CHAPTER 4

Section 4.1

1. (a) Using integration by parts twice, the integral can be written as

$$\begin{aligned}\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx &= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx \\ &= -x^2\cos x + 2x\sin x + 2\cos x + C \\ &= 2x\sin x - \left(x^2-2\right)\cos x + C\end{aligned}$$

(b) Making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\begin{aligned}\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{2x}{1+\left(x^2\right)^2}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du &= \frac{1}{2}\tan^{-1}u + C \\ &= \frac{1}{2}\tan^{-1}x^2 + C\end{aligned}$$

(c) Using partial fraction expansion, we can write

$$\begin{aligned}\int \frac{1}{(x-1)(x-2)}\,dx = \int\left(-\frac{1}{x-1}+\frac{1}{x-2}\right)dx &= -\int\frac{dx}{x-1}+\int\frac{dx}{x-2} \\ &= -\ln(x-1)+\ln(x-2)+C \\ &= \ln\frac{x-2}{x-1}+C\end{aligned}$$

(d) Making the substitution $u = \sqrt{x-1}$ so that $2udu = dx$, the integral can be written as

$$\begin{aligned}\int \frac{1}{1+\sqrt{x-1}}\,dx = 2\int\frac{u}{1+u}\,du &= 2\int\frac{-1+1+u}{1+u}\,du \\ &= 2\int\left(-\frac{1}{1+u}+1\right)du \\ &= -2\int\frac{du}{1+u}+2\int du \\ &= -2\ln(1+u)+2u+C \\ &= 2\left[\sqrt{x-1}-\ln\left(1+\sqrt{x-1}\right)\right]+C\end{aligned}$$

2. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,dx = \int_0^1 \cos^2\theta\,d\theta = \frac{1}{2}\int_0^1 (1+\cos 2\theta)\,d\theta &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\sin 2\theta}{4}\right|_0^{\pi/2} \\ &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\cos\theta\sin\theta}{2}\right|_0^{\pi/2} = \frac{\pi}{4}\end{aligned}$$

(b) Using the identity $\sin mx \sin nx = (1/2)\cos[(m-n)x] - (1/2)\cos[(m+n)x]$, the integral can be written as

$$\int_0^\pi \sin 2x \sin 3x\, dx = \frac{1}{2}\int_0^\pi (\cos x - \cos 5x)\ dx = \frac{1}{2}\int_0^\pi \cos x\, dx - \frac{1}{2}\int_0^\pi \cos 5x\, dx$$
$$= \left.\frac{\sin x}{2}\right|_0^\pi - \left.\frac{\sin 5x}{10}\right|_0^\pi = 0$$

(c) Using integration by parts twice, the integral can be written as

$$\int_0^1 \left(2x^2 - 3x + 1\right) e^x\, dx = \left.\left(2x^2 - 3x + 1\right) e^x\right|_0^1 - \int_0^1 (4x-3)\, e^x\, dx$$
$$= \left.\left(2x^2 - 3x + 1\right) e^x\right|_0^1 - \left.(4x-3)\, e^x\right|_0^1 + \int_0^1 4e^x\, dx$$
$$= \left.\left(2x^2 - 3x + 1\right) e^x\right|_0^1 - \left.(4x-3)\, e^x\right|_0^1 + \left.4e^x\right|_0^1 = 3e - 8$$

(d) Using integration by parts, the fact that $(d/dx)\tan^{-1} x = 1/(1+x^2)$ and making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\int_0^1 \tan^{-1} x\, dx = \left.x\tan^{-1} x\right|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx = \left.x\tan^{-1} x\right|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx$$
$$= \left.x\tan^{-1} x\right|_0^1 - \frac{1}{2}\int_0^1 \frac{du}{1+u}$$
$$= \left.x\tan^{-1} x\right|_0^1 - \left.\frac{\ln(1+u)}{2}\right|_0^1$$
$$= \left.x\tan^{-1} x\right|_0^1 - \left.\frac{\ln(1+x^2)}{2}\right|_0^1 = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$$

3. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 0^+}\int_b^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 0^+}\int_b^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta = \lim_{b\to 0^+}\int_b^{\pi/2} d\theta$$
$$= \lim_{b\to 0^+} \left.\theta\right|_b^{\pi/2}$$
$$= \lim_{b\to 0^+}\left(\frac{\pi}{2} - b\right) = \frac{\pi}{2}$$

