

# CHAPTER 2

## Section 2.4

1. Some examples are:

volume of a cylinder :  $\pi r^2 h$

surface area of a cone :  $\pi r (l + r)$

volume of a cuboid :  $lwh$

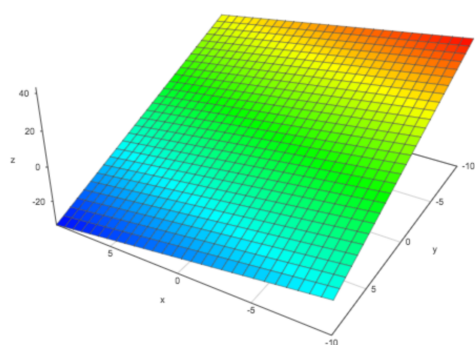


Figure 1:  $z = 3 - x - 3y$

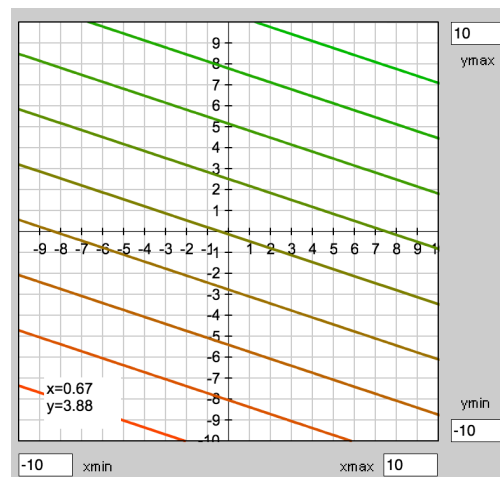


Figure 2:  $z = 3 - x - 3y$

2. (a)

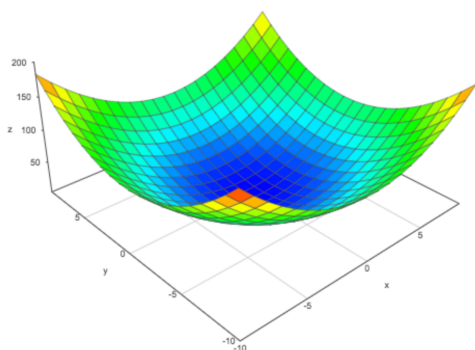


Figure 3:  $z = x^2 + y^2 + 1$

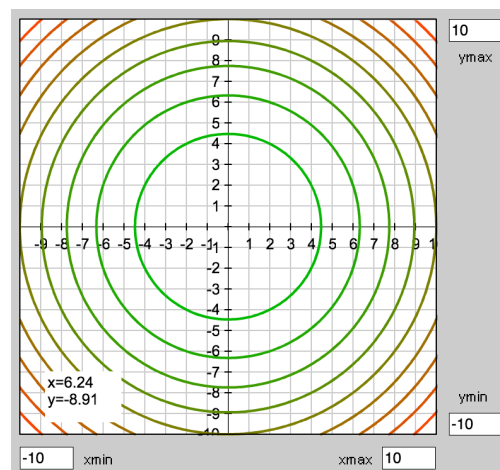


Figure 4:  $z = x^2 + y^2 + 1$

(b)

