Double Pendulum

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Abstract

We demonstrate that a double pendulum follows simple harmonic motion for small initial angles of less than 30° and certain initial conditions but starts acting chaotic for larger angles. We wrote a simulation in Python that models a double pendulum in order to analyze the effect that the initial conditions have on the evolution of the system. Due to the assumptions made in order to implment the Euler method for solving differential equations numerically, the thresholds for initial conditions in our simulations are small. We found that the best results are when $L_1 = 1, L_2 = 5, m_1 = 1, m_2 = 2$, but they can be varied slightly without breaking the code. For future simulations of this system, the Runge-Katta method for solving differential equations would yield more flexible initial conditions.

I. INTRODUCTION

In this report we look to discover how initial conditions of a double pendulum affect the position of the two masses attached to the ends of the two strings. Because of its sensitivity to these initial conditions, the double pendulum can exhibit chaotic motion. Chaotic systems can be found everywhere in nature. Between weather prediction, biological populations, and many economic patterns, Chaos Theory is used to describe numerous complex systems. Industries like robotics, chemistry, and cryptography are using chaos dynamics to create more efficient processes [1]. Edward Lorenz was one of the first to discuss Chaos Theory at the Massachusetts Institute of Technology when he created an elaborate simulation to predict weather patterns. He found that when he rounded values from six decimal places to three the result of the simulation was completely different [2]. The issue with research on chaotic systems is the difficulty of reproducing results due to the complexity of each experiment.

Double pendulums are a great example of a relatively simple chaotic system. We wrote a simulation in Python to mimic the motion of a double pendulum. After doing this we graphed the position of each pendulum as a function of time and also transformed that into the frequency domain to see how changing the initial conditions affects the evolution of the system. From our data we came to the conclusion that at small angles, the double pendulum acts like a simple harmonic oscillator, and at large angles it acts like a chaotic system.

II. THEORY

A double pendulum is just what it sounds like, one simple pendulum attached to the bottom of a second simple pendulum [3]. Although seemingly simple, as it can be modeled by a system of ordinary differential equations, the motion of a double pendulum is chaotic. An example of such a setup can be seen in Figure 1.

By neglecting a damping force we can create equations of motion to calculate the positions

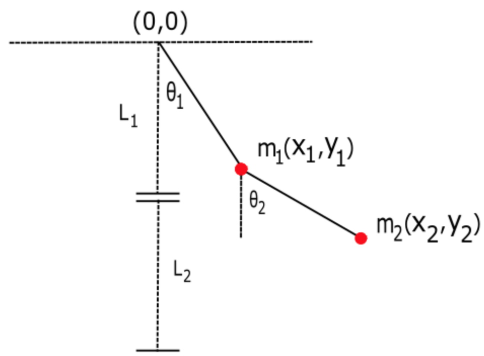

FIG. 1: Setup of a double pendulum

of the two pendulums as a function of time. The position of the masses can be expressed as:

$$\begin{aligned} x_1 &= L_1 sin(\theta_1) & x_2 &= x_1 + L_2 \sin(\theta_2) \\ y_1 &= -L_1 cos(\theta_1) & y_2 &= y_1 - L_2 \sin(\theta_2) \end{aligned} \tag{1}$$

Due to the time dependence of the equations in 1, we can express velocities by taking time derivatives of them:

$$\begin{aligned} \dot{x}_1 &= L_1 \cos(\theta_1)\dot{\theta}_1 & \dot{x}_2 &= L_2 \cos(\theta_2)\dot{\theta}_2 \\ \dot{y}_1 &= L_2 \sin(\theta_1)\dot{\theta}_1 & \dot{y}_2 &= L_2 \sin(\theta_2)\dot{\theta}_2 \end{aligned} \tag{2}$$

Using the equations in 2 we can write the Lagrangian of this system:

$$L = T - V \tag{3}$$

where T and V are kinetic and potential energy, respectively. For our purposes, we use Lagrange equations of the second kind:

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}_{1,2}} - \frac{dL}{d\theta_{1,2}} = 0 \tag{4}$$

