CHAPTER 10

GREEN'S FUNCTIONS

In contrast to the linear differential operators that have been our main concern when formulating problems as differential equations, we now turn to methods involving integral operators, and in particular to those known as **Green's functions**. Green's-function methods enable the solution of a differential equation containing an inhomogeneous term (often called a **source term**) to be related to an integral operator containing the source. As a preliminary and elementary example, consider the problem of determining the potential $\psi(\mathbf{r})$ generated by a charge distribution whose charge density is $\rho(\mathbf{r})$. From the Poisson equation, we know that $\psi(\mathbf{r})$ satisfies

$$-\nabla^2\psi(\mathbf{r}) = \frac{1}{\varepsilon_0}\rho(\mathbf{r}). \tag{10.1}$$

We also know, applying Coulomb's law to the potential at $\mathbf{r}_1$ produced by each element of charge $\rho(\mathbf{r}_2)d^3r_2$, and assuming the space is empty except for the charge distribution, that

$$\psi(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0}\int d^3r_2 \frac{\rho(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{10.2}$$

Here the integral is over the entire region where $\rho(\mathbf{r}_2) \neq 0$. We can view the right-hand side of Eq. (10.2) as an integral operator that converts ρ into ψ, and identify the **kernel** (the function of two variables, one of which is to be integrated) as the Green's function for this problem. Thus, we write

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi\varepsilon}\,\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{10.3}$$

$$\psi(\mathbf{r}_1) = \int d^3r_2\, G(\mathbf{r}_1, \mathbf{r}_2)\rho(\mathbf{r}_2), \tag{10.4}$$

assigning our Green's function the symbol G (for "Green").

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This example is preliminary because the response of more general problems to an inhomogeneous term will depend on the boundary conditions. For example, an electrostatics problem may include conductors whose surfaces will contain charge layers with magnitudes that depend on ρ and which will also contribute to the potential at general $\mathbf{r}$. It is elementary because the form of the Green's function will also depend on the differential equation to be solved, and often it will not be possible to obtain a Green's function in a simple, closed form.

The essential feature of any Green's function is that it provides a way to describe the response of the differential-equation solution to an arbitrary source term (in the presence of the boundary conditions). In our present example, $G(\mathbf{r}_1, \mathbf{r}_2)$ gives us the contribution to ψ at the point $\mathbf{r}_1$ produced by a point source of unit magnitude (a delta function) at the point $\mathbf{r}_2$. The fact that we can determine ψ everywhere by an integration is a consequence of the fact that our differential equation is linear, so each element of the source contributes additively. In the more general context of a PDE that depends on both spatial and time coordinates, Green's functions also appear as responses of the PDE solution to impulses at given positions and times.

The aim of this chapter is to identify some general properties of Green's functions, to survey methods for finding them, and to begin building connections between differential-operator and integral-operator methods for the description of physics problems. We start by considering problems in one dimension.

10.1 ONE-DIMENSIONAL PROBLEMS

Let's consider the second-order self-adjoint inhomogeneous ODE

$$\mathcal{L}y \equiv \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y = f(x), \tag{10.5}$$

which is to be satisfied on the range $a \le x \le b$ subject to homogeneous boundary conditions at $x = a$ and $x = b$ that will cause $\mathcal{L}$ to be Hermitian.[1] Our Green's function for this problem needs to satisfy the boundary conditions and the ODE

$$\mathcal{L}G(x,t) = \delta(x-t), \tag{10.6}$$

so that $y(x)$, the solution to Eq. (10.5) with its boundary conditions, can be obtained as

$$y(x) = \int_a^b G(x,t)\,f(t)\,dt. \tag{10.7}$$

To verify Eq. (10.7), simply apply $\mathcal{L}$:

$$\mathcal{L}y(x) = \int_a^b \mathcal{L}\,G(x,t)\,f(t)\,dt = \int_a^b \delta(x-t)\,f(t)\,dt = f(x).$$

[1] A **homogeneous** boundary condition is one that continues to be satisfied if the function satisfying it is multiplied by a scale factor. Most of the more commonly encountered types of boundary conditions are homogeneous, e.g., $y = 0$, $y' = 0$, even $c_1 y + c_2 y' = 0$. However, $y = c$ with c a nonzero constant is not homogeneous.

General Properties

To gain an understanding of the properties $G(x,t)$ must have, we first consider the result of integrating Eq. (10.6) over a small range of x that includes $x = t$. We have

$$\int_{t-\varepsilon}^{t+\varepsilon} \frac{d}{dx}\left[p(x)\frac{dG(x,t)}{dx}\right]dx + \int_{t-\varepsilon}^{t+\varepsilon} q(x)\,G(x,t)\,dx = \int_{t-\varepsilon}^{t+\varepsilon} \delta(t-x)\,dx,$$

which, carrying out some of the integrations, simplifies to

$$p(x)\frac{dG(x,t)}{dx}\bigg|_{t-\varepsilon}^{t+\varepsilon} + \int_{t-\varepsilon}^{t+\varepsilon} q(x)\,G(x,t)\,dx = 1. \tag{10.8}$$

It is clear that Eq. (10.8) cannot be satisfied in the limit of small ε if $G(x,t)$ and $dG(x,t)/dx$ are both continuous (in x) at $x = t$, but we can satisfy that equation if we require $G(x,t)$ to be continuous but accept a discontinuity in $dG(x,t)/dx$ at $x = t$. In particular, continuity in G will cause the integral containing $q(x)$ to vanish in the limit $\varepsilon \to 0$, and we are left with the requirement

$$\lim_{\varepsilon\to 0+}\left[\frac{dG(x,t)}{dx}\bigg|_{x=t+\varepsilon} - \frac{dG(x,t)}{dx}\bigg|_{x=t-\varepsilon}\right] = \frac{1}{p(t)}. \tag{10.9}$$

Thus, the discontinuous impulse at $x = t$ leads to a discontinuity in the x derivative of $G(x,t)$ at that x value. Note, however, that because of the integration in Eq. (10.7), the singularity in dG/dx does not lead to a similar singularity in the overall solution $y(x)$ in the usual case that $f(x)$ is continuous.

