CHAPTER 4

Section 4.1

1. (a) Using integration by parts twice, the integral can be written as

$$\begin{aligned}\int x^2 \sin x\, dx &= -x^2 \cos x + \int 2x \cos x\, dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x\, dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C = 2x \sin x - \left(x^2 - 2\right) \cos x + C\end{aligned}$$

(b) Making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\int \frac{x}{1+x^4}\, dx = \frac{1}{2} \int \frac{2x}{1+\left(x^2\right)^2}\, dx = \frac{1}{2} \int \frac{1}{1+u^2}\, du = \frac{1}{2} \tan^{-1} u + C = \frac{1}{2} \tan^{-1} x^2 + C$$

(c) Using partial fraction expansion, we can write

$$\begin{aligned}\int \frac{1}{(x-1)(x-2)}\, dx &= \int \left(-\frac{1}{x-1} + \frac{1}{x-2}\right) dx = -\int \frac{dx}{x-1} + \int \frac{dx}{x-2} \\ &= -\ln(x-1) + \ln(x-2) + C = \ln \frac{x-2}{x-1} + C\end{aligned}$$

(d) Making the substitution $u = \sqrt{x-1}$ so that $2udu = dx$, the integral can be written as

$$\begin{aligned}\int \frac{1}{1+\sqrt{x-1}}\, dx &= 2 \int \frac{u}{1+u}\, du = 2 \int \frac{-1+1+u}{1+u}\, du = 2 \int \left(-\frac{1}{1+u} + 1\right) du \\ &= -2 \int \frac{du}{1+u} + 2 \int du = -2 \ln(1+u) + 2u + C \\ &= 2\left[\sqrt{x-1} - \ln\left(1+\sqrt{x-1}\right)\right] + C\end{aligned}$$

2. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\, dx = \int_0^1 \cos^2\theta\, d\theta = \frac{1}{2} \int_0^1 (1+\cos 2\theta)\, d\theta &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\sin 2\theta}{4}\right|_0^{\pi/2} \\ &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\cos\theta \sin\theta}{2}\right|_0^{\pi/2} = \frac{\pi}{4}\end{aligned}$$

(b) Using the identity $\sin mx \sin nx = (1/2)\cos[(m-n)x] - (1/2)\cos[(m+n)x]$, the integral can be written as

$$\begin{aligned}\int_0^\pi \sin 2x \sin 3x\, dx = \frac{1}{2} \int_0^\pi (\cos x - \cos 5x)\, dx &= \frac{1}{2} \int_0^\pi \cos x\, dx - \frac{1}{2} \int_0^\pi \cos 5x\, dx \\ &= \left.\frac{\sin x}{2}\right|_0^\pi - \left.\frac{\sin 5x}{10}\right|_0^\pi = 0\end{aligned}$$

(c) Using integration by parts twice, the integral can be written as

$$\begin{aligned}\int_0^1 \left(2x^2-3x+1\right)e^x\,dx &= \left(2x^2-3x+1\right)e^x\Big|_0^1 - \int_0^1 (4x-3)\,e^x\,dx \\ &= \left(2x^2-3x+1\right)e^x\Big|_0^1 - (4x-3)\,e^x\Big|_0^1 + \int_0^1 4e^x\,dx \\ &= \left(2x^2-3x+1\right)e^x\Big|_0^1 - (4x-3)\,e^x\Big|_0^1 + 4e^x\Big|_0^1 = 3e-8\end{aligned}$$

(d) Using integration by parts, the fact that $(d/dx)\tan^{-1}x = 1/(1+x^2)$ and making the substitution $u=x^2$ so that $du = 2xdx$, the integral can be written as

$$\begin{aligned}\int_0^1 \tan^{-1}x\,dx &= x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx \\ &= x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{du}{1+u} = x\tan^{-1}x\Big|_0^1 - \frac{\ln(1+u)}{2}\Bigg|_0^1 \\ &= x\tan^{-1}x\Big|_0^1 - \frac{\ln(1+x^2)}{2}\Bigg|_0^1 = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\end{aligned}$$

3. (a) Making the substitution $x=\sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta+\cos^2\theta = 1$, the integral can be written as

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{b\to0^+}\int_b^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to0^+}\int_b^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = \lim_{b\to0^+}\int_b^{\pi/2} d\theta \\ &= \lim_{b\to0^+}\theta\Big|_b^{\pi/2} = \lim_{b\to0^+}\left(\frac{\pi}{2}-b\right) = \frac{\pi}{2}\end{aligned}$$

(b) Making the substitution $u=-x$ so that $du=-dx$, the integral can be written as

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to-\infty} -\int_0^b e^u\,du = \lim_{b\to-\infty} -e^u\Big|_0^b = \lim_{b\to-\infty}\left(-e^b+1\right) = 1$$

(c) Using integration by parts, the integral can be written as

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{b\to0^+}\int_b^1 \ln x\,dx = \lim_{b\to0^+} x\ln x\Big|_b^1 - \lim_{b\to0^+}\int_b^1 dx = \lim_{b\to0^+}(x\ln x - x)\Big|_b^1 \\ &= \lim_{b\to0^+}(-1-b\ln b+b) = -1 - \lim_{b\to0^+} b\ln b = -1\end{aligned}$$

where the last step follows from the fact that

$$\lim_{b\to0^+} b\ln b = \lim_{b\to0^+}\frac{\ln b}{1/b} \stackrel{LH}{=} \lim_{b\to0^+}\frac{1/b}{-1/b^2} = \lim_{b\to0^+} -b = 0$$

using L'Hopital's rule.

(d) Making the substitutions $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, $2u = \theta$ so that $2du = d\theta$, $v = \cos u$ so that $dv = -\sin u du$ and $w = \sin u$ so that $dw = \cos u du$ and the identities $1 + \tan^2\theta = \sec^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, the integral can be written as

$$\begin{aligned}
\int_1^\infty \frac{dx}{x\sqrt{1+x^2}} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x\sqrt{1+x^2}} = \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}}\,d\theta \\
&= \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{d\theta}{\sin\theta} = \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{2du}{\sin 2u} = \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{du}{\sin u\cos u} \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin^2 u + \cos^2 u}{\sin u\cos u}\,du \\
&= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin u}{\cos u}\,du + \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\cos u}{\sin u}\,du \\
&= \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2+\sqrt{2}}/2}^b -\frac{dv}{v} + \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2-\sqrt{2}}/2}^b \frac{dw}{w} \\
&= \lim_{b\to\sqrt{2}/2} -\ln v\Big|_{\sqrt{2+\sqrt{2}}/2}^b + \lim_{b\to\sqrt{2}/2} \ln w\Big|_{\sqrt{2-\sqrt{2}}/2}^b \\
&= \lim_{b\to\sqrt{2}/2} -\ln b + \ln\frac{\sqrt{2+\sqrt{2}}}{2} + \lim_{b\to\sqrt{2}/2} \ln b - \ln\frac{\sqrt{2-\sqrt{2}}}{2} \\
&= \ln\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \frac{1}{2}\ln\left(3+2\sqrt{2}\right) = \frac{1}{2}\ln\left(1+\sqrt{2}\right)^2 = \ln\left(1+\sqrt{2}\right)
\end{aligned}$$

(e) Using integration by parts twice and making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\begin{aligned}
\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx &= \lim_{b\to-\infty}\int_0^b -u^2 e^u\,du \\
&= \lim_{b\to-\infty} -u^2 e^u\Big|_0^b + \lim_{b\to-\infty}\int_0^b 2ue^u\,du \\
&= \lim_{b\to-\infty} -u^2 e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty}\int_0^b 2e^u\,du \\
&= \lim_{b\to-\infty} -u^2 e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty} 2e^u\Big|_0^b \\
&= \lim_{b\to-\infty}\left(-b^2e^b + 2be^b - 2e^b + 2\right) = 2
\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to-\infty} -b^2e^b = \lim_{b\to-\infty} -\frac{b^2}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty}\frac{2b}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty} -\frac{2}{e^{-b}} = 0$$

(f) Using integration by parts, the integral can be written as

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} -\frac{\ln x}{x}\bigg|_1^b + \lim_{b\to\infty}\int_1^b \frac{dx}{x^2}$$
$$= \lim_{b\to\infty} -\frac{\ln x}{x}\bigg|_1^b - \lim_{b\to\infty}\frac{1}{x}\bigg|_1^b = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right) = 1$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to\infty} -\frac{\ln b}{b} \overset{LH}{=} \lim_{b\to\infty} -\frac{1/b}{1} = \lim_{b\to\infty} -\frac{1}{b} = 0$$

4. (a)

$$\int_{-1}^1 \frac{dx}{x^{1/3}} = \int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = \lim_{b\to 0^-}\int_{-1}^b \frac{dx}{x^{1/3}} + \lim_{b\to 0^+}\int_b^1 \frac{dx}{x^{1/3}}$$
$$= \lim_{b\to 0^-}\frac{3}{2}x^{2/3}\bigg|_{-1}^b + \lim_{b\to 0^+}\frac{3}{2}x^{2/3}\bigg|_b^1$$
$$= \lim_{b\to 0^-}\frac{3}{2}\left(b^{2/3}-1\right) + \lim_{b\to 0^+}\frac{3}{2}\left(1-b^{2/3}\right) = 0$$

(b)

$$\int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} = \lim_{b\to 0^-}\int_{-1}^b \frac{dx}{x^3} + \lim_{b\to 0^+}\int_b^1 \frac{dx}{x^3}$$
$$= \lim_{b\to 0^-} -\frac{1}{4x^4}\bigg|_{-1}^b + \lim_{b\to 0^+}\frac{1}{4x^4}\bigg|_b^1$$
$$= \lim_{b\to 0^-}\frac{1}{4}\left(-b^{-4}+1\right) + \lim_{b\to 0^+}\frac{1}{4}\left(1-b^{-4}\right) = -\infty$$

Hence, the integral is divergent.

(c) Making the substitution $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, the integral can be written as

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\pi/2}\int_0^b \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \lim_{b\to\pi/2}\int_0^b d\theta = \lim_{b\to\pi/2}\theta\big|_0^b$$
$$= \lim_{b\to\pi/2} b = \frac{\pi}{2}$$

(d) Using a partial fraction expansion, the integral can be written as

$$\int_0^\infty \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx + \lim_{b\to\infty}\int_1^b \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx$$
$$= \lim_{b\to 0^+}\int_b^1\left[\frac{4x-2}{3\left(x^2-x+1\right)} - \frac{1}{3\left(x+1\right)} - \frac{1}{x}\right]dx + \ldots$$
$$= \lim_{b\to 0^+}\frac{1}{3}\int_b^1 \frac{4x-2}{x^2-x+1}\,dx - \lim_{b\to 0^+}\frac{1}{3}\int_b^1 \frac{dx}{x+1} - \lim_{b\to 0^+}\int_b^1 \frac{dx}{x} + \ldots$$

It is clear to see that the first two integrals (obtained by a partial fraction expansion) belonging to the first partial integral converge. However, the third diverges:

$$\lim_{b \to 0^+} \int_b^1 \frac{dx}{x} = \lim_{b \to 0^+} \ln x \Big|_b^1 = - \lim_{b \to 0^+} \ln b = \infty$$

Hence, since the first partial integral diverges, we conclude that the original integral is divergent.

(e)

$$\int_0^\infty \sin x \, dx = \lim_{b \to \infty} \int_0^b \sin x \, dx = \lim_{b \to \infty} - \cos x \Big|_0^b = \lim_{b \to \infty} (-\cos b + 1)$$

Since $\lim_{b \to \infty} \cos b$ does not exist the integral is divergent.