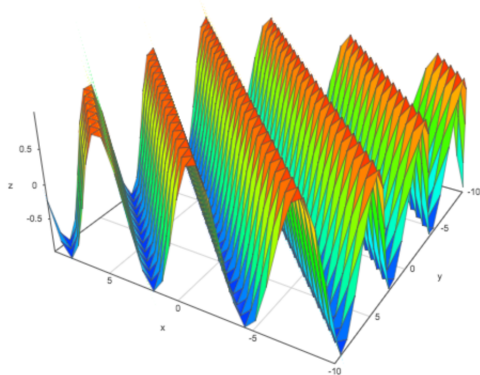


Figure 5:  $z = \sin(x + y)$

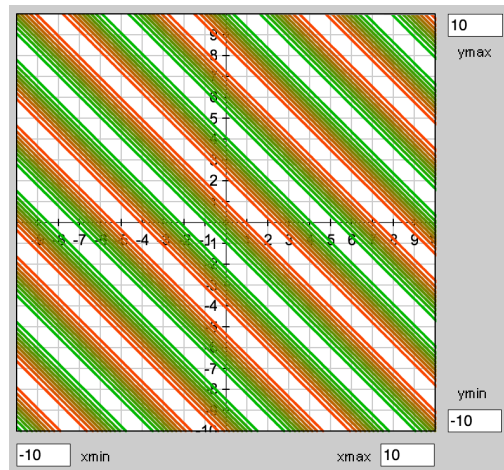


Figure 6:  $z = \sin(x + y)$

(c)

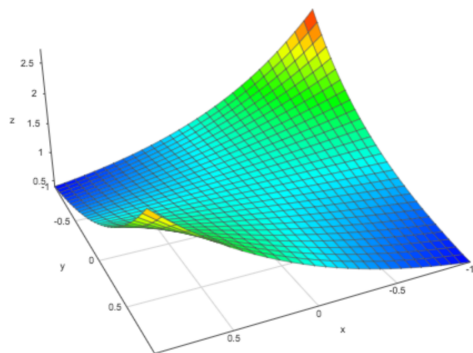


Figure 7:  $z = e^{xy}$

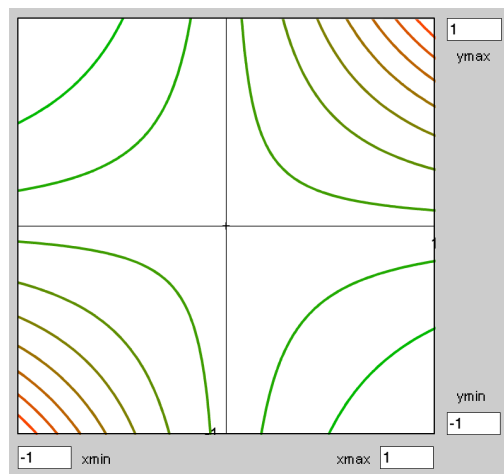


Figure 8:  $z = e^{xy}$

(d)

3. (a) The level surfaces of  $u = x^2 + y^2 + z^2$  are spheres centered at the point  $(x, y, z) = (0, 0, 0)$  and with radius  $r = \sqrt{u}$ .
- (b) The level surfaces of  $u = x + y + z$  are planes, where a particular value for  $u$  denotes the point of intersection of the plane with the  $x$ ,  $y$  and  $z$  axes.
- (c) The level surfaces of  $w = x^2 + y^2 - z$  are hyperbolic paraboloids with the saddle point located at point  $(x, y, z) = (0, 0, -w)$ .
- (d) The level surface of  $w = x^2 + y^2$  is a hyperbolic paraboloid with its saddle point located at at point  $(x, y, z) = (0, 0, 0)$ .

4. (a)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{1 + x^2 + y^2} = \frac{0 + 0}{1 + 0 + 0} = 0$$

(b) Let  $x = y$ . Then the limit becomes

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x}{2x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{1}{2x} = \infty$$

Next, let  $x = 0$ . Then the limit becomes

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{0}{0 + y^2} = \frac{0}{0 + 0} = 0$$

Hence, the limit does not exist.

(c)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(1 + y^2) \sin x}{x} = \left( \lim_{y \rightarrow 0} 1 + y^2 \right) \lim_{x \rightarrow 0} \frac{\sin x}{x} = (1)(1) = 1$$

To show that  $\lim_{x \rightarrow 0} \sin x / x = 1$  we use the sandwich theorem:

$$\sin x \leq x \leq \tan x \quad \rightarrow \quad 1 \leq \frac{x}{\sin x} \leq \frac{\tan x}{\sin x} \quad \rightarrow \quad 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Next, note that

$$\lim_{x \rightarrow 0} \frac{1}{\cos x} = \frac{1}{1} = 1$$

Hence,

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1 \quad \Rightarrow \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