As a next step toward reaching understanding of the properties of Green's functions, let's expand $G(x,t)$ in the eigenfunctions of our operator $\mathcal{L}$, obtained subject to the boundary conditions already identified. Since $\mathcal{L}$ is Hermitian, its eigenfunctions can be chosen to be orthonormal on (a,b), with

$$\mathcal{L}\varphi_n(x) = \lambda_n\varphi_n(x), \quad \langle\varphi_n|\varphi_m\rangle = \delta_{nm}. \tag{10.10}$$

Expanding both the x and the t dependence of $G(x,t)$ in this orthonormal set (using the complex conjugates of the φ_n for the t expansion),

$$G(x,t) = \sum_{nm} g_{nm}\varphi_n(x)\varphi_m^*(t). \tag{10.11}$$

We also expand $\delta(x-t)$ in the same orthonormal set, according to Eq. (5.27):

$$\delta(x-t) = \sum_m \varphi_m(x)\varphi_m^*(t). \tag{10.12}$$

Inserting both these expansions into Eq. (10.6), we have before any simplification

$$\mathcal{L}\sum_{nm} g_{nm}\varphi_n(x)\varphi_m^*(t) = \sum_m \varphi_m(x)\varphi_m^*(t). \tag{10.13}$$

Applying $\mathcal{L}$, which operates only on $\varphi_n(x)$, Eq. (10.13) reduces to

$$\sum_{nm} \lambda_n g_{nm} \varphi_n(x) \varphi_m^*(t) = \sum_m \varphi_m(x) \varphi_m^*(t).$$

Taking scalar products in the x and t domains, we find that $g_{nm} = \delta_{nm}/\lambda_n$, so $G(x,t)$ must have the expansion

$$G(x,t) = \sum_n \frac{\varphi_n^*(t)\varphi_n(x)}{\lambda_n}. \tag{10.14}$$

The above analysis fails in the case that any λ_n is zero, but we shall not pursue that special case further.

The importance of Eq. (10.14) does not lie in its dubious value as a computational tool, but in the fact that it reveals the symmetry of G:

$$G(x,t) = G(t,x)^*. \tag{10.15}$$

Form of Green's Function

The properties we have identified for G are sufficient to enable its more complete identification, given a Hermitian operator $\mathcal{L}$ **and its boundary conditions**. We continue with the study of problems on an interval (a,b) with one homogeneous boundary condition at each endpoint of the interval.

Given a value of t, it is necessary for x in the range $a \le x < t$ that $G(x,t)$ have an x dependence $y_1(x)$ that is a solution to the homogeneous equation $\mathcal{L} = 0$ and that also satisfies the boundary condition at $x = a$. The most general $G(x,t)$ satisfying these conditions must have the form

$$G(x,t) = y_1(x)h_1(t), \quad (x < t), \tag{10.16}$$

where $h_1(t)$ is presently unknown. Conversely, in the range $t < x \le b$, it is necessary that $G(x,t)$ have the form

$$G(x,t) = y_2(x)h_2(t), \quad (x > t), \tag{10.17}$$

where y_2 is a solution of $\mathcal{L} = 0$ that satisfies the boundary condition at $x = b$. The symmetry condition, Eq. (10.15), permits Eqs. (10.16) and (10.17) to be consistent only if $h_2^* = A\,y_1$ and $h_1^* = A\,y_2$, with A a constant that is still to be determined. Assuming that y_1 and y_2 can be chosen to be real, we are led to the conclusion that

$$G(x,t) = \begin{cases} A\,y_1(x)y_2(t), & x < t, \\ A\,y_2(x)y_1(t), & x > t, \end{cases} \tag{10.18}$$

where $\mathcal{L}y_i = 0$, with y_1 satisfying the boundary condition at $x = a$ and y_2 satisfying that at $x = b$. The value of A in Eq. (10.18) depends, of course, on the scale at which the y_i have been specified, and must be set to a value that is consistent with Eq. (10.9). As applied here, that condition reduces to

$$A\Big[\,y_2'(t)y_1(t) - y_1'(t)y_2(t)\Big] = \frac{1}{p(t)},$$

equivalent to

$$A = \big(p(t)\,[y_2'(t)y_1(t) - y_1'(t)y_2(t)]\big)^{-1}. \tag{10.19}$$

Despite its appearance, A does not depend on t. The expression involving the y_i is their Wronskian, and it has a value proportional to $1/p(t)$. See Exercise 7.6.11.

It is instructive to verify that the form for $G(x,t)$ given by Eq. (10.18) causes Eq. (10.7) to generate the desired solution to the ODE $\mathcal{L}y = f$. To this end, we obtain an explicit form for $y(x)$:

$$y(x) = A\,y_2(x)\int_a^x y_1(t)f(t)\,dt + A\,y_1(x)\int_x^b y_2(t)f(t)\,dt. \tag{10.20}$$

From Eq. (10.20) it is easy to verify that the boundary conditions on $y(x)$ are satisfied; if $x = a$ the first of the two integrals vanishes, and the second is proportional to y_1; corresponding remarks apply at $x = b$.

It remains to show that Eq. (10.20) yields $\mathcal{L}y = f$. Differentiating with respect to x, we first have

$$\begin{aligned} y'(x) &= A\,y_2'(x)\int_a^x y_1(t)f(t)\,dt + A\,y_2(x)y_1(x)f(x) \\ &\quad + A\,y_1'(x)\int_x^b y_2(t)f(t)\,dt - A\,y_1(x)y_2(x)f(x) \\ &= A\,y_2'(x)\int_a^x y_1(t)f(t)\,dt + A\,y_1'(x)\int_x^b y_2(t)f(t)\,dt. \end{aligned} \tag{10.21}$$

Proceeding to $(py')'$:

$$\begin{aligned} \Big[p(x)y'(x)\Big]' &= A\Big[p(x)y_2'(x)\Big]'\int_a^x y_1(t)f(t)\,dt + A\Big[p(x)y_2'(x)\Big]y_1(x)f(x) \\ &\quad + A\Big[p(x)y_1'(x)\Big]'\int_x^b y_2(t)f(t)\,dt - A\Big[p(x)y_1'(x)\Big]y_2(x)f(x). \end{aligned} \tag{10.22}$$

Combining Eq. (10.22) and $q(x)$ times Eq. (10.20), many terms drop because $\mathcal{L}y_1 = \mathcal{L}y_2 = 0$, leaving

$$\mathcal{L}y(x) = A\,p(x)\Big[y_2'(x)y_1(x) - y_1'(x)y_2(x)\Big]f(x) = f(x), \tag{10.23}$$

where the final simplification took place using Eq. (10.19).

Example 10.1.1 SIMPLE SECOND-ORDER ODE

Consider the ODE

$$-y'' = f(x),$$

with boundary conditions $y(0) = y(1) = 0$. The corresponding homogeneous equation $-y'' = 0$ has general solution $y_0 = c_0 + c_1 x$; from these we construct the solution $y_1 = x$ that satisfies $y_1(0) = 0$ and the solution $y_2 = 1 - x$, satisfying $y_2(1) = 0$. For this ODE, the coefficient $p(x) = -1$, $y_1'(x) = 1$, $y_2'(x) = -1$, and the constant A in the Green's function is

$$A = \Big[(-1)[(-1)(x) - (1)(1-x)]\Big]^{-1} = 1.$$

Our Green's function is therefore

$$G(x,t) = \begin{cases} x(1-t), & 0 \le x < t, \\ t(1-x), & t < x \le 1. \end{cases}$$

Assuming we can perform the integral, we can now solve this ODE with boundary conditions for any function $f(x)$. For example, if $f(x) = \sin \pi x$, our solution would be

$$y(x) = \int_0^1 G(x,t)\, \sin \pi t\, dt = (1-x) \int_0^x t \sin \pi t\, dt + x \int_x^1 (1-t) \sin \pi t\, dt$$

$$= \frac{1}{\pi^2}\, \sin \pi x.$$

The correctness of this result is easily checked.