A full derivation of Lagrange's equations can be found in [4]. The equations of motion then become

$$A\ddot{\theta}_1 + B\ddot{\theta}_2 + C = 0 \tag{5}$$

and

$$D\ddot{\theta}_2 + B\ddot{\theta}_1 + E = 0, \tag{6}$$

where

$$
\begin{aligned}
&A = m_1 L_1^2 + m_2 L_1^2 \\
&B = m_2 L_1 L_2 cos(\theta_1 - \theta_2) \\
&C = m_2 L_1 L_2 sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (m_1 + m_2) g L_1 sin(\theta_1) \\
&D = m_2 L_2^2 \\
&E = m_2 g L_2 sin(\theta_2) - m_2 L_1 L_2 sin(\theta_1 - \theta_2)\dot{\theta}_1^2
\end{aligned}
\tag{7}
$$

In order to model the system and solve for θ_1 and θ_2 as a function of time, we need differential equations of the first and second order that isolate $\ddot{\theta}_1, \ddot{\theta}_2, \dot{\theta}_1$, and $\dot{\theta}_2$. The reason for this will become evident as we present the approach we used to solve the following system of four differential equations, where the first two equations simply state that $\dot{\theta}_1$ and $\dot{\theta}_2$ are time derivatives of θ_1 and θ_2, respectively, while the latter two were derived by solving for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ from 5 and 6:

$$\dot{\theta}_1 = \frac{d\theta_1}{dt} \tag{8}$$

$$\dot{\theta}_2 = \frac{d\theta_2}{dt} \tag{9}$$

$$\ddot{\theta}_1 = \frac{ABE - B^2C}{A^2D - AB} - \frac{C}{A} \tag{10}$$

$$\ddot{\theta}_2 = \frac{BC + AE}{DA - B} \tag{11}$$

Finding an analytical solution for θ_1 and θ_2 from the four differential equations presented above is not possible, due to the complexity of the system. Therefore, we have to solve the system numerically.

There are a few different methods that can be used in order to do this, but the one we are going to focus on is the Euler method. This approach takes the system that has to be in an initial known state and steps through time in small increments, calculating all values at each time step and using that information to output values for the next time step [5]. Since this tool is heavily dependent on initial conditions, it is going to be useful in finding the angles that lead to a chaotic system.

Double pendulums are a great focus of Chaos theory. Chaos Theory is used to describe complex systems with multiple components. These systems are extremely dependent on initial conditions. In most chaotic systems, one event depends on the event before it so a small change in initial events can lead to a profound difference in the outcome of the system [1]. When coupled with the idea of uncertainty, it states that after a certain number of iterations, a chaotic system is completely random and cannot be predicted.

Once we solved for these positions using the Euler method, we used Fourier Transforms to observe the relationship between frequency and initial angles. A Fourier function breaks any function into the sum of sinusoidal functions using the following equation [6]:

$$F(\omega) = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-i\omega t} \, dt \right\} \tag{12}$$

The result of this transform is a function of frequency. Using this equation we can create a spectrum of initial angles versus frequency. At small angles, as can later be seen in our Data and Analysis section, double pendulums act like Simple Harmonic Oscillators. Simple Harmonic Motion can be modeled using a periodic function:

$$y = Asin(\omega t) \tag{13}$$

Common examples of harmonic motion are simple pendulums, spring systems, and uniform circular motion. The graph of a periodic function oscillates between positive and negative extrema over time. When transformed into the frequency domain, there are two distinctive spikes that correspond to the frequency of the function, which can be seen in Figure 2. We will further discuss this pattern in the Data and Analysis section.

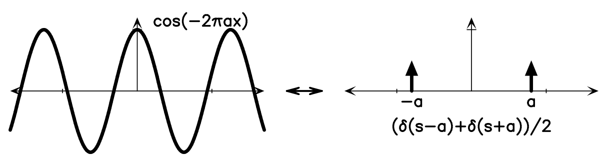

FIG. 2: Plot of a periodic function in both the time domain and the frequency domain.

III. COMPUTATIONAL METHODS

We used Python to simulate a double pendulum using the equations and conditions presented in the Theory section. After defining all variables and creating the system of differential equations, we used the Euler method to solve it and generate various plots which will be presented and discussed later in the report.

In order to use the Euler method, we had to input initial values for θ_1, θ_2, $\dot{\theta}_1$, as well as $\dot{\theta}_2$. Using a for loop, we were able to take the initial values and using time increments dt, we calculated $\ddot{\theta}_1$, $\ddot{\theta}_2$ using equations 4 and 5, as well as $\dot{\theta}_1$, $\dot{\theta}_2$, θ_1, and θ_2 using simple kinematics, for each time step.