(f) Making the substitution $u = \cosh x$ so that $du = \sinh x dx$, the integral can be written as

$$\begin{aligned}
\int_0^\infty (1 - \tanh x) \, dx = \lim_{b \to \infty} \int_0^b (1 - \tanh x) \, dx &= \lim_{b \to \infty} \int_0^b dx - \lim_{b \to \infty} \int_0^b \tanh x \, dx \\
&= \lim_{b \to \infty} x \Big|_0^b - \lim_{b \to \infty} \int_0^b \frac{\sinh x}{\cosh x} \, dx \\
&= \lim_{b \to \infty} x \Big|_0^b - \lim_{b \to \infty} \int_1^{\cosh b} \frac{du}{u} \\
&= \lim_{b \to \infty} x \Big|_0^b - \lim_{b \to \infty} \ln u \Big|_1^{\cosh b} \\
&= \lim_{b \to \infty} (b - \ln \cosh b) = \ln 2
\end{aligned}$$

where the last step follows from the fact that

$$\begin{aligned}
\lim_{b \to \infty} (b - \ln \cosh b) = \lim_{b \to \infty} \left[\ln e^b - \ln \left(e^b + e^{-b}\right) + \ln 2\right] &= \lim_{b \to \infty} \left[\ln 2 - \ln \left(1 + e^{-2b}\right)\right] \\
&= \ln 2
\end{aligned}$$

5. (a) The curves $y = 0$, $y = 1 - x^2$ intersect at the point $(-1, 0)$, $(1, 0)$. Hence, the area between the curves is given by

$$A = \int_{-1}^1 \left(1 - x^2\right) \, dx = \left[x - \frac{x^3}{3}\right]_{-1}^1 = \frac{4}{3}$$

(b) The curves $y = x^3$, $y = x^{1/3}$ intersect at the points $(-1, -1)$, $(0, 0)$, $(1, 1)$. Hence the area between the curves is given by

$$A = 2 \int_0^1 \left(x^{1/3} - x^3\right) \, dx = \left[\frac{3x^{4/3}}{2} - \frac{x^4}{2}\right]_0^1 = 1$$

Note that we have used the fact that the intersection of the two curves is anti-symmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x = 0$ to $x = 1$ and multiply the result by two.

(c) The curves $y = 6\sin^{-1} x$, $y = \pi \sin \pi x$ intersect at the points $(-1/2, -\pi)$, $(0, 0)$, $(1/2, \pi)$. Hence, the area between the curves is given by

$$\begin{aligned}
A &= 2\int_0^{1/2} \left(\pi \sin \pi x - 6\sin^{-1} x\right)\, dx \\
&= 2\pi \int_0^{1/2} \sin \pi x \, dx - 12\int_0^{1/2} \sin^{-1} x \, dx \\
&= -2\cos \pi x\Big|_0^{1/2} - 12x\sin^{-1} x\Big|_0^{1/2} + \int_0^{1/2} \frac{12x}{\sqrt{1-x^2}}\, dx \\
&= 2 - \pi - 6\int_1^{3/4} \frac{du}{\sqrt{u}} \\
&= 2 - \pi - 6\int_1^{3/4} u^{-1/2}\, du \\
&= 2 - \pi - 12\sqrt{u}\Big|_1^{3/4} = 14 - \pi - 6\sqrt{3}
\end{aligned}$$

where we have used integration by parts and the substitution $u = 1 - x^2$ so that $du = -2xdx$ in order to solve the second integral. Note that we have used the fact that the intersection of the two curves is anti-symmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x = 0$ to $x = 1/2$ and multiply the result by two.

6. (a)

$$\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \sin x \, dx = -\frac{2\cos x}{\pi}\Bigg|_0^{\pi/2} = \frac{2}{\pi}$$

(b)

$$\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_{-\pi/2}^{0} \sin x \, dx = -\frac{2\cos x}{\pi}\Bigg|_{-\pi/2}^{0} = \frac{2\cos x}{\pi}\Bigg|_0^{-\pi/2} = -\frac{2}{\pi}$$

(c) Using the identities $\sin^2 x + \cos^2 x = 1$, $\cos 2x = 2\cos^2 x - 1$, we find

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^2 x \, dx &= \frac{2}{\pi}\int_0^{\pi/2} \left(1 - \cos^2 x\right)\, dx \\
&= \frac{1}{\pi}\int_0^{\pi/2} (1 - \cos 2x)\, dx \\
&= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}
\end{aligned}$$

(d)

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{x_2-x_1}\int_{x_1}^{x_2}(ax+b)\,dx = \frac{1}{x_2-x_1}\left[\frac{ax^2}{2}+bx\right]_{x_1}^{x_2}$$
$$= b+\frac{a}{2}(x_1+x_2)$$

7. Let $f(x)$ and $g(x)$ be continuous for $a \le x \le b$ and $|g(x)-f(x)| \le \epsilon$ for $a \le x \le b$. Defining $h(x) = g(x)-f(x)$ so that $|h(x)| \le \epsilon$ and using (4.6) we then find

$$\left|\int_a^b h(x)\,dx\right| = \left|\int_a^b [g(x)-f(x)]\,dx\right| = \left|\int_a^b g(x)\,dx - \int_a^b f(x)\,dx\right| \le \epsilon\,(b-a)$$

8. (a)

$$\int_0^1 \sin x^2\,dx \approxeq \int_0^1\left(x^2-\frac{x^6}{6}\right)dx = \left[\frac{x^3}{3}-\frac{x^7}{42}\right]_0^1 = \frac{13}{42} \approxeq 0.3095$$

The worst error is approximately 0.0081 at the point $x=1$.

(b)

$$\int_0^1 e^{-x^2}\,dx \approxeq \int_0^1\left(1-x^2+\frac{x^4}{2}\right)dx = \left[x-\frac{x^3}{3}+\frac{x^5}{10}\right]_0^1 = \frac{23}{30} \approxeq 0.7667$$

The worst error is approximately 0.1321 at the point $x=1$.

9. Let $f(x)$ be continuous for $0 \le x \le 1$. Then (4.20) may be used to approximate the integral of $f(x)$ numerically:

$$\int_0^1 f(x)\,dx \sim \frac{1}{2n}\left[f(0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(1)\right]$$

where $0 < x_1 < x_2 \cdots < x_{n-1} < 1$. If we then let $n \to \infty$ and choose $x_1 = 1/n$, $x_2 = 2/n, \ldots, x_{n-1} = (n-1)/n$, $x_n = n/n$ such that the endpoints converge to $x=0$ and $x=1$ respectively, while at the same time choosing an infinite number of equally spaced, but infinitely close interior points $x_1, x_2, \ldots, x_{n-1}$ the finite sum converges to:

$$\lim_{n\to\infty}\frac{1}{2n}\left[f(0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(1)\right] =$$
$$\lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n-1}{n}\right)+f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\,dx$$

Note that the end points of the first and second limits differ by a factor of $1/2$. However, since $2\infty = \infty$ this difference is of no importance.

10. (a)

$$\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2} = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}+\frac{n}{n}\right) = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

(b)

$$\begin{aligned}
\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3} &= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2+\left(\frac{n}{n}\right)^2\right] \\
&= \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}
\end{aligned}$$

(c) Provided that $P \geq 0$

$$\begin{aligned}
\lim_{n\to\infty}\frac{1^P+2^P+\cdots+n^P}{n^{P+1}} &= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^P+\left(\frac{2}{n}\right)^P+\cdots+\left(\frac{n-1}{n}\right)^P+\left(\frac{n}{n}\right)^P\right] \\
&= \int_0^1 x^P\,dx = \left.\frac{x^{P+1}}{P+1}\right|_0^1 = \frac{1}{P+1}
\end{aligned}$$

(d) Taking the natural log of both sides of the equation gives $\ln(4/e) = \ln 4 - 1$ and

$$\ln\left\{\lim_{n\to\infty}\frac{1}{n}\left[(n+1)(n+2)\ldots(2n)\right]^{1/n}\right\} = \mathfrak{L}$$

Then, manipulating the left-hand side further, we find

$$\begin{aligned}
\mathfrak{L} &= \lim_{n\to\infty}\ln\left\{\frac{1}{n}\left[(n+1)(n+2)\ldots(2n)\right]^{1/n}\right\} = \lim_{n\to\infty}\frac{1}{n}\ln\left[\frac{1}{n^n}(n+1)(n+2)\ldots(2n)\right] \\
&= \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{n+1}{n}\frac{n+2}{n}+\cdots+\frac{2n}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{n}\left(\ln\frac{n+1}{n}+\ln\frac{n+2}{n}+\cdots+\ln\frac{n+n}{n}\right) \\
&= \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{n-1}{n}\right)+\ln\left(1+\frac{n}{n}\right)\right] \\
&= \int_0^1 \ln(1+x)\,dx = x\ln(1+x)\,\Big|_0^1 - \int_0^1\frac{x}{1+x}\,dx = \ln 2 - \int_1^2\frac{u-1}{u}\,du \\
&= \ln 2 - \int_1^2\left(1-\frac{1}{u}\right)du = \ln 2 - [u-\ln u]_1^2 = \ln 4 - 1 = \ln\frac{4}{e}
\end{aligned}$$

where we have used integration by parts to solve the first integral and the substitution $u = 1 + x$ so that $du = dx$ to solve the second integral.

11. Let $f(x)$ be a continuous function for $a \leq x \leq b$ and let it be a given fact that

$$\int_{a_1}^{b_1} f(x)\,dx = 0$$

for every interval $a_1 \le x \le b_1$ contained in the interval $a \le x \le b$. Next, let us choose a fixed point x_0 such that $a_1 \le x_0$, $x_0 + \delta \le b_1$, where $\delta > 0$. Then by (4.13) we find

$$\int_{x_0}^{x_0+\delta} f(x)\,dx = f(x^*)\,\delta = 0 \qquad \text{for } x_0 \le x^* \le x_0 + \delta$$

Now if we let $\delta \to 0$ we get

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{x_0}^{x_0+\delta} f(x)\,dx = f(x_0) = 0$$

As x_0 was chosen arbitrarily within $a_1 \le x \le b_1$, we conclude that $f(x) \equiv 0$.

12. Let $f(x)$ be a continuous function for $a \le x \le b$, $f(x) \ge 0$ on the interval and

$$\int_a^b f(x)\,dx = 0$$

Next, let c be such that $a < c < b$. Then the integrals $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ are either positive or zero. However, since their sum must be zero, the only option is that in fact they both are zero. The interval of each partial integral thus obtained can in turn be subdivided into smaller intervals over which to individually integrate $f(x)$ and since again $f(x)$ is either positive or zero on this new sub interval, but the total integral over $a \le x \le b$ must be zero, we conclude that each partial integral must be zero over the relevant sub interval. We can continue to apply this argument indefinitely for every smaller sub interval obtained from a larger sub interval and so we conclude that $\int_{a_1}^{b_1} f(x)\,dx = 0$ for every choice a_1, b_1 on the interval $a \le x \le b$. Hence, by Problem 11, $f(x) \equiv 0$.