One advantage of the Green's function formalism is that we do not need to repeat most of our work if we change the function $f(x)$. If we now take $f(x) = \cos \pi x$, we get

$$y(x) = \frac{1}{\pi^2} \Big(2x - 1 + \cos \pi x\Big).$$

Note that our solution takes full account of the boundary conditions. ■

Other Boundary Conditions

Occasionally one encounters problems other than the Hermitian second-order ODEs we have been considering. Some, but not always all of the Green's-function properties we have identified, carry over to such problems.

Consider first the possibility that we may have nonhomogeneous boundary conditions, such as the problem $\mathcal{L}y = f$ with $y(a) = c_1$ and $y(b) = c_2$, with one or both c_i nonzero. This problem can be converted into one with homogeneous boundary conditions by making a change of the dependent variable from y to

$$u = y - \frac{c_1(b-x) + c_2(x-a)}{b-a}.$$

In terms of u, the boundary conditions are homogeneous: $u(a) = u(b) = 0$. A nonhomogeneous condition on the derivative, e.g., $y'(a) = c$, can be treated analogously.

Another possibility for a second-order ODE is that we may have two boundary conditions at one endpoint and none at the other; this situation corresponds to an initial-value problem, and has lost the close connection to Sturm-Liouville eigenvalue problems. The result is that Green's functions can still be constructed by invoking the condition of continuity in $G(x,t)$ at $x = t$ and the prescribed discontinuity in $\partial G/\partial x$, but they will no longer be symmetric.

Example 10.1.2 INITIAL VALUE PROBLEM

Consider

$$\mathcal{L}y = \frac{d^2y}{dx^2} + y = f(x), \tag{10.24}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$. This operator $\mathcal{L}$ has $p(x) = 1$.

We start by noting that the homogeneous equation $\mathcal{L}y = 0$ has the two linearly independent solutions $y_1 = \sin x$ and $y_2 = \cos x$. However, the only linear combination of these solutions that satisfies the boundary condition at $x = 0$ is the trivial solution $y = 0$, so our Green's function for $x < t$ can only be $G(x,t) = 0$. On the other hand, for the region $x > t$ there are no boundary conditions to serve as constraints, and in that region we are free to write

$$G(x,t) = C_1(t)y_1 + C_2(t)y_2, \quad \text{or} \quad G(x,t) = C_1(t)\sin x + C_2(t)\cos x, \quad x > t.$$

We now impose the requirements

$$G(t_-,t) = G(t_+,t) \longrightarrow 0 = C_1(t)\sin t + C_2(t)\cos t,$$
$$\frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} = 1 \longrightarrow C_1(t)\cos t - C_2(t)\sin t - (0) = 1.$$

These equations can now be solved, yielding $C_1(t) = \cos t$, $C_2(t) = -\sin t$, so for $x > t$

$$G(x,t) = \cos t \sin x - \sin t \cos x = \sin(x - t).$$

Thus, the complete specification of $G(x,t)$ is

$$G(x,t) = \begin{cases} 0, & x < t, \\ \sin(x-t), & x > t. \end{cases} \tag{10.25}$$

The lack of correspondence to a Sturm-Liouville problem is reflected in the lack of symmetry of the Green's function. Nevertheless, the Green's function can be used to construct

the solution to Eq. (10.24) subject to its initial conditions:

$$y(x) = \int_0^\infty G(x,t)\, f(t)\, dt$$

$$= \int_0^x \sin(x-t) f(t) dt. \tag{10.26}$$

Note that if we regard x as a time variable, our solution at "time" x is only influenced by source contributions from times t prior to x, so Eq. (10.24) obeys causality.

We conclude this example by observing that we can verify that $y(x)$ as given by Eq. (10.26) is the correct solution to our problem. Details are left as Exercise 10.1.3. ■

Example 10.1.3 BOUNDARY AT INFINITY

Consider

$$\left(\frac{d^2}{dx^2} + k^2\right) \psi(x) = g(x), \tag{10.27}$$

an equation essentially similar to one we have already studied several times, but now with boundary conditions that correspond (when multiplied by $e^{-i\omega t}$) to an outgoing wave.

The general solution to Eq. (10.27) with $g = 0$ is spanned by the two functions

$$y_1 = e^{-ikx} \quad \text{and} \quad y_2 = e^{+ikx}.$$

The outgoing wave boundary condition means that for large positive x we must have the solution y_2, while for large negative x the solution must be y_1. This information suffices to indicate that the Green's function for this problem must have the form

$$G(x,x') = \begin{cases} A y_1(x') y_2(x), & x > x', \\ A y_2(x') y_1(x), & x < x'. \end{cases}$$

We find the coefficient A from Eq. (10.19), in which $p(x) = 1$:

$$A = \frac{1}{y_2'(x) y_1(x) - y_1'(x) y_2(x)} = \frac{1}{ik + ik} = -\frac{i}{2k}.$$

Combining these results, we reach

$$G(x,x') = -\frac{i}{2k} \exp\left(i|x - x'|\right). \tag{10.28}$$

This result is yet another illustration that the Green's function depends on boundary conditions as well as on the differential equation.

Verification that this Green's function yields the desired problem solution is the topic of Exercise 10.1.8. ■

Relation to Integral Equations

Consider now an eigenvalue equation of the form

$$\mathcal{L}y(x) = \lambda y(x), \tag{10.29}$$

where we assume $\mathcal{L}$ to be self-adjoint and subject to the boundary conditions $y(a) = y(b) = 0$. We can proceed formally by treating Eq. (10.29) as an inhomogeneous equation whose right-hand side is the particular function $\lambda y(x)$. To do so, we would first find the Green's function $G(x,t)$ for the operator $\mathcal{L}$ and the given boundary conditions, after which, as in Eq. (10.7), we could write

$$y(x) = \lambda \int_a^b G(x,t)\, y(t)\, dt. \tag{10.30}$$

Equation (10.30) is not a solution to our eigenvalue problem, since the unknown function $y(x)$ appears on both sides and, moreover, it does not tell us the possible values of the eigenvalue λ. What we have accomplished, however, is to convert our eigenvalue ODE and its boundary conditions into an **integral equation** which we can regard as an alternate starting point for solution of our eigenvalue problem.