The run-time of the code changes depending on the increments of time that we choose. We found that for a time of 20 seconds that was split into 100,000 increments, the code only took about 15 seconds to fully run and generate five plots.

Due to the fact that Euler's method does not generate perfect results, since it assumes that during an interval of time, the value of the time change in position divided by the change in time does not change, the code has limits in terms of stability, which will be discussed in the Data and Analysis section.

In order to generate Figures 3-8, we held θ_2 constant with a value of $5°$, while changing θ_1 in $5°$ increments, starting with $5°$ and going up to $30°$ for Figures 5-4, and $60°$ for Figure 8. Furthermore, we kept the initial $\dot{\theta}_1$ and $\dot{\theta}_2$ the same for all of our runs, $\dot{\theta}_1 = \dot{\theta}_2 = \frac{\pi}{30°}$ For consistency, we made L_1 and m_1 equal to 1, in order to find the thresholds of the simulation for L_2 and m_2. The figures that we discuss in this report were created using $L_2 = 5$ and $m_2 = 2$.

IV. DATA AND ANALYSIS

To start, we took data with both initial angles θ_1 and θ_2 at small values of less than 30 degrees. We ran multiple trials at small angles and obtained figures 4-5 . You can see from these graphs that at small angles, the double pendulum oscillates and therefore follows Simple Harmonic Motion. The graphs of θ_1 and θ_2 versus time, which are presented in Figure 3 and 4, respectively, both demonstrate periodic motion.

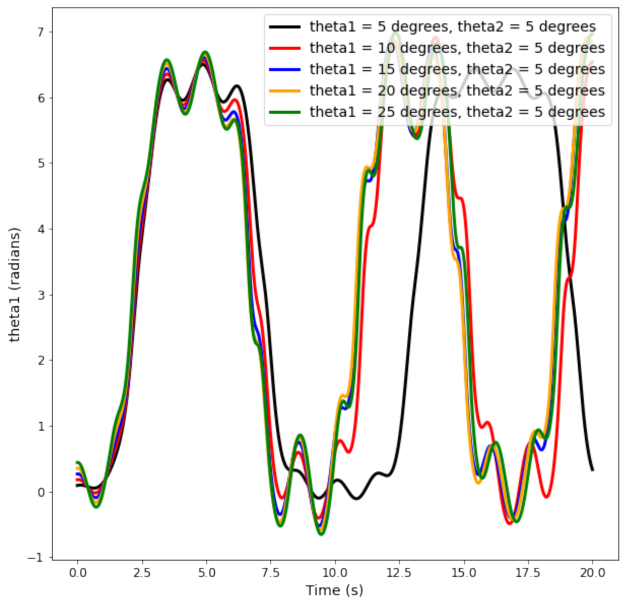

FIG. 3: Plot of θ_1 (radians) vs. time (seconds). Here, we varied θ_1 while keeping θ_2 constant ($\theta_2 = 5°$). The different colors correspond to different initial values of θ_1.

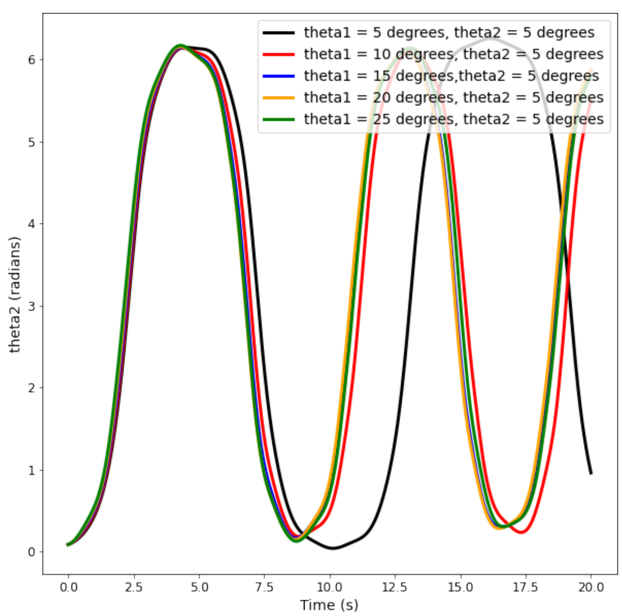

FIG. 4: Plot of θ_2 (radians) vs. time (seconds). Here, we varied θ_1 while keeping θ_2 constant ($\theta_2 = 5°$). The different colors correspond to different initial values of θ_1.

