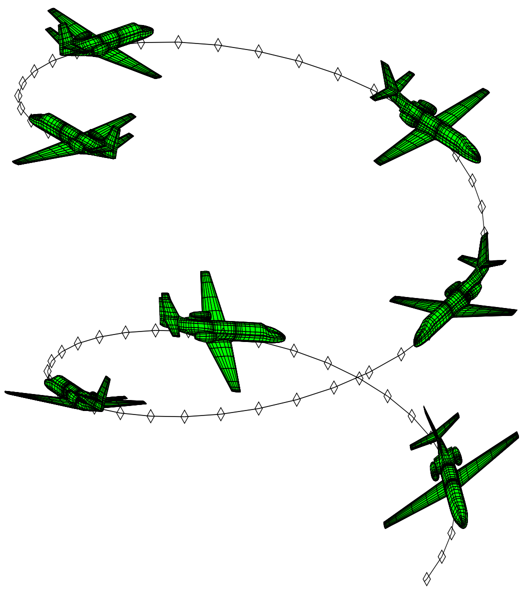
Flight Dynamics Assignment AE3212-II

V1.4 February 2020 A.C. in 't Veld & T.J. Mulder

This assignment aims to bridge the gap between theory and practice. The purpose of the assignment is to obtain a better understanding of the performance, stability and control of aircraft and at the same time practice verification and validation on an aerospace engineering case study. Combining scientific measurement with real flying experience has proved to provide new insights and develop a 'feeling' of theoretical equations and graphs. The knowledge and equations needed are discussed during the lectures 'Introduction to Aerospace Engineering I' AE1110-I, 'Flight and Orbital Mechanics' AE2230-I and 'Aerospace Flight Dynamics & Simulation' AE3212-I.



The general idea for this assignment is that you work for an airplane manufacturer who is designing a new business jet and verify and validate this model. Your team is responsible for flight dynamics and stability and as such you need to build a simulation model of the aircraft to investigate the dynamic behavior of the new jet. Using your model, you need to predict the static and dynamic stability properties of the new aircraft. You have to show how you verify the model and the code using several tests and using the analytical model. You will then use the data you gathered during a test flight with the (prototype) aircraft to derive a number of aerodynamic coefficients and improve the stability parameters of your model, after which you then validate your model. You need to show that the model results can be used for further development of the aircraft.

The test data consists of two sets of stationary measurements and one set of dynamic measurements. The data from the stationary measurements are used to create your analytical and numerical model. The data from the dynamic maneuvers provide you with the exact inputs on the airplane controls and the actual dynamic response data of the aircraft. Feeding the recorded control inputs into your numerical model allows you to validate your model by comparing the simulated responses to the measured aircraft responses.

Analytical Model: Derivation of EOM

As a starting point for the analytical model you should choose a set of equations that can be used to describe the dynamic response of the aircraft to disturbances and control inputs. At this point in the project you will probably not yet have the data from your test flight, so you can use the data from the Tables in Appendix C. The results you obtain using this data should not be incorporated into the report, only the results obtained from the reference data and your actual flight data should be in the report.

Numerical Model: Derivation of EOM

In order to build the simulation model, it is useful to rewrite the equations of motion in state-space form, as explained in Appendix D.

Complete the provided *Cit_par.py* file with the values from the Tables in Appendix C and calculate the matrices A_s, B_s, C_s and D_s and A_a, B_a, C_a and D_a for the symmetric and asymmetric case.

Verification: Natural Frequencies and eigenmodes

Use a simplified model that you can elaborate *by hand* to verify your numerical model.

As explained in the lecture notes of the course AE3212-I *Aerospace Flight Dynamics,* the eigenvalues of the A-matrices determine whether the solution of the state space equations will converge, diverge or be indifferent, whether it will have a periodic or an aperiodic nature, and whether the solution is damped or not. Therefore, the eigenvalues are key parameters for checking your model.

Model response to inputs and disturbances

Using the linear simulation model, it is possible to calculate the aircraft's response to a certain control input. E.g. the Python Control Systems Library *control* contains functions that can solve state space systems, for example `forced_response`, `initial_respose`, `step_response` or `impulse_response`. The first step is to create a continuous-time state-space (SS) model by applying the function ss as follows:

```
sys = control.ss(A, B, C, D)
```

The second step is then to calculate the responses to an input by using `forced_response` in case of an arbitrary input signal, `step_response` in case of a stepwise input, `impulse_response` in case of a pulse shaped input, and `initial_response` in case of zero input with an initial deflection from the equilibrium situation. The command `forced_response` is the most general command and can also be used in any of the other more specific cases.

Calculate the aircraft responses by choosing control inputs that trigger the relevant eigenmodes of your model and comment on the observed behavior and dynamics (in) stability.

Data processing tasks

Up until this point you used the data from the tables in Appendix C, but we can do better. The data you gathered during the test flight can be used to improve a number of the parameters in your simulation model.

First Stationary Measurement Series

The data from the test flight allows you to derive a number of aerodynamic properties. From the first stationary measurements series the $C_L - C_D$ curve in two configurations can be found, enabling you to enhance the analytical and numerical model of the aircraft, as explained in Appendix B. Write a computer program that performs all the necessary calculations and plots.

Second Stationary Measurement Series

The data from this measurement series will be used to calculate two longitudinal aircraft stability parameters, C_{m_α} and C_{m_δ}. These will also be used to improve your analytical and numerical model.

As explained in Appendix B, the measurement data need to be reduced to standard conditions to make them comparable to data from other flights. Appendix F explains how to treat thrust calculations.

Show all reduction computations for one measurement point of the elevator trim and control force curve. Make sure your computations are clear; explain what is happening, and also fill in the numerical values.

To investigate the static stability both stick-fixed and stick-free, the elevator trim curve and elevator control force curve have to be investigated.

From the trim curve you are able to calculate C_{m_α} and the elevator effectiveness C_{m_δ} can be determined from the shift in center of gravity that was deliberately applied by moving one of the observers. Discuss the found stick fixed and stick-free stability of the aircraft.

Improving the model

Complete the provided *Cit_par.py* file with the values you have determined and calculate the matrices A_s, B_s, C_s and D_s and A_a, B_a, C_a and D_a for the symmetric and asymmetric case.

Validation: Comparing linear model to the flight data

Now it is time to validate the improved model to the dynamic measurements you did during the test flight. The data file on Brightspace contains a recording of a number of parameters that were recorded during the test flight where the eigenmotions were demonstrated.

Provide one or more characteristic plots of each of the eigenmotions recorded during the flight. Include <u>on the same time scale</u> the control surface inputs that were used to induce the motion in this series of plots. The validation can be done in terms of the parameters P and $T_{\frac{1}{2}}$ for the *short period, phugoid* and *Dutch rol* and the eigenvalues for the symmetric and asymmetric equations of motion.

Validate the eigenvalues from the linear model to those found from the recorded data.

Validation and model matching

Make plots of the eigenmotions in which you compare the measured response to the simulated response of the aircraft. Explain the results and relate them to the assumptions you made in the modelling. Two eigenmodes will diverge considerably. Investigate and explain this. The stability coefficients can be changed such that the simulated and measured responses match. Which of the stability coefficients do you think are the most effective in this case? Using your best judgment, by trial and error, change the stability coefficients to obtain the best possible match with the flight data. Show this in your report.

Report Requirements

The report should contain the following items:

- A calculation of the current mass and center of gravity location of the aircraft for each measurement point
- A completed mass and balance form.
- All equations that are used, values of constants, SI units etc.
- The equilibrium equations for the horizontal, stationary and symmetric flight. Clearly state the assumptions used, and discuss (if possible) their effect on the results.
- Plots of $C_L - C_D$,and $C_L - \alpha$, labelled with aircraft configuration, Mach number range, and Reynolds number range.
- In general, discuss how the values of parameters compare to the theoretically expected values. Does each graph have the correct and expected slope and curvature? This is a form of verification.
- An analytical derivation of the eigenvalues of A_S and A_a. State any simplifications you made to the EOM for the analytical derivation (if any).
- Provide the following results for the reference data, provided on Brightspace: C_{L_α}, C_{m_α}, C_{m_δ}, and the eigenvalues.
- The eigenvalues for the A_S and A_a matrices as calculated using a computer and an explanation what these values tell you about the characteristics of the eigenmodes. Eigenvalues for matrices constructed based on the reference data on Brightspace <u>and</u> for the matrices using your flight test data.
- Plots of the response of the relevant aircraft states of the numerical model to the different control inputs as demonstrated during the test flight (eigenmotions triggered by control input). Use a relevant time scale.
- Plots of the response of the relevant aircraft states of the numerical model to a disturbance in of the states (eigenmotions demonstrated as an initial value problem).
- Computer generated plots of the elevator trim curve $\delta_e^* - \tilde{V}_e$ and the elevator control force curve $F_e^* - \tilde{V}_e$.
- Validate eigenvalues from the numerical model with those from the measurements
- To save space in the report, a list of symbols is <u>not</u> required. You can refer to the list of symbols in the AE3212-I Lecture Notes instead. However, if you use symbols that cannot be found in the Lecture Notes, these symbols should be listed or explained in your report.
- All plots should be discussed in the text.