Our generation of Eq. (10.30) shows that it is implied by Eq. (10.29). If we can also show that we can connect these equations in the reverse order, namely that Eq. (10.30) implies Eq. (10.29), we can then conclude that they are equivalent formulations of the same eigenvalue problem. We proceed by applying $\mathcal{L}$ to Eq. (10.30), labeling it $\mathcal{L}_x$ to make clear that it is an operator on x, not t:

$$\begin{aligned}\mathcal{L}_x y(x) &= \lambda \mathcal{L}_x \int_a^b G(x,t) y(t)\, dt \\ &= \lambda \int_a^b \mathcal{L}_x G(x,t) y(t)\, dt = \lambda \int_a^b \delta(x-t) y(t)\, dt \\ &= \lambda y(x). \end{aligned} \tag{10.31}$$

The above analysis shows that under rather general circumstances we will be able to convert an eigenvalue equation based on an ODE into an entirely equivalent eigenvalue equation based on an integral equation. Note that to specify completely the ODE eigenvalue equation we had to make an explicit identification of the accompanying boundary conditions, while the corresponding integral equation appears to be entirely self-contained. Of course, what has happened is that the effect of the boundary conditions has influenced the specification of the Green's function that is the **kernel** of the integral equation.

Conversion to an integral equation may be useful for two reasons, the more practical of which is that the integral equation may suggest different computational procedures for solution of our eigenvalue problem. There is also a fundamental mathematical reason why an integral-equation formulation may be preferred: It is that integral operators, such as that in Eq. (10.30), are **bounded** operators (meaning that their application to a function y of

finite norm produces a result whose norm is also finite). On the other hand, differential operators are **unbounded**; their application to a function of finite norm can produce a result of unbounded norm. Stronger theorems can be developed for operators that are bounded.

We close by making the now obvious observation that Green's functions provide the link between differential-operator and integral-operator formulations of the same problem.

Example 10.1.4 DIFFERENTIAL VS. INTEGRAL FORMULATION

Here we return to an eigenvalue problem we have already treated several times in various contexts, namely

$$-y''(x) = \lambda y(x),$$

subject to boundary conditions $y(0) = y(1) = 0$. In Example 10.1.1 we found the Green's function for this problem to be

$$G(x,t) = \begin{cases} x(1-t), & 0 \le x < t, \\ t(1-x), & t < x \le 1, \end{cases}$$

and, following Eq. (10.30), our eigenvalue problem can be rewritten as

$$y(x) = \lambda \int_0^1 G(x,t)\, y(t)\, dt. \tag{10.32}$$

Methods for solution of integral equations will not be discussed until Chapter 21, but we can easily verify that the well-known solution set for this problem,

$$y = \sin n\pi x, \quad \lambda_n = n^2\pi^2, \quad n = 1,\ 2,\ \ldots,$$

also solves Eq. (10.32). ■

Exercises

10.1.1 Show that

$$G(x,t) = \begin{cases} x, & 0 \le x < t, \\ t, & t < x \le 1, \end{cases}$$

is the Green's function for the operator $\mathcal{L} = -d^2/dx^2$ and the boundary conditions $y(0) = 0$, $y'(1) = 0$.

10.1.2 Find the Green's function for

(a) $\mathcal{L}y(x) = \dfrac{d^2y(x)}{dx^2} + y(x), \quad \begin{cases} y(0) = 0, \\ y'(1) = 0. \end{cases}$

(b) $\mathcal{L}y(x) = \dfrac{d^2y(x)}{dx^2} - y(x), \quad y(x)$ finite for $-\infty < x < \infty$.

10.1.3 Show that the function $y(x)$ defined by Eq. (10.26) satisfies the initial-value problem defined by Eq. (10.24) and its initial conditions $y(0) = y'(0) = 0$.

10.1.4 Find the Green's function for the equation

$$-\frac{d^2y}{dx^2} - \frac{y}{4} = f(x),$$

with boundary conditions $y(0) = y(\pi) = 0$.

ANS. $G(x,t) = \begin{cases} 2\sin(x/2)\cos(t/2), & 0 \le x < t, \\ 2\cos(x/2)\sin(t/2), & t < x \le \pi. \end{cases}$

10.1.5 Construct the Green's function for

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - 1)y = 0,$$

subject to the boundary conditions $y(0) = 0$, $y(1) = 0$.

10.1.6 Given that

$$\mathcal{L} = (1 - x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$$

and that $G(\pm 1, t)$ remains finite, show that no Green's function can be constructed by the techniques of this section.

Note. The solutions to $\mathcal{L} = 0$ needed for the regions $x < t$ and $x > t$ are linearly dependent.

10.1.7 Find the Green's function for

$$\frac{d^2\psi}{dt^2} + k\frac{d\psi}{dt} = f(t),$$

subject to the initial conditions $\psi(0) = \psi'(0) = 0$, and solve this ODE for $t > 0$ given $f(t) = \exp(-t)$.

10.1.8 Verify that the Green's function

$$G(x,x') = -\frac{i}{2k}\exp\left(ik|x - x'|\right)$$

yields an outgoing wave solution to the ODE

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = g(x).$$

Note. Compare with Example 10.1.3.

10.1.9 Construct the 1-D Green's function for the modified Helmholtz equation,

$$\left(\frac{d^2}{dx^2} - k^2\right)\psi(x) = f(x).$$

The boundary conditions are that the Green's function must vanish for $x \to \infty$ and $x \to -\infty$.

$$ANS. \quad G(x_1, x_2) = -\frac{1}{2k} \exp\Big(-k|x_1 - x_2|\Big).$$

10.1.10 From the eigenfunction expansion of the Green's function show that

(a) $$\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin n\pi x \, \sin n\pi t}{n^2} = \begin{cases} x(1-t), & 0 \le x < t, \\ t(1-x), & t < x \le 1. \end{cases}$$

(b) $$\frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{\sin(n+\frac{1}{2})\pi x \, \sin(n+\frac{1}{2})\pi t}{(n+\frac{1}{2})^2} = \begin{cases} x, & 0 \le x < t, \\ t, & t < x \le 1. \end{cases}$$

10.1.11 Derive an integral equation corresponding to

$$y''(x) - y(x) = 0, \quad y(1) = 1, \quad y(-1) = 1,$$

(a) by integrating twice.