Section 4.2

1. (a) Let $f(x) = x$ and $F(x) = \int_0^x x\,dx$. Then

x	x	$\int_x^{x+1} t\,dt$	$\int_0^x t\,dt = F(x)$
0	0	0.5	0.0
1	1	1.5	0.5
2	2	2.5	2.0
3	3	3.5	4.5
4	4	4.5	8.0
5	5	5.5	12.5
6	6	6.5	18.0
7	7	7.5	24.5
8	8	8.5	32.0
9	9	9.5	40.5
10	10		50.0

(b) Let $f(x) = e^{-x^2}$ and $F(x) = \int_0^1 e^{-x^2}\,dx$. Then

x	e^{-x^2}	$\int_x^{x+0.1} e^{-t^2}\,dt$	$\int_0^x e^{-t^2}\,dt = F(x)$
0	1.0	0.100	0.00
0.1	0.99	0.098	0.100
0.2	0.96	0.094	0.197
0.3	0.91	0.088	0.291
0.4	0.85	0.082	0.379
0.5	0.78	0.074	0.460
0.6	0.70	0.066	0.534
0.7	0.61	0.057	0.600
0.8	0.53	0.049	0.657
0.9	0.44	0.041	0.705
1.0	0.37		0.746

(c) Let $f(x) = \cos x$ and $F(x) = \int_0^1 \cos x\,dx$. Then

x	$\cos x$	$\int_x^{x+0.1} \cos t\,dt$	$\int_0^x \cos t\,dt = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.099	0.100
0.2	0.98	0.097	0.199
0.3	0.95	0.094	0.295
0.4	0.92	0.090	0.389
0.5	0.88	0.085	0.479
0.6	0.83	0.080	0.564
0.7	0.76	0.073	0.644
0.8	0.70	0.066	0.717
0.9	0.62	0.058	0.783
1.0	0.54		0.841

(d) Let $f(x) = 1/(1+x^3)$ and $F(x) = \int_0^1 dx/(1+x^3)$. Then

x	$1/(1+x^3)$	$\int_x^{x+0.1} dt/(1+t^3)$	$\int_0^x dt/(1+t^3) = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.100	0.100
0.2	0.99	0.098	0.200
0.3	0.97	0.096	0.298
0.4	0.94	0.091	0.393
0.5	0.89	0.086	0.485
0.6	0.82	0.078	0.570
0.7	0.75	0.070	0.649
0.8	0.66	0.062	0.719
0.9	0.58	0.054	0.781
1.0	0.50		0.835

(e) Let $f(x) = \sqrt{1-x^3}$ and $F(x) = \int_0^0 .5\sqrt{1-x^3}\,dx$. Then

x	$\sqrt{1-x^3}$	$\int_x^{x+0.1} \sqrt{1-t^3}\,dt$	$\int_0^x \sqrt{1-t^3}\,dt = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.100	0.100
0.2	1.00	0.099	0.200
0.3	0.99	0.098	0.299
0.4	0.97	0.095	0.397
0.5	0.04		0.492

2. Let $f(x)$ be continuous for $a \le x \le b$

(a) By (4.15) and (4.9) we find

$$\frac{d}{dx}\int_x^b f(t)\;dt = \frac{d}{dx}\left[-\int_b^x f(t)\;dt\right] = \frac{d}{dx}\left[-F(x)\right] = -\frac{dF}{dx} = -f(x)$$

(b) Let us make the substitution $u = x^2$ and note that $a \le u \le b$. Then

$$\frac{d}{dx}\int_a^{x^2} f(t)\;dt = \frac{d}{dx}\int_a^u f(t)\;dt = \frac{d}{dx}F(u) = \frac{dF}{du}\frac{du}{dx} = 2xf(u) = 2xf\left(x^2\right)$$

(c) Let us make the substitution $u = x^2$ and note that $a \le u \le b$. Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^b f(t)\;dt = \frac{d}{dx}\left[-\int_b^{x^2} f(t)\;dt\right] = \frac{d}{dx}\left[-\int_b^u f(t)\;dt\right] &= \frac{d}{dx}\left[-F(u)\right]\\ &= -\frac{dF}{du}\frac{du}{dx}\\ &= -2xf(u) = -2xf\left(x^2\right)\end{aligned}$$

(d) Let us make the substitutions $u = x^2$, $v = x^3$ and note that $a \le u, v \le b$. Furthermore, let c be a fixed point $x^2 \le c \le x^3$. Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{x^3} f(t)\;dt &= \frac{d}{dx}\left(\int_c^{x^3} f(t)\;dt + \int_{x^2}^c f(t)\;dt\right)\\ &= \frac{d}{dx}\int_c^v f(t)\;dt + \frac{d}{dx}\int_u^c f(t)\;dt\\ &= \frac{d}{dx}\int_c^v f(t)\;dt - \frac{d}{dx}\int_c^u f(t)\;dt\\ &= \frac{dF}{dv}\frac{dv}{dx} - \frac{dF}{du}\frac{du}{dx} = 3x^2 f\left(x^3\right) - 2xf\left(x^2\right)\end{aligned}$$

3. (a) By (4.20) we find approximately

$$\ln 1 = \int_1^1 \frac{dt}{t} \sim \frac{1-1}{4}\,[1+1] = 0$$
$$\ln 2 = \int_1^2 \frac{dt}{t} \sim \frac{1}{20}\,(1 + 1.818 + 1.667 + 1.538 + \cdots + 1.053 + 0.5) \approxeq 0.694$$
$$\ln 0.5 = \int_1^{1/2} \frac{dt}{t} = -\int_{1/2}^1 \frac{dt}{t} \sim -\frac{1}{40}\,(2 + 3.636 + 3.333 + \cdots + 2.105 + 1) \approxeq -0.694$$

(b) Using the definition

$$\ln x = \int_1^x \frac{dt}{t} \qquad x > 0$$

let $F(x) = \ln x$. Then by (4.10) we find

$$\frac{dF}{dx} = \frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x} \qquad x > 0$$

Hence, the first derivative of $\ln x$, $x > 0$ exists and as such, $\ln x$ is defined and continuous for $0 < x < \infty$.

(c) Let $F(x) = \ln ax - \ln x$ for $a > 0$ and $x > 0$. Next, let us make the substitution $u = ax$ so that $du = adx$. Then by (b) we find

$$\begin{aligned}\frac{dF}{dx} = \frac{d}{dx}(\ln ax - \ln x) = a\frac{d}{du}\ln u - \frac{d}{dx}\ln x &= a\frac{d}{du}\int_1^u \frac{dt}{t} - \frac{d}{dx}\int_1^x \frac{dt}{t} \\ &= \frac{a}{u} - \frac{1}{x} = \frac{a}{ax} - \frac{1}{x} = 0\end{aligned}$$

Hence, $F'(x) \equiv 0$ so that $F(x) \equiv \text{const} = \ln a$. And so

$$F(x) = \ln a = \ln ax - \ln x \qquad \Longrightarrow \qquad \ln ax = \ln a + \ln x \quad \text{for } a, x > 0$$

4. Let an ellipse be given by the parametric equations: $x = a\cos\phi$, $y = b\sin\phi$, $b > a > 0$. Then by (3.53) the element of arc ds on the curve traced out by the ellipse is defined as $ds^2 = dx^2 + dy^2$. Hence, the length of arc from $\phi = 0$ to $\phi = \alpha$ is given by

$$\begin{aligned}s = \int_0^\alpha ds = \int_0^\alpha \sqrt{dx^2 + dy^2} &= \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,d\phi \\ &= \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2\left(1 - \sin^2\phi\right)}\,d\phi \\ &= \int_0^\alpha \sqrt{b^2 - (b^2 - a^2)\sin^2\phi}\,d\phi \\ &= \int_0^\alpha b\sqrt{1 - \frac{b^2 - a^2}{b^2}\sin^2\phi}\,d\phi = b\int_0^\alpha \sqrt{1 - k^2\sin^2\phi}\,d\phi\end{aligned}$$

5. (a) Let $F(x)$ be as in (4.24). Then by (4.10) we find

$$\frac{dF}{dx} = \frac{d}{dx}\int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \frac{1}{\sqrt{1-k^2\sin^2 x}}$$

Hence, the first derivative of $F(x)$, $0 < k^2 < 1$ exists for all x and as such, $F(x)$ is defined and continuous for all x.

(b) Let $x_2 > x_1$. Then since $F'(x) > 0$ for $0 < k^2 < 1$ it follows from the very definition of the derivative that $F(x_2) > F(x_1)$. Hence, we conclude that as x increases, $F(x)$ increases.

(c) Let $F(x)$ be as in (4.24). Then to show that $F(x+\pi) - F(x) = \text{const}$ we use (4.10) to find

$$\begin{aligned}
\frac{d}{dx}\left[F\left(x+\pi\right) - F\left(x\right)\right] &= \frac{d}{dx}\int_0^{x+\pi} \frac{dt}{\sqrt{1-k^2\sin^2 t}} - \frac{d}{dx}\int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= \frac{1}{\sqrt{1-k^2\sin^2\left(x+\pi\right)}} - \frac{1}{\sqrt{1-k^2\sin^2 x}} \\
&= \frac{1}{\sqrt{1-k^2\left(-\sin x\right)^2}} - \frac{1}{\sqrt{1-k^2\sin^2 x}} = 0
\end{aligned}$$

Hence, since $F'(x+\pi) - F'(x) \equiv 0$, we conclude that the quantity $F(x+\pi) - F(x) = 2K$, where $K > 0$ is some positive constant. The fact that K must be positive and non-zero follows from (b).

(d) We know from (b) that as x increases, $F(x)$ increases. Furthermore, since $F(x) \geq 0$ for $0 < k^2 < 1$ it then follows that $\lim_{x\to\infty} F(x) = \infty$. Next, to show that $\lim_{x\to-\infty} F(x) = -\infty$ we write

$$\begin{aligned}
\lim_{x\to-\infty} F\left(x\right) = \lim_{x\to-\infty}\int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}} &= \lim_{x\to-\infty}\int_x^0 -\frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= \lim_{x\to\infty}\int_0^x -\frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= -\lim_{x\to\infty} F\left(x\right) = -\infty
\end{aligned}$$

6. Let $x = am(y)$ be the inverse of the function $y = F(x)$ of (4.24).

(a) Let $y_2 > y_1$. Furthermore, from (a) and (b) of Problem 5 we know that $F'(x) > 0$ for $0 < k^2 < 1$ so that $F(x_2) > F(x_1)$ for $x_2 > x_1$. Since $y = F(x)$ this implies that $y_2 = F(x_2)$, $y_1 = F(x_1)$. Then noting that $x = am(y)$ is defined as the inverse of $y = F(x)$ we can write $am(y_2) = x_2$, $am(y_1) = x_1$. Now since $x_2 > x_1$ we conclude that $am(y_2) > am(y_1)$ for $y_2 > y_1$.

(b) From (c) of Problem 5 we know that $F(x+\pi) = F(x) + 2K = y + 2K$. Using the fact that $x = am(y)$ is defined as the inverse of $y = F(x)$ we then find

$$am\,(y + 2K) = am\,[F\,(x+\pi)] = x + \pi = am\,(y) + \pi$$

(c) From (a) of Problem 5 we know that $dF/dx = dy/dx = 1/\sqrt{1 - k^2 \sin^2 x}$. Since $(dy/dx)(dx/dy) \equiv 1$ we thus conclude that $am'(y) = dx/dy = \sqrt{1 - k^2 \sin^2 x}$. Hence, the first derivative of $am(y)$, $0 < k^2 < 1$ exists for all y and as such, $am(y)$ is defined and continuous for all y.

7. Let the functions $sn(y)$, $cn(y)$, $dn(y)$ be defined in terms of the function of Problem 6:

$$sn\,(y) = \sin\,[am\,(y)] \qquad cn\,(y) = \cos\,[am\,(y)] \qquad dn\,(y) = \sqrt{1 - k^2 \sin^2 y}$$

Then

(a)

$$sn^2\,(y) + cn^2\,(y) = \sin^2\,[am\,(y)] + \cos^2\,[am\,(y)] = \sin^2 x + \cos^2 x = 1$$

(b)

$$\begin{aligned}\frac{d}{dy} sn\,(y) = \frac{d}{dy} \sin\,[am\,(y)] = \cos\,[am\,(y)]\,\frac{d}{dy} am\,(y) &= cn\,(y)\,\frac{dx}{dy} \\ &= cn\,(y)\,\sqrt{1 - k^2 \sin^2 x} \\ &= cn\,(y)\,dn\,(y)\end{aligned}$$

(c)

$$\begin{aligned}\frac{d}{dy} cn\,(y) = \frac{d}{dy} \cos\,[am\,(y)] = -\sin\,[am\,(y)]\,\frac{d}{dy} am\,(y) &= -sn\,(y)\,\frac{dx}{dy} \\ &= -sn\,(y)\,\sqrt{1 - k^2 \sin^2 x} \\ &= -sn\,(y)\,dn\,(y)\end{aligned}$$

(d)

$$sn\,(y + 4K) = \sin\,[am\,(y + 4K)] = \sin\,[am\,(y) + 2\pi] = \sin\,[am\,(y)] = sn\,(y)$$

(e)

$$cn\,(y + 4K) = \cos\,[am\,(y + 4K)] = \cos\,[am\,(y) + 2\pi] = \cos\,[am\,(y)] = cs\,(y)$$

(f)

$$
\begin{aligned}
dn\,(y+2K) = \sqrt{1-k^2 sn^2\,(y+2K)} &= \sqrt{1-k^2\sin^2\left[am\,(y+2K)\right]} \\
&= \sqrt{1-k^2\sin^2\left[am\,(y)+\pi\right]} \\
&= \sqrt{1-k^2\left(-\sin\left[am\,(y)\right]\right)^2} \\
&= \sqrt{1-k^2\sin^2\left[am\,(y)\right]} \\
&= \sqrt{1-k^2 sn^2\,(y)} = dn\,(y)
\end{aligned}
$$

8. Let the error function $y = erf(x)$ be defined by the equation

$$
y = erf\,(x) = \int_0^x e^{-t^2}\,dt
$$

(a) Using the definition of $y = erf(x)$ form above, then by (4.10) we find

$$
\frac{dy}{dx} = \frac{d}{dx} erf\,(x) = \frac{d}{dx}\int_0^x e^{-t^2}\,dt = e^{-x^2}
$$

Hence, the first derivative of $y = erf(x)$ exists for all x and as such, $y = erf(x)$ is defined and continuous for all x.