In the frequency section of both Figure 5 and Figure 6, we can see that the Fourier

Transform of the spectra of both angles look like the frequency graph of a simple harmonic function in Figure 2.

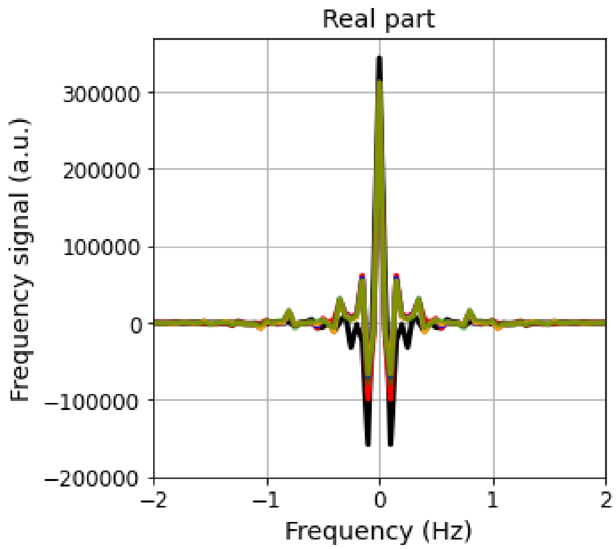

FIG. 5: Frequency spectrum of θ_1

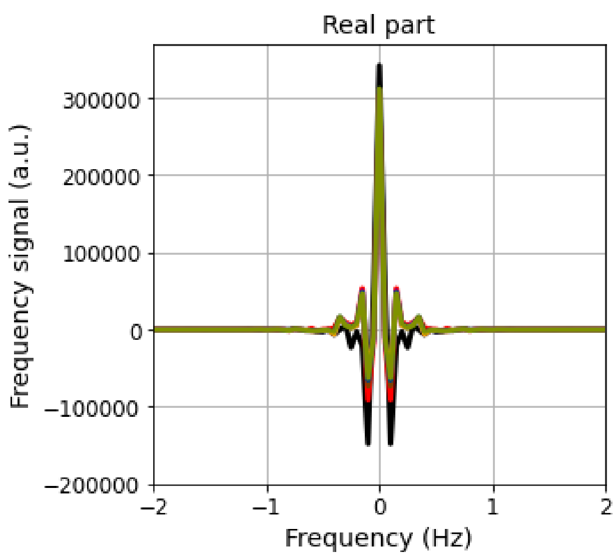

FIG. 6: Frequency spectrum of θ_2

As you increase from 5 degrees to 25 to see simple harmonic motion, it is clear from the frequency spectra in Figures 5 and 6 that the frequency increases from $150,000$ a.u. to

$300,000$ a.u. when the angle increases. When you graph θ_2 as a function of θ_1 like in figure 7, it is clear that the graph is linear, which also points to simple harmonic motion.

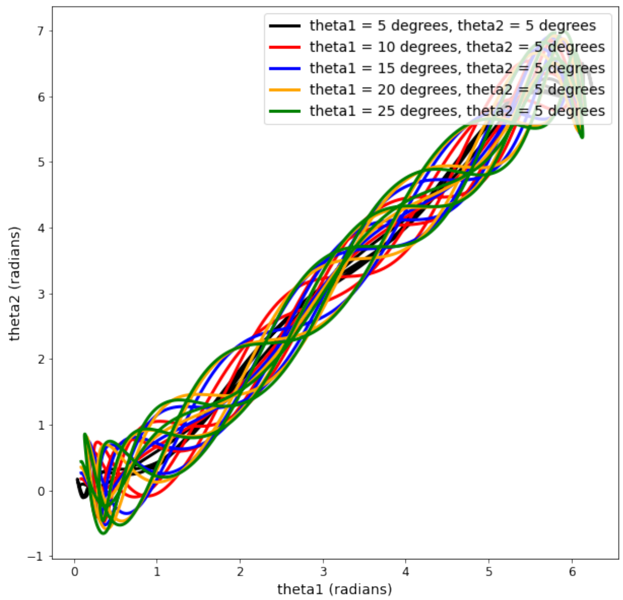

FIG. 7: Plot of θ_1 (radians) vs. θ_2 (radians). The different colors correspond to different initial values for θ_1. The same initial value of $\theta_2 = 5°$ was used for all runs.

