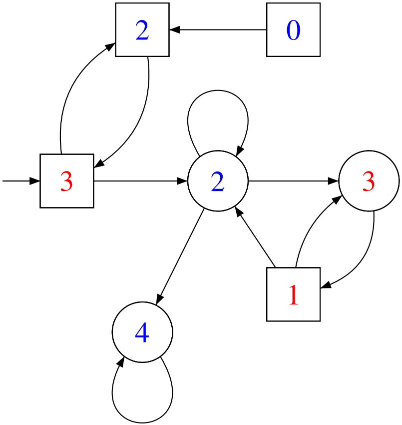
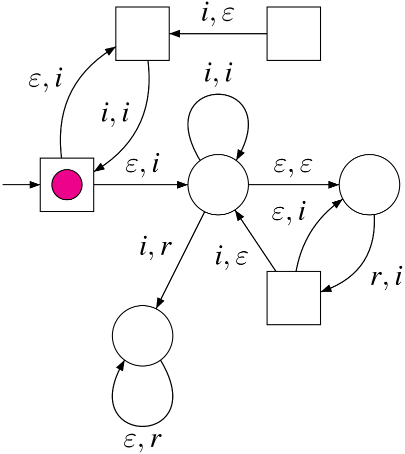


parity condition:

the minimal priority seen infinitely often is even



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ε : nothing

i : increment

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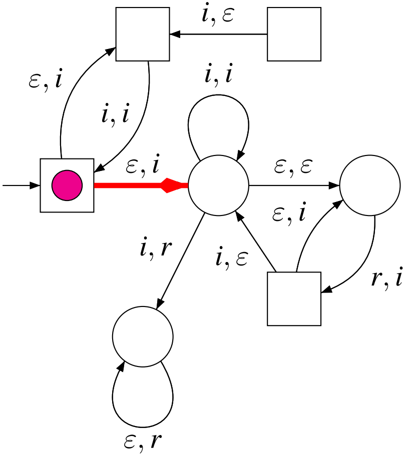

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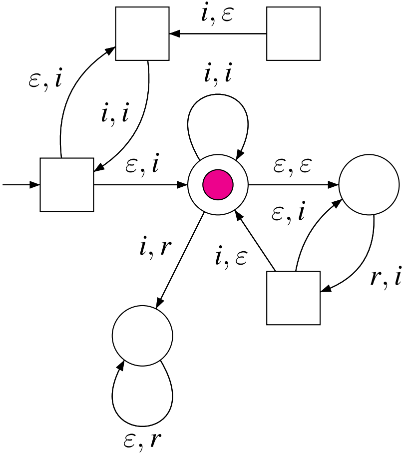

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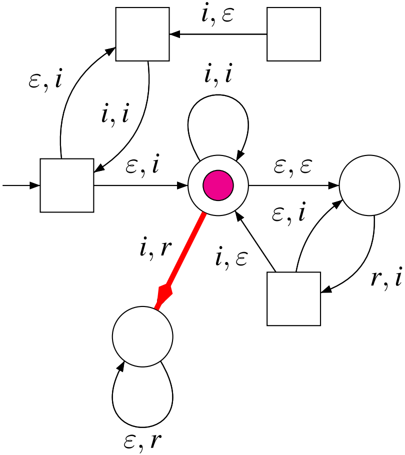

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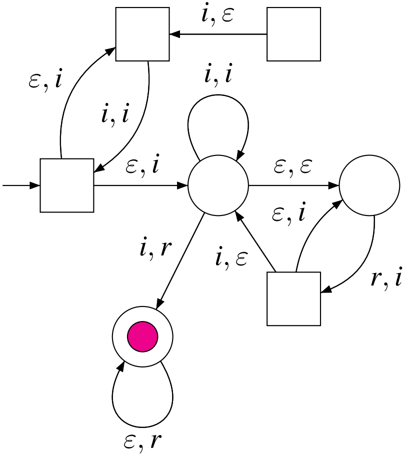