(d)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{1 + x - y}{x^2 + y^2} = \frac{1 + 0 - 0}{0 + 0} = \infty$$

5. (a) Let us consider the limit

$$\lim_{(x,y) \rightarrow (0,0)} z = \lim_{(x,y) \rightarrow (0,0)} \frac{x}{x - y} = \lim_{x \rightarrow 0} \left( \lim_{y \rightarrow 0} \frac{x}{x - y} \right) = \lim_{x \rightarrow 0} \frac{x}{x - 0} = \lim_{x \rightarrow 0} 1 = 1$$

However

$$\lim_{(x,y) \rightarrow (0,0)} z = \lim_{(x,y) \rightarrow (0,0)} \frac{x}{x - y} = \lim_{y \rightarrow 0} \left( \lim_{x \rightarrow 0} \frac{x}{x - y} \right) = \lim_{y \rightarrow 0} \frac{0}{0 - y} = 0$$

Hence, the limit at  $(x, y) = (0, 0)$  does not exist and so the function  $z$  is discontinuous at that point.

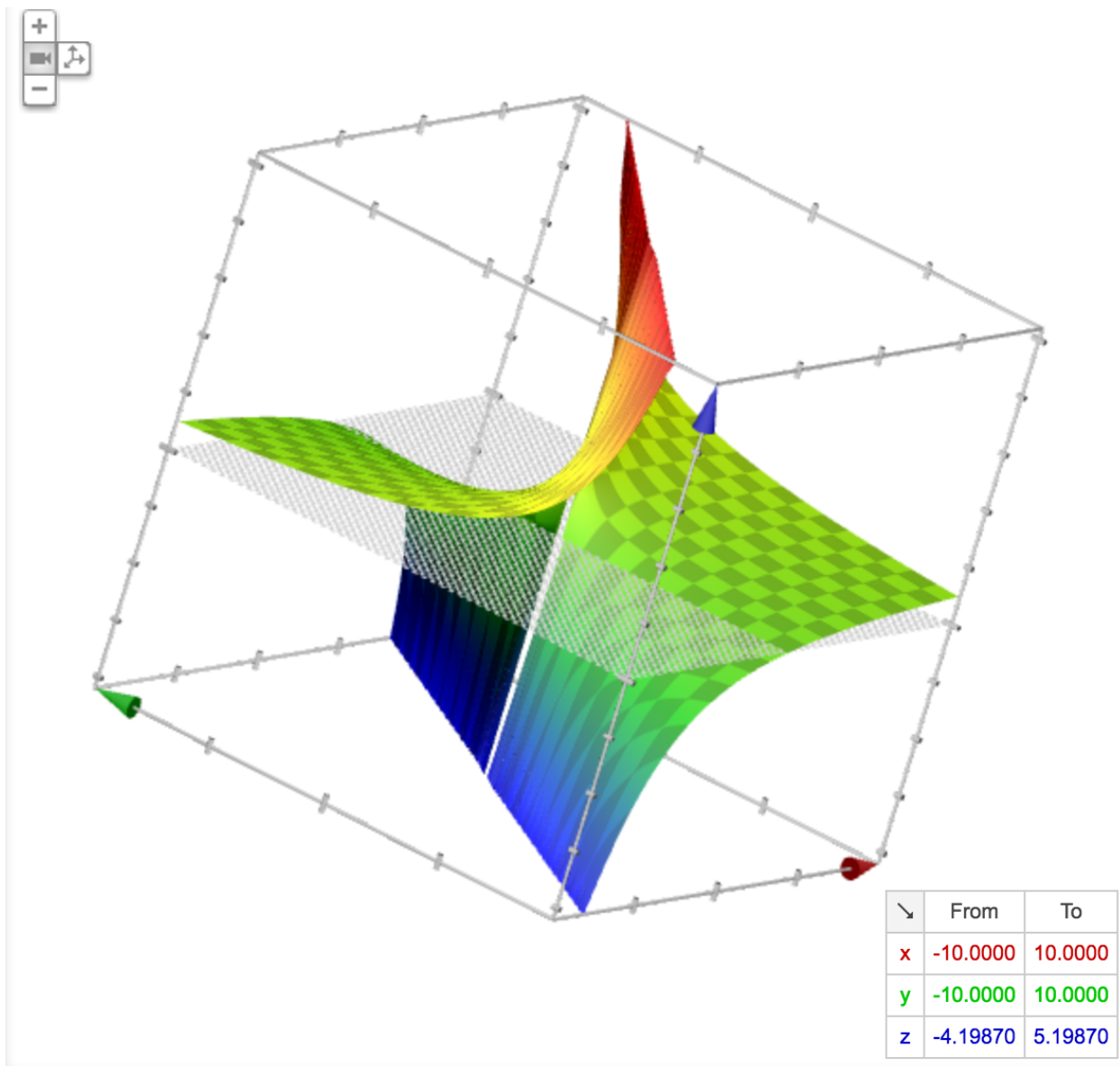


Figure 9:  $z = x/(x - y)$

(b) Let us consider the limit

$$\lim_{(x,y) \rightarrow (0,0)} z = \lim_{(x,y) \rightarrow (0,0)} \ln(x^2 + y^2) = \lim_{x \rightarrow 0} \left( \lim_{y \rightarrow 0} \ln(x^2 + y^2) \right) = \lim_{x \rightarrow 0} \ln x^2 = -\infty$$

However, the point  $(x, y) = (0, 0)$  is not in the domain of  $z$ , as the function is not defined there. Hence, strictly speaking  $z$  is continuous over its domain of definition  $x, y \in (0, \infty)$  and has an infinite discontinuity at the point  $(x, y) = (0, 0)$  as it is not defined there.

