University of Glasgow

School of Mathematics and Statistics

Level-4 Honours project

The automorphisms of the lamplighter group

Author:
Ryan Lamy
2518376L

Supervisor:
Dr. Jim Belk

Declaration of Originality
I confirm that this assignment is my own work and that I have:

- *Read and understood the guidance on plagiarism in the Student Handbook, including the University of Glasgow Statement on Plagiarism*
- *Not made use of the work of any other student(s) past or present without acknowledgment. This includes any of my own work, that has been previously, or concurrently, submitted for assessment, either at this or any other educational institution, including school*
- *Not sought or used the services of any professional agencies to produce this work*
- *In addition, I understand that any false claim in respect of this work will result in disciplinary action in accordance with University regulations*
- *I am aware of and understand the University's policy on plagiarism and I certify that this assignment is my own work, except where indicated by referencing, and that I have followed the good academic practices noted above.*

Monday 2nd December, 2024

Abstract

We offer a step-by-step proof for the computation of the automorphism group of the lamplighter group L_n. The method given is based on the proof sketch by Luc Guyot found in [1]. We ultimately prove that the group $\mathrm{Aut}(L_n)$ *is isomorphic to a semidirect product of the form* $\mathbb{Z}_n[X^{\pm 1}]^+ \rtimes (U_n \rtimes \mathbb{Z}_2)$*, where* U_n *and* $\mathbb{Z}_n[X^{\pm 1}]^+$ *denote the unit group and additive group of the Laurent polynomial ring* $\mathbb{Z}_n[X^{\pm 1}]$ *respectively.*

Contents

1 Introduction

The lamplighter group denoted L_n is a special kind of wreath product, namely $\mathbb{Z}_n \wr \mathbb{Z}$. In this paper we give the reader every tool necessary to understand how this infinite group works, and how we can go about determining its automorphisms. In the paper [5] by Melanie Stein, Jennifer Taback and Peter Wong, an advanced method is given for how to compute the automorphism group of any group whose Cayley graph is a Diestel-Leader graph, and this includes L_n. As such, it offers the most general method available today on how to compute the automorphism group of the lamplighter group.

However, in the interest of accessibility, we offer a simplified approach which focuses solely on the lamplighter group. Said approach is outlined in the proof sketch by Luc Guyot on his MathOverflow post [1], which draws from [3], [4], [5] and [6] (in particular, it is based on a specialization of [5, Theorem 3.2], which is in the paper discussed above). The method used in question is appealing as it does not rely on any knowledge about Diestel-Leader graphs, since it doesn't take the lamplighter group's Cayley graph into consideration. This paper draws from said proof sketch and expands it to a fully-fledged, self-contained proof which is built from the ground up. The requirements from group theory are made explicit and accessible at an undergraduate level, as every step given in [1] is broken down in detail.

In Section 2, we give the reader a complete introduction on wreath products, and every tool necessary to understand them. This is because the lamplighter group L_n is itself a wreath product.

In Section 3, we properly introduce L_n by offering the standard analogy of the lamplighter which can help build an intuition for the group. We also define the lamplighter group formally as the wreath product $L_n := \mathbb{Z}_n \wr \mathbb{Z}$, and show how it relates to the lamplighter analogy.

In Section 4, we go over some important definitions and results from group theory, which are required to understand every section that follows.

In Section 5, we examine the torsion subset of the lamplighter group, and show that it is a characteristic, abelian normal subgroup of L_n. This fact is vital as it is used routinely in the remainder of the paper.

In Section 6, we offer a very useful group presentation for the lamplighter group, which allows us to express any element in L_n in terms of two generators. We also go over the effect this has on the torsion subgroup, which concludes the set up required for the last section.

Finally, in Section 7, we go over three special kinds of automorphisms of L_n, namely ϕ_P, ψ_Q and τ, as they allow us to build every possible automorphism of L_n. We eventually obtain an explicit description of $\mathrm{Aut}(L_n)$ by proving that it is isomorphic to the semidirect product

$$\mathbb{Z}_n[X^{\pm 1}]^+ \rtimes_{\hat{\Theta}} (U_n \rtimes_{\Theta} \mathbb{Z}_2),$$

where $\mathbb{Z}_n[X^{\pm 1}]$ is the Laurent polynomial ring with coefficients in $\mathbb{Z}_n$, U_n is said ring's unit group, and $\mathbb{Z}_n[X^{\pm 1}]^+$ is its additive group. The homomorphisms Θ and $\hat{\Theta}$ are also given.

Note that, while the overarching proof method is based on the approach in [1], the proof of each separate proposition, lemma, theorem, and corollary is original, save for Proposition 4.9, which is based on Lemma 1 and Proposition 3 of [7, Chapter 4].

2 Wreath products

In this section, we will go over all the necessary tools needed to understand wreath products of groups. This will allow us to understand how the lamplighter group L_n is constructed, because it is a wreath product, namely $\mathbb{Z}_n \wr \mathbb{Z}$.

2.1 Automorphisms

We begin with a basic review of automorphisms.

Definition 2.1. *Let G be a group. We say γ is an automorphism of G if it is an isomorphism from G to itself.*

It is easy to verify that the set of all automorphisms of a group G, denoted $\mathrm{Aut}(G)$, is actually a group itself. More specifically, it is a group under the operation of composition. Indeed, we have that

1. compositions of automorphisms are automorphisms,
2. composition is associative,
3. the identity map on G is an automorphism, and
4. each automorphism has an inverse by virtue of being a a bijection.

The fact that $\mathrm{Aut}(G)$ is a group is one of the reasons that taking a semidirect product is possible in the first place. Indeed, it requires the construction of a homomorphism whose image is an automorphism group, as will be seen in Section 2.2.

Furthermore, the end goal of this paper is to determine all the possible automorphisms of L_n. Hence, the fact that they will build a group allows us to describe $\mathrm{Aut}(L_n)$ through an isomorphism. This will be done in the very last theorem of this paper, Theorem 7.13.

2.2 Semidirect products

We now introduce a kind of group called the semidirect product of two groups.

Let G and H be groups. The semidirect product denoted $G \rtimes_\Omega H$ is a group whose underlying set is simply the Cartesian product $G \times H$, but whose binary operation is defined as follows: given two elements $(g_1, h_1), (g_2, h_2) \in G \rtimes_\Omega H$, we have

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \omega_{h_1}(g_2), h_1 h_2)$$

where $\omega_{h_1} = \Omega(h_1)$ for some homomorphism Ω given by:

$$\begin{aligned} \Omega : H &\longrightarrow \mathrm{Aut}(G) \\ h &\longmapsto \omega_h. \end{aligned}$$

In other words, the operation in $G \rtimes_\Omega H$ is the same as the operation for direct products, except we permute the element g_2 within G in some way (dictated by what Ω and h_2 are) before multiplying it by g_1.

2.3 Direct sums

There is one last kind of group required to understand wreath products, that being direct sums of groups. For our purposes, we will only consider direct sums of a group with itself, even though in general it is possible to take the direct sum of several different groups.

Given a group G and an indexing set I, we denote the direct sum of $|I|$ copies of G by

$$\left(\bigoplus_{i\in I}(G)_i\right).$$

Let H denote the Cartesian product of G with itself $|I|$ times. The underlying set of the group $\left(\bigoplus_{i\in I}(G)_i\right)$ is the set consisting of every tuple in H which satisfies that only finitely many entries in said tuple are non-identity elements. The group operation is given as follows: for elements $\underline{g}, \underline{\hat{g}} \in \left(\bigoplus_{i\in I}(G)_i\right)$, we have

$$\underline{g}\,\underline{\hat{g}} = (g_1, g_2, ...)\,(\hat{g}_1, \hat{g}_2, ...) = (g_1\hat{g}_1, g_2\hat{g}_2, ...).$$

2.4 Wreath products

Now we are in a position to describe what a wreath product is. The following explanation is based on [10, p.156]. If G and H are two groups, then we define the wreath product of G with H as follows:

$$G \wr H = \left(\bigoplus_{h\in H}(G)_h\right) \rtimes_{S_H} H.$$

Now, in order to discuss the binary operation for this semidirect product, we must first define the shift map s_h. Let $h \in H$, let $\underline{g} \in \left(\bigoplus_{h\in H}(G)_h\right)$, and let h_i represent the indices of the entries in $\underline{g}$. Now we can write

$$\underline{g} = (g_{h_1}, g_{h_2}, ...).$$

What the shift map s_h does to $\underline{g}$ is that it permutes all of its entries by replacing each index h_i with h_ih. In other words, for all $i \in I$, it moves the entry g_{h_i} to the position indexed by h_ih. The reader may convince themselves that this is indeed an automorphism of $\left(\bigoplus_{h\in H}(G)_h\right)$.

Now we can return to the wreath product $G \wr H$ and define its binary operation. Given two elements $(\underline{g_1}, h_1), (\underline{g_2}, h_2) \in G \wr H$, we have

$$(\underline{g_1}, h_1) \cdot (\underline{g_2}, h_2) = (\underline{g_1} s_{h_1}(\underline{g_2}), h_1h_2),$$

where $s_{h_1} = S_H(h_1)$ and S_H is the homomorphism

$$\begin{aligned} S_H : H &\longrightarrow Aut\left(\bigoplus_{h\in H}(G)_h\right) \\ h &\longmapsto \; s_h. \end{aligned}$$

3 What is the lamplighter group L_n?

3.1 The lamplighter analogy

The following analogy is taken from [2, Office Hour Fifteen]. Picture an infinite road with lampposts installed along the sidewalk. The number of lampposts is countably infinite, as they line the entire road. Now imagine a lamplighter walking from lamp to lamp, whose job it is to turn certain lamps on or off as they walk past them.

The street starts off with all lamps turned off. The lamplighter starts their journey from a given, base lamppost, and can walk in either direction on the street, turning some but not all lamps on. If they happen to walk by a lamp they've already turned on earlier, they can decide to turn it back off, but they don't have to. Finally, the lamplighter can end their journey wherever they want. What picture are we left with here? We have an infinite array of lamps, with some finite number of them turned on, and the rest turned off, and, we have the final position of the lamplighter.

This "picture" is precisely what an element of the lamplighter group L_2 looks like. It corresponds to two pieces of information: the first is describing which lamps are switched on on the street, and the second is describing where the lamplighter ended their journey.

It is easy to generalize this "picture" to elements of L_n. In the case of L_2, we had exactly 2 states for each lamppost, those being on or off. Well, for L_n, everything will remain the same, except that our lampposts will now have n configurations. For instance, with $n = 3$, one can imagine a lamppost which can be either turned to a low brightness, a high brightness, or off. For the sake of simplicity, we will always assume that exactly one of the n states corresponds to the lamppost being completely off.

We can now summarize the lamplighter analogy thus: the elements of L_n each correspond to two specific pieces of information:

1. the state of each lamp after the lamplighter's journey, and

2. the lamplighter's final position.

Note here, we can already intuitively tell that certain distinct journeys along the infinite street can correspond to the same final outcome, so that they will be equal as far as L_n is concerned.

Finally, we can specify what the group operation is in L_n. Simply put, multiplying two journeys together corresponds to doing the first journey, and then starting the second journey from wherever the lamplighter left off in the first journey. Mathematically, this operation is described at the end of Section 3.2.

3.2 The lamplighter group L_n is a wreath product

In order to encode the state of each lamp and the position of the lamplighter simultaneously, we consider the wreath product

$$\mathbb{Z}_n \wr \mathbb{Z} = \left(\bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}_n)_i\right) \rtimes_{S_\mathbb{Z}} \mathbb{Z}.$$

The elements inside this wreath product can be written as $(\underline{x}, k)$, where $k \in \mathbb{Z}$, and $\underline{x} = (..., x_{-1}, x_0, x_1, x_2, ...) \in \left(\bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}_n)_i\right)$, given $x_i \in \mathbb{Z}_n$ for all $i \in \mathbb{Z}$. Notice that $\underline{x}$ is an infinite tuple which can represent the state of each lamp on the street, as it stretches infinitely in both directions, and k can encode the final position of the lamplighter on said street.