(b) by forming the Green's function.

$$ANS. \quad y(x) = 1 - \int_{-1}^{1} K(x,t)\, y(t)\, dt,$$

$$K(x,t) = \begin{cases} \frac{1}{2}(1-x)(t+1), & x > t, \\ \frac{1}{2}(1-t)(x+1), & x < t. \end{cases}$$

10.1.12 The general second-order linear ODE with constant coefficients is

$$y''(x) + a_1 y'(x) + a_2 y(x) = 0.$$

Given the boundary conditions $y(0) = y(1) = 0$, integrate twice and develop the integral equation

$$y(x) = \int_{0}^{1} K(x,t)\, y(t)\, dt,$$

with

$$K(x,t) = \begin{cases} a_2 t(1-x) + a_1(x-1), & t < x, \\ a_2 x(1-t) + a_1 x, & x < t. \end{cases}$$

Note that $K(x,t)$ is symmetric and continuous if $a_1 = 0$. How is this related to self-adjointness of the ODE?

10.1.13 Transform the ODE

$$\frac{d^2 y(r)}{dr^2} - k^2 y(r) + V_0 \frac{e^{-r}}{r}\, y(r) = 0$$

and the boundary conditions $y(0) = y(\infty) = 0$ into an integral equation of the form

$$y(r) = -V_0 \int_0^\infty G(r,t)\,\frac{e^{-t}}{t}\,y(t)\,dt.$$

The quantities V_0 and k^2 are constants. The ODE is derived from the Schrödinger wave equation with a mesonic potential:

$$G(r,t) = \begin{cases} -\dfrac{1}{k}\,e^{-kt}\sinh kr, & 0 \le r < t, \\ -\dfrac{1}{k}\,e^{-kr}\sinh kt, & t < r < \infty. \end{cases}$$

10.2 Problems in Two and Three Dimensions

Basic Features

The principles, but unfortunately not all the details of our analysis of Green's functions in one dimension, extend to problems of higher dimensionality. We summarize here properties of general validity for the case where $\mathcal{L}$ is a linear second-order differential operator in two or three dimensions.

1. A homogeneous PDE $\mathcal{L}\psi(\mathbf{r}_1) = 0$ **and its boundary conditions** define a **Green's function** $G(\mathbf{r}_1, \mathbf{r}_2)$, which is the solution of the PDE

   $$\mathcal{L}G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)$$

   subject to the relevant boundary conditions.
2. The inhomogeneous PDE $\mathcal{L}\psi(\mathbf{r}) = f(\mathbf{r})$ has, subject to the boundary conditions of Item 1, the solution

   $$\psi(\mathbf{r}_1) = \int G(\mathbf{r}_1, \mathbf{r}_2)\, f(\mathbf{r}_2)\, d^3 r_2,$$

   where the integral is over the entire space relevant to the problem.
3. When $\mathcal{L}$ and its boundary conditions define the Hermitian eigenvalue problem $\mathcal{L}\psi = \lambda\psi$ with eigenfunctions $\varphi_n(\mathbf{r})$ and corresponding eigenvalues λ_n, then

   - $G(\mathbf{r}_1, \mathbf{r}_2)$ is symmetric, in the sense that

     $$G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1), \text{ and}$$

   - $G(\mathbf{r}_1, \mathbf{r}_2)$ has the eigenfunction expansion

     $$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \frac{\varphi_n^*(\mathbf{r}_2)\varphi_n(\mathbf{r}_1)}{\lambda_n}.$$

4. $G(\mathbf{r}_1, \mathbf{r}_2)$ will be continuous and differentiable at all points such that $\mathbf{r}_1 \neq \mathbf{r}_2$. We cannot even require continuity in a strict sense at $\mathbf{r}_1 = \mathbf{r}_2$ (because our Green's function may become infinite there), but we can have the weaker condition that G remain continuous in regions that surround, but do not include $\mathbf{r}_1 = \mathbf{r}_2$. G must have more serious singularities in its first derivatives, so that the second-order derivatives in $\mathcal{L}$ will generate the delta-function singularity characteristic of G and specified in Item 1.

What does not carry over from the 1-D case are the explicit formulas we used to construct Green's functions for a variety of problems.

Self-Adjoint Problems

In more than one dimension, a second-order differential equation is self-adjoint if it has the form

$$\mathcal{L}\psi(\mathbf{r}) = \nabla \cdot \Big[p(\mathbf{r})\nabla\psi(\mathbf{r}) \Big] + q(\mathbf{r})\psi(\mathbf{r}) = f(\mathbf{r}), \tag{10.33}$$

with $p(\mathbf{r})$ and $q(\mathbf{r})$ real. This operator will define a Hermitian problem if its boundary conditions are such that $\langle\varphi|\mathcal{L}\psi\rangle = \langle\mathcal{L}\varphi|\psi\rangle$. See Exercise 10.2.2.

Assuming we have a Hermitian problem, consider the scalar product

$$\Big\langle G(\mathbf{r}, \mathbf{r}_1) \Big| \mathcal{L}G(\mathbf{r}, \mathbf{r}_2) \Big\rangle = \Big\langle \mathcal{L}G(\mathbf{r}, \mathbf{r}_1) \Big| G(\mathbf{r}, \mathbf{r}_2) \Big\rangle. \tag{10.34}$$

Here the scalar product and $\mathcal{L}$ both refer to the variable $\mathbf{r}$, and the Hermitian property is responsible for this equality. The points $\mathbf{r}_1$ and $\mathbf{r}_2$ are arbitrary. Noting that $\mathcal{L}G$ results in a delta function, we have, from the left-hand side of Eq. (10.34),

$$\Big\langle G(\mathbf{r}, \mathbf{r}_1) \Big| \mathcal{L}G(\mathbf{r}, \mathbf{r}_2) \Big\rangle = \Big\langle G(\mathbf{r}, \mathbf{r}_1) \Big| \delta(\mathbf{r} - \mathbf{r}_2) \Big\rangle = G^*(\mathbf{r}_2, \mathbf{r}_1). \tag{10.35}$$

But, from the right-hand side of Eq. (10.34),

$$\Big\langle \mathcal{L}G(\mathbf{r}, \mathbf{r}_1) \Big| G(\mathbf{r}, \mathbf{r}_2) \Big\rangle = \Big\langle \delta(\mathbf{r} - \mathbf{r}_1) \Big| G(\mathbf{r}, \mathbf{r}_2) \Big\rangle = G(\mathbf{r}_1, \mathbf{r}_2). \tag{10.36}$$

Substituting Eqs. (10.35) and (10.36) into Eq. (10.34), we recover the symmetry condition $G(\mathbf{r}_1, \mathbf{r}_2) = G^*(\mathbf{r}_2, \mathbf{r}_1)$.