(b)

$$
erf\,(-x) = \int_0^{-x} e^{-t^2}\,dt = -\int_{-x}^{0} e^{-t^2}\,dt = -\int_0^x e^{-t^2}\,dt = -erf\,(x)
$$

(c) Let us first consider the case where $x \geq 0$. Then we find

$$
\begin{aligned}
erf\,(x) = \int_0^x e^{-t^2}\,dt \leq \int_0^1 dt + \int_1^x te^{-t^2}\,dt &= 1 - \frac{1}{2}\int_1^x \frac{d}{dt}e^{-t^2}\,dt \\
&= 1 - \frac{1}{2}\frac{d}{dt}\int_1^x e^{-t^2}\,dt \\
&= 1 - \frac{e^{-x^2}}{2} + \frac{e^{-1}}{2} < 1
\end{aligned}
$$

Note that we can move the differentiation operation outside of the integral, since the limits of integration are assumed constant relative to the integration variable t. From (b) it then follows immediately that $-1 < erf(x) < 1$.

Section 4.5

1. (a) If R is a triangle with vertices $(0,0)$, $(1,0)\,(1,1)$, so that $0 \le x \le 1$, $0 \le y \le x$, then

$$\iint_R \left(x^2+y^2\right)\,dx\,dy = \int_0^1 \int_0^x \left(x^2+y^2\right)\,dy\,dx = \int_0^1 \left[x^2y+\frac{y^3}{3}\right]_0^x dx$$
$$= \int_0^1 \left(x^3+\frac{x^3}{3}\right) dx$$
$$= \left[\frac{x^4}{4}+\frac{x^4}{12}\right]_0^1 = \frac{1}{3}$$

(b) Let R be the region: $u^2+v^2 \le 1$, $0 \le w \le 1$, so that $-1 \le x \le 1$, $-\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$, $0 \le w \le 1$. Then

$$\begin{aligned}\iiint_R u^2v^2w\,du\,dv\,dw &= \int_{-1}^1 \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_0^1 u^2v^2w\,dw\,dv\,du \\ &= 4\int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^1 u^2v^2w\,dw\,dv\,du \\ &= 2\int_0^1 \int_0^{\sqrt{1-u^2}} u^2v^2w^2\Big|_0^1\,dv\,du = 2\int_0^1 \int_0^{\sqrt{1-u^2}} u^2v^2\,dv\,du \\ &= \frac{2}{3}\int_0^1 u^2v^3\Big|_0^{\sqrt{1-u^2}}\,du = \frac{2}{3}\int_0^1 u^2\left(1-u^2\right)^{3/2}\,du \\ &= \frac{2}{3}\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta = \frac{1}{12}\int_0^{\pi/2} \sin^2 2\theta\,(1+2\cos 2\theta)\;d\theta \\ &= \frac{1}{48}\int_0^{\pi/2} (2+\cos 2\theta - 2\cos 4\theta - \cos 6\theta)\;d\theta \\ &= \frac{1}{48}\left[2\theta+\frac{\sin 2\theta}{2}-\frac{\sin 4\theta}{2}-\frac{\sin 6\theta}{6}\right]_0^{\pi/2} = \frac{\pi}{48}\end{aligned}$$

where we have used the identities: $2\sin^2\theta = 1-\cos 2\theta$, $2\cos^2\theta = 1+\cos 2\theta$, $2\cos A\cos B = \cos(A+B)+\cos(A-B)$.

(c) If R is the region: $1 \le r \le 2$, $(\pi/4) \le \theta \le \pi$, then

$$\iint_R r^3\cos\theta\,dr\,d\theta = \int_1^2 \int_{\pi/4}^{\pi} r^3\cos\theta\,d\theta\,dr = \int_1^2 r^3\sin\theta\Big|_{\pi/4}^{\pi}\,dr = -\int_1^2 \frac{\sqrt{2}}{2}r^3\,dr$$
$$= -\frac{\sqrt{2}}{8}r^4\Big|_1^2 = -\frac{15\sqrt{2}}{8}$$

(d) Let R be a tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$. To determine the x and y limits we can use the triangle in the xy-plane with vertices

$(0,0)$, $(1,0)$, $(0,2)$, since this is the projection of the tetrahedron onto the xy-plane. Hence, we find $0 \le x \le 1$, $0 \le y \le 2-2x$. To find the z limit we need to find the equation for the plane passing through the points $(1,0,0)$, $(0,2,0)$, $(0,0,3)$ as this gives the top surface of the tetrahedron. To this end, we first form the two planar vectors $\mathbf{u} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 3\mathbf{k}$. Next, to find the normal to the plane we compute $\mathbf{u} \times \mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Then by (1.23) the equation for the plane is given by $z = 3 - 3x - (3/2)y$. Hence,

$$\begin{aligned}
\iiint_R (x+z)\ dV &= \int_0^1 \int_0^{2-2x} \int_0^{3-3x-(3/2)y} (x+z)\ dz\,dy\,dx \\
&= \int_0^1 \int_0^{2-2x} \left[xz + \frac{z^2}{2}\right]_0^{3-3x-(3/2)y} dy\,dx \\
&= \frac{1}{8}\int_0^1 \int_0^{2-2x} \left(12x^2 + 24xy - 48x + 9y^2 - 36y + 36\right)\ dy\,dx \\
&= \frac{1}{8}\int_0^1 \left[12x^2y + 12xy^2 - 48xy + 3y^3 - 18y^2 + 36y\right]_0^{2-2x}\ dx \\
&= 3\int_0^1 (-x+1)^2\ dx = 1
\end{aligned}$$

2. (a) Let $z = f(x,y) = e^x \cos y$ and $0 \le x \le 1$, $0 \le y \le \pi/2$. Then

$$\begin{aligned}
V = \iint_R f(x,y)\ dx\,dy = \int_0^1 \int_0^{\pi/2} e^x \cos y\,dy\,dx = \int_0^1 e^x \sin y\Big|_0^{\pi/2} dx = \int_0^1 e^x\,dx \\
= e^x\Big|_0^1 = e - 1
\end{aligned}$$

(b) Let $z = f(x,y) = x^2 e^{-x-y}$ and $0 \le x \le 1$, $0 \le y \le 2$. Next, let $u = e^{-x}$ so that $-u^{-1}du = dx$. Then

$$\begin{aligned}
V = \iint_R f(x,y)\ dx\,dy &= \int_0^1 \int_0^2 x^2 e^{-x-y}\,dy\,dx = -\int_0^1 x^2 e^{-x-y}\Big|_0^2 dx \\
&= \int_0^1 x^2\left(e^{-x} - e^{-x-2}\right)\ dx = \left(e^{-2} - 1\right)\int_1^{e^{-1}} \ln^2 u\,du \\
&= \left(e^{-2} - 1\right)\left[u \ln^2 u\Big|_1^{e^{-1}} - 2\int_1^{e^{-1}} \ln u\,du\right] \\
&= \left(e^{-2} - 1\right)\left(e^{-1} - 2u \ln u\Big|_1^{e^{-1}} + 2\int_1^{e^{-1}} du\right) \\
&= \left(e^{-2} - 1\right)\left(3e^{-1} + 2u\Big|_1^{e^{-1}}\right) = \left(1 - e^{-2}\right)\left(2 - 5e^{-1}\right)
\end{aligned}$$

(c) Let $z = f(x,y) = x^2 y$ and $0 \le x \le 1$, $x+1 \le y \le x+2$. Then

$$
\begin{aligned}
V = \iint_R f(x,y)\; dx\,dy = \int_0^1 \int_{x+1}^{x+2} x^2 y\, dy\, dx &= \frac{1}{2}\int_0^1 x^2 y^2 \Big|_{x+1}^{x+2} dx \\
&= \frac{1}{2}\int_0^1 \left(2x^3 + 3x^2\right)\, dx \\
&= \frac{1}{4}\left[x^4 + 2x^3\right]_0^1 = \frac{3}{4}
\end{aligned}
$$

(d) Let $z = f(x,y) = \sqrt{x^2 - y^2}$ and $x^2 - y^2 \ge 0$, $0 \le x \le 1$. Next, let $\sin u = y/x$ so that $\cos u\, du = dy/x$. Then

$$
\begin{aligned}
V = \iint_R f(x,y)\; dx\,dy &= \int_0^1 \int_{-x}^{x} \sqrt{x^2 - y^2}\, dy\, dx \\
&= \int_0^1 \int_{-x}^{x} x^2 \cos^2 u\, du\, dx = \frac{1}{2}\int_0^1 x^2 \int_{-x}^{x} (1 + \cos 2u)\; du\, dx \\
&= \frac{1}{2}\int_0^1 x^2 \left[u + \frac{\sin 2u}{2}\right]_{-x}^{x} dx \\
&= \frac{1}{2}\int_0^1 x^2 \left[\sin^{-1}\frac{y}{x} + \frac{1}{2}\sin\left(2\sin^{-1}\frac{y}{x}\right)\right]_{-x}^{x} = \frac{\pi}{2}\int_0^1 x^2\, dx \\
&= \frac{\pi}{6}x^2\Big|_0^1 = \frac{\pi}{6}
\end{aligned}
$$

3. (a) Let $1 \le x \le 2$, $1 - x \le y \le 1 + x$. Then

$$
\iint_R f(x,y)\; dA = \int_1^2 \int_{1-x}^{1+x} f(x,)\; dy\, dx =
$$

(b) Let $y^2 + x(x-1) \le 0$. Then

$$
\iint_R f(x,y)\; dA = \int_0^1 \int_{-\sqrt{x(1-x)}}^{\sqrt{x(1-x)}} f(x,y)\; dy\, dx
$$

4. (a) Let R be the cube of vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$,

$(0,1,1)$, $(1,1,1)$.

$$\iiint_R f(x,y,z)\,dV = \int_0^1\int_0^1\int_0^1 \sqrt{x+y+z}\,dz\,dy\,dx = \int_0^1\int_0^1\int_0^1 \underbrace{\sqrt{u}}_{u=x+y+z}\,du\,dy\,dx$$
$$= \frac{2}{3}\int_0^1\int_0^1 (x+y+z)^{3/2}\big|_0^1\,dy\,dx$$
$$= \frac{2}{3}\int_0^1\int_0^1 \left[(1+x+y)^{3/2}-(x+y)^{3/2}\right]\,dy\,dx$$
$$= \frac{4}{15}\int_0^1 \left[(1+x+y)^{5/2}-(x+y)^{5/2}\right]_0^1\,dx$$
$$= \frac{4}{15}\int_0^1 \left[(2+x)^{5/2}-2\,(1+x)^{5/2}+x^{5/2}\right]\,dx$$
$$= \frac{8}{105}\left[(2+x)^{7/2}-2\,(1+x)^{7/2}+x^{7/2}\right]_0^1 = \frac{8}{35}\left(9\sqrt{3}-8\sqrt{2}+1\right)$$