Appendix A: Airspeed Calibration Table

SECTION IV - PERFORMANCE
STANDARD CHARTS

MODEL 550

AIRSPEED AND MACHMETER CALIBRATION
PILOT'S AND COPILOT'S SYSTEMS

AIRSPEED CALIBRATION

CONDITIONS:

All Flight Configurations

KIAS	KCAS
80	78
85	83
90	88
95	93
100	98
105	103
110	108
115	113
120	118
125	123
130	128
135	133
140	138
145	143
150	148
155	153
180	158
165	163
170	168
175	173
180	178
185	183
190	188
195	193
200	198
205	203
210	208
215	213
220	218
225	223
230	228
235	233
240	238
245	243
250	248
255	253
260	258
265	263
270	268
275	273
277	275

GROUND AIRSPEED CALIBRATION

CONDITIONS:

All Flap Positions

KIAS	KCAS
50	52
55	56
60	61
65	65
70	70
75	75
80	79
85	84
90	88
95	93
100	98
105	102
110	107
115	111
120	116
125	121
130	125

MACHMETER CALIBRATION

CONDITIONS:

All Altitudes
All Flight Configurations

INDICATED MACH. NO.	CALIBRATED MACH. NO.
0.400	0.393
0.410	0.403
0.420	0.413
0.430	0.423
0.440	0.433
0.450	0.443
0.460	0.453
0.470	0.463
0.480	0.473
0.490	0.483
0.500	0.493
0.510	0.503
0.520	0.513
0.530	0.523
0.540	0.533
0.550	0.543
0.560	0.553
0.570	0.563
0.580	0.573
0.590	0.583
0.600	0.693
0.610	0.603
0.620	0.613
0.630	0.623
0.640	0.633
0.650	0.643
0.660	0.653
0.670	0.663
0.680	0.673
0.690	0.683
0.700	0.693
0.705	0.698

Appendix B: Analysis of stationary measurement data

The data from the first stationary measurements is used to construct the drag polar $C_L - C_D$, the lift curve $C_L - \alpha$ and the drag curve $C_D - \alpha$. From these curves aerodynamic parameters are derived that are used to improve the numerical model.

NOTE:

To be able to derive the zero lift drag coefficient and the Oswald factor for a single configuration from the data, use the fact that $C_D = C_{D_0} + \dfrac{C_L^2}{\pi A e}$ is a straight line if you plot C_D versus C_L^2. Make use of the calculated values for the Oswald factors and C_{D_0}'s to calculate the correct $C_L - C_D$ curves, and then also plot the measurement points. The lift drag polar can be used as input for your analytical and numerical model.

Constructing the reduced elevator trim and control force curves from measurement data

During the flight, several parameters have been measured so that we can reconstruct the elevator trim and elevator control force curves. These measurements are of course almost always conducted under non-standard conditions, i.e. every measurement result is influenced by uncontrollable variations in atmospheric conditions and in aircraft and engine settings. For the results to be predictable and comparable, it is necessary to reduce (or correct) test conditions to some (arbitrary) standard value. For this purpose, we identify three different independent variables [Perkins *et al.*1955]:

1. Uncontrollable variables - these are not normally controllable during flight; e.g. the outside air temperature cannot be controlled when the measurement is to be conducted at a specific altitude.

2. Adjustable variables - these can be adjusted in flight, but not exactly to a predetermined value; the aircraft mass can for example be set to an approximate value by estimating how much fuel will be burned before the test sequence starts, and taking aboard an appropriate amount of fuel before take-off.

3. Fully controllable variables - these variables can be adjusted in flight to any predetermined value within their working range; the airspeed is a good example of a controllable variable.

The uncontrollable and adjustable variables should be corrected to standard values for the reasons stated above. The fully controllable variables should be set precisely to their predetermined values during the test, so that they do not hamper the predictability and comparability of the results. They will therefore be mentioned as parameters in the presentation of the results, whereas the uncontrollable and adjustable variables can be left out; their standard values can be mentioned once in the accompanying text.

Drawing the reduced elevator trim curve

From a careful observation of the general equation for the required elevator deflection for equilibrium:

$$\delta_{e_{eq}} = -\frac{1}{C_{m_\delta}}\left\{C_{m_0} + C_{m_\alpha}\left(\alpha - \alpha_0\right) + C_{m_{\delta_f}}\delta_f + C_{m_{T_c}}T_c + C_{m_{lg}}\Big|_{\text{lg down}}\right\}$$

Note that for the thrust coefficient you would need the characteristic diameter of a jet engine. However, in the end it does not matter which dimension you use, as long as it is used consistent.

It follows that the equilibrium elevator angle for steady symmetrical horizontal flight is - directly or indirectly - a function of several independent variables:

$$\delta_{e_{eq}} = f(\rho, V, x_{cg}, W, \delta_{t_e}, T_C, M, \delta_f, lg\,p)$$

with δ_{t_e} the elevator trim tab angle, T_C the dimensionless thrust coefficient, M the Mach number, δ_f the wing flap angle and '*lgp*' the landing gear position. Since the *stabilizer angle of incidence* i_h is fixed on the Citation II, it is not included in this treatment.

The *flap angle* and the *landing gear position* are typical examples of controllable variables and therefore they are not reduced to standard values. These variables are mentioned in the resulting graphs. The influence of the *elevator trim tab angle* on the trim curve is so small that it is discarded. Moreover, it has no influence on control position stability, see the *AE3212-I Flight Dynamics* lecture notes.

The airspeed for a given engine setting (e.g. *N1* rpm) during steady symmetric horizontal flight is dependent on the air density ρ, the aircraft mass W and the undisturbed Mach number *M*, the three of which are at the most adjustable variables. Therefore, these variables are all reduced to standard values by working with a reduced equivalent airspeed, as will be shown later on.

The longitudinal abscissa of the c.g. position x_{cg}, is an adjustable variable, which we will not reduce to a standard value, since it is of prime importance to the control position stability. Hiding this influence in a reduction prevents us from interpreting the impact it has on longitudinal static stability. Moreover, it will be used to compute the elevator effectiveness C_{m_δ}. The value of x_{cg} depends on the distribution of mass over the aircraft and thus changes when fuel is consumed by the engines. The computation of $x_{cg} = f(t, \textit{mass distribution})$ is quite straightforward, as is shown in Appendix E, an extract from the aircraft's mass and balance manual.

The engine thrust cannot be regarded as a totally controllable variable, since the actual thrust depends on atmospheric circumstances, as mentioned above. Therefore, we need to reduce these variations around the standard thrust, so that the elevator trim curve becomes reproducible. Engine settings - more specifically, *N1* rpm or fuel flow - should be mentioned

on the resulting graphs, however. The engine thrust can be derived from the measured fuel flow $\dot{m}_f$ and the computed Mach number *M* using the figures in Appendix E.

The reduction of the measured airspeed

When instrument and position errors are discarded, the airspeed as displayed on the pilot's airspeed indicator is the calibrated airspeed V_c. By converting V_c to the equivalent airspeed V_e, the atmospheric variables are reduced to ISA values. This conversion is carried out most elegantly via the Mach number and true airspeed [Ruijgrok 1990], as will be shown below.

The Mach number follows directly from V_c and the static pressure *p*, measured via the pressure altitude h_p. The calibration formula used for the altimeter is:

$$p = p_0 \left[1 + \frac{\lambda h_p}{T_0}\right]^{-\frac{g_0}{\lambda R}}$$

This expression delivers the last unknown in the Mach number equation:

$$M = \sqrt{\frac{2}{\gamma - 1}\left[\left(1 + \frac{p_0}{p}\left\{\left(1 + \frac{\gamma - 1}{2\gamma}\frac{\rho_0}{p_0}V_c^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right\}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]}$$

with γ the air ratio of specific heats, and p_0 and ρ_0 the ISA values for sea-level static air pressure and air density, respectively.

Furthermore, the speed of sound is obtained from $a = \sqrt{\gamma R T}$, with *R* the specific gas constant of air and *T* the static air temperature. We have to correct the measured total air temperature for ram rise via:

$$T = \frac{T_m}{1 + \frac{\gamma - 1}{2}M^2}$$

The true airspeed can now be obtained from $V_t = M \cdot a$, and the equivalent airspeed follows from its definition:

$$V_e = V_t \sqrt{\frac{\rho}{\rho_0}}$$

with ρ obtainable from the perfect gas law.