Furthermore, the binary operation for this wreath product is given by the homomorphism $S_\mathbb{Z}$, which in our case, is quite simple to conceptualize. Indeed, we can explicitly describe $S_\mathbb{Z}$ in the following way:

$$\begin{aligned} S_\mathbb{Z} : \mathbb{Z} &\longrightarrow \mathrm{Aut}\left(\bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}_n)_i\right) \\ k &\longmapsto s_k : \left(\bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}_n)_i\right) \longrightarrow \left(\bigoplus_{i\in\mathbb{Z}} (\mathbb{Z}_n)_i\right) \\ &\qquad (...x_{-1}, x_0, x_1...) \longmapsto (...y_{-1}, y_0, y_1...) \end{aligned}$$

where $y_i = x_{i-k}$ for all $i \in \mathbb{Z}$. With this structure in place, we can understand the binary operation in the lamplighter group in a rigorous way. Indeed, given two elements $(\underline{x}, k), (\underline{x}', k') \in \mathbb{Z}_n \wr \mathbb{Z}$, we have

$$(\underline{x}, k) \cdot (\underline{x}', k') = (\underline{x} + s_k(\underline{x}'), k + k').$$

We can see that multiplying two elements in L_n indeed corresponds to performing one journey along the infinite street, then performing the next from the new starting point given by k, which is why we shift the entries in $\underline{x}'$ by k.

4 Preliminary results in group theory

In this section we provide certain definitions and propositions (along with their proofs) to do with general group theory. These will be vital in understanding the results and proofs in every section that follows this one (in particular Section 7, which describes the group $\mathrm{Aut}(L_n)$).

Definition 4.1. *Let G be a group. The torsion subset $T(G)$ of G is defined as the set of all elements in G with finite order.*

Definition 4.2. *Let G be a group. A subgroup H of G is characteristic if, for all automorphisms γ of G, we have that γ preserves H, that is, $\gamma(H) = H$.*

Proposition 4.3. *Let G be a group such that $T(G)$ is a subgroup. Then $T(G)$ is characteristic.*

Proof. Let γ be an automorphism of the group G with $T(G) \leq G$. We know that for all $g \in T(G)$, there exists $n \in \mathbb{N}$ such that

$$g^n = 1,$$

and hence we can write

$$\gamma(g)^n = \gamma(g^n) = \gamma(1) = 1.$$

Thus, $\gamma(g)$ has finite order, meaning that $\gamma(g) \in T(G)$. So, $\gamma(T(G)) \subseteq T(G)$.

Now let $h \in T(G)$. To say h is in $\gamma(T(G))$ would be to say that the preimage of h under γ has finite order. This is what we will show. Indeed, for some $m \in \mathbb{N}$, we have

$$\gamma^{-1}(h)^m = \gamma^{-1}(h^m) = \gamma^{-1}(1) = 1,$$

and hence, $h \in \gamma(T(G))$. So, $T(G) \subseteq \gamma(T(G))$.

Thus we have shown by double inclusion that $\gamma(T(G)) = T(G)$, so $T(G)$ is characteristic. □

Proposition 4.4. *Let G be a group and let N be a characteristic subgroup of G. Then, N is normal.*

Proof. Let $g \in G$ be arbitrary and consider the automorphism ρ_g given by

$$\begin{aligned} \rho_g : G &\longrightarrow G \\ x &\longmapsto gxg^{-1}. \end{aligned}$$

Since N is characteristic, it must be the case that $\rho_g(n) \in N$ for all $n \in N$, which means $gng^{-1} \in N$. Since $g \in G$ was arbitrary, we conclude N is normal. □

Proposition 4.5. *Let G be a group, let $N \trianglelefteq G$ be characteristic, and let γ be an automorphism of G. Then, the induced map*

$$\begin{aligned}\gamma_N : G/N &\longrightarrow G/N \\ gN &\longmapsto \gamma(g)N\end{aligned}$$

is an automorphism of G/N.

Proof. We begin by proving that γ_N is well-defined. Let $g, h \in G$ and assume $gN = hN$. Then,

$$\begin{aligned}
& & gN \cdot (hN)^{-1} &= N \\
&\implies & (gh^{-1})N &= N \\
&\implies & gh^{-1} &\in N \\
&\implies & \gamma(gh^{-1}) &\in N \quad \text{because } \gamma \text{ preserves } N, \\
&\implies & (\gamma(g)\gamma(h)^{-1})N &= N \\
&\implies & \gamma(g)N \cdot (\gamma(h)N)^{-1} &= N \\
&\implies & \gamma(g)N &= \gamma(h)N \\
&\implies & \gamma_N(gN) &= \gamma_N(hN).
\end{aligned}$$

Next, we will show that γ_N is a homomorphism. Indeed, for $g, h \in G$, we have

$$\begin{aligned}\gamma_N(gN \cdot hN) &= \gamma_N((gh)N)) = \gamma(gh)N = (\gamma(g)\gamma(h))N \\ &= \gamma(g)N \cdot \gamma(h)N = \gamma_N(gN) \cdot \gamma_N(hN).\end{aligned}$$

Lastly, we will show that γ_N is a bijection. For injectivity, we note that

$$\begin{aligned}\ker(\gamma_N) &= \{gN \in G/N \mid \gamma_N(gN) = \gamma(g)N = N\} \\ &= \{gN \in G/N \mid \gamma(g) \in N\} \\ &= \{gN \in G/N \mid g \in N\} \text{ because } \gamma \text{ preserves } N, \\ &= \{N\},\end{aligned}$$

and hence $\ker(\gamma_N)$ is trivial, meaning γ_N is injective. As for surjectivity, let $g \in G$ be a representative of an arbitrary coset $gN \in G/N$. Write $\gamma^{-1}(g) = h \in G$, and notice that

$$\gamma_N(hN) = \gamma(h)N = \gamma(\gamma^{-1}(g))N = gN,$$

meaning γ_N is surjective. We conclude γ_N is an automorphism of G/N. □

Proposition 4.6. *Let G, H be cyclic groups and let $f : G \longrightarrow H$ be an isomorphism. Define the following sets:*

$$\begin{aligned}V_G &= \{g \in G \mid \langle g \rangle = G\}, \\ V_H &= \{h \in H \mid \langle h \rangle = H\}.\end{aligned}$$

Then, $f(V_G) = V_H$.

Proof. Let $g \in V_G$ and $h \in H$ be arbitrary. Then, since f is surjective, there exists $\hat{g} \in G$ such that

$$\begin{aligned}h &= f(\hat{g}) \\ &= f(g^n) \text{ for some } n \in \mathbb{Z} \text{ because } g \text{ generates } G, \\ &= f(g)^n.\end{aligned}$$

Now since $h \in H$ was arbitrary, we obtain $H \leq \langle f(g) \rangle$, but since $f(g) \in H$, this must imply $H = \langle f(g) \rangle$. Hence $f(g) \in V_H$, and since $g \in V_G$ was arbitrary, this yields $f(V_G) \subseteq V_H$.

Now, using the same argument for f^{-1} gives us $f^{-1}(V_H) \subseteq V_G$, and notice that

$$\begin{aligned} & f^{-1}(V_H) \subseteq V_G \\ \implies \quad & f(f^{-1}(V_H)) \subseteq f(V_G) \\ \implies \quad & V_H \subseteq f(V_G). \end{aligned}$$

Thus by double inclusion we have $f(V_G) = V_H$. $\square$

Definition 4.7. *Let G, H be groups, and let $X \subseteq G$ generate G. Let $\ell : X \longrightarrow H$ be any map. We say ℓ extends to a homomorphism λ if there exists a homomorphism $\lambda : G \longrightarrow H$ such that the following diagram commutes:*

$$\begin{tikzcd} & G \arrow[dd, "\lambda"] \\ X \arrow[ur, hook, "\iota"] \arrow[dr, "\ell"'] & \\ & H \end{tikzcd}$$

where ι denotes the inclusion map.

In order to better appreciate the definition above, it is important to give a non-example. Indeed, such a map λ may not be a homomorphism. Consider the case wherein $G = \mathbb{Z}_2$, $H = \mathbb{Z}$, and ℓ is the map

$$\begin{aligned} \ell : \{1\} &\longrightarrow \mathbb{Z} \\ 1 &\longmapsto 1. \end{aligned}$$

In this case, the corresponding map $\lambda : \mathbb{Z}_2 \longrightarrow \mathbb{Z}$ is not well a homomorphism, as

$$\lambda(1) + \lambda(1) + \lambda(1) = \lambda(\iota(1)) + \lambda(\iota(1)) + \lambda(\iota(1)) = \ell(1) + \ell(1) + \ell(1) = 1 + 1 + 1 = 3,$$

and yet

$$\lambda(1 + 1 + 1) = \lambda(1) = \lambda(\iota(1)) = \ell(1) = 1.$$

In Proposition 4.9, we will establish the exact conditions needed for λ to be a homomorphism in this construction.

Definition 4.8. *Let G be a group, let $F(X)$ denote the free group with basis X and suppose $X \subseteq G$ generates G. Then, for any word $x_1^{v_1} x_2^{v_2} ... x_k^{v_k} \in F(X)$, where $x_i \in X$ and $v_i \in \mathbb{Z}$ for all $i \in \{1, ..., k\}$, we write $[x_1^{v_1} x_2^{v_2} ... x_k^{v_k}]_G$ to denote the corresponding product within G.*

The proof of the following proposition is based on the treatment in Lemma 1 and Proposition 3 of [7, Chapter 4], but certain details were added.

Proposition 4.9. *Let G,H be groups. Suppose G has the presentation $G = \langle X \mid R \rangle$, where $X \subseteq G$ generates G and $R \subseteq G$ is the set of relators in G, which can be written as products $[x_1^{v_1} x_2^{v_2} ... x_k^{v_k}]_G$ using the alphabet X, where $x_i \in X$ and $v_i \in \mathbb{Z}$ for all $i \in \{1, ..., k\}$. Let $\ell : X \longrightarrow H$. Then ℓ extends to a homomorphism λ if and only if $\ell(x_1)^{v_1} \ell(x_2)^{v_2} ... \ell(x_k)^{v_k} = 1_H$ for all $r = [x_1^{v_1} x_2^{v_2} ... x_k^{v_k}]_G \in R$.*

Proof. First, we prove the forward implication. Suppose ℓ extends to a homomorphism λ. Then, for all words $x_1^{v_1} x_2^{v_2} ... x_k^{v_k} \in R$, we have

$$\ell(x_1)^{v_1} \ell(x_2)^{v_2} ... \ell(x_k)^{v_k} = \lambda(x_1^{v_1} x_2^{v_2} ... x_k^{v_k}) = \lambda(1_G) = 1_H.$$

Now, we prove the far less trivial backward implication. Let λ' be the map

$$\begin{aligned} \lambda' : F(X) &\longrightarrow H \\ x_1^{v_1}x_2^{v_2}...x_k^{v_k} &\longmapsto [\ell(x_1)^{v_1}\ell(x_2)^{v_2}...\ell(x_k)^{v_k}]_H. \end{aligned}$$

By the universal property of free groups, λ' is a homomorphism. Now let ν : be the natural map

$$\begin{aligned} \nu : F(X) &\longrightarrow G \\ x_1x_2...x_k &\longmapsto [x_1^{v_1}x_2^{v_2}...x_k^{v_k}]_G, \end{aligned}$$

which is also a homomorphism. We first wish to show there exists a well-defined map $\lambda : G \longrightarrow H$ satisfying $\lambda' = \lambda \circ \nu$, i.e. satisfying that the following diagram commutes.

$$\begin{tikzcd} & G \arrow[d, "\lambda"] \\ F(X) \arrow[ur, two heads, "\nu"] \arrow[r, two heads, "\lambda'"'] & H \end{tikzcd}$$