(b) Let R be the pyramid of vertices $(\pm1,\pm1,0)$ and $(0,0,1)$. In order to determine the limits of integration we will use the cross-section method. Let $0\le z\le 1$. Then a plane perpendicular to the z-axis whose boundaries satisfy the inequalities $-1+z\le y\le 1-z$, $-1+z\le x\le 1-z$ will represents the cross-sectional area of the pyramid for a given value of z. Hence,

$$\iiint_R f(x,y,z)\;dV = \int_0^1\int_{-1+z}^{1-z}\int_{-1+z}^{1-z}\left(x^2+z^2\right)\,dx\,dy\,dz$$
$$= \int_0^1\int_{-1+z}^{1-z}\left[\frac{x^3}{3}+xz^2\right]_{-1+z}^{1-z}\,dy\,dz$$
$$= \frac{2}{3}\int_0^1\left(1-3z+6z^2-4z^3\right)\int_{-1+z}^{1-z}\,dy\,dz$$
$$= \frac{2}{3}\int_0^1\left(1-3z+6z^2-4z^3\right)y\big|_{-1+z}^{1-z}\,dz$$
$$= \frac{4}{3}\int_0^1\left(1-4z+9z^2-10z^3+4z^4\right)\,dz$$
$$= \frac{4}{3}\left[z-2z^2+3z^3-\frac{5}{2}z^4+\frac{4}{5}z^5\right]_0^1 = \frac{2}{5}$$

5. (a) Let the integral

$$\int_{1/2}^1\int_0^{1-x} f(x,y)\;dy\,dx$$

be given. It then follows that $1/2\le x\le 1$, $0\le y\le 1-x$ and so the region R is the triangle with vertices $(1/2,0)$, $(1,0)$, $(1/2,1/2)$ in the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_0^{1/2}\int_{1/2}^{1-y} f(x,y)\ dx\, dy$$

(b) Let the integral

$$\int_0^1\int_0^{\sqrt{1-x^2}} f(x,y)\ dy\, dx$$

be given. It then follows that $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$ and so the region R is the quarter circle in the positive quadrant of the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\ dx\, dy$$

(c) Let the integral

$$\int_0^1\int_{y-1}^{0} f(x,y)\ dx\, dy$$

be given. It then follows that $y-1 \le x \le 0$, $0 \le y \le 1$ and so the region R is the triangle with vertices $(0,0)$, $(-1,0)$, $(0,1)$ in the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_{-1}^0\int_0^{x+1} f(x,y)\ dy\, dx$$

(d) Let the integral

$$\int_0^1\int_{1-x}^{1+x} f(x,y)\ dy\, dx$$

be given. It then follows that $0 \le x \le 1$, $1-x \le y \le 1+x$ and so the region R is the triangle with vertices $(1,0)$, $(0,1)$, $(1,1)$ in the xy-plane. Hence, interchanging the order of integration, the integral can also be written as

$$\int_0^2\int_{|y-1|}^{1} f(x,y)\ dx\, dy$$

6. (a) First of all, let us define a diametral plane of a sphere as a plane that cuts the sphere into two equal halves, hence, containing the midpoint of the sphere. Next, let $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$, where $\mathbf{N} = A\mathbf{i}+B\mathbf{j}+C\mathbf{k}$, be a unit vector normal to the diametral plane and let $P = (x_0, y_0, z_0)$ be an arbitrary point of the plane so that by (1.23) the equation for the plane can be written as $Ax + By + Cz + D = 0$, where $D = -Ax_0 - By_0 - Cz_0$ and x, y, z are expected to be on the place. Then the distance of the point P to another arbitrary point $Q = (x_1, y_1, z_1)$ is the length of

the projection of the vector $\overrightarrow{PQ}$ onto the normal unit vector $\mathbf{n}$, or in other words by (1.12), as the component of $\overrightarrow{PQ}$ in the direction of $\mathbf{n}$:

$$\begin{aligned} d = \operatorname{comp}_{\mathbf{n}} \overrightarrow{PQ} = |\overrightarrow{PQ}| \cos\alpha = |\overrightarrow{PQ} \cdot \mathbf{n}| &= \frac{|A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

Recognizing that the point Q was chosen arbitrarily then by (2.84) the distance of any point to the diametral plane can be written as

$$d(\rho, \phi, \theta) = \frac{|A\rho \sin\phi \cos\theta + B\rho \sin\phi \sin\theta + C\rho \cos\phi + D|}{\sqrt{A^2 + B^2 + C^2}}$$

in spherical coordinates. Hence, the mass of a sphere whose density is proportional to the distance from one diametral plane is given by

$$\begin{aligned} M &= k \iiint_R d(\rho, \phi, \theta)\ dV = k \int_0^{2\pi} \int_0^{\pi} \int_0^{R} d(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \kappa \int_0^{2\pi} \int_0^{\pi} \int_0^{R} |A\rho \sin\phi \cos\theta + B\rho \sin\phi \sin\theta + C\rho \cos\phi + D| \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

where R is the radius of the sphere, $\kappa = k/\sqrt{A^2 + B^2 + C^2}$, k is the constant of proportionality and $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

(b) By (4.54), the coordinates of the center of mass of the sphere of part (a) are given by

$$\begin{aligned} \bar{x} &= \frac{k}{M} \iiint_R x d(\rho, \phi, \theta)\ dV = \frac{k}{M} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} d(\rho, \phi, \theta)\, \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta \\ &= \frac{\kappa}{M} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} |A\rho \sin\phi \cos\theta + B\rho \sin\phi \sin\theta + C\rho \cos\phi + D| \rho^3 \sin^2\phi \cos\theta \, d\rho \, d\phi \, d\theta \\ \bar{y} &= \frac{k}{M} \iiint_R y d(\rho, \phi, \theta)\ dV = \frac{k}{M} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} d(\rho, \phi, \theta)\, \rho^3 \sin^2\phi \sin\theta \, d\rho \, d\phi \, d\theta \\ &= \frac{\kappa}{M} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} |A\rho \sin\phi \cos\theta + B\rho \sin\phi \sin\theta + C\rho \cos\phi + D| \rho^3 \sin^2\phi \sin\theta \, d\rho \, d\phi \, d\theta \end{aligned}$$

$$\begin{aligned} \bar{z} &= \frac{k}{M} \iiint_R z d(\rho, \phi, \theta)\ dV = \frac{k}{M} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} d(\rho, \phi, \theta)\, \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta \\ &= \frac{\kappa}{M} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} |A\rho \sin\phi \cos\theta + B\rho \sin\phi \sin\theta + C\rho \cos\phi + D| \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

where again R is the radius of the sphere, $\kappa = k/\sqrt{A^2 + B^2 + C^2}$, k is the constant of proportionality and $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

(c) The moment of inertia about the x-axis of a solid filling the region $0 \leq z \leq 1-x^2-y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 1-x$ and having density proportional to xy is, using (4.55), given by

$$I_x = \iiint_R \left(y^2+z^2\right) f(x,y,z)\, dx\, dy\, dz = k \int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} xy\left(y^2+z^2\right)\, dz\, dy\, dx$$

where k is the constant of proportionality.

7. The moment of inertia of a solid about an arbitrary line L is defined as

$$I_L = \iiint\limits_R d^2 f(x,y,z)\, dx\, dy\, dz$$

where f is density and d is the distance from a general point (x, y, z) of the solid to the line L. Next, let $\bar{L}$ be a line parallel to L that coincides with the z-axis, such that the center of mass is located at the origin. Since $\bar{L}$ and L are parallel, we can define the distance between $\bar{L}$ and L by the constant h. We may assume, without loss of generality, that in a Cartesian coordinate system the distance between the lines L and $\bar{L}$ lies along the x-axis. Hence, the square of the distance from a general point $(x, y, z$ of the solid to the line L may be written as $d^2 = (x+h)^2 + y^2$. Substituting for this in the equation for I_L then gives

$$\begin{aligned}
I_L &= \iiint\limits_R \left[(x+h)^2 + y^2\right] f(x,y,z)\, dx\, dy\, dz \\
&= \iiint\limits_R \left(x^2+y^2+h^2+2xh\right) f(x,y,z)\, dx\, dy\, dz \\
&= \underbrace{\iiint\limits_R \left(x^2+y^2\right) f(x,y,z)\, dx\, dy\, dz}_{(4.55)} + h^2 \underbrace{\iiint\limits_R f(x,y,z)\, dx\, dy\, dz}_{(4.52)} \\
&\quad + \underbrace{2h \iiint\limits_R x f(x,y,z)\, dx\, dy\, dz}_{(4.54)} = I_{\bar{L}} + Mh^2 + 2Mh\bar{x} = I_{\bar{L}} + Mh^2
\end{aligned}$$

Note that the term $2Mh\bar{x} = 0$, since this is a multiple of the x-coordinate of the center of mass, which is located at the origin.

8. Let L be a line through the origin O with directions cosines l, m, n. Then we can define the vector $\mathbf{n} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ as a unit vector parallel to the line L. Furthermore, we can also define the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as the position vector of a point in space. The shortest distance from an arbitrary point in space P to the line L is then

given by the length of the perpendicular line from P to the line L, which may be expressed as $d = |\mathbf{r}| \sin\theta$, where $0 \le \theta \le \pi$ is the angle between the vectors $\mathbf{n}$ and $\mathbf{r}$. Since $\mathbf{n}$ is a unit vector (i.e. a vector with magnitude 1), we note that by (1.16) $|\mathbf{r}| \sin\theta = |\mathbf{r}||\mathbf{n}| \sin\theta = |\mathbf{r} \times \mathbf{n}|$. The moment of inertia of a solid about the line L can then be defined as

$$\begin{aligned}
I_L &= \iiint\limits_R d^2 f(x,y,z)\; dx\, dy\, dz = \iiint\limits_R |\mathbf{r} \times \mathbf{n}|^2 f(x,y,z)\; dx\, dy\, dz \\
&= \iiint\limits_R \underbrace{\left[(ny-mz)^2 + (lz-nx)^2 + (mx-ly)^2\right]}_{(1.20)}\; dx\, dy\, dz \\
&= l^2 \iiint\limits_R \left(y^2+z^2\right)\, dx\, dy\, dz + m^2 \iiint\limits_R \left(x^2+z^2\right)\, dx\, dy\, dz + n^2 \iiint\limits_R \left(x^2+y^2\right)\, dx\, dy\, dz \\
&\quad - 2lm \iiint\limits_R xy\, dx\, dy\, dz - 2mn \iiint\limits_R yz\, dx\, dy\, dz - 2ln \iiint\limits_R xz\, dx\, dy\, dz \\
&= I_x l^2 + I_y m^2 + I_z n^2 - 2I_{xy} lm - 2I_{yz} mn - 2I_{xz} ln
\end{aligned}$$

9. Let us consider a tetrahedron whose base (i.e. one of its sides) is in the xy-plane such that the centroid of its base is located at the origin O and let the three vertices of the base be given by the points (x_1, y_1), (x_2, y_2), (x_3, y_3). As such, the centroid of the triangular base satisfies the equations

$$\frac{x_1 + x_2 + x_3}{3} = 0 \qquad\qquad \frac{y_1 + y_2 + y_3}{3} = 0$$

Next, the z-coordinate of the only vertex that is *not* in the xy-plane can be defined as the constant $z = h$. Then the coordinates of the tetrahedron are given by $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, $(x_3, y_3, 0)$, (x_4, y_4, h). Next, let us consider the line L passing through the origin and the point (x_4, y_4, h). The centroid of the tetrahedron thus is given by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4}{4} \qquad \bar{y} = \frac{y_1 + y_2 + y_3 + y_4}{4} \qquad \bar{z} = \frac{z_1 + z_2 + z_3 + z_4}{4}$$