Reducing the non-standard aircraft mass

In the AE3212-I lecture notes it is explained that for a steady symmetrical horizontal flight, $\delta_{e_{eq}} = f(W, V, x_{cg}, aerodyn.\,parameters)$. This shows us that when we want to reduce W to W_s, while keeping x_{cg} and the flow conditions constant, we will have to adjust the definition of *V* so that $\delta_{e_{eq}}$ is not affected. To this end, we use an expression for the equilibrium of vertical forces during steady symmetric horizontal flight at any two arbitrary conditions i, j:

$$C_N = \frac{W}{\frac{1}{2}\rho V^2 S} = \frac{W_i}{\frac{1}{2}\rho_0 V_{e_i}^2 S} = \frac{W_j}{\frac{1}{2}\rho_0 V_{e_j}^2 S}, \text{ so that } V_{e_j} = V_{e_i}\sqrt{\frac{W_j}{W_i}}$$

Let us now define the reduced equivalent airspeed as:

$$\tilde{V}_e = V_e\sqrt{\frac{W_s}{W}}$$

When we use $\tilde{V}_e$ in the elevator trim curve instead of V_c, mass, airflow and compressibility reductions are implicitly included.

Reduction for non-standard engine thrust

Let us start with combining the general equation for the required elevator deflection for equilibrium:

$$\delta_{e_{eq}} = -\frac{1}{C_{m_\delta}}\left\{C_{m_0} + C_{m_\alpha}\left(\alpha - \alpha_0\right) + C_{m_{\delta_f}}\delta_f + C_{m_{T_c}}T_c + C_{m_g}\Big|_{\text{1g down}}\right\} \qquad \text{(C.1)}$$

with the equation for vertical equilibrium:

$$C_N \approx C_{N_\alpha}\left(\alpha - \alpha_0\right) \approx \frac{W}{\frac{1}{2}\rho V^2 S}$$

This leads to the following expression for the elevator angle required for longitudinal moment equilibrium, [see AE3212-I lecture notes for a more elaborate derivation]:

$$\delta_{e_{eq}} = -\frac{1}{C_{m_\delta}}\left\{C_{m_0} + \frac{C_{m_\alpha}}{C_{N_\alpha}}\frac{W}{\frac{1}{2}\rho V^2 S} + C_{m_{\delta_f}}\cdot\delta_f + C_{m_{T_c}}\cdot T_c + C_{m_g}\Big|_{\text{1g } down}\right\}$$

When measurement errors are neglected, this is also the measured elevator position in flight. When we now define the *reduced elevator deflection* as:

$$\delta^*_{e_{eq}} = -\frac{1}{C_{m_\delta}}\left\{C_{m_0} + \frac{C_{m_\alpha}}{C_{N_\alpha}}\frac{W}{\frac{1}{2}\rho \tilde{V}_e^2 S} + C_{m_{\delta_f}}\cdot\delta_f + C_{m_{T_c}}\cdot T_{c_s} + C_{m_g}\Big|_{\text{1g } down}\right\}$$

we can see that:

$$\delta^*_{e_{eq}} = \delta_{e_{eq_{meas}}} - \frac{1}{C_{m_\delta}}C_{m_{T_c}}\left(T_{c_s} - T_c\right)$$

in which $C_{m_{T_c}}$ is the dimensionless thrust moment arm (given in tables C.1 – C.3.) and T_{c_s} is the dimensionless standard thrust coefficient, with a value close to the real measured thrust coefficient value. We cannot choose T_{c_s} arbitrarily, since turbofan thrust decreases with airspeed, see Appendix F. To minimize this error, we choose its value at the thrust "normally" required to sustain horizontal flight at the measurement conditions. Actually we define a standard engine setting (in case of the Citation II, a standard fuel flow $\dot{m}_{f_s}$) so that the thrust variations due to varying atmospheric conditions over different measurement series are discarded.

Now, we can draw the elevator trim curves as $\delta^*_{e_{eq}}$ versus $\tilde{V}_e$, with on the graphs remarks on the values of the fully controllable parameters pertaining to those particular measurements. All standard values for the adjustable and uncontrollable parameters shall be mentioned as well, of course, but since they are the same for all graphs, they can be mentioned at some central point in the text. The values to be used for reduced parameters are given in Table B.1.

Parameter	**Notation**	**Standard value**
standard aircraft mass	W_s	60500 N (≅13600 lbs)
standard engine fuel flow per engine	$\dot{m}_{f_s}$	0.048 kg/sec
standard air density	ρ_0	1.225 kg/m^3

Table B.1

Drawing the reduced elevator control force curve

Analogous to what has been said about the trim curves, the control force curve measurement data have to be reduced to standard conditions as well. To this end, we divide the measured control force into three parts, [see also AE3212-I lecture notes]:

$$F_{e_{meas}} = F_{e_{aer}} + F_{e_f}$$

with:

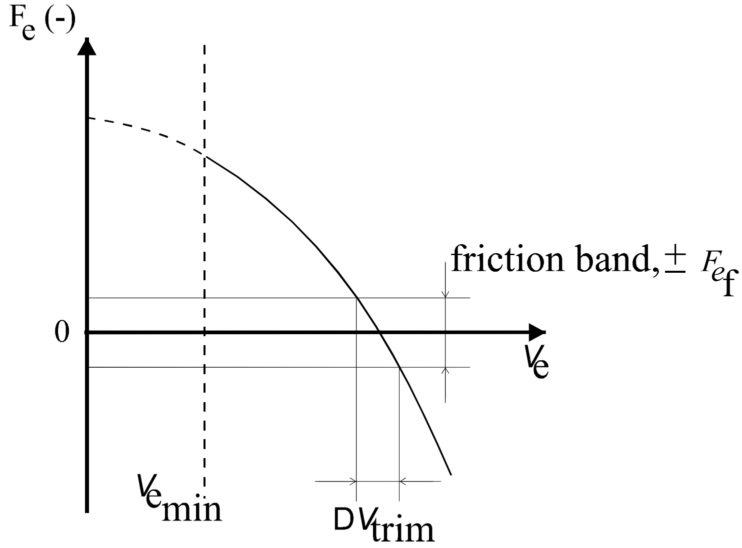
$F_{e_{meas}}$ is the measured elevator control force;

$F_{e_{aer}}$ is the force required to counteract the aerodynamic hinge moment;

F_{e_f} is the force required to overcome friction in the control mechanism;

The friction component F_{e_f} is, under normal circumstances, unknown. The aircraft manufacturer has however put much effort into minimizing this friction, since the trim speed is not well defined when F_{e_f} is large, see figure C.1.

To reduce this unwanted effect, F_{e_f} is minimized and the control force gradient is given a certain minimum value. Furthermore, any airframe vibrations during flight have a dithering effect and thus will further reduce friction. Therefore, we will ignore F_{e_f} and - from necessity - accept the fact that the measurements are accurate only to within $\pm F_{e_f}$.



What remains is the aerodynamic component in the elevator control force, $F_{e_{aer}}$. Inspection of [see AE3212-I lecture notes]:

$$F_{e_{eq}} = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left\{ \frac{C_{h_\delta}}{C_{m_\delta}} \frac{x_{cg} - x_{n_{free}}}{\bar{c}} \frac{W}{S} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e0}}\right) \right\}$$

reveals that $F_{e_{aer}} = f(W, x_{cg}, \frac{1}{2}\rho V^2, \delta_{t_e})$. We can draw the following conclusions with respect to data reduction: the reduction to V_e and W_s are combined when we use $\tilde{V}_e$, as with the trim curve. Since $F_{e_{aer}}$ is proportional to V^2, the reduced aerodynamic elevator control force component $F^*_{e_{aer}}$ is defined as:

$$F^*_{e_{aer}} = F_{e_{aer}} \cdot \frac{W_s}{W}$$

Since x_{cg} and δ_{t_e} both have influence on V_{tr} for a given aircraft mass, we cannot reduce these to a standard value. Moreover, they are at least adjustable variables. Therefore, they are both mentioned in every elevator control force curve graph.

The reduced elevator control force curve can now be drawn as $F^*_e = F^*_{e_{aer}}$ versus $\tilde{V}_e$.