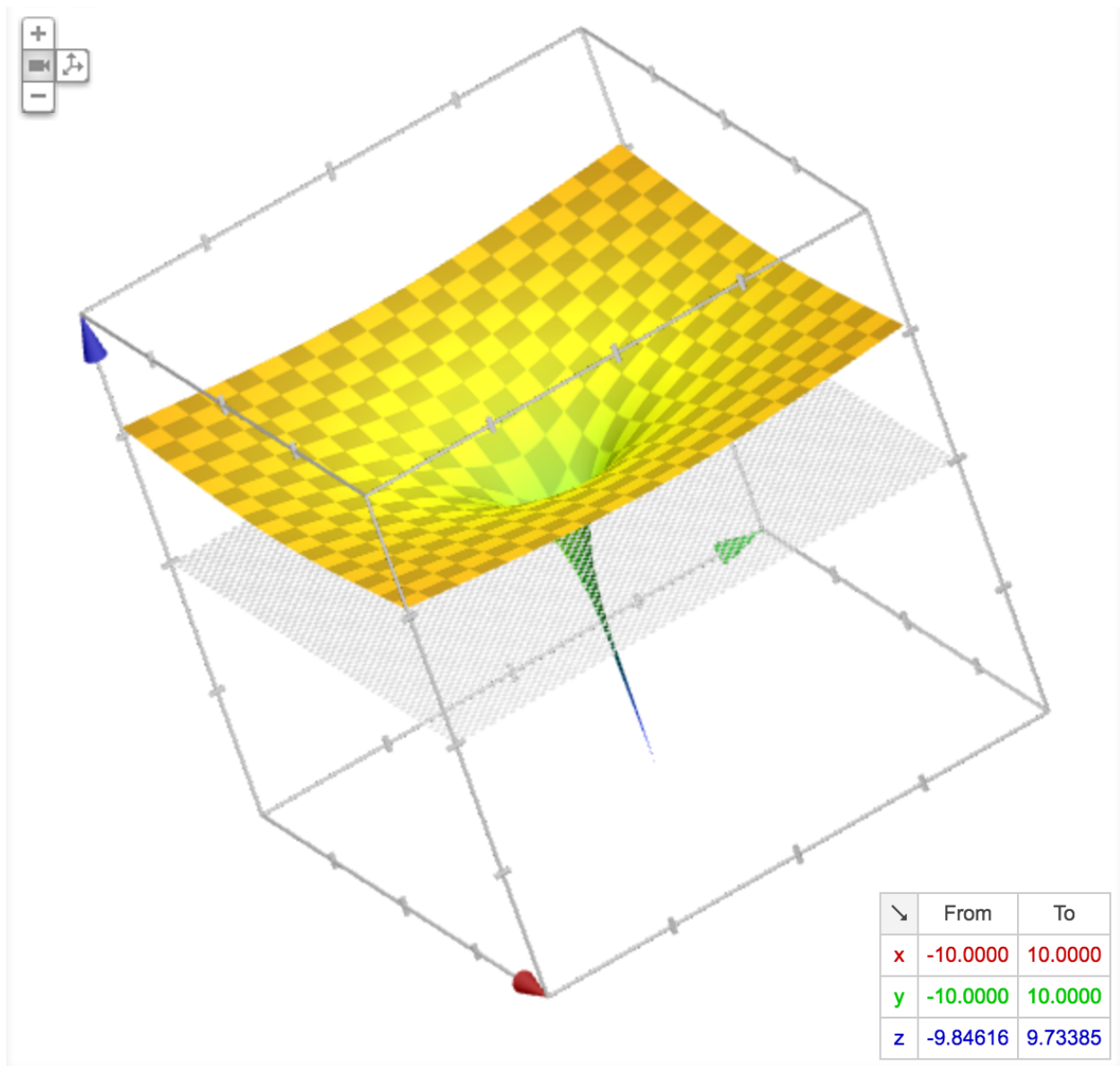


Figure 10:  $z = x/(x - y)$

6. (a) The function  $e^a$  where  $a$  is an arbitrary real valued scalar is defined for any  $a$ , positive or negative. Hence, the domain may be formally defined as  $\{x, y \in \mathbb{R} \mid \infty < x, y < \infty\}$ .
- (b) The domain for the function  $z = \ln(x^2 + y^2 - 1)$  is given by  $\{x, y \in \mathbb{R} \mid x^2 + y^2 > 1\}$ .
- (c) The set in which the function  $z = \sqrt{1 - x^2 - y^2}$  is defined is the closed region  $\{x, y \in \mathbb{R} \mid x^2 + y^2 \leq 1\}$ .
- (d) The set in which the function  $u = xy/z$  is defined is an open set, excluding the points lying in the  $xy$  plane (i.e.  $z = 0$ ). It is not a domain, since not all points

in the open set can be joined by a broken line.