Using the fact that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = z_1 + z_2 + z_3 = 0$ the three equations above reduce to $\bar{x} = x_4/4$, $\bar{y} = y_4/4$, $\bar{z} = h/4$. Denoting the vertex not in the xy-plane by $P = (x_4, y_4, h)$ and the centroid of the tetrahedron by $G = (1/4)(x_4, y_4, h)$ we then find that

$$P - G = (x_4, y_4, h) - \frac{1}{4}(x_4, y_4, h) = \frac{3}{4}(x_4, y_4, h) = \frac{3}{4}(P - O)$$

10. (a) Let $\mathbf{F}(t) = t^2\mathbf{i} - e^t\mathbf{j} + \mathbf{k}/(1+t)$. Then

$$\int_0^1 \mathbf{F}(t)\, dt = \int_0^1 t^2\, dt\mathbf{i} - \int_0^1 e^t\, dt\mathbf{j} + \int_0^1 \frac{dt}{1+t}\mathbf{k} = \left.\frac{t^3}{3}\right|_0^1 \mathbf{i} - \left.e^t\right|_0^1\mathbf{j} + \ln(1+t)\Big|_0^1\mathbf{k}$$

$$= \frac{1}{3}\mathbf{i} + (1-e)\,\mathbf{j} + \ln 2\mathbf{k}$$

(b) Let R be the triangular region enclosed by the triangle of vertices $(0,0)$, $(1,0)$, $(0,1)$ and $\mathbf{F}(x,y) = x^2y\mathbf{i} + xy^2\mathbf{j}$. Then

$$\begin{aligned}
\iint_R \mathbf{F}(x,y)\, dA &= \int_0^1\int_0^{1-x} x^2y\, dy\, dx\mathbf{i} + \int_0^1\int_0^{1-x} xy^2\, dy\, dx\mathbf{j} \\
&= \frac{1}{2}\int_0^1 \left.x^2y^2\right|_0^{1-x} dx\mathbf{i} + \frac{1}{3}\int_0^1 \left.xy^3\right|_0^{1-x} dx\mathbf{j} \\
&= \frac{1}{2}\int_0^1 x^2(1-x)^2\, dx\mathbf{i} + \frac{1}{3}\int_0^1 x(1-x)^3\, dx\mathbf{j} \\
&= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right]_0^1\mathbf{i} + \frac{1}{3}\left[\frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}\right]_0^1\mathbf{j} = \frac{1}{60}(\mathbf{i}+\mathbf{j})
\end{aligned}$$

11. Let $\mathbf{F}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be continuous for $a \le t \le b$ and let $\mathbf{q}$ be a constant vector. Then

(a)

$$\begin{aligned}
\int_a^b \mathbf{q}\cdot\mathbf{F}(t)\, dt &= \int_a^b (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k})\cdot[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]\, dt \\
&= \int_a^b [q_xf(t) + q_yg(t) + q_zh(t)]\, dt \\
&= \int_a^b q_xf(t)\, dt + \int_a^b q_yg(t)\, dt + \int_a^b q_zh(t)\, dt \\
&= q_x\int_a^b f(t)\, dt + q_y\int_a^b g(t)\, dt + q_z\int_a^b h(t)\, dt \\
&= (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k})\cdot\underbrace{\left[\int_a^b f(t)\, dt\mathbf{i} + \int_a^b g(t)\, dt\mathbf{j} + \int_a^b h(t)\, dt\mathbf{k}\right]}_{(4.57)} \\
&= \mathbf{q}\cdot\int_a^b \mathbf{F}(t)\, dt
\end{aligned}$$

(b)

$$
\begin{aligned}
\int_a^b \mathbf{q} \times \mathbf{F}(t)\ dt &= \int_a^b (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) \times [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]\ dt \\
&= \int_a^b [(q_y h(t) - q_z g(t))\,\mathbf{i} + (q_z f(t) - q_x h(t))\,\mathbf{j} + (q_x g(t) - q_y f(t))\,\mathbf{k}]\ dt \\
&= \int_a^b [q_y h(t) - q_z g(t)]\ dt\mathbf{i} + \int_a^b [q_z f(t) - q_x h(t)]\ dt\mathbf{j} \\
&\quad + \int_a^b [q_x g(t) - q_y f(t)]\ dt\mathbf{k} \\
&= \left[q_y \int_a^b h(t)\ dt - q_z \int_a^b g(t)\ dt\right]\mathbf{i} + \left[q_z \int_a^b f(t)\ dt - q_x \int_a^b h(t)\ dt\right]\mathbf{j} \\
&\quad + \left[q_x \int_a^b g(t)\ dt - q_y \int_a^b f(t)\ dt\right]\mathbf{k} \\
&= (q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}) \times \left[\int_a^b f(t)\ dt\mathbf{i} + \int_a^b g(t)\ dt\mathbf{j} + \int_a^b h(t)\ dt\mathbf{k}\right] \\
&= \mathbf{q} \times \int_a^b \mathbf{F}(t)\ dt
\end{aligned}
$$

Section 4.6

1. (a) Let $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$. Using the identity $2\cos^2\theta = 1 + \cos 2\theta$, then

$$
\begin{aligned}
\int_0^1 \left(1 - x^2\right)^{3/2}\ dx = \int_0^{\pi/2} \cos^4\theta\, d\theta &= \frac{1}{4}\int_0^{\pi/2} (1 + \cos 2\theta)^2\ d\theta \\
&= \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \cos^2\theta\right)\ d\theta \\
&= \frac{1}{8}\int_0^{\pi/2} (3 + 4\cos 2\theta + \cos 4\theta)\ d\theta \\
&= \frac{1}{32}\left[12\theta + 8\sin 2\theta + \sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{16}
\end{aligned}
$$

(b) Let $x = u^2 - 1$ so that $dx = 2u\, du$ and $v = 1 + u$ so that $dv = du$. Then

$$
\begin{aligned}
\int_0^1 \frac{dx}{1 + \sqrt{1+x}} = 2\int_1^{\sqrt{2}} \frac{u}{1+u}\, du = 2\int_2^{1+\sqrt{2}} \frac{v-1}{v}\, dv &= 2\left[v - \ln v\right]_2^{1+\sqrt{2}} \\
&= 2\sqrt{2} - 2 + 2\ln\left(2\sqrt{2} - 2\right)
\end{aligned}
$$

(c) Let $t = \tan(x/2)$ so that $dt = (1/2)\sec^2 x\, dx$ and $\sqrt{2}u = t + 1$ so that $\sqrt{2}du = dt$.

Then

$$\begin{aligned}
\int_0^{\pi/2} \frac{dx}{\sin x + \cos x + 2} &= \int_0^1 \frac{2\cos^2(x/2)}{\sin x + \cos x + 2}\, dt \\
&= \int_0^1 \frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2) + 2\cos^2(x/2) + 1}\, dt \\
&= \int_0^1 \frac{2\cos^2(x/2)}{2\tan(x/2)\cos^2(x/2) + 2\cos^2(x/2) + 1}\, dt \\
&= \int_0^1 \frac{2\cos^2(x/2)}{2\tan(x/2)\cos^2(x/2) + 3\cos^2(x/2) + \sin^2(x/2)}\, dt \\
&= \int_0^1 \frac{2}{\tan^2(x/2) + 2\tan(x/2) + 3}\, dt = 2\int_0^1 \frac{dt}{t^2 + 2t + 3} \\
&= 2\int_0^1 \frac{dt}{(t+1)^2 + 2} = \sqrt{2}\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{du}{u^2+1} = \sqrt{2}\tan^{-1} u\Big|_{\sqrt{2}/2}^{\sqrt{2}} \\
&= \sqrt{2}\left(\tan^{-1}\sqrt{2} - \tan^{-1}\frac{\sqrt{2}}{2}\right)
\end{aligned}$$

(d) Let $t = 1 + x\cos x$ so that $dt = (\cos x - x\sin x)\, dx$. Then

$$\int_0^{\pi/4} \frac{x\cos x\,(x\sin x - \cos x)}{1 + x\cos x}\, dx = \int_1^{1+c} \left(\frac{1}{t} - 1\right) dt = [\ln t - t]_1^{1+c} = \ln(1+c) - c$$

where $c = \pi/(4\sqrt{2})$.

2. Let $\phi(u)$ be a function of u. Next, we consider the integral

$$\int_{u_1}^{u_2} \phi'(u)\, du = \int_{u_1}^{u_2} \frac{d\phi}{du}\, du = \int_{u_1}^{u_2} d\phi = \phi(u_2) - \phi(u_1)$$

which is equal to (4.60) when setting $f(x) = f[x(u)] \equiv 1$.

3. (a) Let $f(x)$ be continuous for $x_1 \le x \le x_2$. Next, let us choose a u_0 and $x_0 = x(u_0)$ such that $u_1 < u_0 < u_2$. Replacing u_2 for u_0 in (4.60) then gives

$$\int_{x_1}^{x_0} f(x)\, dx = \int_{u_1}^{u_0} f[x(u)]\, \frac{dx}{du}\, du$$

which remains to hold. Then if we let $u_0 \to u_2$ we get

$$\begin{aligned}
\lim_{u_0 \to u_2} \int_{u_1}^{u_0} f[x(u)]\, \frac{dx}{du}\, du = \lim_{u_0 \to u_2} \left(F[x(u_0)] - F[x(u_1)]\right) &= F[x(u_2)] - F[x(u_1)] \\
&= F(x_2) - F(x_1)
\end{aligned}$$

which is the same as the value of the left-hand side of (4.60). Hence, if the right-hand side of (4.60) has a limit, then the other side has a limit also and the two limits are equal.

(b) Let $u = 1/x$ so that $du = -dx/x^2$. Then

$$\int_1^\infty \frac{1}{x^2}\sinh\frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\sinh\frac{1}{x}\,dx = \int_0^1 \sinh u\,du = \cosh u\big|_0^1 = \cosh 1 - 1$$

(c) Let $u = \tanh x$ so that $du = (1-\tanh^2 x)\,dx$. Then

$$\int_0^\infty (1-\tanh x)\;dx = \lim_{b\to\infty}\int_0^b (1-\tanh x)\;dx = \int_0^1 \frac{du}{1+u} = \ln(1+u)\big|_0^1 = \ln 2$$

4. (a) Let R_{xy} be the region $x^2+y^2 \le 1$ and $x = r\cos\theta$, $y = r\sin\theta$. Then by (4.64) we find

$$\begin{aligned}\iint_{R_{xy}} \left(1-x^2-y^2\right)\,dx\,dy &= \int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(1-x^2-y^2\right)\,dy\,dx \\ &= \int_0^1\int_0^{2\pi}\left(1-r^2\right)r\,d\theta\,dr = \int_0^1 \left(1-r^2\right)r\theta\big|_0^{2\pi}\,dr \\ &= 2\pi\int_0^1 \left(1-r^2\right)r\,dr = \pi\left[r^2-\frac{r^4}{2}\right]_0^1 = \frac{\pi}{2}\end{aligned}$$

(b) Let R be the region $1 \le x \le 2$, $0 \le y \le x$ and $x = r\cos\theta$, $y = r\sin\theta$. Furthermore, let $u = \sec^3\theta$ so that $du = 3\sec^3\theta\tan\theta\,d\theta$. Then by (4.64) we find

$$\begin{aligned}\iint_R \frac{y\sqrt{x^2+y^2}}{x}\,dx\,dy &= \int_1^2\int_0^x \frac{y\sqrt{x^2+y^2}}{x}\,dy\,dx = \int_0^{\pi/4}\int_{\sec\theta}^{2\sec\theta} r^2\tan\theta\,dr\,d\theta \\ &= \frac{1}{3}\int_0^{\pi/4} r^3\tan\theta\big|_{\sec\theta}^{2\sec\theta}\,d\theta = \frac{7}{9}\int_0^{\pi/4}\sec^3\theta\tan\theta\,d\theta \\ &= \frac{7}{9}\int_1^{2\sqrt{2}} du = \frac{7}{9}u\big|_1^{2\sqrt{2}} = \frac{14\sqrt{2}-7}{9}\end{aligned}$$