Determining the elevator effectiveness, C_{m_δ}

All variables on the right-hand side of this expression are known, except for the elevator effectiveness, C_{m_δ}. We can determine the value of this parameter for our flight conditions as follows: let us rewrite eq. (C.1) for the longitudinal moment equilibrium and reduce it to relevant terms:

$$C_m = C_m\big|_{\delta_e=0} + C_{m_\delta}\ \delta_e = 0$$

This expression tells us that when C_m changes, the pilot must apply a certain $\Delta\delta_e$ to maintain the longitudinal moment equilibrium:

$$\Delta C_m = C_{m_\delta} \cdot \Delta\delta_e$$

This implies that when we generate a known ΔC_m and then measure the $\Delta\delta_e$ required to maintain longitudinal moment equilibrium, we can assess C_{m_δ}. We could for example generate a ΔC_m by shifting the c.g. to a new location, x_{cg_2}. The aerodynamic normal force C_N, acting through the centre of pressure, remains constant; the aircraft mass W now acts through x_{cg_2}. To achieve a c.g. shift, the experiment coordinator could move from this seat to a new position in the cabin, with a known abscissa, see also Appendix D. Thus, an unbalancing moment comes into being:

$$\Delta C_m = C_N \frac{x_{cg_2} - x_{cg_1}}{\bar{c}}$$

This moment can be overcome by the pilot using the elevator:

$$C_N \frac{x_{cg_2} - x_{cg_1}}{\bar{c}} + C_{m_\delta}(\delta_{e_2} - \delta_{e_1}) = 0$$

so that:

$$C_{m_\delta} = -\frac{1}{\Delta\delta_e} C_N \frac{\Delta x_{cg}}{\bar{c}}$$

with $C_N = \dfrac{W}{\frac{1}{2}\rho V^2 S}$ in steady horizontal flight. The change ΔC_N due to the changed elevator angle is neglected, as is the change in C_{m_δ} caused by the c.g. shift.

Determining the longitudinal stability, C_{m_α}

Now that we have determined the value of C_{m_δ} we can use the elevator trim curve to determine the value of the longitudinal stability C_{m_α}. Recall the expression for the required elevator deflection:

$$\delta_{e_{eq}} = -\frac{1}{C_{m_\delta}} \left\{ C_{m_0} + C_{m_\alpha}\left(\alpha - \alpha_0\right) + C_{m_{\delta_f}} \delta_f + C_{m_{T_c}} T_c + C_{m_{lg}} \Big|_{\text{lg down}} \right\}$$

If we take the derivative with respect to α we get:

$$\frac{d\delta_e}{d\alpha} = -\frac{C_{m_\alpha}}{C_{m_\delta}}$$

So, by determining the slope of the elevator trim curve we can calculate C_{m_α}.

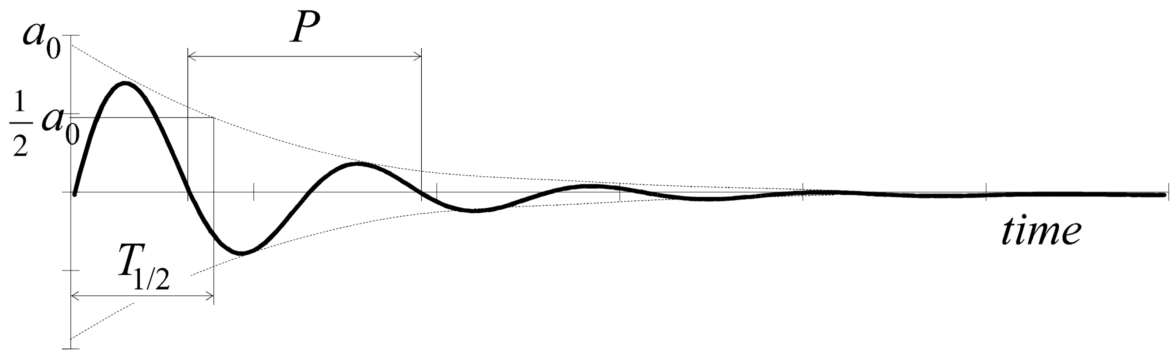
Appendix C: Deriving characteristic motion parameters from registered flight data

As described in the AE3212-I *Flight Dynamics* lecture notes, several parameters have been used to attach numerical values to aperiodic and oscillatory characteristic motions. Many of these parameters can be determined by graphical construction. Since the data set contains recorded time histories of the laboratory aircraft's characteristic motions, we can construct a few descriptive parameters for each registration.

For aperiodic motions, we determine the time constant τ as defined by [see AE3212-I lecture notes]:

$$x(t+\tau) = \frac{1}{e}x(t)$$

For oscillatory motions, we can determine the time to half amplitude $T_{\frac{1}{2}}$ and the period P from the recorded time histories, as shown in figure C.1.



Since for a complex eigenvalue $\lambda_c = \xi_c + i\eta_c$ the following holds [see AE3212-I *Flight Dynamics* lecture notes]:

$$T_{\frac{1}{2}} = \frac{ln\frac{1}{2}}{\xi_c}\frac{\bar{c}}{V} \text{ and } P = \frac{2\pi}{\eta_c}\frac{\bar{c}}{V}$$

For the asymmetric eigenvalues, read *b* for *c* and $\bar{c}$

By applying some simple mathematics, we can also compute estimates for the damping ratio ζ and the eigenfrequency ω_n.

Comparing the measured time responses to simulated time histories

A very important aspect of using linearized models of aircraft dynamics is how closely they resemble the real aircraft motions. When deriving linear models, the engineer must make quite a few simplifying assumptions about both the aircraft and its dynamical behavior. All these assumptions introduce errors in the model. In the AE3212-I lecture notes we remark that "comparison of time histories generated with linearized equations of motions and (small) real aircraft motions has shown that the differences are sufficiently small to warrant the use of linearized equations". How small are these differences then? This will be answered by the validation of the numerical model.

The particular method for validating is sometimes also ccalled Proof of Match [Baarspul 1989]. Usually associated with the certification of flight simulators, this technique demonstrates the match of a mathematical model with the real aircraft dynamics. To this end, a series of flight test maneuvers is performed during which both the model input variables (e.g. the control surface deflections) and the corresponding output variables are measured and recorded. Back on the ground, the recorded input variables are fed into the model, so that the model reacts on exactly the same input as the real aircraft. The model output is then compared to the recorded output variables, to reveal the differences between model dynamics and aircraft dynamics.

During the flight test eigenmotions demonstration, the flight test instrumentation system will record the control surface deflections and the relevant motion parameters (i.e. the dimension-having equivalents of the elements of x_c), at different recording frequencies. These recordings are available via the Brightspace internet site (http://brightspace.tudelft.nl), and are ready to be used with e.g. Python. With a pre-programmed command file, we can generate a proof-of-match (POM) plot for each of the eigenmotions of the Citation II aircraft. What we now need is a linear model of the symmetric and asymmetric Citation motions to compare with the recorded flight test data.

This model can be obtained from tables C.1, C.2 and C.3. It is inherent to the linearity of a model that it is usable only in a very small part of the flight envelope, the so-called nominal point. Although the model has a limited validity, it will have to do for our POM check since it is impossible to give the stability derivatives for all combinations of nominal flight conditions that may occur during the flight test. However, you will see that the model will show surprising similarity with the real flight under a variety of flight conditions.

First of all, it must be realized that any linear model of the Citation II flight dynamics, describing small deviations from a stationary flight condition, will depend on the center of gravity location, the mass, the moments of inertia, the air density and the airspeed in the stationary flight condition. In tables C.1, C.2 and C.3 however the aircraft mass, the air density and the trim speed ($V_{t,0}$) are not specified. Their influence will only act through C_L, C_{X_0} and C_{Z_0}, which can be calculated from the flight test data and are defined as:

$$C_L = \frac{W}{\frac{1}{2}\rho V_{t_0}^2 S}$$

$$C_{X_0} = \frac{W\sin(\theta_0)}{\frac{1}{2}\rho V_{t_0}^2 S}$$

$$C_{Z_0} = -\frac{W\cos(\theta_0)}{\frac{1}{2}\rho V_{t_0}^2 S}$$

The moments of inertia mentioned in table C.1 represent mean values for the flight test conditions and will not be varied.

Apart from C_L, C_{X_0} and C_{Z_0}, all stability derivatives are assumed to be independent of true airspeed V_{t_0} and of pressure altitude h_p in the stationary flight condition.

Symmetric motion

The linear model for symmetric aircraft motions, as described in the *AE3212-I* lecture notes is as follows:

$$\begin{bmatrix} C_{X_u}-2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha}+\left(C_{Z_{\dot\alpha}}-2\mu_c\right)D_c & -C_{X_0} & C_{Z_q}+2\mu_c \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha}+C_{m_{\dot\alpha}}D_c & 0 & C_{m_q}-2\mu_c K_{yy}^2 D_c \end{bmatrix} \cdot \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \dfrac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{X_\delta} \\ -C_{Z_\delta} \\ 0 \\ -C_{m_\delta} \end{bmatrix} \cdot \delta_e$$

All parameters, both dependent and independent, are dimensionless, whereas we have recorded dimension-having parameters during the flight. This means that we have to perform a transformation on the recorded parameters:

$\hat{u}=\dfrac{V_t-V_{t0}}{V_{t0}}$, with the index 0 indicating the stationary nominal condition

$$\frac{q\bar{c}}{V}=q_{meas}\cdot\frac{\bar{c}}{V_{t_0}}$$

When rewriting the equations, bear in mind that the dimensionless differential operator is defined as:

$$D_c=\frac{\bar{c}}{V_{t_0}}\frac{d}{dt}$$

Another complication is that the linear model is defined with respect to the stability axis system [see AE3212-I *Flight Dynamics* lecture notes], while the measurements are calibrated with respect to the body axis system. We transform the measurements to the stability axis system as follows:

$$\alpha_{stab}=\alpha_{body}-\alpha_0$$
$$\theta_{stab}=\theta_{body}-\theta_0$$

with the index 0 denoting the parameter value at t = 0, provided that the registration starts during stationary flight.