7. Let  $f(x, y)$  be defined in domain  $D$  and continuous at the point  $(x_1, y_1)$  of  $D$ , so that  $\lim_{(x,y) \rightarrow (x_1,y_1)} f(x, y) = c = f(x_1, y_1)$ . Substituting  $\epsilon = (1/2)f(x_1, y_1)$  in (2.3) then gives

$$|f(x, y) - f(x_1, y_1)| < \frac{1}{2}f(x_1, y_1)$$

which is equivalent to stating that there is a neighbourhood of  $(x_1, y_1)$  in which  $f(x, y) > (1/2)f(x_1, y_1) > 0$ . Rewriting the absolute inequality gives

$$\begin{aligned} -\frac{1}{2}f(x_1, y_1) &< f(x, y) - f(x_1, y_1) < \frac{1}{2}f(x_1, y_1) \\ 0 &< f(x, y) - \frac{1}{2}f(x_1, y_1) < f(x_1, y_1) \\ f(x_1, y_1) &> f(x, y) - \frac{1}{2}f(x_1, y_1) > 0 \end{aligned}$$

Focusing on the second inequality we find

$$f(x, y) > \frac{1}{2}f(x_1, y_1)$$

And since the starting assumption was that  $f(x_1, y_1) > 0$  clearly  $(1/2)f(x_1, y_1) > 0$  as well.