Now let $r \in R$ be arbitrary, and write $r = [x_1^{v_1}x_2^{v_2}...x_k^{v_k}]_G$. we have that

$$\begin{aligned} & \ell(x_1)^{v_1}\ell(x_2)^{v_2}...\ell(x_k)^{v_k} = 1_H \\ \implies \quad & \lambda'(x_1^{v_1}x_2^{v_2}...x_k^{v_k}) = 1_H \\ \implies \quad & \lambda'(r) = 1_H. \end{aligned}$$

Next, write $\bar{R} = \langle frf^{-1} \mid r \in R,\ f \in F(X)\rangle$ to denote the normal closure of R in $F(X)$. Notice that, letting $\bar{r} \in \bar{R}$ be arbitrary, we have

$$\begin{aligned} \lambda'(\bar{r}) &= \lambda'(\textstyle\prod_{i\in I} f_i r_i f_i^{-1}) \text{ for some elements } f_i \in F(X) \text{ and } r_i \in R, \\ &= \textstyle\prod_{i\in I}(\lambda'(f_i)\lambda'(r_i)\lambda'(f_i^{-1})) \\ &= \textstyle\prod_{i\in I}(\lambda'(f_i)\lambda'(f_i)^{-1}) \\ &= 1_H, \end{aligned}$$

Which shows that $\bar{R} \subseteq \ker(\lambda')$. Furthermore, by definition of G and ν, we must have $\ker(\nu) = \bar{R}$, and so we obtain $\ker(\nu) \subseteq \ker(\lambda')$. Now, note that ν is surjective, so we may define $\lambda : G \longrightarrow H$ to be the map given by $\lambda(g) = \lambda'(f)$, where f is any element in $F(X)$ such that $\nu(f) = g$. We will show λ is well defined. Let $g_1, g_2 \in G$ be arbitrary. We may choose $f_1, f_2 \in F(X)$ such that $g_1 = \nu(f_1)$ and $g_2 = \nu(f_2)$. Then,

$$\begin{array}{cccccc} & g_1 = g_2 & \implies & \nu(f_1) = \nu(f_2) & \implies & 1_G = \nu(f_2)\nu(f_1)^{-1} \\ \implies & 1_G = \nu(f_2f_1^{-1}) & \implies & f_2f_1^{-1} \in \ker(\nu) & \implies & f_2f_1^{-1} \in \ker(\lambda') \\ \implies & 1_H = \lambda'(f_2f_1^{-1}) & \implies & \lambda'(f_1) = \lambda'(f_2) & \implies & \lambda(g_1) = \lambda(g_2). \end{array}$$

Next, we show that λ is a homomorphism. Taking arbitrary $g_1, g_2 \in G$ and $f_1, f_2 \in F(X)$ as before, we have

$$\begin{aligned} \lambda(g_1g_2) &= \lambda(\nu(f_1)\nu(f_2)) = \lambda(\nu(f_1f_2)) = \lambda'(f_1f_2) \\ &= \lambda'(f_1)\lambda'(f_2) = \lambda(\nu(f_1))\lambda(\nu(f_2)) = \lambda(g_1)\lambda(g_2). \end{aligned}$$

Now, consider the following diagram:

$$
\begin{tikzcd}
 & & G \arrow[dd, "\lambda"] \\
X \arrow[urr, hook, "\iota"] \arrow[r, hook, "\iota'"] \arrow[drr, "\ell"'] & F(X) \arrow[ur, "\nu"'] \arrow[dr, "\lambda'"] & \\
 & & H
\end{tikzcd}
$$

where ι and ι' are inclusions. we have already shown that a well defined homomorphism λ exists such that the rightmost triangle commutes. Notice also that, by definition of λ' and ν, the other two inner triangles commute. Now, since all three inner triangles commute, we conclude that the outer triangle commutes as well, and hence it can be said that ℓ extends to a homomorphism λ. $\square$

Proposition 4.10. *Let G be a group, let $N \trianglelefteq G$ and let $H \leq G$ such that $N \cap H$ is trivial. Define the map ϱ as follows:*

$$
\begin{aligned}
\varrho : H \longrightarrow & Aut(N) \\
h \longmapsto \ & \rho_h : N \longrightarrow N \\
& \qquad n \longmapsto hnh^{-1}.
\end{aligned}
$$

Then, NH is isomorphic to $N \rtimes_\varrho H$.

Proof. Consider the map Υ given by

$$
\begin{aligned}
\Upsilon : NH &\longrightarrow N \rtimes_\varrho H \\
nh &\longmapsto (n, h).
\end{aligned}
$$

We will show that Υ is an isomorphism. First, we show Υ is well-defined. Assume $n_1, n_2 \in N$ and $h_1, h_2 \in H$ are such that $n_1h_1 = n_2h_2$. Then we have $h_1h_2^{-1} = n_1^{-1}n_2 \in H \cap N$, so that $h_1h_2^{-1}$ and $n_1^{-1}n_2$ are both equal to the identity by assumption. Thus,

$$n_1 = n_2 \text{ and } h_1 = h_2,$$

or in other words, $(n_1, h_1) = (n_2, h_2)$, so Υ is well-defined.

Next we show Υ is a homomorphism. Letting $n_1, n_2 \in N$ and $h_1, h_2 \in H$, we know there exists $n_3 \in N$ such that $n_3 = h_1n_2h_1^{-1} = \rho_{h_1}(n_2)$, because N is normal. Hence,

$$\Upsilon((n_1h_1)(n_2h_2)) = \Upsilon(n_1(h_1n_2h_1^{-1})h_1h_2) = \Upsilon((n_1n_3)(h_1h_2)) = (n_1n_3, h_1h_2) = (n_1\rho_{h_1}(n_2), h_1h_2).$$

Moreover, we have that

$$\Upsilon(n_1h_1)\Upsilon(n_2h_2) = (n_1, h_1) \cdot (n_2, h_2) = (n_1\rho_{h_1}(n_2), h_1h_2),$$

and therefore Υ is a homomorphism. It is straightforward to verify injectivity and surjectivity, so we conclude Υ is an isomorphism. $\square$

Proposition 4.11. *Let $G, G', H,$ and H' be groups. Suppose $\alpha : G \longrightarrow G'$ and $\beta : H \longrightarrow H'$ are isomorphisms. Consider the isomorphism ξ given by*

$$
\begin{aligned}
\xi : \mathrm{Aut}(G) &\longrightarrow \mathrm{Aut}(G') \\
\omega &\longmapsto \alpha \circ \omega \circ \alpha^{-1},
\end{aligned}
$$

and suppose the following diagram commutes:

$$
\begin{tikzcd}
H \arrow[r, "\Omega"] \arrow[d, hook, two heads, "\beta"'] & \mathrm{Aut}(G) \arrow[d, hook, two heads, "\xi"] \\
H' \arrow[r, "\Theta"] & \mathrm{Aut}(G')
\end{tikzcd}
$$

where Ω and Θ are homomorphisms. Then, $G \rtimes_\Omega H$ is isomorphic to $G' \rtimes_\Theta H'$.

Proof. Consider the map κ given by

$$
\begin{aligned}
\kappa : G \rtimes_\Omega H &\longrightarrow G' \rtimes_\Theta H' \\
(g, h) &\longmapsto (\alpha(g), \beta(h)).
\end{aligned}
$$

We will show κ is an isomorphism. Since α and β are well-defined, so is κ. Write $\Omega(h) = \omega_h$ for all $h \in H$, and $\Theta(h') = \theta_{h'}$ for all $h' \in H'$. Let $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We have that

$$
\begin{aligned}
\kappa((g_1, h_1) \cdot (g_2, h_2)) &= \kappa(g_1\omega_{h_1}(g_2), h_1h_2) \\
&= (\alpha(g_1\omega_{h_1}(g_2)), \beta(h_1h_2)) \\
&= (\alpha(g_1)\alpha(\omega_{h_1}(g_2)), \beta(h_1)\beta(h_2)),
\end{aligned}
$$

and

$$
\begin{aligned}
\kappa(g_1, h_1) \cdot \kappa(g_2, h_2) &= (\alpha(g_1), \beta(h_1)) \cdot (\alpha(g_2), \beta(h_2)) \\
&= (\alpha(g_1)\theta_{\beta(h_1)}(\alpha(g_2)), \beta(h_1)\beta(h_2)).
\end{aligned}
$$

Now, if we can prove that $\alpha(\omega_{h_1}(g_2)) = \theta_{\beta(h_1)}(\alpha(g_2))$, then we will have shown that κ is a homomorphism. This essentially boils down to proving that this diagram commutes:

$$
\begin{tikzcd}
G \arrow[r, hook, two heads, "\omega_{h_1}"] \arrow[d, hook, two heads, "\alpha"'] & G \arrow[d, hook, two heads, "\alpha"] \\
G' \arrow[r, hook, two heads, "\theta_{\beta(h_1)}"] & G'
\end{tikzcd}
$$

Recall that by assumption, $\xi \circ \Omega = \Theta \circ \beta$. Notice that

$$
\begin{aligned}
\xi \circ \Omega(h_1) = \Theta \circ \beta(h_1) \quad &\Longrightarrow \quad \xi(\omega_{h_1}) = \theta_{\beta(h_1)} \\
\Longrightarrow \quad \alpha \circ \omega_{h_1} \circ \alpha^{-1} = \theta_{\beta(h_1)} \quad &\Longrightarrow \quad \alpha \circ \omega_{h_1} = \theta_{\beta(h_1)} \circ \alpha,
\end{aligned}
$$

which is the result we were seeking. Finally, κ is a bijection because α and β are bijections, so we conclude κ is an isomorphism. $\square$

The proof is now complete, but it is interesting to summarise the obtained result by noting that, for all $h \in H$, the following diagram commutes:

$$
\begin{tikzcd}
G \arrow[r, hook, two heads, "\omega_h"] \arrow[d, hook, two heads, "\alpha"'] & G \arrow[r, hook, "\mu"] \arrow[d, hook, two heads, "\alpha"] & G \rtimes_\Omega H \arrow[r, two heads, "\eta"] \arrow[d, hook, two heads, "\kappa"] & H \arrow[r, "\Omega"] \arrow[d, hook, two heads, "\beta"] & \mathrm{Aut}(G) \arrow[d, hook, two heads, "\xi"] \\
G' \arrow[r, hook, two heads, "\theta_{\beta(h)}"] & G' \arrow[r, hook, "\mu'"] & G' \rtimes_\Theta H' \arrow[r, two heads, "\eta'"] & H' \arrow[r, "\Theta"] & \mathrm{Aut}(G')
\end{tikzcd}
$$

where μ and η are given by

$$
\begin{aligned}
\mu : G &\longrightarrow G \rtimes_\Omega H \\
g &\longmapsto (g, 1_H)
\end{aligned}
\qquad\qquad
\begin{aligned}
\eta : G \rtimes_\Omega H &\longrightarrow H \\
(g, h) &\longmapsto h,
\end{aligned}
$$

and μ', η' are defined similarly. Indeed, κ was defined in such a way that the two middle squares commute, but in order for it to become an isomorphism, we saw that the leftmost square has to commute, and this can be obtained by ensuring that the rightmost square commutes, as seen in the proof.

Proposition 4.12. *Let R be a ring. Then the set of all units in R is a group under multiplication. we call this group the unit group of R.*

Proof. Let $R^* \subseteq R$ denote the set of units in R. Firstly, multiplication in R^* is associative because it is associative in R. Secondly, Let $u, v \in R^*$, and notice that, by definition, there exist inverses $u^{-1}, v^{-1} \in R^*$, and we must have that

$$(uv)(v^{-1}u^{-1}) = uu^{-1} = 1,$$

meaning that $(uv) \in R^*$. Lastly, $1 \in R^*$, so R^* is a group under multiplication. $\square$

This result will be implicitly used when we introduce the unit group U_n of the Laurent polynomial ring $\mathbb{Z}_n[X^{\pm 1}]$, starting in Section 7.1.

5 The torsion subgroup of L_n

In this section, we will determine certain properties of the torsion subset $T(L_n)$ of L_n which will become useful in due course.

Lemma 5.1. *We have that $T(L_n) = \{(\underline{x}, 0) \mid \underline{x} \in \left(\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i\right)\}$ is the torsion subset of L_n.*

Proof. We know that within the lamplighter analogy, every element of L_n has the lamplighter ending their journey either back in their original position, or in a different position. We will argue that the elements that fit into the former category are precisely the ones with finite order in L_n.