There are some limitations to our program when it comes to obtaining the simple harmonic motion associated with small angles of a double pendulum. Due to the variance in evolution of the system we have to keep the time interval relatively small. If the time interval is much larger than $110,000$ points over 20 seconds or much smaller than $100,000$ points over 20 seconds, the results become chaotic and no longer follow periodic motion at small angles.

The limitations for our code could be due to the inaccuracy of the Euler Method. Small differences in initial values almost anywhere in the simulation can lead to a much larger difference in result, as previously discussed in the Theory section. A potential solution to improve this limitation issue would be to use the Runge-Katta method to solve the system of four differential equations presented in Equations 8, 9, 10, and 11. This would lead to less error in numerical approximation and overall more accurate results, as well as more flexibility in choosing the initial values.

As we ran the simulation with larger angles, it was clear that the system became chaotic.

The graph of θ_1 versus θ_2 made seemingly random turns, as seen in Figure 8. The non-linearity of this graph indicates that the system no longer follows simple harmonic motion and instead displays a chaotic pattern of motion.

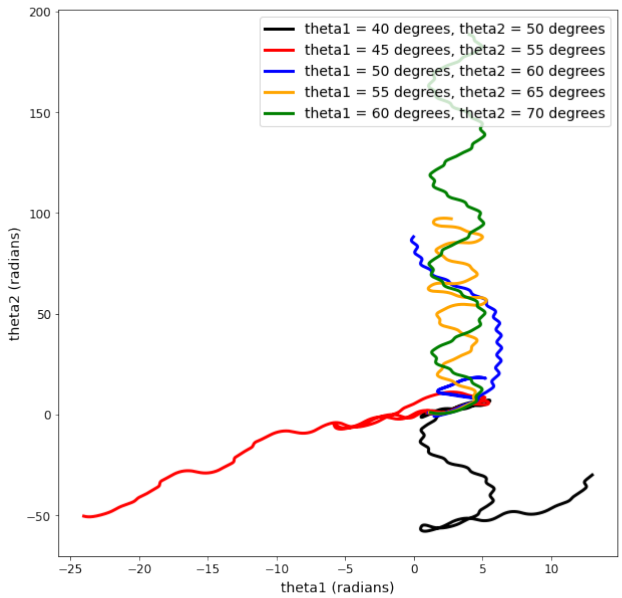

FIG. 8: θ_1 vs θ_2

On top of using a specific time interval, the lengths and masses of the pendulum also need to be within a certain threshold. L_1 and m_1, for this simulation are held at 1, for consistency. L_2 could vary between 5 and 5.5 and m_2 can fluctuate between 2 and 2.4. Any other value yields a chaotic result as we hold L_1 and m_1 constant and equal to 1.

We compared our results with those of Shinbrot, et. al in Chaos in a Double Pendulum, [7], and Levine, et. al in Double pendulum: An experiment in chaos [8] due to the similarity of their experiments to our simulation. While these experiments had a few differences in methodology, such as the use of the Runge-Katta method, and the use of Lyapunov exponents, they found similar results in terms of small and large angle motion. Both experiments agreed that at small angles a double pendulum demonstrates Simple Harmonic Motion but at large angles the system becomes chaotic.

V. CONCLUSION

This experiment shows that at small initial angles of under 30°, a double pendulum system acts as a simple harmonic oscillator, but at larger initial angles it follows chaotic motion. There were a fair amount of limitations on this simulation due to the inaccuracy of the Euler method for solving differential equations. These limitations show the sensitivity to initial conditions of the double pendulum system. A possible improvement for this simulation would be to use the Runge-Katta method rather than the Euler method in order to get more accurate results when solving the system of differential equations. A potentially useful application for this simulation is the crane problem, since bridge cranes with large payloads act like double pendulums [9].

VI. REFERENCES

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