Both transformations must be carried out in your computer code.

Asymmetric motion

The linear model for asymmetric aircraft motions, as described in the AE3212-I *Flight Dynamics* lecture notes is repeated here for clarity:

$$\begin{bmatrix} C_{Y_\beta}+\left(C_{Y_{\dot\beta}}-2\mu_b\right)D_b & C_L & C_{Y_p} & C_{Y_r}-4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{\ell_\beta} & 0 & C_{\ell_p}-4\mu_b K_{xx}^2 D_b & C_{\ell_r}+4\mu_b K_{xz} D_b \\ C_{n_\beta}+C_{n_{\dot\beta}}D_b & 0 & C_{n_p}+4\mu_b K_{xz} D_b & C_{n_r}-4\mu_b K_{zz}^2 D_b \end{bmatrix} \cdot \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} =$$

$$= \begin{bmatrix} -C_{y_{\delta_a}} \\ 0 \\ -C_{\ell_{\delta_a}} \\ -C_{n_{\delta_a}} \end{bmatrix} \cdot \delta_a + \begin{bmatrix} -C_{y_{\delta_r}} \\ 0 \\ -C_{\ell_{\delta_r}} \\ -C_{n_{\delta_r}} \end{bmatrix} \cdot \delta_r$$

Again, all recorded parameters are dimension-having, so we have to transform some state variables to the same dimensions as the recorded variables. In this case we have to substitute:

$$\frac{pb}{2V} = p_{meas} \cdot \frac{b}{2V_{t_0}}$$

$$\frac{rb}{2V} = r_{meas} \cdot \frac{b}{2V_{t_0}}$$

When rewriting the equations to obtain the dimension-having state vector $\left[\beta \quad \varphi \quad p \quad r\right]^T$, bear in mind that the dimensionless differential operator of asymmetric motion is defined as

$$D_b = \frac{b}{V_{t_0}} \frac{d}{dt}$$

Table C.1. Linear model parameters – symmetric flight

aircraft: Cessna Citation II **(C550)** **configuration:** Clean cruise (flaps up, gear up) $x_{cg} = 0.25\overline{c}$	
Aircraft dimensions $S = 30.00\ m^2$ $\overline{c} = 2.0569\ m$ $b = 15.911\ m$ Aerodynamic coefficients $C_{D_0} = 0.04$ $C_{L_\alpha} = 5.084$ $e = 0.8$	Aircraft inertia $K_{xx}^2 = 0.019$ $K_{yy}^2 = 1.3925$ $K_{zz}^2 = 0.042$ $K_{xz} = 0.002$

Table C.2. Linear model stability derivatives – symmetric flight

aircraft: Cessna Citation II **(C550)** **configuration:** Clean cruise (flaps up, gear up)		
Longitudinal force derivatives	Normal force derivatives	Pitch moment derivatives
$C_{X_u} = -0.0279$	$C_{Z_u} = -0.3762$	$C_{m_0} = 0.0297$
$C_{X_\alpha} = -0.4797$	$C_{Z_\alpha} = -5.7434$	$C_{m_u} = 0.0699$
$C_{X_{\dot{\alpha}}} = 0.0833$	$C_{Z_{\dot{\alpha}}} = -0.0035$	$C_{m_\alpha} = -0.5626$
$C_{X_q} = -0.2817$	$C_{Z_q} = -5.6629$	$C_{m_{\dot{\alpha}}} = 0.1780$
$C_{X_\delta} = -0.0373$	$C_{Z_\delta} = -0.6961$	$C_{m_q} = -8.7941$
		$C_{m_\delta} = -1.1642$
		$C_{m_{T_c}} = -0.0064$

Table C.3. Linear model stability derivatives – asymmetric flight

aircraft: Cessna Citation II **(C550)**		
configuration: Clean cruise (flaps up, gear up)		
Lateral force derivatives	Roll moment derivatives	Yaw moment derivatives
$C_{Y_\beta} = -0.7500$	$C_{l_\beta} = -0.1026$	$C_{n_\beta} = 0.1348$
$C_{Y_{\dot\beta}} = 0$	$C_{l_p} = -0.7108$	$C_{n_{\dot\beta}} = 0$
$C_{Y_p} = -0.0304$	$C_{l_r} = 0.2376$	$C_{n_p} = -0.0602$
$C_{Y_r} = 0.8495$	$C_{l_{\delta_a}} = -0.2309$	$C_{n_r} = -0.2061$
$C_{Y_{\delta_a}} = -0.0400$	$C_{l_{\delta_r}} = 0.0344$	$C_{n_{\delta_a}} = -0.0120$
$C_{Y_{\delta_r}} = 0.2300$		$C_{n_{\delta_r}} = -0.0939$

Appendix D: Rewriting the EOM in state-space format

The symmetric equations of motion as derived in "Flight Dynamics" are:

$$\begin{bmatrix} C_{X_u}-2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha}+\left(C_{Z_{\dot\alpha}}-2\mu_c\right)D_c & -C_{X_0} & C_{Z_q}+2\mu_c \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha}+C_{m_{\dot\alpha}}D_c & 0 & C_{m_q}-2\mu_c K_Y^2 D_c \end{bmatrix} \cdot \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{X_{\delta_e}} \\ -C_{Z_{\delta_e}} \\ 0 \\ -C_{m_{\delta_e}} \end{bmatrix} \cdot \delta_e$$

The asymmetric equations of motion, derived in "Flight Dynamics" as well, are:

$$\begin{bmatrix} C_{Y_\beta}+\left(C_{Y_{\dot\beta}}-2\mu_b\right)D_b & C_L & C_{Y_p} & C_{Y_r}-4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{\ell_\beta} & 0 & C_{\ell_p}-4\mu_b K_X^2 D_b & C_{\ell_r}+4\mu_b K_{XZ} D_b \\ C_{n_\beta}+C_{n_{\dot\beta}}D_b & 0 & C_{n_p}+4\mu_b K_{XZ} D_b & C_{n_r}-4\mu_b K_Z^2 D_b \end{bmatrix} \cdot \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} =$$

$$= \begin{bmatrix} -C_{Y_{\delta_a}} \\ 0 \\ -C_{\ell_{\delta_a}} \\ -C_{n_{\delta_a}} \end{bmatrix} \cdot \delta_a + \begin{bmatrix} -C_{Y_{\delta_r}} \\ 0 \\ -C_{\ell_{\delta_r}} \\ -C_{n_{\delta_r}} \end{bmatrix} \cdot \delta_r$$

In these equations D_c and D_b are differential operators, $D_c = \frac{\bar{c}}{V}\frac{d}{dt}$ and $D_b = \frac{b}{V}\frac{d}{dt}$

$\hat{u}$ is the dimensionless speed, $\hat{u} = \frac{u}{V}$ with V the true airspeed of the aircraft in the stationary flight condition.

The equations can be rewritten using the absolute deviation u from the nominal airspeed V instead of the relative airspeed $\hat{u}$, the normal angular pitch rate q, instead of the dimensionless version $\frac{q\bar{c}}{V}$, the normal angular roll rate p, instead of the dimensionless version $\frac{pb}{2V}$ and the normal angular yaw rate r, instead of the dimensionless version $\frac{rb}{2V}$.