As far as the elements of the wreath product $\mathbb{Z}_n \wr \mathbb{Z}$ are concerned, it is easy to recognize that elements in the first category are of the form $(\underline{x}, 0)$, and ones in the second category are of the form $(\underline{x}, k)$, where $k \neq 0$.

Now write $\underline{x} = (..., x_{-1}, x_0, x_1, ...)$. We know that each $x_i \in \mathbb{Z}_n$ has finite order n_i because $\mathbb{Z}_n$ is a finite group. Then, pick $m = \mathrm{lcm}_{i\in\mathbb{Z}}(n_i)$ to yield

$$m\underline{x} = \underline{0}.$$

Moreover, recalling that s_k is the shifting map which takes the entry in $\underline{x}$ at the i^{th} position and moves it to the $(i+k)^{th}$ position for all $i \in \mathbb{Z}$, we can see that $s_0(\underline{x}) = \underline{x}$. Hence, we have that

$$(\underline{x}, 0) \cdot (\underline{x}, 0) = (\underline{x} + s_0(\underline{x}), 0 + 0) = (\underline{x} + \underline{x}, 0), \tag{5.1}$$

and as a result,

$$(\underline{x}, 0)^m = (m\underline{x}, 0) = (\underline{0}, 0).$$

Thus, $(\underline{x}, 0)$ has finite order. Now suppose $k \in \mathbb{Z}$ and $k \neq 0$. Then k has infinite order, meaning that for all $\underline{x} \in \left(\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i\right)$ and $m \in \mathbb{Z}\backslash\{0\}$,

$$(\underline{x}, k)^m = (\underline{y}, mk) \neq (\underline{0}, 0)$$

for some $\underline{y} \in \left(\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i\right)$, which means $(\underline{x}, k)$ does not have finite order.

Therefore, we have shown that the elements in $T(L_n)$ are precisely the elements of the form $(\underline{x}, 0)$. $\square$

In abelian groups, the torsion subset is always a subgroup, but in non-abelian groups, this isn't necessarily true. That being said, in the case of L_n, the torsion subset $T(L_n)$ is not only a group, but also happens to be normal, abelian, and characteristic. This will be shown in the following theorem:

Theorem 5.2. *We have that $T(L_n)$ is a characteristic, abelian normal subgroup of L_n.*

Proof. We will begin by using the subgroup test on $T(L_n)$ to prove it is a subgroup of L_n. Let $(\underline{x}, 0), (\underline{y}, 0) \in T(L_n)$. Write $\underline{x} = (..., x_{-1}, x_0, x_1, ...)$. We know that for all $i \in \mathbb{Z}$, $x_i \in \mathbb{Z}_n$ has an inverse $-x_i \in \mathbb{Z}_n$, and so we can write $-\underline{x} = (..., -x_{-1}, -x_0, -x_1, ...)$. As a result,

$$(\underline{x}, 0) \cdot (-\underline{x}, 0) = (\underline{x} + s_0(-\underline{x}), 0 + 0) = (\underline{x} - \underline{x}, 0 + 0) = (\underline{0}, 0),$$

which means

$$(\underline{x}, 0)^{-1} = (-\underline{x}, 0) \in T(L_n). \tag{5.2}$$

Furthermore, notice that

$$(\underline{x}, 0) \cdot (\underline{y}, 0) = (\underline{x} + s_0(\underline{y}), 0 + 0) = (\underline{x} + \underline{y}, 0) \in T(L_n).$$

Thus we have shown that $T(L_n)$ is closed under addition and inverses, meaning $T(L_n)$ is a subgroup. Furthermore, since $\mathbb{Z}_n$ is abelian, so is $\left(\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}_n)_i\right)$, which means, for all $\underline{x}, \underline{y} \in \left(\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}_n)_i\right)$,

$$(\underline{x}, 0) \cdot (\underline{y}, 0) = (\underline{x} + \underline{y}, 0) = (\underline{y} + \underline{x}, 0) = (\underline{y}, 0) \cdot (\underline{x}, 0),$$

so $T(L_n)$ is abelian. Finally, the fact $T(L_n)$ is characteristic simply follows from Proposition 4.3, and the fact it is normal then follows from Proposition 4.4. □

Remark 5.3. Note that we can use equations (5.1) and (5.2) to deduce a very useful rule for elements $(\underline{x}, 0) \in T(L_n)$, that being, in general,

$$(\underline{x}, 0)^c = (c\underline{x}, 0)$$

for any $c \in \mathbb{Z}$. This rule will be used frequently in the rest of this paper.

6 Expressing elements in L_n in terms of generators

If we go back to the lamplighter analogy from Section 3, it is tempting to describe each possible path taken by the lamplighter in terms of two generators:

1. the act of moving from one lamppost to the next in a rightward direction, and
2. the act of switching a lamppost to its next state.

We will symbolize the former with the letter t, and the latter with the letter a. Intuitively, we can see that with these actions alone, the lamplighter can go anywhere on the infinite street (keeping in mind the act of going one lamppost to the left can be symbolized by t^{-1}), and they can turn any lamp to any of its n states.

More formally, this means any element of L_n could in theory be expressed as a word using the letters a, t only. We will prove this in Section 6.3, where we show that the following group

$$\mathcal{L}_n \coloneqq \langle\, a, t \,|\, a^n = 1,\ [t^k a t^{-k},\ t^j a t^{-j}] = 1\ \forall k, j \in \mathbb{Z} \,\rangle$$

is isomorphic to L_n.

Before that can be done, we must first establish, in Section 6.2, an algorithm to simplify arbitrary words in $\mathcal{L}_n$ into words of the form $a^P t^k$. This notation is defined and explained in Section 6.1, with the introduction of the Laurent polynomial ring $\mathbb{Z}_n[X^{\pm 1}]$.

In Section 6.4, we revisit the torsion subgroup $T(L_n)$ and express all of its elements in terms of generators. This will aid us in determining what the quotient $L_n/T(L_n)$ is isomorphic to. Finally, in Section 6.5 we prove two extremely helpful identities which apply to all elements of $T(L_n)$. These will be used frequently in Section 7, where we compute the group $\mathrm{Aut}(L_n)$.

6.1 The Laurent polynomial ring $\mathbb{Z}_n[X^{\pm 1}]$

When thinking about all the possible configurations for the lampposts on the infinite street, it is helpful to encode the relevant information with a polynomial whose coefficients describe the state of each lamppost, and whose exponents describe the position of said lampposts. Crucially, we know that on a given lamplighter's journey, only a finite number of lampposts can be turned to a nontrivial state, in precisely the same way that polynomials cannot be infinite sums.

Intuitively, the coefficients of such a polynomial must be in $\mathbb{Z}_n$, the same way that elements $\underline{x} \in \left(\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i\right)$ have entries $x_i \in \mathbb{Z}_n$. Furthermore, since the street stretches infinitely in both directions, the exponents in the polynomial must live in $\mathbb{Z}$, and this is where the Laurent polynomial ring $\mathbb{Z}_n[X^{\pm 1}]$ becomes of interest. Polynomials $P(X)$ in this ring are of the form

$$P(X) = ...c_{-2}X^{-2} + c_{-1}X^{-1} + c_0 + c_1X + c_2X^2... = \sum_{i\in\mathbb{Z}} c_iX^i,$$

where there is only a finite number of nonzero coefficients $c_i \in \mathbb{Z}_n$.

This correspondence will help us in our goal to express elements in L_n only in terms of a and t. In order to do so, we consider the following notational convention.

Definition 6.1. *Let $P \in \mathbb{Z}_n[X^{\pm 1}]$ be a Laurent polynomial with nonzero coefficients $c_i \in \mathbb{Z}_n$, where i is indexed by a finite subset I of $\mathbb{Z}$. Let $g \in \mathcal{L}_n$. We define the following notation:*

$$g^P \coloneqq \prod_{i\in I} t^i g^{c_i} t^{-i}.$$

Now, notice that we may easily encode the state of each lamppost on the infinite street with a word of the form a^P, where $P \in \mathbb{Z}_n[X^{\pm 1}]$, and we may specify the final position of the lamplighter with a word of the form t^k for some $k \in \mathbb{Z}$. All in all, it seems natural to express any element in L_n with a word of the form a^Pt^k. In the following section, we prove that all words in $\mathcal{L}_n$ can in fact be written like this, and in Section 6.3, we show that $\mathcal{L}_n \cong L_n$, giving us grounds to actually express any element in L_n in this way.

6.2 An algorithm for computing the canonical form of words in $\mathcal{L}_n$

Recall that $\mathcal{L}_n$ is the group given by

$$\mathcal{L}_n \coloneqq \langle\, a, t \,|\, a^n = 1,\ [t^kat^{-k},\ t^jat^{-j}] = 1\ \forall k, j \in \mathbb{Z}\,\rangle.$$

In this section, we present an algorithm which rearranges any unreduced word $g \in \mathcal{L}_n$ into a word of the canonical form a^Pt^k for some $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $k \in \mathbb{Z}$.

Theorem 6.2. *Let $g \in \mathcal{L}_n$. Then, $g = a^P t^k$ for some $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $k \in \mathbb{Z}$.*

Proof. Write $g = t^{v_1}a^{w_1}t^{v_2}a^{w_2}...t^{v_r}a^{w_r}$ where for all $s \in \{1, ...r\}$, $v_s \in \mathbb{Z}$, and, if we decide to partially reduce g by only applying the relation $a^n = 1$, then we can declare that $w_s \in \{0, ...n-1\}$.

Before we start, note that for any $v \in \mathbb{Z}$ and $w \in \{0, ...n-1\}$, we can always write

$$t^v a^w t^{-v} = \prod_{1 \leq i \leq w} t^v a t^{-v}.$$

This means that by the relation $[t^k a t^{-k},\, t^j a t^{-j}] = 1$, terms of the form $t^v a^w t^{-v}$ commute with each other. This fact will be used in step 3 of the algorithm. Now we are in a position to begin:

1. First, write

   $$g = (t^{v_1}a^{w_1}t^{-v_1})t^{v_1}t^{v_2}a^{w_2}...t^{v_r}a^{w_r}$$

   and let $\tilde{v}_2 = v_1 + v_2$ to simplify the expression to

   $$g = (t^{v_1}a^{w_1}t^{-v_1})t^{\tilde{v}_2}a^{w_2}...t^{v_r}a^{w_r}.$$

2. Next, repeat step 1 but for $\tilde{v}_2$ and write $\tilde{v}_3 = \tilde{v}_2 + v_3$ to yield

   $$g = (t^{v_1}a^{w_1}t^{-v_1})(t^{\tilde{v}_2}a^{w_2}t^{-\tilde{v}_2})t^{\tilde{v}_3}a^{w_3}...t^{v_r}a^{w_r}.$$

   Reiterating this pattern eventually yields

   $$g = (t^{\tilde{v}_1}a^{w_1}t^{-\tilde{v}_1})(t^{\tilde{v}_2}a^{w_2}t^{-\tilde{v}_2})(t^{\tilde{v}_3}a^{w_3}t^{-\tilde{v}_3})...(t^{\tilde{v}_r}a^{w_r}t^{-\tilde{v}_r})t^{\tilde{v}_r}.$$