(b) Making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\int_0^\infty e^{-x}\, dx = \lim_{b\to\infty}\int_0^b e^{-x}\, dx = \lim_{b\to-\infty} -\int_0^b e^u\, du = \lim_{b\to-\infty} \left.-e^u\right|_0^b = \lim_{b\to-\infty}\left(-e^b + 1\right) = 1$$

(c) Using integration by parts, the integral can be written as

$$\int_0^1 \ln x\,dx = \lim_{b\to 0^+}\int_b^1 \ln x\,dx = \lim_{b\to 0^+} x\ln x\Big|_b^1 - \lim_{b\to 0^+}\int_b^1 dx = \lim_{b\to 0^+}(x\ln x - x)\Big|_b^1$$
$$= \lim_{b\to 0^+}(-1 - b\ln b + b)$$
$$= -1 - \lim_{b\to 0^+} b\ln b = -1$$

where the last step follows from the fact that

$$\lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{1/b} \overset{LH}{=} \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = \lim_{b\to 0^+} -b = 0$$

using L'Hopital's rule.

(d) Making the substitutions $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, $2u = \theta$ so that $2du = d\theta$, $v = \cos u$ so that $dv = -\sin u du$ and $w = \sin u$ so that $dw = \cos u du$ and the identities $1 + \tan^2\theta = \sec^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, the integral can be written as

$$\begin{aligned}
\int_1^\infty \frac{dx}{x\sqrt{1+x^2}} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x\sqrt{1+x^2}} \\
&= \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}}\,d\theta \\
&= \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{d\theta}{\sin\theta} \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{2du}{\sin 2u} \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{du}{\sin u\cos u} \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin^2 u + \cos^2 u}{\sin u\cos u}\,du \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin u}{\cos u}\,du + \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\cos u}{\sin u}\,du \\
&= \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2+\sqrt{2}}/2}^b -\frac{dv}{v} + \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2-\sqrt{2}}/2}^b \frac{dw}{w} \\
&= \lim_{b\to\sqrt{2}/2} -\ln v\Big|_{\sqrt{2+\sqrt{2}}/2}^b + \lim_{b\to\sqrt{2}/2}\ln w\Big|_{\sqrt{2-\sqrt{2}}/2}^b \\
&= \lim_{b\to\sqrt{2}/2} -\ln b + \ln\frac{\sqrt{2+\sqrt{2}}}{2} + \lim_{b\to\sqrt{2}/2}\ln b - \ln\frac{\sqrt{2-\sqrt{2}}}{2} \\
&= \ln\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \frac{1}{2}\ln\left(3+2\sqrt{2}\right) = \frac{1}{2}\ln\left(1+\sqrt{2}\right)^2 = \ln\left(1+\sqrt{2}\right)
\end{aligned}$$

(e) Using integration by parts twice and making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\begin{aligned}
\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx &= \lim_{b\to-\infty}\int_0^b -u^2 e^u\,du \\
&= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty}\int_0^b 2ue^u\,du \\
&= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty}\int_0^b 2e^u\,du \\
&= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty} 2e^u\Big|_0^b \\
&= \lim_{b\to-\infty}\left(-b^2e^b + 2be^b - 2e^b + 2\right) = 2
\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to-\infty} -b^2e^b = \lim_{b\to-\infty} -\frac{b^2}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty}\frac{2b}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty} -\frac{2}{e^{-b}} = 0$$

(f) Using integration by parts, the integral can be written as

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} -\frac{\ln x}{x}\Big|_1^b + \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} \\
&= \lim_{b\to\infty} -\frac{\ln x}{x}\Big|_1^b - \lim_{b\to\infty}\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right) = 1
\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to\infty} -\frac{\ln b}{b} \overset{LH}{=} \lim_{b\to\infty} -\frac{1/b}{1} = \lim_{b\to\infty} -\frac{1}{b} = 0$$