Implementing the mentioned substitutions in both the symmetric and the asymmetric equations yields the following equation:

$$C_1\dot{\overline{x}} + C_2\overline{x} + C_3\overline{u} = \overline{0}$$

Here $\overline{u}$ is the input vector and $\overline{x}$ is the state vector;

In the symmetric case $\overline{u}$ and $\overline{x}$ are defined as:

$$\overline{u} = [\delta_e]$$
$$\overline{x} = [u \quad \alpha \quad \theta \quad q]^T$$

And in the asymmetric case:

$$\overline{u} = [\delta_a \quad \delta_r]^T$$
$$\overline{x} = [\beta \quad \varphi \quad p \quad r]^T$$

The form of the equations we are aiming at is the *state space representation* of the system:

$$\dot{\overline{x}} = A\overline{x} + B\overline{u}$$
$$\overline{y} = C\overline{x} + D\overline{u}$$

With $\overline{x}$ the state vector, $\overline{u}$ the input vector and $\overline{y}$ the output vector. The system matrices A and B can be determined from:

$$A = -C_1^{-1}C_2$$
$$B = -C_1^{-1}C_3$$

Appendix E: Extract from the Citation II mass and balance manual

In order to reconstruct the aircraft mass $W(t)$ and c.g. position $x_{cg}(t)$ as functions of time during the flight test, we start with the aircraft as it stands on the platform, with all persons on board and the engines not yet started. We can divide the aircraft *ramp mass* into (see figure E.1):

- *Basic Empty Mass (BEM)*: this is the mass of the aircraft structure, engines, systems, furnishings (i.e. seats and carpets) and all other optional equipment. It further includes unusable fuel, engine oil, hydraulic fluids, fire extinguishers and the Flight Test Instrumentation System (FTIS).
- *Fuel*: the fuel quantity is not measured in litres, but in pounds. One pound is 0.453592 kg.
- *Payload*: this is the mass of the crew, passengers, baggage and cargo.

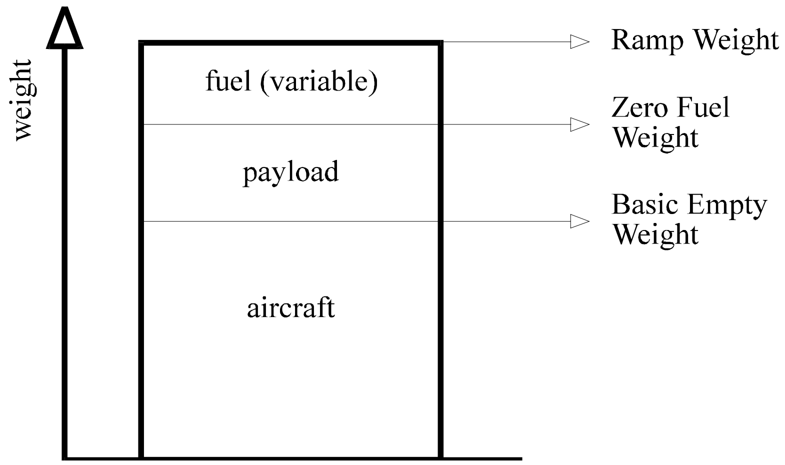


Figure D.1. Breakdown of aircraft mass

The BEM has been determined by weighing the aircraft with FTIS installed, and will be passed on to the student-observers on the day of the flight test. All crew and passengers will stand on a weighing balance before they enter the aircraft. The amount of fuel on board is carefully read off the indicators before engine start-up.

When the engines are started, they begin consuming fuel, which is registered by the fuel flow indicators on FTIS ($\dot{m}_{f_{\ell,r}}$). Within FTIS, these signals are integrated to compute the fuel consumed, $\int \dot{m}_{f_{\ell+r}}\, \mathrm{d}t$. The momentary aircraft mass is now easily computed as:

$$W(t_1) = \text{ramp mass - } \int_0^{t_1} \dot{m}_{f_{\ell+r}}\, \mathrm{d}t$$

The c.g. position $x_{cg}(t)$ is influenced by both the mass and the distance to some datum line of any mass, which is part of the aircraft operating mass $W(t)$. To this end, the aircraft manufacturer has defined a datum line with corresponding *fuselage stations*, as depicted in figure E.2. As usual in the aerospace world, the measurement units used are inches, with 1 inch = 2.54 cm.

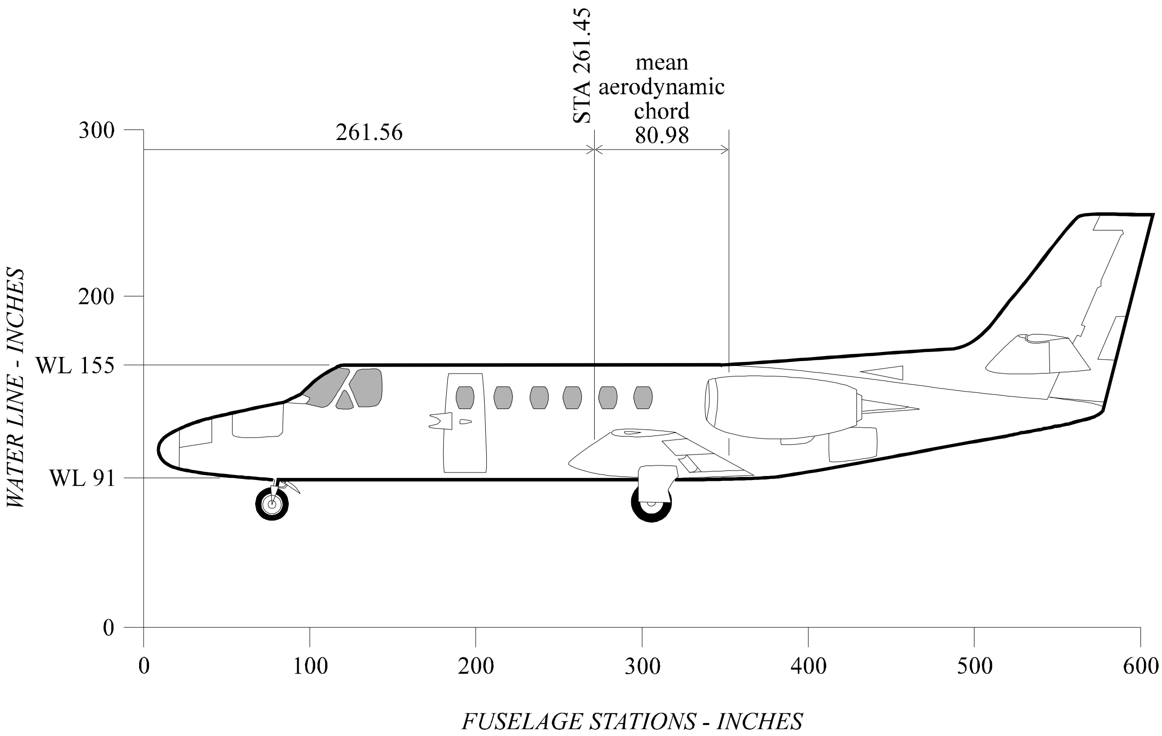


Figure E.2. Citation *II* fuselage stations

All components of the aircraft, the fuel and the passenger and crew seats are assigned fuselage stations according to the figure above. By multiplying the mass of each of these items with the corresponding fuselage station (functioning as a moment arm), we obtain their moment with respect to an arbitrary point on the datum line. We then add all these moments to get the grand total moment of the aircraft including fuel and payload. If we now divide this total aircraft moment by *W*, we obtain the moment arm of the aircraft including fuel and payload. This moment arm is the horizontal distance between the datum line and the line of action of the gravity vector acting on the aircraft, i.e. $x_{cg_{datum}}$. The aircraft manufacturer has specified that at maximum gross mass, $x_{cg_{datum}}$ should lie between STA 276.10 and STA 285.80, in order to maintain acceptable flying qualities.

The procedure for assessing $x_{cg}(t)$ is quite straightforward: we use the Technical weighing report (figure E.4.) to determine the A/C empty mass [lbs] and moment [Inch-lbs/100]. From these numbers the basic empty weight [N] and moment [Ninch] can be calculated. It is now possible to fill in a mass and balance form (Table E.1) for the ramp mass, i.e. $t = 0$, and then apply corrections for both the mass and the cg shift of the fuel used at any measurement time point. The positions of the seats and baggage areas in the cabin are depicted in figure E.3. The moment arm of the fuel varies when the fuel is burnt off, so we have to use Table E.2 for the fuel moment. This table is copied from the Cessna Citation II Airplane Flight Manual (AFM).

Table E.1. Citation II

Mass and Balance form

payload computations			
crew and pax	$x_{cg_{datum}}$ *[inches]*	*mass [pounds]*	*moment [inch-pound]*
seat 1	131		
seat 2	131		
seat 3	214		
seat 4	214		
seat 5	251		
seat 6	251		
seat 7	288		
seat 8	288		
seat 10	170		
baggage			
Nose	74		
aft cabin	321		
	338		
payload			

mass and balance computations		
Item	*mass [pound]*	*moment [inch-pounds]*
basic empty mass $x_{cg_{datum}}$ at BEM = ____		
Payload		
zero fuel mass $x_{cg_{datum}}$ at ZFM = ____		
fuel load		
ramp mass $x_{cg_{datum}}$ at RM = ____		

Note: a heavy line above a field means that the preceding fields have to be summed.