3. Now, if any of the exponents for t are equal anywhere, i.e. there exist $s, \hat{s} \in \{1, ..., r\}$ such that $\tilde{v}_s = \tilde{v}_{\hat{s}}$, then use the relation $[t^k a t^{-k},\, t^j a t^{-j}] = 1$ to make $(t^{\tilde{v}_{\hat{s}}}a^{w_{\hat{s}}}t^{-\tilde{v}_{\hat{s}}})$ commute with all the other bracketed terms in g until it sits right next to $(t^{\tilde{v}_s}a^{w_s}t^{-\tilde{v}_s})$. Then we obtain

   $$\begin{aligned} g &= (t^{\tilde{v}_1}a^{w_1}t^{-\tilde{v}_1})(t^{\tilde{v}_2}a^{w_2}t^{-\tilde{v}_2})...(t^{\tilde{v}_s}a^{w_s}t^{-\tilde{v}_s})(t^{\tilde{v}_{\hat{s}}}a^{w_{\hat{s}}}t^{-\tilde{v}_{\hat{s}}})...(t^{\tilde{v}_r}a^{w_r}t^{-\tilde{v}_r})t^{\tilde{v}_r} \\ &= (t^{\tilde{v}_1}a^{w_1}t^{-\tilde{v}_1})(t^{\tilde{v}_2}a^{w_2}t^{-\tilde{v}_2})...(t^{\tilde{v}_s}a^{w_s+w_{\hat{s}}}t^{-\tilde{v}_{\hat{s}}})...(t^{\tilde{v}_r}a^{w_r}t^{-\tilde{v}_r})t^{\tilde{v}_r}. \end{aligned}$$

   Reiterate this process until there are no more repeats in the exponents for t. Let $\tilde{I} \subset \mathbb{Z}$ denote the set of remaining unique exponents of t up to sign. Let $c_{\tilde{i}} = (\sum_{s \in S_{\tilde{i}}} w_s) \bmod(n)$, where $S_{\tilde{i}} = \{s \in \{1, ...r\} \mid \tilde{v}_s = \tilde{i} \in \tilde{I}\}$. It may be the case that $c_{\tilde{i}} = 0$ for certain $\tilde{i} \in \tilde{I}$, so let I denote the set of remaining unique exponents of t for which that is not the case. Finally, let $k = \tilde{v}_r$. We can now write

   $$g = \left(\prod_{i \in I} t^i a^{c_i} t^{-i} \right) t^k = a^P t^k,$$

   where $P(X) = \sum_{i \in I} c_i X^i$ is a Laurent polynomial in $\mathbb{Z}_n[X^{\pm 1}]$ which is simplified so that it has no repeated exponents. □

6.3 An efficient group presentation for L_n

In this section, we would like to prove that the lamplighter group L_n has the same group presentation as $\mathcal{L}_n$. This will allow us to express elements in L_n in terms of only two generators a and t, as well as use the relations $a^n = 1$ and $[t^k a t^{-k},\, t^j a t^{-j}] = 1$ for all $k, j \in \mathbb{Z}$, which will prove extremely useful throughout the rest of this paper. The objective here then, is to show that $\mathcal{L}_n$ is isomorphic to L_n.

Note, there are alternate ways of proving that L_n has the desired group presentation. These include [8, Theorem 1.5], where the general group presentation of semidirect products is considered, and [9, Theorem 2.2.1], which involves the free group on two generators. However, the following proof adopts a different approach.

Theorem 6.3. *We have that $\mathcal{L}_n$ is isomorphic to L_n.*

Proof. Let $Y = \{a, t\}$ and consider the map ℓ given by

$$\begin{aligned} \ell : Y &\longrightarrow L_n \\ a &\longmapsto (\underline{\mathsf{a}}, 0) \\ t &\longmapsto (\underline{0}, 1), \end{aligned}$$

where $\underline{\mathsf{a}} = (...0, 1, 0...)$ has all entries equal to $0 \in \mathbb{Z}_n$ except for its 0^{th} entry, which is equal to $1 \in \mathbb{Z}_n$. We will show ℓ extends to a homomorphism $\lambda : \mathcal{L}_n \longrightarrow L_n$.

Firstly notice that, since $1 \in \mathbb{Z}_n$ has order n, we have

$$\ell(a)^n = (\underline{\mathsf{a}}, 0)^n = (n\underline{\mathsf{a}}, 0) = (\underline{0}, 0) = 1. \tag{6.1}$$

Furthermore, since $\ell(a) = (\underline{\mathsf{a}}, 0) \in T(L_n)$, and $T(L_n)$ is normal by Theorem 5.2, we must also have that $D_k := \ell(t)^k \ell(a) \ell(t)^{-k} = \ell(t)^k \ell(a)(\ell(t)^k)^{-1} \in T(L_n)$ for all $k \in \mathbb{Z}$. By the same theorem, $T(L_n)$ is abelian, so elements of the form D_k commute, and thus we obtain

$$[D_k, D_j] = 1 \tag{6.2}$$

for all $k, j \in \mathbb{Z}$.

Then, by Proposition 4.9, equations (6.1) and (6.2) imply that ℓ extends to a homomorphism $\lambda : \mathcal{L}_n \longrightarrow L_n$. Moreover, letting $(\underline{x}, k) \in L_n$ be arbtirary, we have that

$$\underline{x} = \sum_{i \in I} s_i(c_i \underline{\mathsf{a}}),$$

where the nonzero entries in $\underline{x}$ are denoted by $c_i \in \mathbb{Z}_n$. Now, for all $i \in I$, we have that

$$(s_i(c_i\underline{\mathsf{a}}), 0) = (\underline{0}, i) \cdot (c_i\underline{\mathsf{a}}, 0) \cdot (\underline{0}, -i) = (\underline{0}, 1)^i \cdot (\underline{\mathsf{a}}, 0)^{c_i} \cdot (\underline{0}, 1)^{-i},$$

meaning that

$$\begin{aligned} \lambda\left(\left(\prod_{i \in I} t^i a^{c_i} t^{-i}\right) t^k\right) &= \left(\prod_{i \in I} (\underline{0}, 1)^i \cdot (\underline{\mathsf{a}}, 0)^{c_i} \cdot (\underline{0}, 1)^{-i}\right) \cdot (\underline{0}, 1)^k \\ &= \left(\prod_{i \in I} (s_i(c_i\underline{\mathsf{a}})), 0)\right) \cdot (\underline{0}, 1)^k \\ &= (\underline{x}, 0) \cdot (\underline{0}, k) = (\underline{x}, k). \end{aligned}$$

Thus, λ is surjective.

Finally, let $g \in \mathcal{L}_n$ be arbitrary. By Theorem 6.2, we may write $g = a^P t^k$ for some $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $k \in \mathbb{Z}$. Suppose that $\lambda(g) = \lambda(a^P t^k) = (\underline{0}, 0)$. Then, writing $P(X) = \sum_{i \in I} c_i X^i$,

$$\begin{aligned} \lambda(a^P t^k) &= \lambda\left(\left(\prod_{i \in I} t^i a^{c_i} t^{-i}\right) t^k\right) \\ &= \left(\prod_{i \in I} (s_i(c_i\underline{\mathsf{a}})), 0)\right) \cdot (\underline{0}, 1)^k \\ &= (\textstyle\sum_{i \in I} s_i(c_i\underline{\mathsf{a}}), k) = (\underline{0}, 0), \end{aligned}$$

which implies firstly that $k = 0$, but also that $c_i = 0$ for all $i \in I$, meaning $P(X) = 0$, and hence g is the identity in $\mathcal{L}_n$. Thus $\ker(\lambda)$ is trivial and we conclude λ is injective. □

Now that we have established the isomorphism $\mathcal{L}_n \cong L_n$, we will adopt the abuse of notation wherein we will often write $a^{P_{\underline{x}}} t^k = (\underline{x}, k) \in L_n$, where $P_{\underline{x}} \in \mathbb{Z}_n[X^{\pm 1}]$ is the Laurent polynomial given by $P_{\underline{x}}(X) = \sum_{i \in I} x_i X^i$ and x_i are the nonzero entries in $\underline{x}$.

6.4 Consequences on the torsion subgroup $T(L_n)$

The properties of the torsion subgroup $T(L_n)$ are essential in understanding one of the more important results in Section 7, that being Lemma 7.5. Recalling that $T(L_n) = \{(\underline{x}, 0) \mid \underline{x} \in \left(\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i\right)\}$ by Lemma 5.1, we can use our updated notation to translate this to

$$T(L_n) = \{a^{P_{\underline{x}}}t^0 \mid \underline{x} \in (\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i)\} = \{a^P \mid P \in \mathbb{Z}_n[X^{\pm 1}]\}. \tag{6.3}$$

Now, we have already shown that $T(L_n)$ is normal in L_n via Theorem 5.2, so the natural question to ask is what the quotient $L_n/T(L_n)$ looks like. We will determine this by using the newly acquired group presentation for L_n.

Theorem 6.4. *We have that $L_n/T(L_n)$ is isomorphic to $\mathbb{Z}$.*

Proof. Recall that we have

$$L_n \cong \langle\, a, t \,|\, a^n = 1,\ [t^k a t^{-k},\ t^j a t^{-j}] = 1\ \forall k, j \in \mathbb{Z}\,\rangle,$$

which means that we can write, by (6.3),

$$L_n/T(L_n) \cong \langle\, a, t \,|\, a^n = 1,\ [t^k a t^{-k},\ t^j a t^{-j}] = 1\ \forall k, j \in \mathbb{Z},\ a^P = 1\ \forall P \in \mathbb{Z}[X^{\pm 1}]\,\rangle.$$

Now, we will show that

$$a^P = 1\ \forall P \in \mathbb{Z}[X^{\pm 1}] \iff a = 1.$$

Suppose $a^P = 1$ for all $P \in \mathbb{Z}[X^{\pm 1}]$. Then, picking $P_0(X) = 1$ yields $a = t^0 a^1 t^{-0} = a^{P_0} = 1$. Conversely, if we suppose $a = 1$, then for all $P \in \mathbb{Z}[X^{\pm 1}]$, denoting the nonzero coefficients in P by $c_i \in \mathbb{Z}_n$, we have $a^P = \prod_{i\in I} t^i a^{c_i} t^{-i} = \prod_{i\in I} t^i t^{-i} = 1$.

Hence, we can perform the following Tietze transformation on this group presentation to simplify:

$$L_n/T(L_n) \cong \langle\, a, t \,|\, a^n = 1,\ [t^k a t^{-k},\ t^j a t^{-j}] = 1\ \forall k, j \in \mathbb{Z},\ a = 1\,\rangle.$$

Notice now that if $a = 1$, then the relations $a^n = 1$ and $[t^k a t^{-k},\ t^j a t^{-j}] = 1\ \forall k, j \in \mathbb{Z}$ are redundant, and thus, we have grounds to perform a couple more Tietze transformations to yield:

$$L_n/T(L_n) \cong \langle\, a, t \,|\, a = 1\,\rangle \cong \langle\, t \,|\, -\,\rangle \cong \mathbb{Z}.$$

□

6.5 Useful identities

In this subsection we introduce two helpful identities on the elements of $T(L_n)$ which will be used frequently in Section 7.

Theorem 6.5. *Let $P_1, P_2 \in \mathbb{Z}_n[X^{\pm 1}]$. Then,*

1. $a^{P_1} a^{P_2} = a^{P_1+P_2}$, *and*
2. $(a^{P_1})^{P_2} = a^{P_1 P_2}$.

Proof. Let $\underline{x}_1, \underline{x}_2 \in (\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i)$ be the tuples whose entries equal the coefficients of the Laurent polynomials P_1 and P_2 respectively. For *1*, we have that

$$a^{P_1} a^{P_2} = (\underline{x}_1, 0) \cdot (\underline{x}_2, 0) = (\underline{x}_1 + \underline{x}_2, 0).$$

Now, notice that the polynomial with coefficients given by the entries of $\underline{x}_1 + \underline{x}_2 \in (\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i)$ is precisely the polynomial $P_1 + P_2$. Hence,

$$a^{P_1} a^{P_2} = (\underline{x}_1 + \underline{x}_2, 0) = a^{P_1+P_2}.$$

As for *2*, Firstly notice that

$$P_1(X)P_2(X) = \left(\sum_{i\in I} c_i X^i\right)\left(\sum_{j\in J} d_j X^j\right) = \sum_{i\in I,\, j\in J} c_i d_j X^{i+j}.$$

Now, we have

$$\begin{aligned}(a^{P_1})^{P_2} &= \prod_{j\in J} t^j (a^{P_1})^{d_j} t^{-j} = \prod_{j\in J} t^j (\underline{x}_1, 0)^{d_j} t^{-j}\\ &= \prod_{j\in J} (\underline{0}, j)(d_j\underline{x}_1, 0)(\underline{0}, -j) = \prod_{j\in J} (s_j(d_j\underline{x}_1), 0).\end{aligned}$$

Notice that for any $\underline{y} \in (\bigoplus_{i\in\mathbb{Z}}(\mathbb{Z}_n)_i)$ and $l \in \mathbb{Z}$, we can write,

$$\begin{aligned}(s_l(\underline{y}), 0) &= \left(s_l\left(\textstyle\sum_{k\in K} s_k(e_k\underline{\mathsf{a}})\right), 0)\right) = \left(\textstyle\sum_{k\in K} s_l(s_k(e_k\underline{\mathsf{a}})), 0\right)\\ &= \left(\textstyle\sum_{k\in K} s_{l+k}(e_k\underline{\mathsf{a}}), 0\right) = \prod_{k\in K} (s_{k+l}(e_k\underline{\mathsf{a}})), 0),\end{aligned} \tag{6.4}$$

for some finite subset $K \subset \mathbb{Z}$, where e_k denotes the nonzero entries in $\underline{y}$. Note that the third equality follows directly from the fact that $S_{\mathbb{Z}}$ is a homomorphism.