4. (a)

$$\begin{aligned}
\int_{-1}^1 \frac{dx}{x^{1/3}} = \int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} &= \lim_{b\to0^-}\int_{-1}^b \frac{dx}{x^{1/3}} + \lim_{b\to0^+}\int_b^1 \frac{dx}{x^{1/3}} \\
&= \lim_{b\to0^-}\frac{3}{2}x^{2/3}\Big|_{-1}^b + \lim_{b\to0^+}\frac{3}{2}x^{2/3}\Big|_b^1 \\
&= \lim_{b\to0^-}\frac{3}{2}\left(b^{2/3}-1\right) + \lim_{b\to0^+}\frac{3}{2}\left(1-b^{2/3}\right) = 0
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} &= \lim_{b\to0^-}\int_{-1}^b \frac{dx}{x^3} + \lim_{b\to0^+}\int_b^1 \frac{dx}{x^3} \\
&= \lim_{b\to0^-} -\frac{1}{4x^4}\Big|_{-1}^b + \lim_{b\to0^+}\frac{1}{4x^4}\Big|_b^1 \\
&= \lim_{b\to0^-}\frac{1}{4}\left(-b^{-4}+1\right) + \lim_{b\to0^+}\frac{1}{4}\left(1-b^{-4}\right) = -\infty
\end{aligned}$$

Hence, the integral is divergent.

(c) Making the substitution $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, the integral can be written as

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\pi/2}\int_0^b \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \lim_{b\to\pi/2}\int_0^b d\theta = \lim_{b\to\pi/2}\theta\Big|_0^b$$
$$= \lim_{b\to\pi/2} b = \frac{\pi}{2}$$

(d) Using a partial fraction expansion, the integral can be written as

$$\begin{aligned}\int_0^\infty \frac{x^2-x-1}{x(x^3+1)}\,dx &= \lim_{b\to0^+}\int_b^1 \frac{x^2-x-1}{x(x^3+1)}\,dx + \lim_{b\to\infty}\int_1^b \frac{x^2-x-1}{x(x^3+1)}\,dx \\ &= \lim_{b\to0^+}\int_b^1 \left[\frac{4x-2}{3(x^2-x+1)} - \frac{1}{3(x+1)} - \frac{1}{x}\right]dx + \ldots \\ &= \lim_{b\to0^+}\frac{1}{3}\int_b^1 \frac{4x-2}{x^2-x+1}\,dx - \lim_{b\to0^+}\frac{1}{3}\int_b^1 \frac{dx}{x+1} - \lim_{b\to0^+}\int_b^1 \frac{dx}{x} + \ldots\end{aligned}$$

It is clear to see that the first two integrals (obtained by a partial fraction expansion) belonging to the first partial integral converge. However, the third diverges:

$$\lim_{b\to0^+}\int_b^1 \frac{dx}{x} = \lim_{b\to0^+}\ln x\Big|_b^1 = -\lim_{b\to0^+}\ln b = \infty$$

Hence, since the first partial integral diverges, we conclude that the original integral is divergent.

(e)

$$\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty} -\cos x\Big|_0^b = \lim_{b\to\infty}(-\cos b + 1)$$

Since $\lim_{b\to\infty}\cos b$ does not exist the integral is divergent.

(f) Making the substitution $u = \cosh x$ so that $du = \sinh x dx$, the integral can be written as

$$\begin{aligned}\int_0^\infty (1-\tanh x)\,dx = \lim_{b\to\infty}\int_0^b (1-\tanh x)\,dx &= \lim_{b\to\infty}\int_0^b dx - \lim_{b\to\infty}\int_0^b \tanh x\,dx \\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_0^b \frac{\sinh x}{\cosh x}\,dx \\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_1^{\cosh b}\frac{du}{u} \\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\ln u\Big|_1^{\cosh b} \\ &= \lim_{b\to\infty}(b - \ln\cosh b) = \ln 2\end{aligned}$$

where the last step follows from the fact that

$$\begin{aligned}\lim_{b\to\infty}(b - \ln\cosh b) = \lim_{b\to\infty}\left[\ln e^b - \ln\left(e^b + e^{-b}\right) + \ln 2\right] &= \lim_{b\to\infty}\left[\ln 2 - \ln\left(1 + e^{-2b}\right)\right] \\ &= \ln 2\end{aligned}$$

5. (a)