8. Suppose that the domain  $D$  could consist of two open sets  $E_1$  and  $E_2$  with no point in common. Next let us choose point  $P$  in  $E_1$  and  $Q$  in  $E_2$  and join them by a broken line in  $D$ . We will regard this line as a path from point  $P$  to  $Q$  and let  $s$  be the distance from  $P$  along the path so that the path is given by continuous functions  $x = x(s)$  and  $y = y(s)$ , where  $0 \leq s \leq L$ , with  $s = 0$  at point  $P$  and  $s = L$  at point  $Q$ . Now consider a function  $f(s)$  and let  $f(s) = -1$  if  $(x(s), y(s))$  is in  $E_1$  and let  $f(s) = 1$  if  $(x(s), y(s))$  is in  $E_2$ . Furthermore, let this function  $f(s)$  be some linear combination of  $x(s)$  and  $y(s)$ , i.e.  $f(s) = ax(s) + by(s)$ , where  $a$  and  $b$  are arbitrary scalars. Now since both  $x(s)$  and  $y(s)$  are continuous for  $0 \leq s \leq L$  then according to (2.7) so will be  $f(s)$ . Next, we apply the *intermediate value theorem*: If  $f(x)$  is continuous for  $a \leq x \leq b$  and  $f(a) < 0$ ,  $f(b) > 0$ , then  $f(x) = 0$  for some  $x$  between  $a$  and  $b$ . Hence, since  $f(s)$  is continuous for  $0 \leq s \leq L$  and  $f(0) = -1 < 0$  and  $f(L) = 1 > 0$ , then  $f(s) = 0$  for some  $s = s_0$  between 0 and  $L$ . But  $f(s_0) = 0$  does not correspond to a point  $(x(s_0), y(s_0))$  lying in either  $E_1$  or  $E_2$ . In other words, a section of the path representing the broken line connecting points  $P$  and  $Q$  and given by continuous functions  $x(s)$  and  $y(s)$  doesn't belong to either  $E_1$  or  $E_2$ . But this contradicts the definition of a domain  $D$ , which states that two points  $P$  and  $Q$  belonging to two different non-overlapping open sets  $E_1$  and  $E_2$  cannot be joined by a broken line.
9. Let the set  $A$  consist of all points  $(x, y)$  for which the continuous function  $f(x, y) > 0$  in domain  $D$ . Let  $(x_1, y_1)$  be such a point. We can choose  $f(x_1, y_1)$  arbitrarily small

as long as  $f(x_1, y_1) > 0$ . Then with the help of the answer to Problem 7 we can verify that there is a neighborhood of  $f(x_1, y_1)$  in which  $f(x, y) > (1/2)f(x_1, y_1) > 0$ . In other words, no matter how small  $f(x_1, y_1)$  is, as long as  $f(x_1, y_1) > 0$  and  $f(x, y)$  is continuous, there will always exist a neighborhood of  $(x_1, y_1)$  of radius  $\delta$  where  $f(x, y) > (1/2)f(x_1, y_1) > 0$ . Hence, the set  $A$  is an open set. A similar reasoning can be applied to conclude that the set  $B$  is an open set. Together,  $A$  and  $B$  form two non-overlapping open sets. Next, imagine choosing a point  $(x, y) = P$  in  $A$  and a point  $(x, y) = Q$  in  $B$  and join them by a continuous (broken) line in  $D$ . Let  $s$ ,  $0 \leq s \leq L$  denote the distance along the path in the same way as for Problem 8, i.e. the path is from  $P$  to  $Q$  and is given by the continuous functions  $x = x(s)$  and  $y = y(s)$ . Now we apply the *intermediate value theorem*; since  $f(s)$  is continuous for  $0 \leq s \leq L$  and  $f(0) > 0$  and  $f(L) < 0$ , then  $f(s) = 0$  for some  $s = s_0$  between  $s = 0$  and  $s = L$ . Let us suppose the opposite; that  $f(s) \neq 0$  for any  $s$ . This would imply that  $D$  consists of the two non-overlapping open sets  $A$  and  $B$  only, which as we concluded in Problem 8 contradicts the definition of a domain  $D$ ; stating that two points  $P$  and  $Q$  belonging to two different non-overlapping open sets  $A$  and  $B$  cannot be joined by a (broken) line.