In our case, we have that the nonzero entries in $s_j(d_j\underline{x}_1)$ are indexed by I and are equal to $c_i d_j$ for each $i \in I$, so we may write, by fact (6.4),

$$\begin{aligned}(s_j(d_j\underline{x}_1), 0) &= \prod_{i\in I} (s_{i+j}(c_i d_j \underline{\mathsf{a}}), 0)\\ &= \prod_{i\in I} (\underline{0}, i+j)\cdot(c_i d_j\underline{\mathsf{a}}, 0)\cdot(\underline{0}, -i-j)\\ &= \prod_{i\in I} t^{i+j} a^{c_i d_j} t^{-i-j}.\end{aligned}$$

Thus, we conclude

$$(a^{P_1})^{P_2} = \prod_{j\in J} (s_j(d_j\underline{x}_1), 0) = \prod_{j\in J}\left(\prod_{i\in I} t^{i+j} a^{c_i d_j} t^{-i-j}\right) = \prod_{i\in I,\, j\in J} t^{i+j} a^{c_i d_j} t^{-i-j} = a^{P_1 P_2}. \qquad \square$$

Corollary 6.5.1. *Let $P \in \mathbb{Z}_n[X^{\pm 1}]$. Then we can write $(a^P)^{-1} = a^{-P}$.*

Proof. This simply follows from statement *1* in the above theorem, because

$$a^P a^{-P} = a^{P-P} = a^0 = 1. \qquad \square$$

7 The automorphism group of L_n

We will now study the automorphisms of L_n. In doing so, we will cover several useful preliminary lemmas and theorems, whose statements are loosely taken and inspired from the proof sketch in [1], but proven rigorously in this section. They all culminate in Theorem 7.13, which will specify what the automorphism group of L_n looks like.

7.1 Three special kinds of automorphisms of L_n

Let U_n denote the unit group of the Laurent polynomial ring $\mathbb{Z}_n[X^{\pm 1}]$. We will not provide an explicit description of this group, as this lies outside the scope of this paper, but the interested reader can go to [11, Theorem 1].

Definition 7.1. *Let $Y = \{a, t\}$ be the usual set of generators of L_n. Let $P \in \mathbb{Z}_n[X^{\pm 1}]$ and let $Q \in U_n$. We define the maps $\tilde{\phi}_P$, $\tilde{\psi}_Q$ and $\tilde{\tau}$ as follows:*

$$\begin{aligned} \tilde{\phi}_P : Y &\longrightarrow L_n \\ a &\longmapsto a \\ t &\longmapsto a^P t \end{aligned} \qquad \begin{aligned} \tilde{\psi}_Q : Y &\longrightarrow L_n \\ a &\longmapsto a^Q \\ t &\longmapsto t \end{aligned} \qquad \begin{aligned} \tilde{\tau} : Y &\longrightarrow L_n \\ a &\longmapsto a \\ t &\longmapsto t^{-1}. \end{aligned}$$

Lemma 7.2. *Let $P \in \mathbb{Z}_n[X^{\pm 1}]$ and let $Q \in U_n$. We have that $\tilde{\phi}_P$, $\tilde{\psi}_Q$ and $\tilde{\tau}$ extend to homomorphisms ϕ_P, ψ_Q and τ of L_n respectively.*

Proof. Let $P \in \mathbb{Z}_n[X^{\pm 1}]$ and let $Q \in U_n$. Recall the group presentation

$$L_n \cong \langle\, a, t \,|\, a^n = 1,\ [t^k a t^{-k},\ t^j a t^{-j}] = 1\ \forall k, j \in \mathbb{Z}\,\rangle.$$

Firstly, for $\tilde{\phi}_P$, notice that

$$\tilde{\phi}_P(a)^n = a^n = 1. \tag{7.1}$$

Furthermore, for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} A_k &:= \tilde{\phi}_P(t)^k \tilde{\phi}_P(a) \tilde{\phi}_P(t)^{-k} \\ &= (a^P t)^k a (a^P t)^{-k} \\ &= (a^P t)^k a ((a^P t)^k)^{-1} \in T(L_n), \end{aligned}$$

because $T(L_n)$ is normal by Theorem 5.2 and $a \in T(L_n)$. Also, by the very same theorem, $T(L_n)$ is abelian, meaning that elements of the form A_k commute, so for all $k, j \in \mathbb{Z}$,

$$[A_k, A_j] = 1. \tag{7.2}$$

Then, by Proposition 4.9, equations (7.1) and (7.2) imply that $\tilde{\phi}_P$ extends to a homomorphism ϕ_P of L_n.

Secondly, we consider $\tilde{\psi}_Q$. We have that for some $\underline{x} \in \left(\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}_n)_i\right)$,

$$\tilde{\psi}_Q(a)^n = (a^Q)^n = (\underline{x}, 0)^n = (n\underline{x}, 0) = (\underline{0}, 0) = 1, \tag{7.3}$$

because the order of each component $x_i \in \mathbb{Z}_n$ of $\underline{x}$ has order $n_i \in \mathbb{N}$ dividing n by Lagrange's Theorem. Furthermore, for all $k \in \mathbb{Z}$,

$$\begin{aligned} B_k &:= \tilde{\psi}_Q(t)^k \tilde{\psi}_Q(a) \tilde{\psi}_Q(t)^{-k} \\ &= t^k a^Q t^{-k} \\ &= t^k a^Q (t^k)^{-1} \in T(L_n), \end{aligned}$$

again because $T(L_n)$ is normal and $a^Q \in T(L_n)$. So by a similar argument to (7.2), we have, for all $k, j \in \mathbb{Z}$,

$$[B_k, B_j] = 1. \tag{7.4}$$

Then, by Proposition 4.9, equations (7.3) and (7.4) imply that $\tilde{\psi}_Q$ extends to a homomorphism ψ_Q of L_n.

Finally, we consider $\tilde{\tau}$. We have that

$$\tilde{\tau}(a)^n = a^n = 1. \tag{7.5}$$

Furthermore, for all $k \in \mathbb{Z}$,

$$\begin{aligned} C_k &:= \tilde{\tau}(t)^k \tilde{\tau}(a) \tilde{\tau}(t)^{-k} \\ &= (t^{-1})^k a (t^{-1})^{-k} \\ &= t^{k'} a t^{-k'}, \end{aligned}$$

where $k' = -k \in \mathbb{Z}$. Hence, according to the group presentation for L_n, we have that for all $k, j \in \mathbb{Z}$,

$$[C_k, C_j] = 1. \tag{7.6}$$

Once again, by Proposition 4.9, equations (7.5) and (7.6) imply that $\tilde{\tau}$ extends to a homomorphism τ of L_n. $\square$

Lemma 7.3. *Let $P_1, P_2 \in \mathbb{Z}_n[X^{\pm 1}]$ and let $Q_1, Q_2 \in U_n$. we have that*

1. $\phi_{P_1} \circ \phi_{P_2} = \phi_{(P_1+P_2)}$, *and*
2. $\psi_{Q_1} \circ \psi_{Q_2} = \psi_{(Q_1 Q_2)}$.

Proof. For *1*, let $P_1, P_2 \in \mathbb{Z}_n[X^{\pm 1}]$, and let $c_i, d_j \in \mathbb{Z}_n$ denote the nonzero coefficients of P_1, P_2 respectively. Firstly, we trivially have

$$\phi_{P_1} \circ \phi_{P_2}(a) = \phi_{(P_1+P_2)}(a).$$

Secondly,

$$\begin{aligned} \phi_{P_1} \circ \phi_{P_2}(t) &= \phi_{P_1}(a^{P_2} t) \\ &= \left(\prod_{j \in J} \phi_{P_1}(t)^j \phi_{P_1}(a)^{d_j} \phi_{P_1}(t)^{-j} \right) \phi_{P_1}(t) \\ &= \left(\prod_{j \in J} (a^{P_1} t)^j a^{d_j} (a^{P_1} t)^{-j} \right) a^{P_1} t \\ &= \left(\prod_{j \in J} (a^{P_1'}) t^j a^{d_j} ((a^{P_1'}) t^j)^{-1} \right) a^{P_1} t \qquad \text{for some } P_1' \in \mathbb{Z}_n[X^{\pm 1}], \\ &= \left(\prod_{j \in J} (a^{P_1'}) (t^j a^{d_j} t^{-j}) (a^{P_1'})^{-1} \right) a^{P_1} t. \end{aligned}$$

Now, notice that given $j \in J$, the terms $(a^{P_1'})$ and $(t^j a^{d_j} t^{-j})$ both belong to $T(L_n)$, meaning they may commute, and after some cancellation we are left with

$$\phi_{P_1} \circ \phi_{P_2}(t) = \left(\prod_{j \in J} t^j a^{d_j} t^{-j} \right) a^{P_1} t = a^{P_2} a^{P_1} t = a^{P_1+P_2} t = \phi_{(P_1+P_2)}(t)$$

by making use of Theorem 6.5.

Now, for *2*, let $Q_1, Q_2 \in U_n$ and let $c_i, d_j \in \mathbb{Z}_n$ denote the nonzero coefficients of Q_1, Q_2 respectively. Firstly, we trivially have

$$\psi_{Q_1} \circ \psi_{Q_2}(t) = \psi_{(Q_1 Q_2)}(t).$$

Secondly,

$$\begin{aligned}
\psi_{Q_1} \circ \psi_{Q_2}(a) &= \psi_{Q_1}(a^{Q_2}) \\
&= \prod_{j \in J} \psi_{Q_1}(t)^j \psi_{Q_1}(a)^{d_j} \psi_{Q_1}(t)^{-j} \\
&= \prod_{j \in J} t^j (a^{Q_1})^{d_j} t^{-j} \\
&= (a^{Q_1})^{Q_2} = a^{Q_1 Q_2} = \psi_{(Q_1 Q_2)}(a),
\end{aligned}$$

by making use of Theorem 6.5 again. $\square$

Theorem 7.4. *Let $P \in \mathbb{Z}_n[X^{\pm 1}]$ and let $Q \in U_n$. We have that ϕ_P, ψ_Q and τ are automorphisms of L_n.*

Proof. We begin with τ. We have that $a = \tau(a)$ and $t = \tau(t^{-1})$, meaning τ is able to output the generators for L_n, and thus it is surjective. Furthermore, let $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $k \in \mathbb{Z}$ be arbitrary. Let c_i denote the nonzero coefficients in P and let $\tilde{P}$ denote the polynomial obtained when negating all the exponents in P. Then,

$$\begin{aligned}
\tau(a^P t^k) = 1 \quad &\implies \quad \left(\prod_{i \in I} \tau(t)^i \tau(a)^{c_i} \tau(t)^{-i} \right) \tau(t)^k = 1 \\
\implies \quad \left(\prod_{i \in I} t^{-i} a^{c_i} t^i \right) t^{-k} = 1 \quad &\implies \quad a^{\tilde{P}} t^{-k} = 1,
\end{aligned}$$

which implies that $\tilde{P}(X) = 0$ and $k = 0$. This must mean that $P(X) = 0$ too, and so we conclude $\ker(\tau)$ is trivial and thus τ is injective.