Eigenfunction Expansions

We already saw, in 1-D Hermitian problems, that the Green's function of a Hermitian problem can be written as an eigenfunction expansion. If $\mathcal{L}$, with its boundary conditions, has normalized eigenfunctions $\varphi_n(\mathbf{r})$ and corresponding eigenvalues λ_n, our expansion took the form

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \frac{\varphi_n^*(\mathbf{r}_2)\varphi_n(\mathbf{r}_1)}{\lambda_n}. \tag{10.37}$$

It turns out to be useful to consider the somewhat more general equation

$$\mathcal{L}\psi(\mathbf{r}_1) - \lambda\psi(\mathbf{r}_1) = \delta(\mathbf{r}_2 - \mathbf{r}_1), \tag{10.38}$$

where λ is a parameter (not an eigenvalue of $\mathcal{L}$). In this more general case, an expansion in the φ_n yields for the Green's function of the entire left-hand side of Eq. (10.38) the formula

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \frac{\varphi_n^*(\mathbf{r}_2)\varphi_n(\mathbf{r}_1)}{\lambda_n - \lambda}. \tag{10.39}$$

Note that Eq. (10.39) will be well-defined only if the parameter λ is not equal to any of the eigenvalues of $\mathcal{L}$.

Form of Green's Functions

In spaces of more than one dimension, we cannot divide the region under consideration into two intervals, one on each side of a point (here designated $\mathbf{r}_2$), then choosing for each interval a solution to the homogeneous equation appropriate to its outer boundary. A more fruitful approach will often be to obtain a Green's function for an operator $\mathcal{L}$ subject to some particularly convenient boundary conditions, with a subsequent plan to add to it whatever solution to the homogeneous equation $\mathcal{L}\psi(\mathbf{r}) = 0$ that may be needed to adapt to the boundary conditions actually under consideration. This approach is clearly legitimate, as the addition of any solution to the homogeneous equation will not affect the (dis)continuity properties of the Green's function.

We consider first the Laplace operator in three dimensions, with the boundary condition that G vanish at infinity. We therefore seek a solution to the inhomogeneous PDE

$$\nabla_1^2 G(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{10.40}$$

with $\lim_{r_1 \to \infty} G(\mathbf{r}_1, \mathbf{r}_2) = 0$. We have added a subscript "1" to ∇ to remind the reader that it operates on $\mathbf{r}_1$ and not on $\mathbf{r}_2$. Since our boundary conditions are spherically symmetric and at an infinite distance from $\mathbf{r}_1$ and $\mathbf{r}_2$, we may make the simplifying assumption that $G(\mathbf{r}_1, \mathbf{r}_2)$ is a function only of $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$.

Our first step in processing Eq. (10.40) is to integrate it over a spherical volume of radius a centered at $\mathbf{r}_2$:

$$\int_{r_{12}<a} \nabla_1 \cdot \nabla_1 G(\mathbf{r}_1, \mathbf{r}_2)\, d^3 r_1 = 1, \tag{10.41}$$

where we have reduced the right-hand side using the properties of the delta function and written the left-hand side in a form making it ready for the application of Gauss' theorem. We now apply that theorem to the left-hand side of Eq. (10.41), reaching

$$\int_{r_{12}=a} \nabla_1 G(\mathbf{r}_1, \mathbf{r}_2) \cdot d\boldsymbol{\sigma}_1 = 4\pi a^2 \left.\frac{dG}{dr_{12}}\right|_{r_{12}=a} = 1. \tag{10.42}$$

Since Eq. (10.42) must be satisfied for all values of a, it is necessary that

$$\frac{d}{dr_{12}} G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi r_{12}^2},$$

which can be integrated to yield

$$G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{10.43}$$

We do not need to add a constant of integration because this form for G vanishes at infinity.

At this point it may be useful to note that the sign of $G(\mathbf{r}_1, \mathbf{r}_2)$ depends on the sign associated with the differential operator of which it is a Green's function. Some texts (including previous editions of this book) have defined G as produced by a negative delta function so that Eq. (10.43) when associated with $+\nabla^2$ would not need a minus sign. There is, of course, no ambiguity in any physical results because a change in the sign of G must be accompanied by a change in the sign of the integral in which G is combined with the inhomogeneous term of a differential equation.

The Green's function of Eq. (10.43) is only going to be appropriate for an infinite system with $G = 0$ at infinity but, as mentioned already, it can be converted into the Green's functions of another problem by addition of a suitable solution to the homogeneous equation (in this case, Laplace's equation). Since that is a reasonable starting point for a variety of problems, the form given in Eq. (10.43) is sometimes called the **fundamental** Green's function of Laplace's equation (in three dimensions).

Let's now repeat our analysis for the Laplace operator in two dimensions for a region of infinite extent, using circular coordinates $\boldsymbol{\rho} = (\rho, \varphi)$. The integral in Eq. (10.41) is then over a circular area, and the 2-D analog of Eq. (10.42) becomes

$$\int_{\rho_{12}=a} \nabla_1 G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \cdot d\boldsymbol{\sigma}_1 = 2\pi a \left. \frac{dG}{d\rho_{12}} \right|_{\rho_{12}=a} = 1,$$

leading to

$$\frac{d}{d\rho_{12}} G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{1}{2\pi\rho_{12}},$$

which has the indefinite integral

$$G(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{1}{2\pi} \ln|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|. \tag{10.44}$$

The form given in Eq. (10.44) becomes infinite at infinity, but it nevertheless can be regarded as a fundamental 2-D Green's function. However, note that we will generally need to add to it a suitable solution to the 2-D Laplace equation to obtain the form needed for specific problems.

The above analysis indicates that the Green's function for the Laplace equation in 2-D space is rather different than the 3-D result. This observation illustrates the fact that there is a real difference between flatland (2-D) physics and actual (3-D) physics, even when the latter is applied to problems with translational symmetry in one direction.

This is also a good time to note that the symmetry in the Green's function corresponds to the notion that a source at $\mathbf{r}_2$ produces a result (a potential) at $\mathbf{r}_1$ that is the same as the potential at $\mathbf{r}_2$ from a similar source at $\mathbf{r}_1$. This property will persist in more complicated problems so long as their definition makes them Hermitian.

Table 10.1 Fundamental Green's Functions[a]

	Laplace ∇^2	Helmholtz[b] $\nabla^2 + k^2$	Modified Helmholtz[c] $\nabla^2 - k^2$
1-D	$\frac{1}{2}\lvert x_1 - x_2\rvert$	$-\frac{i}{2k}\exp(ik\lvert x_1 - x_2\rvert)$	$-\frac{1}{2k}\exp(-k\lvert x_1 - x_2\rvert)$
2-D	$\frac{1}{2\pi}\ln\lvert\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\rvert$	$-\frac{i}{4}H_0^{(1)}(k\lvert\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\rvert)$	$-\frac{1}{2\pi}K_0(k\lvert\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\rvert)$
3-D	$-\frac{1}{4\pi}\frac{1}{\lvert\mathbf{r}_1 - \mathbf{r}_2\rvert}$	$-\frac{\exp(ik\lvert\mathbf{r}_1 - \mathbf{r}_2\rvert)}{4\pi\lvert\mathbf{r}_1 - \mathbf{r}_2\rvert}$	$-\frac{\exp(-k\lvert\mathbf{r}_1 - \mathbf{r}_2\rvert)}{4\pi\lvert\mathbf{r}_1 - \mathbf{r}_2\rvert}$

[a] Boundary conditions: For the Helmholtz equation, outgoing wave; for modified Helmholtz and 3-D Laplace equations, $G \to 0$ at infinity; for 1-D and 2-D Laplace equation, arbitrary.
[b] H_0^1 is a Hankel function, Section 14.4.
[c] K_0 is a modified Bessel function, Section 14.5.