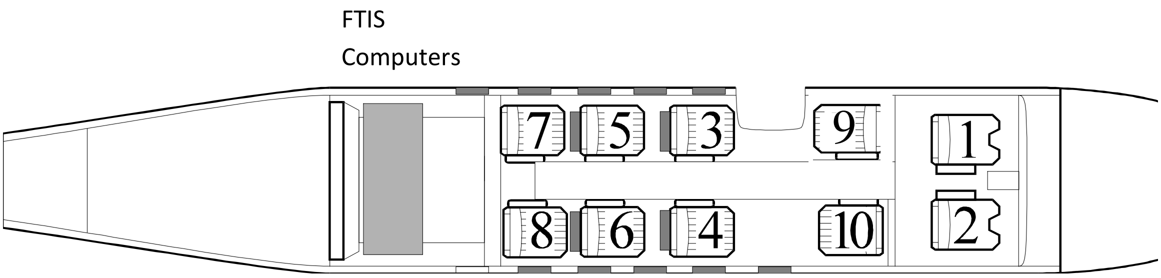


Figure E.3. Citation *II* cabin layout with FTIS installed

The value of $x_{\mathrm{cg}_{\mathrm{datum}}}$ thus found is again expressed in inches with respect to the (arbitrarily chosen) datum line, and we would like to use a more generic unit. Therefore, we convert $x_{\mathrm{cg}_{\mathrm{datum}}}$ to x_{cg}, expressed in meters and with respect to the forward end of the (imaginary) mean aerodynamic chord, STA 261.45 (inches) as defined in figure E.2. We express the abscissa of the neutral point, x_n, in meters and with respect to the forward end of the m.a.c. as well. Another usable unit for x_{cg} is $\%\bar{c}$, which is of course nothing more than $x_{\mathrm{cg}}(m)\cdot 100/\bar{c}$.

During the measurement procedure for C_{m_δ}, the experiment co-ordinator, who usually is seated on seat 10, moves in between seats 7 and 8. He thus has another $x_{\mathrm{cg co\text{-}ordinator}}$, and influences the x_{cg} of the whole aircraft. The calculation of this Δx_{cg} is quite straightforward when Table E.1 is used.

Table E.2. Citation II fuel moments with respect to the datum line

AIRPLANES –0627 AND ON

MASS (POUNDS)	MOMENT/100 ARM VARIES (INCH-POUNDS)	MASS (POUNDS)	MOMENT/100 ARM VARIES (INCH-POUNDS)
100	298.16	2600	7415.33
200	591.18	2700	7699.60
300	879.08	2800	7984.34
400	1165.42	2900	8269.06
500	1448.40	3000	8554.05
600	1732.53	3100	8839.04
700	2014.80	3200	9124.80
800	2298.84	3300	9410.62
900	2581.92	3400	9696.97
1000	2866.30	3500	9983.40
1100	3150.18	3600	10,270.08
1200	3434.52	3700	10,556.84
1300	3718.52	3800	10,843.87
1400	4003.23	3900	11,131.00
1500	4287.76	4000	11,418.20
1600	4572.24	4100	11,705.50
1700	4856.56	4200	11,993.31
1800	5141.16	4300	12,281.18
1900	5425.64	4400	12,569.04
2000	5709.90	4500	12,856.86
2100	5994.04	4600	13,144.73
2200	6278.47	4700	13,432.48
2300	6562.82	4800	13,720.56
2400	6846.96	4900	14,008.46
2500	7131.00	5008	14,320.34

Figure E.4. The Technical weighing report form

TUDelft	DATE OF WEIGHING:	
	PERFORMED BY:	WEIGHING METHOD:

Masss in lbs.					
NOSE					
LEFT WING					
RIGHT WING +					
A/C AS WEIGHED					
FUEL -					
A/C EMPTY MASS					

AIRCRAFT CONFIGURATION:

√ FTIS installed

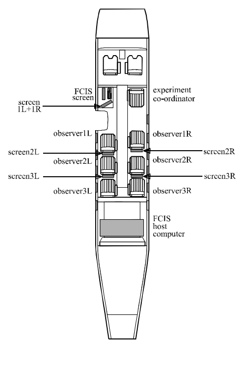
√ no persons on board

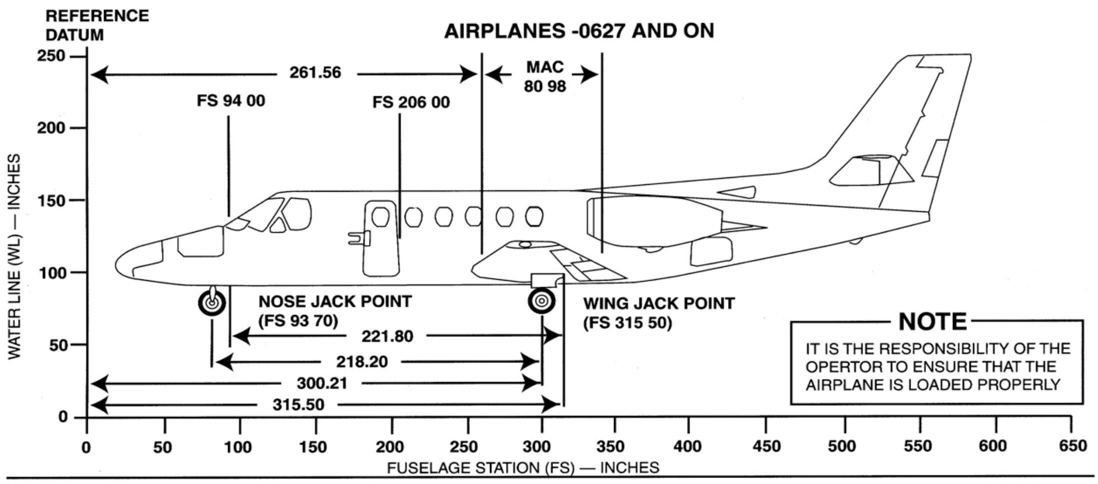
√ 7 seats in cabin

√ unusable fuel in tanks

√ full oil

□ other:





LOCATING CG WITH AIRPLANE ON LANDING GEAR

$$\text{CG arm of airplane (in inches aft of datum)} = 300.21 - \frac{218.20 \times \text{Nose gear weight}}{\text{Nose and main gear weight totaled}}$$

LOCATING CG WITH AIRPLANE ON JACK PADS

$$\text{CG arm of airplane (in inches aft of datum)} = 315.50 - \frac{221.80 \times \text{Nose jack point weight}}{\text{Nose and wing jack point weight totaled}}$$

LOCATION OF CG IN PERCENT MAC

ITEM	MASS					CG ARM					MOMENT						
AIRPLANE AS WEIGHED																	
FUEL -																	
A/C EMPTY MASS																	

Appendix F: Derivation of engine thrust from flight measurements

The thrust of a jet engine is a function of the engine control setting (Γ), the Mach number (M) and the altitude (H) ($T = T(\Gamma, M, H)$) for a given aircraft configuration in the standard atmosphere. If the altitude and the Mach number are known the thrust can be determined directly by using one of the following engine parameters: the number of revolutions per minute of the low or high pressure compressor (N_1 and N_2 respectively), the temperature of the hot gas flow between the high and low pressure turbine (ITT = Interstage Turbine Temperature) or the amount of fuel per time period (fuel-flow) m_f.

For this assignment the last option will be used because the engine manufacturer has provided a so-called "Computer Card Deck" for which the calculated thrust is given as a function of three characteristic engine control settings. These settings are a function of the fuel-flow, Mach number and altitude in the standard atmosphere. Using this data a computer program thrust.exe was developed for this test flight that can determine the thrust per engine in the standard atmosphere for every flight condition by using the momentary fuel-flow, Mach number and pressure altitude. To determine the thrust in the actual (read: non-standard) atmosphere the computer program uses a temperature correction (ΔT_{temp}), which is the difference between the actual outside air temperature (T_{temp}) and the temperature in the standard atmosphere ($T_{temp\ ISA}$):

$$\Delta T_{temp} = T_{temp} - T_{temp,ISA}$$

To avoid confusion the index "*temp*" is added to indicate that the symbol T in this case is used for the temperature and not the thrust. Note that these temperatures are *static* or *true* temperatures and not total temperatures. The thrust.exe is available on Brightspace and you can use it in your computer code.

The Cessna Citation II laboratory aircraft is powered by two Pratt & Whitney Canada JT15D-4 twin-spool turbofan engines. Table F.1 shows relevant data, while figure F.1 depicts a cross-section of a JT15D-4.