Next, we consider ϕ_P. Fix $P_0 \in \mathbb{Z}_n[X^{\pm 1}]$. Firstly,

$$a = \phi_{P_0}(a),$$

and secondly,

$$t = a^{P_0 - P_0} t = \phi_{(P_0 - P_0)}(t) = \phi_{P_0} \circ \phi_{-P_0}(t) = \phi_{P_0}(a^{-P_0} t)$$

by Lemma 7.3, so ϕ_{P_0} is able to output the generators for L_n, meaning it is surjective. Secondly, letting $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $k \in \mathbb{Z}$ be arbitrary, we have that

$$\phi_{P_0}(a^P t^k) = \phi_{P_0}(a^P) \phi_{P_0}(t)^k,$$

and we already know from the proof of *1* in Lemma 7.3 that $\phi_{P_0}(a^P) = a^P$, so we obtain

$$\phi_{P_0}(a^P t^k) = a^P (a^{P_0} t)^k = a^P a^{P_0'} t^k = a^{P + P_0'} t^k$$

for some $P_0' \in \mathbb{Z}_n[X^{\pm 1}]$. Then, if $\phi_{P_0}(a^P t^k) = 1$, this must mean $k = 0$, and we are left with

$$1 = \phi_{P_0}(a^P t^k) = a^P (a^{P_0} t)^0 = a^P,$$

which implies $P(X) = 0$. We conclude $\ker(\phi_{P_0})$ is trivial, and hence, ϕ_{P_0} is injective.

Finally, we consider ψ_Q. Fix $Q_0 \in U_n$. We know that Q_0^{-1} exists because Q_0 is a unit in $\mathbb{Z}_n[X^{\pm 1}]$. We have that

$$t = \psi_{Q_0}(t),$$

and

$$a = a^{Q_0 Q_0^{-1}} = \psi_{(Q_0 Q_0^{-1})}(a) = \psi_{Q_0} \circ \psi_{Q_0^{-1}}(a) = \psi_{Q_0}(a^{Q_0^{-1}})$$

by Lemma 7.3, so ψ_{Q_0} can output the generators for L_n, meaning it is surjective. Furthermore, Let $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $k \in \mathbb{Z}$ be arbitrary, and let c_i denote the nonzero coefficients of P. We have that

$$\psi_{Q_0}(a^P t^k) = \left(\prod_{i \in I} \psi_{Q_0}(t)^i \psi_{Q_0}(a)^{c_i} \psi_{Q_0}(t)^{-i} \right) \psi_{Q_0}(t)^k = \left(\prod_{i \in I} t^i (a^{Q_0})^{c_i} t^{-i} \right) t^k = (a^{Q_0})^P t^k = a^{Q_0 P} t^k$$

by Theorem 6.5. Then, if $\psi_{Q_0}(a^P t^k) = 1$, we must have $k = 0$ and $Q_0 P = 0$, but since Q_0 is a unit, it is not a 0-divisor, and hence $P = 0$. We conclude $\ker(\psi_{Q_0})$ is trivial, and hence, ψ_{Q_0} is injective.

In summary, we have shown that ϕ_P, ψ_Q and τ are bijections, and so, by Lemma 7.2, they are automorphisms. □

7.2 Properties of an arbitrary automorphism of L_n

Lemma 7.5. *Let γ be an automorphism of L_n. Then, $\gamma(t) = a^P t^{\pm 1}$ for some $P \in \mathbb{Z}_n[X^{\pm 1}]$.*

Proof. For ease of notation, write $T \coloneqq T(L_n)$. We first claim that the coset $tT \in L_n/T$ generates L_n/T.

To show this, let $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $k \in \mathbb{Z}$ be arbitrary, and hence let $(a^P t^k)T$ be an arbitrary coset of L_n/T. Since T is normal, we know that left cosets are the same as right cosets, so

$$(a^P t^k)T = T(a^P t^k) = (Ta^P)t^k.$$

But $T = \{a^P \mid P \in \mathbb{Z}_n[X^{\pm 1}]\}$, so $Ta^P = T$, and hence

$$(a^P t^k)T = Tt^k = t^k T = (tT)^k.$$

Next, by Theorem 6.4, $L_n/T \cong \mathbb{Z}$. Hence we can let $\varphi : L_n/T \longrightarrow \mathbb{Z}$ be an isomorphism. Then by Proposition 4.6, we have

$$\varphi(V_{L_n/T}) = V_{\mathbb{Z}} = \{-1, 1\},$$

which means that $|V_{L_n/T}| = 2$ because φ is injective. Now, since tT generates L_n/T, so does $(tT)^{-1} = t^{-1}T$, and thus $V_{L_n/T} = \{tT, t^{-1}T\}$.

Now let γ be an automorphism of L_n, and let γ_T denote the map

$$\begin{aligned} \gamma_T : L_n/T &\longrightarrow L_n/T \\ gT &\longmapsto \gamma(g)T \end{aligned}$$

where $g \in L_n$. By Proposition 4.5 and Theorem 5.2, we know that γ_T is an automorphism of L_n/T. Hence, again by Proposition 4.6, we must have $\gamma_T(V_{L_n/T}) = V_{L_n/T}$, so in particular,

$$\gamma(t)T = \gamma_T(tT) = t^{\pm 1}T,$$

which means $\gamma(t) \in t^{\pm 1}T = Tt^{\pm 1} = \{a^P t^{\pm 1} \mid P \in \mathbb{Z}_n[X^{\pm 1}]\}$. □

Lemma 7.6. *Let γ be an automorphism of L_n. Then, $\gamma(a) = a^Q$ for some $Q \in U_n$.*

Proof. Let γ be an automorphism of L_n. Since $T(L_n)$ is characteristic and $a \in T(L_n)$, we must have $\gamma^{-1}(a) = a^P$ for some $P \in \mathbb{Z}_n[X^{\pm 1}]$, meaning that $\gamma(a^P) = a$. Now let c_i denote the nonzero coefficients in P and notice that

$$\gamma(a^P) = \prod_{i \in I} \gamma(t)^i \gamma(a)^{c_i} \gamma(t)^{-i} = \prod_{i \in I} (a^{P_0} t^{\pm 1})^i \gamma(a)^{c_i} (a^{P_0} t^{\pm 1})^{-i},$$

for some $P_0 \in \mathbb{Z}_n[X^{\pm 1}]$ by Lemma 7.5. Furthermore, we know that $(a^{P_0} t^{\pm 1})^i = (a^{P_0'})^i t^{\pm i}$ for some $P_0' \in \mathbb{Z}_n[X^{\pm 1}]$, so we have

$$\gamma(a^P) = \prod_{i \in I} (a^{P_0'})^i (t^{\pm i} \gamma(a)^{c_i} t^{\mp i}) (a^{P_0'})^{-i},$$

and now we can commute $(a^{P_0'})^i$ with $(t^{\pm i} \gamma(a)^{c_i} t^{\mp i})$ to yield

$$\gamma(a^P) = \prod_{i \in I} (t^{\pm i} \gamma(a)^{c_i} t^{\mp i}) = \gamma(a)^{P'}$$

for some $P' \in \mathbb{Z}_n[X^{\pm 1}]$. Now since $T(L_n)$ is characteristic, we have $\gamma(a) = a^Q$ for some $Q \in \mathbb{Z}_n[X^{\pm 1}]$, so, recalling that $\gamma(a^P) = a$, we obtain

$$a = \gamma(a^P) = \gamma(a)^{P'} = a^{QP'},$$

which is only possible if $QP' = 1$, and therefore Q is a unit. So in conclusion, $\gamma(a) = a^Q$ for some $Q \in U_n$. □

Theorem 7.7. *Let γ be an automorphism of L_n. Then, $\gamma = \phi_P \circ \psi_Q \circ \tau^e$ for some $P \in \mathbb{Z}_n[X^{\pm 1}]$, $Q \in U_n$, and $e \in \{0, 1\}$.*

Proof. Let γ be an automorphism of L_n. Then, by Lemma 7.5 and Lemma 7.6, $\gamma(a) = a^{Q_0}$ and $\gamma(t) = a^{P_0} t^{\pm 1}$ for some $P_0 \in \mathbb{Z}_n[X^{\pm 1}]$ and $Q_0 \in U_n$. Now notice that for all $P \in \mathbb{Z}_n[X^{\pm 1}]$, $Q \in U_n$ and $e \in \{0, 1\}$,

$$\phi_P \circ \psi_Q \circ \tau^e(a) = \phi_P \circ \psi_Q(a) = \phi_P(a^Q) = a^Q$$

by the proof of *1* in Lemma 7.3. So, pick $Q = Q_0$ to yield

$$\phi_P \circ \psi_Q \circ \tau^e(a) = \gamma(a). \tag{7.7}$$

Now, suppose $e = 0$. Then, we can pick $P = P_0$ to obtain

$$\phi_P \circ \psi_Q \circ \tau^e(t) = \phi_P \circ \psi_Q(t) = \phi_P(t) = a^{P_0} t.$$

But, if $e = 1$, then

$$\phi_P \circ \psi_Q \circ \tau^e(t) = \phi_P \circ \psi_Q(t^{-1}) = \phi_P(\psi_Q(t)^{-1}) = \phi_P(t)^{-1} = (a^P t)^{-1} = t^{-1}(a^P)^{-1} = t^{-1} a^{-P}$$

By Corollary 6.5.1. Now, notice that $a^P = (\underline{x}, 0)$ for some $\underline{x} \in \left(\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}_n)_i\right)$ and so we can write

$$t^{-1} a^{-P} = (\underline{0}, -1) \cdot (-\underline{x}, 0) = (s_{-1}(\underline{x}), -1). \tag{7.8}$$

Since $P \in \mathbb{Z}_n[X^{\pm 1}]$ was arbitrary, we may pick a specific P, label it P', such that $\underline{x} = s_1(\underline{x}_0)$, where $\underline{x}_0$ is the element of $\left(\bigoplus_{i \in \mathbb{Z}} (\mathbb{Z}_n)_i\right)$ which satisfies $a^{P_0} = (\underline{x}_0, 0)$. Then, plugging P' into the equality (7.8) yields

$$t^{-1} a^{-P'} = (s_{-1}(s_1(\underline{x}_0)), -1) = (\underline{x}_0, -1) = a^{P_0} t^{-1}. \tag{7.9}$$

In summary, if $\gamma(t) = a^{P_0} t$, then we can pick $P = P_0$ and $e = 0$ to yield

$$\phi_P \circ \psi_Q \circ \tau^e(t) = \gamma(t), \tag{7.10}$$

and if $\gamma(t) = a^{P_0} t^{-1}$, then we can pick $P = P'$ and $e = 1$ to yield the exact same result.

By equations (7.7) and (7.10), we conclude that $\gamma = \phi_P \circ \psi_Q \circ \tau^e$ for some $P \in \mathbb{Z}_n[X^{\pm 1}]$, $Q \in U_n$, and $e \in \{0, 1\}$. □

7.3 Describing the automorphism group $\mathrm{Aut}(L_n)$

Note, in this section, we will consider the additive group of the ring $\mathbb{Z}_n[X^{\pm 1}]$, and so for ease of distinction, we will let $\mathbb{Z}_n[X^{\pm 1}]^+$ denote this group from now on.