10. (a) Let  $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$  in  $V^n$ . If  $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2} < \epsilon$  then

$$|\mathbf{x}|^2 = x_1^2 + \cdots + x_n^2 < \epsilon^2 \quad \implies \quad |x_1| < \epsilon, \dots, |x_n| < \epsilon$$

For  $n = 2$ , the result may be geometrically interpreted by stating that if the length of a vector  $\mathbf{x}$  with origin at point  $(x_1, x_2) = (0, 0)$ , i.e.  $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$  is smaller than some  $\epsilon > 0$ , then there exists a neighborhood of  $(0, 0)$  where  $|x_1| < \epsilon$  and  $|x_2| < \epsilon$ .

- (b) Suppose that  $|x_1| < \delta, \dots, |x_n| < \delta$  then  $x_1^2 < \delta^2, \dots, x_n^2 < \delta^2$  and so

$$x_1^2 + \cdots + x_n^2 < \delta^2 + \cdots + \delta^2 = n\delta^2 \quad n \geq 0$$

Taking square roots next gives

$$\sqrt{x_1^2 + \cdots + x_n^2} = |\mathbf{x}| < \sqrt{n}\delta < n\delta$$

where the right most inequality clearly holds, since  $\sqrt{n} < n$ .

- (c) To show continuity of the mapping  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  at the point  $\mathbf{x}^0$ , where  $\mathbf{x} \in V^n$  and  $\mathbf{y} \in V^m$ , we choose a  $\delta > 0$  for a given  $\epsilon > 0$  small enough such that

$$|f_1(x_1, \dots, x_n) - f_1(x_1^0, \dots, x_n^0)| < \frac{\epsilon}{m}, \dots, |f_m(x_1, \dots, x_n) - f_m(x_1^0, \dots, x_n^0)| < \frac{\epsilon}{m}$$

for  $|\mathbf{x} - \mathbf{x}^0| = \sqrt{(x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0)^2} < \delta$ . Squaring and summing gives

$$[f_1(x_1, \dots, x_n) - f_1(x_1^0, \dots, x_n^0)]^2 + \cdots + [f_m(x_1, \dots, x_n) - f_m(x_1^0, \dots, x_n^0)]^2 < \frac{\epsilon^2}{m}$$

Finally, taking square roots of both sides of the inequality gives

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \frac{\epsilon}{\sqrt{m}} < \epsilon$$

In conclusion, since we have chosen  $\delta$  such that  $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \epsilon/\sqrt{m}$ , it will certainly satisfy  $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \epsilon$ , since  $\epsilon > \epsilon/\sqrt{m}$ .

- (d) Squaring the inequality that signifies continuity for the mapping  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  gives

$$[f_1(x_1, \dots, x_n) - f_1(x_1^0, \dots, x_n^0)]^2 + \dots + [f_m(x_1, \dots, x_n) - f_m(x_1^0, \dots, x_n^0)]^2 < \epsilon^2$$

which implies that

$$[f_1(x_1, \dots, x_n) - f_1(x_1^0, \dots, x_n^0)]^2 < \epsilon^2, \dots, [f_m(x_1, \dots, x_n) - f_m(x_1^0, \dots, x_n^0)]^2 < \epsilon^2$$

Taking square roots of both sides then finally results in

$$|f_1(\mathbf{x}) - f_1(\mathbf{x}^0)| < \epsilon, \dots, |f_m(\mathbf{x}) - f_m(\mathbf{x}^0)| < \epsilon$$

from which we may conclude that each of the functions  $f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)$  is continuous at  $(x_1^0, \dots, x_n^0)$ .

11. Let us for the moment assume that the limit  $P_n \rightarrow P_0$  is not unique so that there exists a  $P_n \rightarrow P'_0$ ,  $P'_0 \neq P_0$  and let's take  $\epsilon = (1/3)d(P_0, P'_0) = (1/3)|P'_0 - P_0|$ . We thus have  $|P_n - P_0| < \epsilon$  and  $|P_n - P'_0| < \epsilon$ . Then

$$|P'_0 - P_0| = \underbrace{|P_n - P_0 + P'_0 - P_n|}_{\text{triangle inequality}} \leq |P_n - P_0| + |P'_0 - P_n| < \frac{2}{3}|P'_0 - P_0|$$

which is clearly a contradiction and so we must conclude that in fact  $P_0 = P'_0$ , i.e. the limit  $P_0$  is unique.