(c) Let R_{xy} be the parallelogram with successive vertices $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$, $(0,\pi)$ and $u = x-y$, $v = x+y$ so that R_{uv} is the square region with vertices (π,π), $(\pi,3\pi)$, $(-\pi,3\pi)$, $(-\pi,\pi)$. Then by (4.61)

$$\begin{aligned}\iint_{R_{xy}} (x-y)^2\sin^2(x+y)\;dx\,dy &= \int_0^{2\pi}\int_{|\pi-x|}^{2\pi-|\pi-x|} (x-y)^2\sin^2(x+y)\;dy\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2\sin^2 v\,dv\,du \\ &= \frac{1}{4}\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2\left(1-\cos 2v\right)\;dv\,du \\ &= \frac{1}{8}\int_{-\pi}^{\pi} u^2\left[2v-\sin 2v\right]_{\pi}^{3\pi}\;du = \frac{\pi}{2}\int_{-\pi}^{\pi} u^2\,du \\ &= \frac{\pi u^3}{6}\bigg|_{-\pi}^{\pi} = \frac{\pi^4}{3}\end{aligned}$$

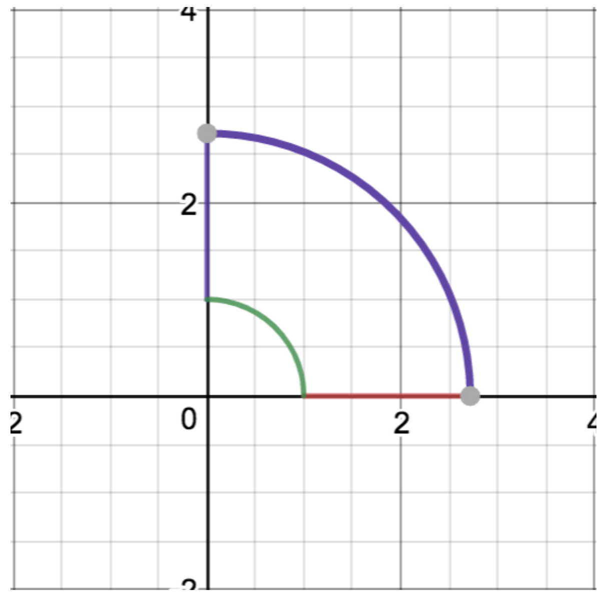
(d) Let R be the trapezoidal region bounded by the lines $x+y=1$, $x+y=2$ in the first quadrant and $u=1+x+y$, $v=x-y$. Then by (4.61)

$$
\begin{aligned}
\iint_R \frac{(x-y)^2}{1+x+y}\,dx\,dy &= \int_0^1 \int_{1-x}^{2-x} \frac{(x-y)^2}{1+x+y}\,dy\,dx + \int_1^2 \int_0^{2-x} \frac{(x-y)^2}{1+x+y}\,dy\,dx \\
&= \frac{1}{2}\int_2^3 \int_{1-u}^{-1+u} \frac{v^2}{u}\,dv\,du = \frac{1}{6}\int_2^3 \left.\frac{v^3}{u}\right|_{1-u}^{-1+u} du \\
&= \frac{1}{3}\int_2^3 \left(u^2-3u+3-u^{-1}\right)\,du = \frac{1}{3}\left[\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right]_2^3 \\
&= \frac{11}{18}+\frac{1}{3}\ln\frac{2}{3}
\end{aligned}
$$

(e) Let R be the region bounded by the ellipse $5x^2+2xy+2y^2=1$ and $x=u+v$, $y=-2u+v$. Then by (4.61)

$$
\begin{aligned}
\iint_R \sqrt{5x^2+2xy+2y^2}\,dx\,dy &= \int_{-\sqrt{2}/3}^{\sqrt{2}/3} \int_{-\left(\sqrt{2-9x^2}-x\right)/2}^{\left(\sqrt{2-9x^2}-x\right)/2} \sqrt{5x^2+2xy+2y^2}\,dy\,dx \\
&= 9\int_{-1/3}^{1/3} \int_{-\sqrt{(1/9)-u^2}}^{\sqrt{(1/9)-u^2}} \sqrt{u^2+v^2}\,dv\,du \\
&= 9\int_0^{2\pi}\int_0^{1/3} r^2\,dr\,d\theta = 3\int_0^{2\pi} r^3\Big|_0^{1/3}\,d\theta \\
&= \frac{3}{27}\int_0^{2\pi} d\theta = \frac{2\pi}{9}
\end{aligned}
$$

5. According to Section 2.7, the transformation $u=e^x\cos y$, $v=e^x\sin y$ can be regarded as a mapping from the xy-plane to the uv-plane. Since the inverse mapping exists and is given by $x=(1/2)\ln(u^2+v^2)$, $y=\tan^{-1}(v/u)$, hence, taking each point of a region in the uv-plane to a unique point in the xy-plane, the mapping is one-to-one. Now let us consider the rectangle $R_{xy}:\ 0\le x\le 1,\ 0\le y\le \pi/2$. Then u and v are defined and continuous for each point $(x,y)\in R_{xy}$ (i.e. their derivatives exist). Hence, the given transformation defines a one-to-one mapping of the rectangle R_{xy} onto a region of the uv-plane. To find the boundaries of the region in the uv-plane we consider each of the sides of the rectangle R_{xy} in turn. For the bottom side of the rectangle given by $0\le x\le 1$, $y=0$ we find $1\le u\le e$, $v=0$. For the top side of the rectangle given by $0\le x\le 1$, $y=\pi/2$ we find $u=0$, $1\le v\le e$. The left side of the rectangle given by $x=0$, $0\le y\le \pi/2$ transforms as $u=\cos y$, $v=\sin y$, whereas the right side of the rectangle given by $x=1$, $0\le y\le \pi/2$ transforms as $u=e\cos y$, $v=e\sin y$. Hence, the rectangle R_{xy} transforms to the ring R_{uv} as shown in the figure below



As such, by (4.61) we find that

$$\iint\limits_{R_{xy}} \frac{e^{2x}}{1+e^{4x}\cos^2 y \sin^2 y}\,dx\,dy = \int_0^{\pi/2}\int_0^1 \frac{e^{2x}}{1+e^{4x}\cos^2 y\sin^2 y}\,dx\,dy$$

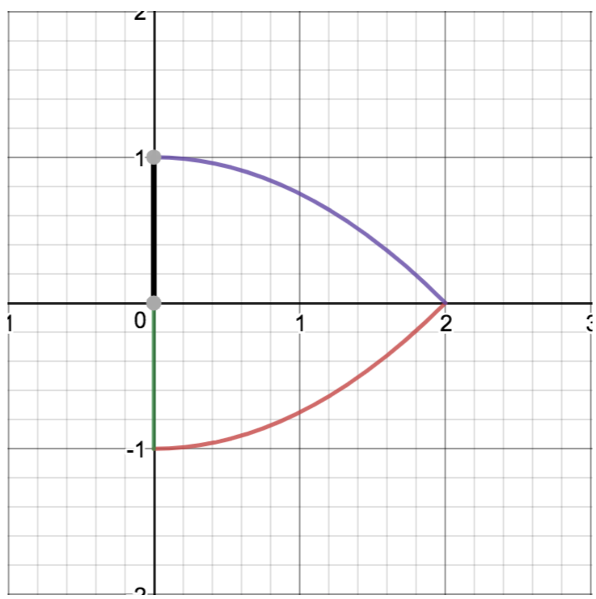
$$= \int_0^1 \int_{\sqrt{1-u^2}}^{\sqrt{e^2-u^2}} \frac{dv\,du}{1+u^2v^2} + \int_1^e \int_0^{\sqrt{e^2-u^2}} \frac{dv\,du}{1+u^2v^2}$$

$$= \iint\limits_{R_{uv}} \frac{du\,dv}{1+u^2v^2}$$

6. The inverse mapping is given by

$$x = \pm\sqrt{(1/2)(v \pm \sqrt{u^2+v^2})} \qquad\qquad y = \pm\sqrt{(1/2)(-v \pm \sqrt{u^2+v^2})}$$

Since the inverse mapping is well-defined, the mapping from the xy-plane to the $uv-$plane is one-to-one. Now we consider the square region $0 \le x \le 1$, $0 \le y \le 1$. Then u and v are defined and continuous for each point (x, y) of the square. Hence, the given transformation defines a one-to-one mapping of the square $0 \le x \le 1$, $0 \le y \le 1$ onto a region of the uv-plane. To find the boundaries of the region in the uv-plane we consider each of the sides of the square in the xy-plane in turn. For the bottom side of the square given by $0 \le x \le 1$, $y = 0$ we find $u = 0$, $v = x^2$. Hence, the bottom side of the square in the xy-plane maps to the vertical line segment with endpoints $(0, 0)$, $(0, 1)$ in the uv-plane. For the top side of the square given by $0 \le x \le 1$, $y = 1$ we find $u = 2x$, $v = x^2 - 1$. Treating this as a parametric equation in the variable t this becomes $u = 2t$, $v = t^2 - 1$ where $0 \le t \le 1$. Then solving for v gives $v = (1/4)u^2 - 1$, $0 \le u \le 2$. Hence, the top side of the square in the xy-plane maps to the line segment given by the aforementioned equation with endpoints $(0, -1)$, $(2, 0)$ in the uv-plane. For the left side of the square given by $x = 0$, $0 \leq y \leq 1$ we find $u = 0$, $v = -y^2$. Hence, the left side of the square in the xy-plane maps to the vertical line segment with endpoints $(0, 0)$, $(0, -1)$ in the uv-plane. Lastly, for the right side of the square given by $x = 1$, $0 \leq y \leq 1$ we find $u = 2y$, $v = 1 - y^2$. Treating this as a parametric equation in t this becomes $u = 2t$, $v = 1 - t^2$ where $0 \leq t \leq 1$. Solving for v then gives $v = 1 - (1/4)u^2$, $0 \leq u \leq 2$. Hence, the right side of the square in the xy-plane maps to the line segment given by the aforementioned equation, having endpoints $(0, 1)$, $(2, 0)$ in the uv-plane. To summarize, the figure below shows the transformed region in the uv-plane.