Lemma 7.8. *We have that*

1. $\Phi := \langle \phi_P \mid P \in \mathbb{Z}_n[X^{\pm 1}] \rangle$ *is isomorphic to* $\mathbb{Z}_n[X^{\pm 1}]^+$*,*
2. $\Psi := \langle \psi_Q \mid Q \in U_n \rangle$ *is isomorphic to* U_n*, and*
3. $\mathfrak{T} := \langle \tau \rangle$ *is isomorphic to* $\mathbb{Z}_2$*.*

Proof. For *1*, consider the map Λ given by

$$\begin{aligned} \Lambda : \Phi &\longrightarrow \mathbb{Z}_n[X^{\pm 1}]^+ \\ \phi_P &\longmapsto P. \end{aligned}$$

It is straightforward to verify that α is well-defined and bijective. Furthermore, We have that $\Lambda(\phi_{P_1}) + \Lambda(\phi_{P_2}) = P_1 + P_2 = \Lambda(\phi_{(P_1+P_2)}) = \Lambda(\phi_{P_1} \circ \phi_{P_2})$, so Λ is an isomorphism and we can write $\Phi \cong \mathbb{Z}_n[X^{\pm 1}]^+$.

Next, for *2*, it is important to recall that U_n is indeed a group by Proposition 4.12. Now consider the map α given by

$$\begin{aligned} \alpha : \Psi &\longrightarrow U_n \\ \psi_Q &\longmapsto Q. \end{aligned}$$

It is straightforward to verify that α is well-defined and bijective. Furthermore, We have that $\alpha(\psi_{Q_1})\alpha(\psi_{Q_2}) = Q_1 Q_2 = \alpha(\psi_{(Q_1Q_2)}) = \alpha(\psi_{Q_1} \circ \psi_{Q_2})$, so α is an isomorphism and we can write $\Psi \cong U_n$.

Lastly, for *3*, notice that $\tau^2 = \mathrm{id} \in \mathrm{Aut}(L_n)$. It follows that $\mathfrak{T} \cong \mathbb{Z}_2$. $\square$

Lemma 7.9. *We have that* $\mathrm{Aut}(L_n) = \Phi \circ (\Psi \circ \mathfrak{T})$*. Additionally,*

1. Φ *is normal in* $\mathrm{Aut}(L_n)$*, and*
2. Ψ *is normal in* $(\Psi \circ \mathfrak{T}) \leq \mathrm{Aut}(L_n)$*.*

Proof. First we establish that $(\Psi \circ \mathfrak{T})$ is indeed a subgroup of $\mathrm{Aut}(L_n)$. This will be done by showing that $(\psi \circ \mathfrak{T}) = (\mathfrak{T} \circ \psi)$. Let $\psi_Q \circ \tau^e \in (\Psi \circ \mathfrak{T})$, where $Q \in U_n$ and $e \in \{0, 1\}$. If $e = 0$, then trivially we have $\psi_Q \circ \tau^e = \tau^e \circ \psi_Q$. for $e = 1$ however, notice that

$$\psi_Q \circ \tau(t) = \psi_Q(t^{-1}) = t^{-1} = \tau \circ \psi_{\tilde{Q}}(t),$$

and

$$\psi_Q \circ \tau(a) = \psi_Q(a) = a^Q = \tau(a^{\tilde{Q}}) = \tau \circ \psi_{\tilde{Q}}(a)$$

where $\tilde{Q}$ denotes the polynomial with the same coefficients as Q but with negated exponents. So from these two equations we gather

$$\psi_Q \circ \tau = \tau \circ \psi_{\tilde{Q}}. \tag{7.11}$$

Now, since Q was arbitrary, so was $\tilde{Q}$, meaning we have double inclusion. Hence, $(\Psi \circ \mathfrak{T}) = (\mathfrak{T} \circ \Psi)$, so we conclude $(\Psi \circ \mathfrak{T}) \leq \mathrm{Aut}(L_n)$. Moreover, we can notice that equation (7.11) also shows normality of Ψ in $(\Psi \circ \mathfrak{T})$, so *2* has been proven.

Next, we have that $\mathrm{Aut}(L_n) = \Phi \circ (\Psi \circ \mathfrak{T})$. The leftwards inclusion follows from Theorem 7.7 trivially, and the rightwards inclusion follows from Theorem 7.4: indeed, since ϕ_P, ψ_Q and τ^e are automorphisms for all $P \in \mathbb{Z}_n[X^{\pm 1}]$, $Q \in U_n$, and $e \in \{0,1\}$, so are compositions thereof, which proves the inclusion.

As a result, for *1*, it will suffice to show that for all $P \in \mathbb{Z}_n[X^{\pm 1}]$ and $Q \in U_n$, we have

$$\psi_Q \circ \phi_P = \phi_{P_0} \circ \psi_Q \tag{7.12}$$

for some $P_0 \in \mathbb{Z}_n[X^{\pm 1}]$, and

$$\phi_P \circ \tau = \tau \circ \phi_{P_0'} \tag{7.13}$$

for some $P_0' \in \mathbb{Z}_n[X^{\pm 1}]$. Notice that

$$\psi_Q \circ \phi_P(t) = \psi_Q(a^P t) = a^{QP} t = \phi_{QP} \circ \psi_Q(t),$$

and

$$\psi_Q \circ \phi_P(a) = \psi_Q(a) = a^Q = \phi_{QP} \circ \psi_Q(a),$$

so result (7.12) is obtained.

Next, notice that

$$\phi_P \circ \tau(t) = \phi_P(t)^{-1} = a^{P'} t^{-1} = \tau \circ \phi_{\tilde{P}'}(t)$$

for some $P' \in \mathbb{Z}_n[X^{\pm 1}]$ by a similar argument to the one in equation (7.9), and

$$\phi_P \circ \tau(a) = \tau \circ \phi_{\tilde{P}'}(a)$$

trivially. Thus result (7.13) is obtained, which concludes the proof. □

We now have all the lemmas and theorems required to explicitly describe $\mathrm{Aut}(L_n)$. The following definitions will facilitate this task.

Definition 7.10. *Define the maps ϱ, $\tilde{\varrho}$ and $\hat{\varrho}$ in the following way:*

$$\begin{aligned} \varrho : \mathfrak{T} &\longrightarrow \mathrm{Aut}(\Psi) \\ \tau^e &\longmapsto \rho_{\tau^e} \end{aligned} \qquad \begin{aligned} \hat{\varrho} : \Psi \rtimes_{\varrho} \mathfrak{T} &\longrightarrow (\Psi \circ \mathfrak{T}) \xrightarrow{\tilde{\varrho}} \mathrm{Aut}(\Phi) \\ (\psi_Q, \tau^e) &\longmapsto \psi_Q \circ \tau^e \longmapsto \rho_{\psi_Q \circ \tau^e}, \end{aligned}$$

where, for any $\gamma \in \mathrm{Aut}(L_n)$, ρ_γ denotes conjugation by γ, as is the case in Proposition 4.10.

Remark 7.11. The automorphisms ρ_{τ^e} and $\rho_{\psi_Q \circ \tau^e}$ are well-defined because Ψ is normal in $(\Psi \circ \mathfrak{T})$ and Φ is normal in $\Phi \circ (\Psi \circ \mathfrak{T})$ by the lemma above. Consequently, the maps ϱ, $\tilde{\varrho}$ and $\hat{\varrho}$ are also well-defined.

Definition 7.12. *Define Θ and $\hat{\Theta}$ to be the unique homomorphisms such that the following diagrams commute:*

$$\begin{tikzcd} \mathfrak{T} \arrow[r, "\varrho"] \arrow[d, hook, two heads, "\beta"'] & \mathrm{Aut}(\Psi) \arrow[d, hook, two heads, "\xi"] \\ \mathbb{Z}_2 \arrow[r, "\Theta"] & \mathrm{Aut}(U_n) \end{tikzcd} \qquad \begin{tikzcd} \Psi \rtimes_{\varrho} \mathfrak{T} \arrow[r, "\hat{\varrho}"] \arrow[d, hook, two heads, "\kappa"'] & \mathrm{Aut}(\Phi) \arrow[d, hook, two heads, "\hat{\xi}"] \\ U_n \rtimes_{\Theta} \mathbb{Z}_2 \arrow[r, "\hat{\Theta}"] & \mathrm{Aut}(\mathbb{Z}_n[X^{\pm 1}]^+) \end{tikzcd}$$

where $\hat{\xi}$ and ξ are analogous to the set up in Proposition 4.11, β is the unique isomorphism from $\mathfrak{T}$ to $\mathbb{Z}_2$, and κ is the isomorphism given by $\kappa(\psi_Q, \tau^e) = (\alpha(\psi_Q), \beta(\tau^e)) = (Q, e)$.

Theorem 7.13. *We have that $\mathrm{Aut}(L_n)$ is isomorphic to $\mathbb{Z}_n[X^{\pm 1}]^+ \rtimes_{\hat{\Theta}} (U_n \rtimes_{\Theta} \mathbb{Z}_2)$.*

The intuition behind this result is that, since every automorphism of L_n can be written as a composition $\phi_P \circ \psi_Q \circ \tau^e$, and the three components therein all separately behave like the elements of $\mathbb{Z}_n[X^{\pm 1}]^+$, U_n and $\mathbb{Z}_2$ respectively, we must have that $\mathrm{Aut}(L_n)$ is isomorphic to some semidirect product of those three groups.

Proof. Recall that $\mathrm{Aut}(L_n) = \Phi \circ (\Psi \circ \mathfrak{T})$ by Lemma 7.9. Now, we will simply apply Proposition 4.10 to Definition 7.10 and Proposition 4.11 to Definition 7.12.

We begin with the former. Notice that since Ψ is normal in $(\Psi \circ \mathfrak{T})$ by Lemma 7.9, and $\Psi \cap (\Psi \circ \mathfrak{T})$ is trivial, we can write

$$(\Psi \circ \mathfrak{T}) \cong \Psi \rtimes_{\varrho} \mathfrak{T}$$

by Proposition 4.10.

Furthermore, since Φ is normal in $\Phi \circ (\Psi \circ \mathfrak{T})$ by Lemma 7.9, and $\Phi \cap (\Psi \circ \mathfrak{T})$ is trivial, we can write

$$\Phi \circ (\Psi \circ \mathfrak{T}) \cong \Phi \rtimes_{\tilde{\varrho}} (\Psi \circ \mathfrak{T}) \cong \Phi \rtimes_{\hat{\varrho}} (\Psi \rtimes_{\varrho} \mathfrak{T})$$

by the same proposition.

Finally, we apply Proposition 4.11 to the first commutative diagram in Definition 7.12 to obtain

$$(\Psi \rtimes_{\varrho} \mathfrak{T}) \cong U_n \rtimes_{\Theta} \mathbb{Z}_2,$$

and we apply the same proposition again to the second commutative diagram in Definition 7.12 to yield the final result,

$$\mathrm{Aut}(L_n) = \Phi \circ (\Psi \circ \mathfrak{T}) \cong \Phi \rtimes_{\hat{\varrho}} (\Psi \rtimes_{\varrho} \mathfrak{T}) \cong \mathbb{Z}_n[X^{\pm 1}]^+ \rtimes_{\hat{\Theta}} (U_n \rtimes_{\Theta} \mathbb{Z}_2). \qquad \square$$

References

[1] L. Guyot, https://mathoverflow.net/q/450702, MathOverflow, (2023).

[2] M. Clay and D. Margalit, Office Hours with a Geometric Group Theorist, Princeton University Press, (2017).

[3] F. Szechtman, The group of outer automorphisms of the semidirect product of the additive group of a ring by a group of units, *Communications in Algebra*, (2004).

[4] C. Weibel, The K-book: An introduction to algebraic K-theory, Graduate Studies in Mathematics, (2013).

[5] M. Stein, J. Taback and P. Wong, Automorphisms of Higher Rank lamplighter Groups, *International journal of algebra and computation*, Volume 25 Issue 8, (2015).

[6] A. Genevois and R. Tessera, A note on morphisms to wreath products, (2022).

[7] D.L. Johnson, Presentations of groups, *London Mathematical Society student texts*, (1990).

[8] A. Saint-Criq, The lamplighter Groups, *Institut de Mathématiques de Toulouse*, (2021).

[9] S. Eckenthal, The lamplighter Groups, *Trinity College Digital Repository*, (2012).

[10] J. Meier, Groups, Graphs, and Trees: an Introduction to the Geometry of Infinite Groups, Cambridge University Press, (2008).

[11] G. Karpilovsky, On finite generation of unit groups of commutative group rings, Archiv der Mathematik, (1983).