Table F.1. JT15D-4 engine data *[Vermeij 1993]*

	take-off (0 m ISA)	max. cruise (0 m ISA)
thrust T [N]	11121	10431
fuel flow $\dot{m}_f$	0.177	0.164
total mass flow $\dot{m}$ [kgs^{-1}]	35.245	
by-pass ratio λ	2.6	

As the figure shows, all intake air passes through the fan. Immediately aft of the fan, the airflow is divided by a concentric duct. Most of the air* passes around the engine core through the outer duct and is expelled from the rear. The air entering the inner duct passes around the engine core through the axial low-pressure compressor stage and is further compressed by the centrifugal high-pressure compressor. The high-pressure air then passes through a diffuser and moves aft to the combustion section. In the combustion section (of the reverse-flow type to reduce the engine size) a portion of the air is ignited by the fuel sprayed into the flow. The remainder is guided along the combustion chamber walls to provide cooling. Electric igniters provide sparks only during the start sequence of the engine, since when the engine is running the combustion becomes self-sustaining.

The hot gases coming from the combustion section run through the high-pressure turbine, which drives the high-pressure centrifugal compressor. The remaining (larger) portion of energy in the air is taken out by the low-pressure turbine, which drives the low-pressure compressor and the fan. The gases are then expelled at the rear of the engine through the exhaust.

Jet engines produce thrust by increasing the momentum $p = \dot{m}V$ of the air that passes through the engine. The core section accelerates a small air mass to high velocity, whereas the by-pass or fan section accelerates a large air mass to a lower velocity. Neglecting the various devices that extract energy from the engines to power aircraft systems, we can define the thrust of one turbo-fan engine as [Ruijgrok 1990]:

$$T_p = (\dot{m}_{core} + \dot{m}_f)\, w_{e_{core}} + \dot{m}_{fan} w_{e_{fan}} - \dot{m}V_0 + A_e(p_e - p_0) \qquad \text{(F.1)}$$

with $\dot{m}_i$ the various airflows, $\dot{m}_f$ the fuel flow, w_{e_i} the various exit speeds directly aft of the nozzles, V_0 the undisturbed airspeed, A_e the nozzle area and p_e and p_0 the exit and ambient static pressures, respectively. The only case in which the exit pressure p_e is not equal to the ambient static pressures, is when the engine nozzle is choked, i.e. when the air is expelled from the engine at the local speed of sound. This is not the case for the JT15D-4, so $p_e = p_0$ and the last term in eq. (F.1) disappears.

Furthermore, we assume that we can neglect the fuel flow $\dot{m}_f$ when compared to the total air mass flow $\dot{m}$ through the engine. The total air mass flow through the JT15D-4 at sea-level static ($V_0 = 0$ at 0 m ISA) take-off thrust is 35.2 kg/sec with a fuel mass flow of 0.18 kg/sec (Table F.1), so the assumption seems to be valid.

When we now denote the by-pass ratio (mentioned in the previous footnote) λ, we see that:

$$\dot{m}_{core} = \frac{1}{\lambda + 1}\dot{m} \text{ and } \dot{m}_{fan} = \frac{\lambda}{\lambda + 1}\dot{m}$$

so that we can re-write eq. (F.1) as:

---

* Actually, 2.5 times as much air passes around the core, compared to the air mass passing through the core. This means that the *by-pass ratio* λ of the engine is 2.5.

$$T_p = \dot{m}\left(\frac{1}{\lambda+1} w_{e_{core}} + \frac{\lambda}{\lambda+1} w_{e_{fan}} - V_0\right)$$

or:

$$T_p = \dot{m}(\overline{w}_e - V_0) \tag{F.2}$$

with $\overline{w}_e = \frac{1}{\lambda+1} w_{e_{core}} + \frac{\lambda}{\lambda+1} w_{e_{fan}}$ a weighed sum of the core and by-pass exit speeds.

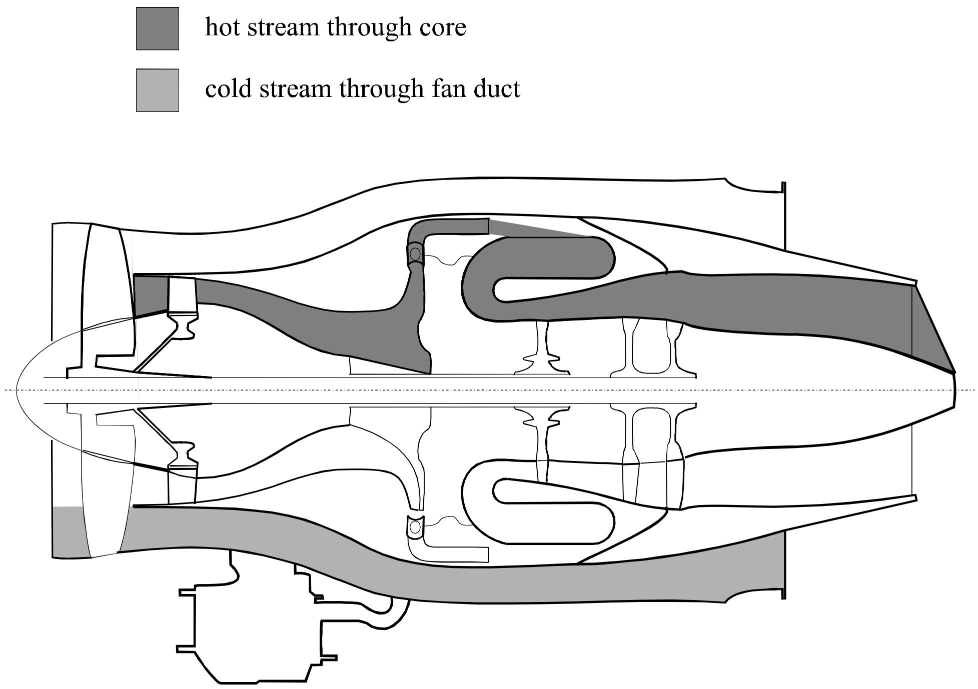


Figure F.1. Schematic view of the hot and cold airflows through the JT15D-4 engine.

The product of $\dot{m}_f$ and the heating value of the fuel H can be regarded as the *thermal input* to the engine. The engine converts part of this input into thrust power (i.e. work), which can be written as $T_p V_0$. The proportion of the input and the output is quantified by the *total engine efficiency*:

$$\eta_{tot} = \frac{T_p V_0}{\dot{m}_f H} = \frac{\dot{m}(\overline{w}_e - V_0)V_0}{\dot{m}_f H}$$

so that:

$$T_p = \frac{\eta_{tot}\dot{m}_f H}{V_0} = \dot{m}(\overline{w}_e - V_0) = f(\eta_{tot}, \dot{m}_f, H, V_0, \dot{m}) \tag{F.3}$$

The fuel controller translates a thrust lever angle (the pilot's input) into a fuel flow, with a restricting factor that the ratio of fuel flow and air through the core should be more or less constant. Thrust, however, depends for a given $\dot{m}_f$ on the conditions of the air entering the intake. We can understand this by noting that the ratio of air mass and air volume, i.e. the specific volume, is not constant. The thrust depends on air *mass*, while the engine inlet can only put through a maximum air *volume*. So, keeping in mind that specific volume is the reciprocal of density ρ and assuming validity of the perfect gas law:

$$T_p = f(\dot{m}_f, p_0, T_0, V_0) \tag{F.4}$$

We implicitly assume H and η_{tot} to be constant, and we assume that the fuel controller keeps the ratio of $\dot{m}$ and $\dot{m}_f$ constant. Note that, as opposed to the conventions used in the normal chapters of this manual, the index 0 means "outside in the undisturbed air". When we substitute V_0 by the undisturbed Mach number $M_0 = V_0\sqrt{\gamma R T_0}$ and p_0 by the pressure altitude h_p, we can re-write eq. (F.4) as:

$$T_p = f(\dot{m}_f, h_p, M_0) \tag{F.5}$$

[Vermeij 1993] has calculated the thrust of the JT15D-4 engine for a range of values of all independent variables. We can find the thrust for any measurement point by interpolating between tables for the specific combination of $\dot{m}_f$, h_p, and M_0 belonging to the measurement point.

The engine thrust of the Cessna Citation II engines can be determined when the Mach number, the altitude and the fuel flow to both the engines are known. The engine manufacturer has made some graphs available in which for three characteristic engine settings the calculated thrust has been given as a function of the fuel flow, the Mach number and the altitude in ISA. The faculty has developed a computer program, named *thrust.exe*, which calculates the thrust per engine by linear interpolation of these graphs for every possible flight condition. This program is available on Brightspace.

For the calculations *thrust.exe* needs the following properties in each stationary measurement point:

- the pressure altitude h_p [m]
- the Mach number M
- the fuel flow of the left jet engine Mf1 [kg/s]
- the fuel flow of the right jet engine Mf2 [kg/s]
- the temperature difference ΔT_{ISA} between the corrected total air temperature and the temperature in ISA, via the equation $T_{ISA} = T_0 + (\lambda \cdot h_p)$.