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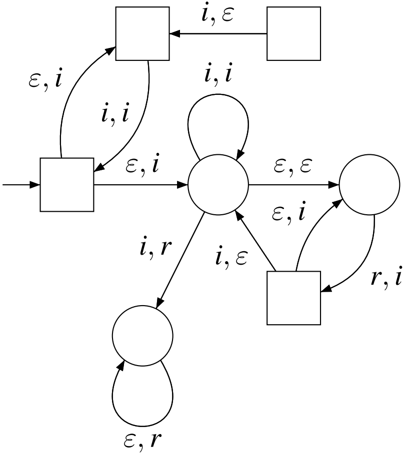

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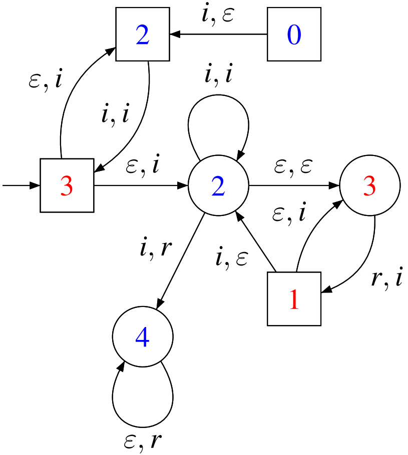

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Uniform versus non-uniform quantification

Eve wins means:

$\exists \sigma$ (strategy for Eve),
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non-uniform
$(\mathrm{MSO} + \mathbb{B})$

uniform
(cost MSO)

Colcombet's Conjecture

Fix a game G and assume Eve wins $B(N) \cap$ Parity.

Observation

Eve has a strategy with N memory states to ensure $B(N) \cap$ Parity.

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The conjecture involves a trade-off between memory and quality:

Conjecture

There exists a function $\alpha : \mathbb{N} \to \mathbb{N}$ and a constant $m \in \mathbb{N}$ such that for all games:

if Eve wins $B(N) \cap \mathit{Parity}$,
then she has a strategy with m memory states to ensure
$B(\alpha(N)) \cap \mathit{Parity}$.

An easy instance of Colcombet's Conjecture

Theorem

for all finite games G:

if Eve wins $B(N) \cap Parity$,

then she has a strategy with 2 memory states to ensure

$B(|G| \cdot N) \cap Parity$.

Some results obtained so far

Theorem (Vanden Boom)

For infinite chronological games:

- *If Eve wins $B(N) \cap$ Büchi, then she has a strategy with 2 memory states to ensure $B(N) \cap$ Büchi.*
- *If Eve wins $\overline{B}(N) \cup$ Büchi, then she has a strategy with 2 memory states to ensure $\overline{B}(N) \cup$ Büchi.*

Corollary (Vanden Boom)

Cost weak MSO is decidable.

Some results obtained so far

Theorem ("Folklore in the regular cost function community")

For infinite chronological games without ε:

- *If Eve wins $B(N) \cap Parity$, then she has a strategy with 2 memory states to ensure $B(N) \cap Parity$.*
- *If Eve wins $\overline{B}(N) \cup Parity$, then she has a strategy with 2 memory states to ensure $\overline{B}(N) \cup Parity$.*

Corollary

MSO + "$|x - y| \leq N$" (called temporal cost MSO) is decidable.

Some results obtained so far

Theorem (F.,Horn,Kuperberg,Skrzypczak)

Colcombet's Conjecture holds for thin tree games (with non-elementary bounds).

Corollary

Cost MSO is decidable over thin trees.

Conclusion

To extend Rabin's theorem to cost MSO via Muller and Schupp's proof, the following three ingredients are required:

1. Determinacy of cost-parity games: ✓
2. Determinization of cost-parity automata over infinite words: ✓
3. Finite-memory determinacy for cost-parity games: ongoing