As such, by (4.61) we find that

$$\iint\limits_{R_{xy}} \sqrt[3]{x^4 - 6x^2y^2 + y^4}\, dx\, dy = \int_0^1 \int_0^1 \sqrt[3]{x^4 - 6x^2y^2 + y^4}\, dx\, dy$$

$$= \frac{1}{4} \int_0^2 \int_{-1+(1/4)u^2}^{1-(1/4)u^2} \frac{\sqrt[3]{v^2 - u^2}}{\sqrt{v^2 + u^2}}\, dv\, du = \frac{1}{4} \iint\limits_{R_{uv}} \frac{\sqrt[3]{v^2 - u^2}}{\sqrt{v^2 + u^2}}\, du\, dv$$

7. (a) Let R_{xy} be the triangular region bounded by the lines $y = x$, $0 \leq x \leq 1$ and $x = 1$, $0 \leq y \leq 1$ in the first quadrant of the xy-plane. And let the transformation $x = u + v$, $y = u - v$ be given. Then the inverse transformation is given by

$$u = \frac{1}{2}(x + y) \qquad\qquad v = \frac{1}{2}(x - y)$$

so that it defines a one-to-one mapping of the triangle R_{xy} onto the triangle R_{uv}, which boundaries are given by the two lines $v = u$, $0 \leq u \leq 1/2$ and $v = 1 - u$, $1/2 \leq u \leq 1$ in the first quadrant of the uv-plane. As such, by (4.61) we find

$$\iint\limits_{Rxy} \ln\left(1+x^2+y^2\right)\,dx\,dy = \int_0^1\int_0^x \ln\left(1+x^2+y^2\right)\,dy\,dx$$

$$= 2\int_0^{1/2}\int_v^{1-v} \ln\left(1+2u^2+2v^2\right)\,du\,dv$$

$$= 2\iint\limits_{R_{uv}} \ln\left(1+2u^2+2v^2\right)\,du\,dv$$

(b) Let R_{xy} be the triangular region bounded by the lines $y = 1-x$, $y = 1+x$, $0 \le x \le 1$ and $x = 1$, $0 \le y \le 2$ in the first quadrant of the xy-plane. And let the transformation $x = u$, $y = u+v$ be given. Then the inverse transformation is given by

$$u = x \qquad\qquad v = y - x$$

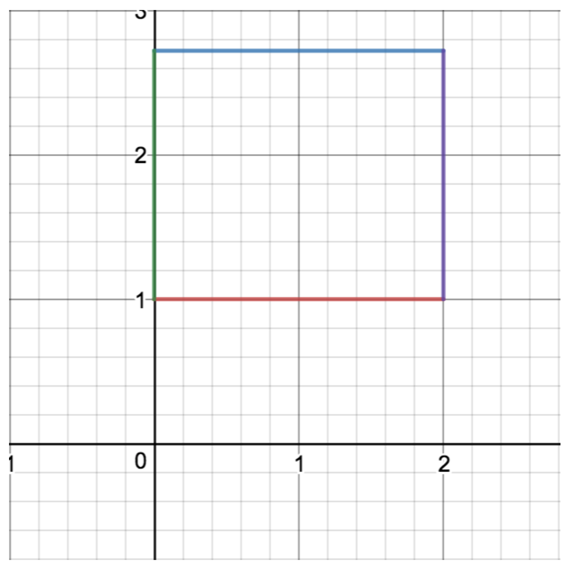
so that it defines a one-to-one mapping of the triangle R_{xy} onto the triangle R_{uv}, which boundaries are given by the lines $v = 1-2u$, $0 \le u \le 1$, $v = 1+u$, $0 \le u \le 1$ and $u = 1$, $-1 \le v \le 1$ in the first and fourth quadrants of the uv-plane. As such, by (4.61) we find

$$\iint\limits_{R_{xy}} \sqrt{1+x^2y^2}\,dx\,dy = \int_0^1\int_{1-x}^{1+x} \sqrt{1+x^2y^2}\,dy\,dx = \int_0^1\int_{1-2u}^{1} \sqrt{1+u^2\left(u+v\right)^2}\,dv\,du$$

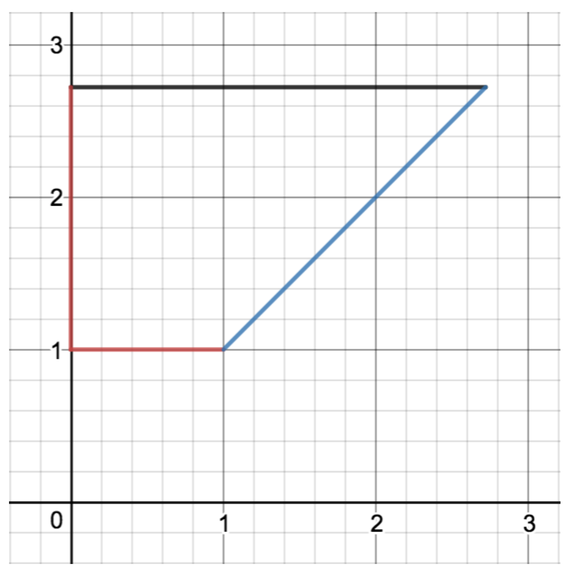
$$= \iint\limits_{R_{uv}} \sqrt{1+u^2\left(u+v\right)^2}\,du\,dv$$

8. Let R_{uv} be the square $0 \le u \le 1$, $0 \le v \le 1$.

(a) Let the transformation $x = u + u^2$, $y = e^v$ be given. It then defines a one-to-one mapping of the square R_{uv} onto the rectangle R_{xy}. The bottom side of the square given by $0 \le u \le 1$, $v = 0$ maps to the line segment $y = 1$, $0 \le x \le 2$ forming the bottom side of the rectangle. The top side of the square given by $0 \le u \le 1$, $v = 1$ maps to the line segment $y = e$, $0 \le x \le 2$ forming the top side of the rectangle. The left side of the square given by $u = 0$, $0 \le v \le 1$ maps to the line segment $x = 0$, $1 \le y \le e$ forming the left side of the rectangle. Finally, the right side of the square given by $u = 1$, $0 \le v \le 1$ maps to the line segment $x = 2$, $1 \le y \le e$ forming the right side of the rectangle. The below figure shows the graph for the rectangle R_{xy}.

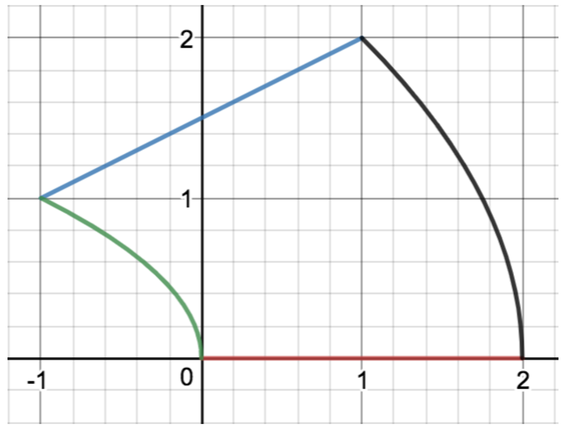


(b) Let the transformation $x = ue^v$, $y = e^v$ be given. The bottom side of the square given by $0 \le u \le 1$, $v = 0$ maps to the line segment $0 \le x \le 1$, $y = 1$. The top side of the square given by $0 \le u \le 1$, $v = 1$ maps to the line segment $0 \le x \le e$, $y = e$. The left side of the square given by $u = 0$, $0 \le v \le 1$ maps to the line segment $x = 0$, $1 \le y \le e$. Finally, the right side of the square given by $u = 1$, $0 \le v \le 1$ maps to the line segment $x = y$, $1 \le x \le e$. The below figure shows the graph for the region R_{xy} in the xy-plane.

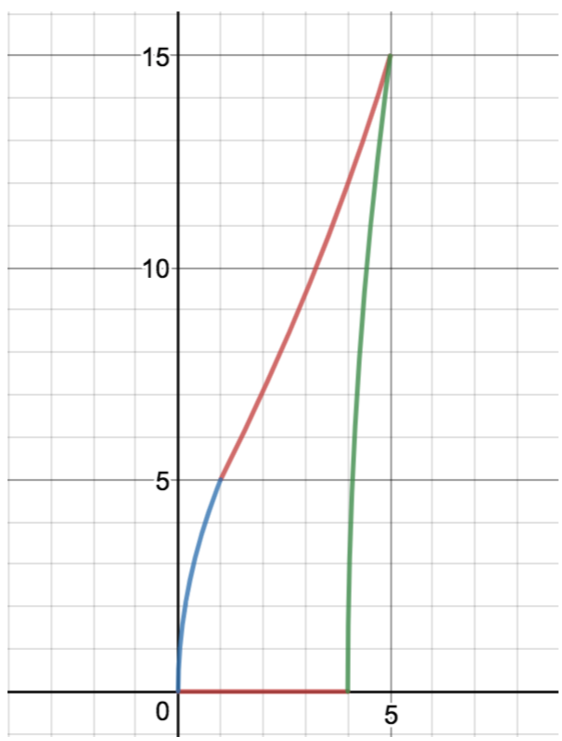


(c) Let the transformation $x = 2u - v^2$, $y = v + uv$ be given. The bottom side of the square maps to the line segment $0 \le x \le 2$, $y = 0$. The top side of the square maps to the line segment $y = (1/2)(3 + x)$, $-1 \le x \le 1$. The left side of the square maps to the line segment $x = -y^2$, $0 \le y \le 1$. Finally, the right side of the square maps to the line segment $x = (1/4)(8 - y^2)$, $1 \le$

$x \leq 2$. The below figure shows the graph for the region R_{xy} in the xy-plane.



(d) Let the transformation $x = 5u - u^2 + v^2$, $y = 5v + 10uv$ be given. The bottom side of the square maps to the line segment $0 \leq x \leq 4$, $y = 0$. The top side of the square maps to the line segment $x = -(1/100)y^2 + (3/5)y - (7/4)$, $5 \leq y \leq 15$. The left side of the square maps to the line segment $x = (1/25)y^2$, $0 \leq y \leq 5$. Finally, the right side of the square maps to the line segment $x = 4 + (1/225)y^2$, $0 \leq y \leq 15$. The below figure shows the graph for the given region R_{xy} in the xy-plane.



9. Cylindrical coordinates are defined by the transformation $x = r\cos\theta$, $y = r\sin\theta$, $z =$

z. Hence, we find

$$J = \left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

And so by (4.66)

$$\iiint\limits_{R_{xyz}} f(x,y,z)\;dx\,dy\,dz = \iiint\limits_{R_{r\theta z}} F(r,\theta,z)\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz = \iiint\limits_{R_{r\theta z}} F(r,\theta,z)\,r\,dr\,d\theta\,dz$$

The element of volume is approximately a rectangular box with sides $r\Delta\theta$, Δr and Δz: $\Delta V \sim r\Delta\theta\Delta r\Delta z$.

10. Spherical coordinates are defined by the transformation $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Hence, we find

$$J = \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi$$

And so by (4.66)

$$\begin{aligned}\iiint\limits_{R_{xyz}} f(x,y,z)\;dx\,dy\,dz &= \iiint\limits_{R_{\rho\phi\theta}} F(\rho,\phi,\theta)\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| d\rho\,d\phi\,d\theta \\ &= \iiint\limits_{R_{\rho\phi\theta}} F(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\end{aligned}$$

The element of volume is approximately a rectangular box with sides $\rho\Delta\phi$, $\Delta\rho$ and $\rho\sin\phi\Delta\theta$: $V \sim \rho^2\sin\phi\Delta\phi\Delta\rho\Delta\theta$.

11. Let cylindrical coordinates be given: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

(a)

$$\begin{aligned}\iiint\limits_{R_{xyz}} x^2y\,dx\,dy\,dz = \int_0^1\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2y\,dx\,dy\,dz &= \int_0^1\int_0^{2\pi}\int_0^1 r^4\cos^2\theta\sin\theta\,dr\,d\theta\,dz \\ &= \iiint\limits_{R_{r\theta z}} r^4\cos^2\theta\sin\theta\,dr\,d\theta\,dz\end{aligned}$$

where R_{xyz} is the region $x^2+y^2 \le 1$, $0 \le z \le 1$.

(b)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1+x+y} \left(x^2 - y^2\right)\, dz\, dy\, dx = \int_0^{\pi/2} \int_0^1 \int_0^{1+r(\cos\theta+\sin\theta)} r^3 \cos 2\theta\, dz\, dr\, d\theta$$

12. Let spherical coordinates be given by $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

(a)

$$\iiint\limits_{R_{xyz}} x^2 y\, dx\, dy\, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^5 \sin^4\phi \cos^2\theta \sin\theta\, d\rho\, d\phi\, d\theta$$

where R_{xyz} is the spherical region $x^2 + y^2 + z^2 \leq a^2$.

(b) From the bounds $-1 \leq x \leq 1$, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ follows $0 \leq \theta \leq 2\pi$. From the bound $\sqrt{x^2+y^2} \leq z \leq 1$ follows $0 \leq \phi \leq \pi/4$, $0 \leq \rho \leq \sec\phi$. Hence,

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \left(x^2 + y^2 + z^2\right)\, dz\, dy\, dx = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta$$

Section 4.7

1. (a)