Because they occur rather frequently, it is useful to have Green's functions for the Helmholtz and modified Helmholtz equations in two and three dimensions (for one dimension these Green's functions were introduced in Example 10.1.3 and Exercise 10.1.9). For the Helmholtz equation, a convenient fundamental form results if we take boundary conditions corresponding to an outgoing wave, meaning that the asymptotic r dependence must be of the form $\exp(+ikr)$. For the modified Helmholtz equation, the most convenient boundary condition (for one, two, and three dimensions) is that G decay to zero in all directions at large r. The one-, two-, and three-dimensional (3-D) fundamental Green's functions for the Laplace, Helmholtz, and modified Helmholtz operators are listed in Table 10.1.

We shall not derive here the forms of the Green's functions for the Helmholtz equations; in fact, for two dimensions, they involve Bessel functions and are best treated in detail in a later chapter. However, for three dimensions, the Green's functions are of relatively simple form, and the verification that they return correct results is the topic of Exercises 10.2.4 and 10.2.6. The fundamental Green's function for the 1-D Laplace equation may not be instantly recognizable in comparison to the formulas we derived in Section 10.1, but consistency with our earlier analysis is the topic of Example 10.2.1

Sometimes it is useful to represent Green's functions as expansions that take advantage of the specific properties of various coordinate systems. The so-called **spherical Green's function** is the radial part of such an expansion in spherical polar coordinates. For the Laplace operator, it takes a form developed in Eqs. (16.65) and (16.66). We write it here only to show that it exhibits the two-region character that provides a convenient representation of the discontinuity in the derivative:

$$-\frac{1}{4\pi}\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}\, g(r_1, r_2)\, P_l(\cos\chi),$$

where χ is the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, P_l is a Legendre polynomial, and the spherical Green's function $g(r_1, r_2)$ is

$$g_l(r_1, r_2) = \begin{cases} -\dfrac{1}{2l+1}\dfrac{r_1^l}{r_2^{l+1}}, & r_1 < r_2, \\ -\dfrac{1}{2l+1}\dfrac{r_2^l}{r_1^{l+1}}, & r_1 > r_2. \end{cases}$$

An explicit derivation of the formula for g_l is given in Example 16.3.2.

In cylindrical coordinates (ρ, φ, z) one encounters an **axial Green's function** $g_m(\rho_1, \rho_2)$, in terms of which the fundamental Green's function for the Laplace operator takes the form (also involving a continuous parameter k)

$$G(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

$$= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty} e^{im(\varphi_1-\varphi_2)}\int_0^{\infty} g_m(k\rho_1, k\rho_2)\cos k(z_1 - z_2)\, dk\,.$$

Here

$$g_m(k\rho_1, k\rho_2) = -I_m(k\rho_<)K_m(k\rho_>),$$

where $\rho_<$ and $\rho_>$ are, respectively, the smaller and larger of ρ_1 and ρ_2. The quantities I_m and K_m are modified Bessel functions, defined in Chapter 14. This expansion is discussed in more detail in Example 14.5.1. Again we note the two-region character.

Example 10.2.1 ACCOMMODATING BOUNDARY CONDITIONS

Let's use the fundamental Green's function of the 1-D Laplace equation,

$$\frac{d^2\psi(x)}{dx^2} = 0, \quad \text{namely} \quad G(x_1, x_2) = \frac{1}{2}|x_1 - x_2|,$$

to illustrate how we can modify it to accommodate specific boundary conditions. We return to the oft-used example with Dirichlet conditions $\psi = 0$ at $x = 0$ and $x = 1$. The continuity of G and the discontinuity in its derivative are unaffected if we add to the above G one or more terms of the form $f(x_1)g(x_2)$, where f and g are solutions of the 1-D Laplace equation, i.e., any functions of the form $ax + b$.

For the boundary conditions we have specified, the Green's function we require has the form

$$G(x_1, x_2) = -\frac{1}{2}(x_1 + x_2) + x_1x_2 + \frac{1}{2}|x_1 - x_2|.$$

The continuous and differentiable terms we have added to the fundamental form bring us to the result

$$G(x_1, x_2) = \begin{cases} -\frac{1}{2}(x_1 + x_2) + x_1x_2 + \frac{1}{2}(x_2 - x_1) = -x_1(1 - x_2), & x_1 < x_2, \\ -\frac{1}{2}(x_1 + x_2) + x_1x_2 + \frac{1}{2}(x_1 - x_2) = -x_2(1 - x_1), & x_2 < x_1. \end{cases}$$

This result is consistent with what we found in Example 10.1.1. ■

Example 10.2.2 QUANTUM MECHANICAL SCATTERING: BORN APPROXIMATION

The quantum theory of scattering provides a nice illustration of Green's function techniques and the use of the Green's function to obtain an integral equation. Our physical picture of scattering is as follows. A beam of particles moves along the negative z-axis toward the origin. A small fraction of the particles is scattered by the potential $V(\mathbf{r})$ and goes off as an outgoing spherical wave. Our wave function $\psi(\mathbf{r})$ must satisfy the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{10.45}$$

or

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = \left[\frac{2m}{\hbar^2}V(\mathbf{r})\psi(\mathbf{r})\right], \quad k^2 = \frac{2mE}{\hbar^2}. \tag{10.46}$$

From the physical picture just presented we look for a solution having the **asymptotic** form

$$\psi(\mathbf{r}) \sim e^{i\mathbf{k_0}\cdot\mathbf{r}} + f_k(\theta,\varphi)\frac{e^{ikr}}{r}, \tag{10.47}$$

where $e^{i\mathbf{k_0}\cdot\mathbf{r}}$ is an incident plane wave[2] with the propagation vector $\mathbf{k}_0$ carrying the subscript 0 to indicate that it is in the $\theta = 0$ (z-axis) direction. The e^{ikr}/r term describes an outgoing spherical wave with an angular and energy-dependent amplitude factor $f_k(\theta,\varphi)$,[3] and its $1/r$ radial dependence causes its asymptotic total flux to be independent of r. This is a consequence of the fact that the scattering potential $V(\mathbf{r})$ becomes negligible at large r.