12. To show that a set  $E$  in the plane is closed if and only if for every convergent sequence of points  $P_n$  in  $E$  the limit of the sequence is in  $E$  we will try to prove the opposite and see that it produces a contradiction. First, suppose  $E$  is closed and  $P_0 \rightarrow P_0$ , with  $P_n$  in  $E$  for all  $n$ , but the limit  $P_0$  not in  $E$  (i.e.  $P_0 \in \mathbb{R} \setminus E$ ). According to section (2.2), since  $E$  is closed,  $\mathbb{R} \setminus E$  is open. Now since  $P_0 \in \mathbb{R} \setminus E$  and the set is open there will exist a neighborhood of  $P_0$  of radius  $\epsilon$  such that  $d(P, P_0) < \epsilon$  which is completely contained in  $\mathbb{R} \setminus E$  and so implies  $P \notin E$ . But this would mean there exists an  $N$  such that for all  $n \geq N$ ,  $P_n \in \mathbb{R} \setminus E$ , which contradicts the assumption that the sequence  $P_n$  is entirely contained in  $E$ .

Next, suppose  $E$  is such that whenever  $P_n \in E$  and  $P_n \rightarrow P_0$ , then  $P_0 \in E$ . To show that  $E$  is closed, we need to prove that  $\mathbb{R} \setminus E$  is open, meaning that a neighborhood of a point  $P \in \mathbb{R} \setminus E$  of radius  $\epsilon > 0$  is contained entirely in  $\mathbb{R} \setminus E$ . Let us suppose the opposite however, that  $P_0$  is a point not in  $E$  ( $P_0 \in \mathbb{R} \setminus E$ ), but has at least



one point of its neighborhood in  $E$ . In other words, suppose a neighborhood of  $P_0$  of arbitrary radius  $\epsilon > 0$  will contain at least one point that lies in  $E$ , in particular consider  $\epsilon = 1, \epsilon = 1/2, \dots, \epsilon = 1/n$ . Let  $P_n \in E$  be such a point and let its distance to  $P_0 \in \mathbb{R} \setminus E$  satisfy the condition  $d(P_n, P_0) < 1/n$ . Then for  $\epsilon = 1/n$  we arrive at  $P_n \rightarrow P_0$  (see Problem 11 for the definition of the limit of a convergent series), which implies  $P_n \in \mathbb{R} \setminus E$ . But this is contradictory to the original assumption that  $P_n \in E$ . Hence, this proves that  $E$  is closed and in conclusion we have proven that a set  $E$  is closed if and only if for every convergent sequence of point  $P_n$  in  $E$ , the limit of the sequence  $P_0$  is in  $E$ .

13. (a) A set is called open if we can form a neighborhood of a point in the set of radius  $\epsilon$  that is contained entirely in the set. In other words, this neighborhood does not contain any elements that are not part of the set. Since by definition the empty set does not contain any elements, the above statement can be applied to it without any problems and so it can be considered to be open.
- (b) To show that a set  $E$  in the plane and its boundary is closed is equivalent to showing that the complement to this is an open set  $\mathbb{R} \setminus \bar{E}$  where the set  $\bar{E}$  denotes the union of  $E$  and its boundary. To show that  $\mathbb{R} \setminus \bar{E}$  is open is equivalent to showing that a neighborhood of a point  $P \in \mathbb{R} \setminus \bar{E}$  of radius  $\epsilon > 0$  is contained entirely in  $\mathbb{R} \setminus \bar{E}$ . We have already proven this as part of the second part of the solution to Problem 12 and so we won't repeat it again. Hence, we may conclude that a set  $E$  in the plane and its boundary are indeed closed.

## Section 2.6

- 1.