Equation (10.45) contains nothing describing the internal structure or possible motion of the scattering center and therefore can only represent **elastic scattering**, so the propagation vector of the incoming wave, $\mathbf{k}_0$, must have the same magnitude, k, as the scattered wave. In quantum mechanics texts it is shown that the differential probability of scattering, called the **scattering cross section**, is given by $|f_k(\theta,\varphi|^2$.

We now need to solve Eq. (10.46) to obtain $\psi(r)$ and the scattering cross section. Our approach starts by writing the solution in terms of the Green's function for the operator on the left-hand side of Eq. (10.46), obtaining an integral equation because the inhomogeneous term of that equation has the form $(2m/\hbar^2)V(\mathbf{r})\psi(\mathbf{r})$:

$$\psi(\mathbf{r}_1) = \int \frac{2m}{\hbar^2}\, V(\mathbf{r}_2)\,\psi(\mathbf{r}_2)\,G(\mathbf{r}_1,\mathbf{r}_2)\,d^3r_2. \tag{10.48}$$

We intend to take the Green's function to be the fundamental form given for the Helmholtz equation in Table 10.1. We then recover the $\exp(ikr)/r$ part of the desired asymptotic form, but the incident-wave term will be absent. We therefore modify our tentative formula, Eq. (10.48), by adding to its right-hand side the term $\exp(i\mathbf{k}_0\cdot\mathbf{r})$, which is legitimate because this quantity is a solution to the homogeneous (Helmholtz) equation. That

[2] For simplicity we assume a continuous incident beam. In a more sophisticated and more realistic treatment, Eq. (10.47) would be one component of a wave packet.

[3] If $V(\mathbf{r})$ represents a central force, f_k will be a function of θ only, independent of the azimuthal angle φ.

approach leads us to

$$\psi(\mathbf{r}_1) = e^{i\mathbf{k_0}\cdot\mathbf{r}_1} - \int \frac{2m}{\hbar^2} V(\mathbf{r}_2)\psi(\mathbf{r}_2)\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|} d^3r_2. \tag{10.49}$$

This integral equation analog of the original Schrödinger wave equation is **exact**. It is called the **Lippmann-Schwinger** equation, and is an important starting point for studies of quantum-mechanical scattering phenomena.

We will later study methods for solving integral equations such as that in Eq. (10.49). However, in the special case that the unscattered amplitude

$$\psi_0(\mathbf{r}_1) = e^{i\mathbf{k}_0\cdot\mathbf{r}_1} \tag{10.50}$$

dominates the solution, it is a satisfactory approximation to replace $\psi(\mathbf{r}_2)$ by $\psi_0(\mathbf{r}_2)$ within the integral, obtaining

$$\psi_1(\mathbf{r}_1) = e^{i\mathbf{k}_0\cdot\mathbf{r}_1} - \int \frac{2m}{\hbar^2} V(\mathbf{r}_2)\frac{e^{ik|\mathbf{r}_1-\mathbf{r}_2|}}{4\pi|\mathbf{r}_1-\mathbf{r}_2|} e^{i\mathbf{k}_0\cdot\mathbf{r}_2} d^3r_2. \tag{10.51}$$

This is the famous **Born approximation**. It is expected to be most accurate for weak potentials and high incident energy. ■

Exercises

10.2.1 Show that the fundamental Green's function for the 1-D Laplace equation, $|x_1 - x_2|/2$, is consistent with the form found in Example 10.1.1.

10.2.2 Show that if

$$\mathcal{L}\psi(\mathbf{r}) \equiv \nabla\cdot\Big[\, p(\mathbf{r})\nabla\psi(\mathbf{r})\Big] + q(\mathbf{r})\psi(\mathbf{r}),$$

then $\mathcal{L}$ is Hermitian for $p(\mathbf{r})$ and $q(\mathbf{r})$ real, assuming Dirichlet boundary conditions on the boundary of a region and that the scalar product is an integral over that region with unit weight.

10.2.3 Show that the terms $+k^2$ in the Helmholtz operator and $-k^2$ in the modified Helmholtz operator do not affect the behavior of $G(\mathbf{r}_1, \mathbf{r}_2)$ in the immediate vicinity of the singular point $\mathbf{r}_1 = \mathbf{r}_2$. Specifically, show that

$$\lim_{|\mathbf{r}_1-\mathbf{r}_2|\to 0} \int k^2 G(\mathbf{r}_1, \mathbf{r}_2) d^3r_2 = -1.$$

10.2.4 Show that

$$-\frac{\exp(ik|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}$$

satisfies the appropriate criteria and therefore is a Green's function for the Helmholtz equation.

10.2.5 Find the Green's function for the 3-D Helmholtz equation, Exercise 10.2.4, when the wave is a standing wave.

10.2.6 Verify that the formula given for the 3-D Green's function of the modified Helmholtz equation in Table 10.1 is correct when the boundary conditions of the problem are that G vanish at infinity.

10.2.7 An electrostatic potential (mks units) is

$$\varphi(\mathbf{r}) = \frac{Z}{4\pi\varepsilon_0}\,\frac{e^{-ar}}{r}.$$

Reconstruct the electrical charge distribution that will produce this potential. Note that $\varphi(r)$ vanishes exponentially for large r, showing that the net charge is zero.

$$ANS. \quad \rho(r) = Z\delta(r) - \frac{Za^2}{4\pi}\,\frac{e^{-ar}}{r}.$$

Additional Readings

Byron, F. W., Jr., and R. W. Fuller, *Mathematics of Classical and Quantum Physics*. Reading, MA: Addison-Wesley (1969), reprinting, Dover (1992). This book contains nearly 100 pages on Green's functions, starting with some good introductory material.

Courant, R., and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1 (English edition). New York: Interscience (1953). This is one of the classic works of mathematical physics. Originally published in German in 1924, the revised English edition is an excellent reference for a rigorous treatment of integral equations, Green's functions, and a wide variety of other topics on mathematical physics.

Jackson, J. D., *Classical Electrodynamics*, 3rd ed. New York: Wiley (1999). Contains applications to electromagnetic theory.

Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics*, 2 vols. New York: McGraw-Hill (1953). Chapter 7 is a particularly detailed, complete discussion of Green's functions from the point of view of mathematical physics. Note, however, that Morse and Feshbach frequently choose a source of $4\pi\delta(\mathbf{r}-\mathbf{r}')$ in place of our $\delta(\mathbf{r}-\mathbf{r}')$. Considerable attention is devoted to bounded regions.

Stakgold, I., *Green's Functions and Boundary Value Problems*. New York: Wiley (1979).