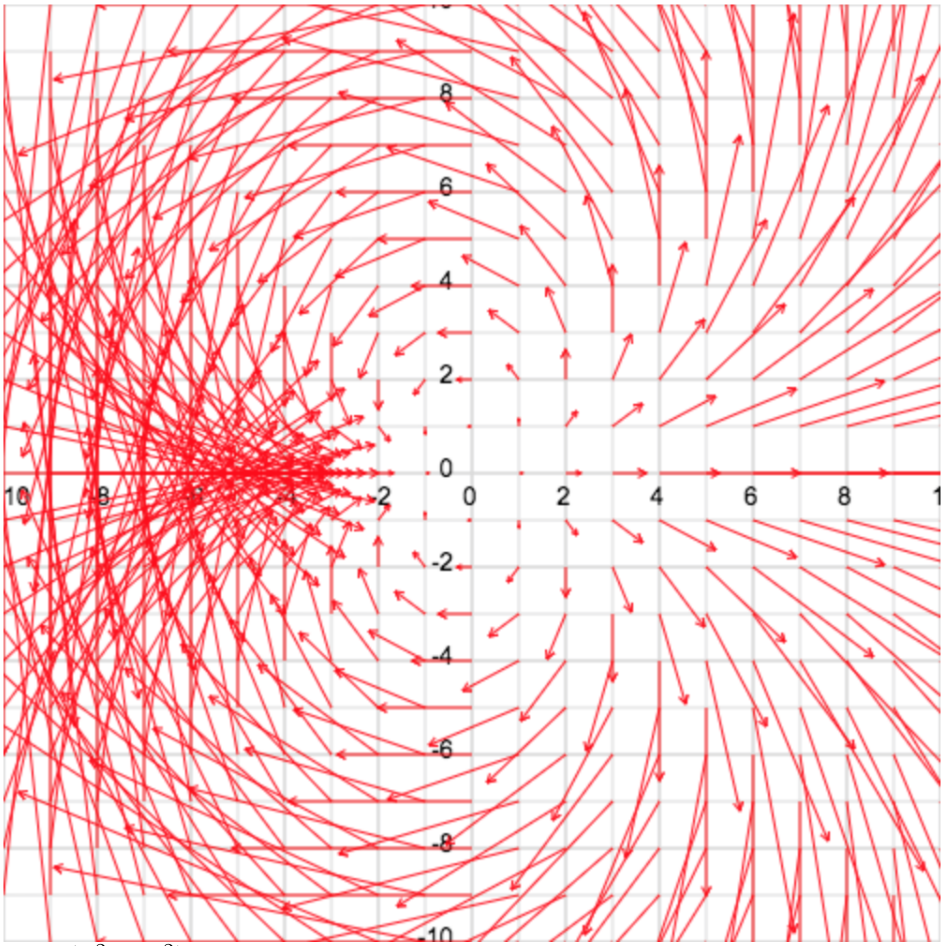
CHAPTER 3

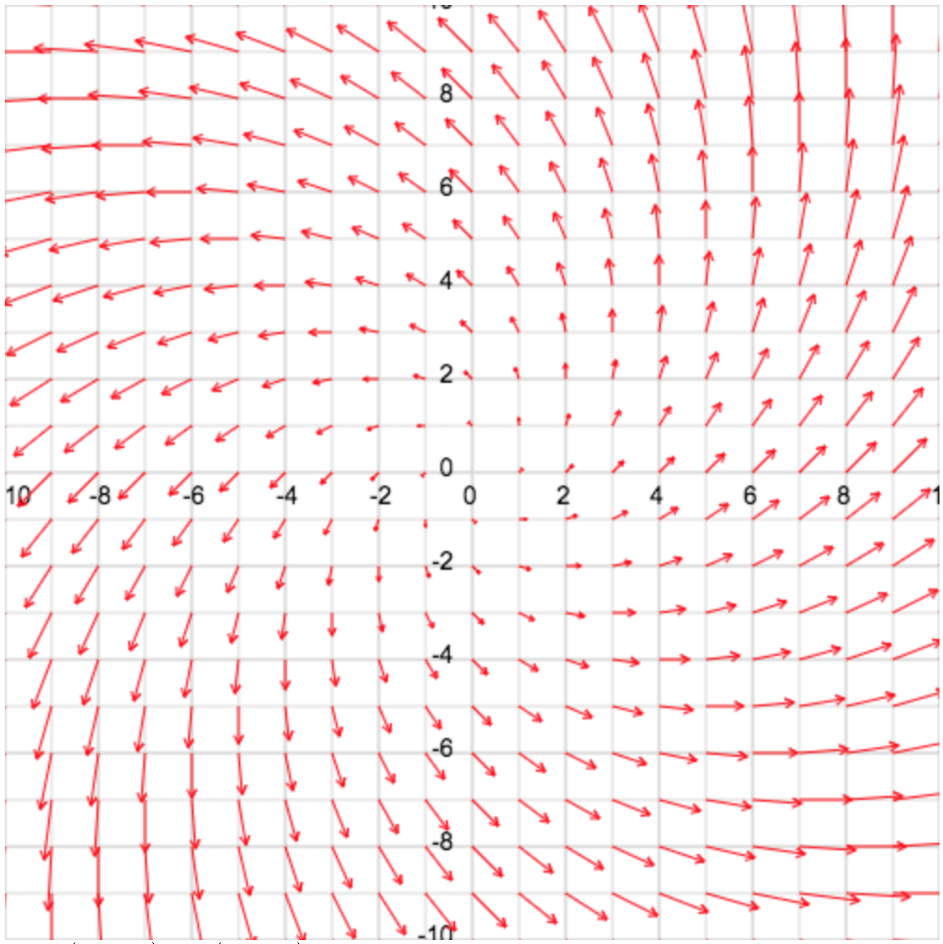
Section 3.3

1. (a)



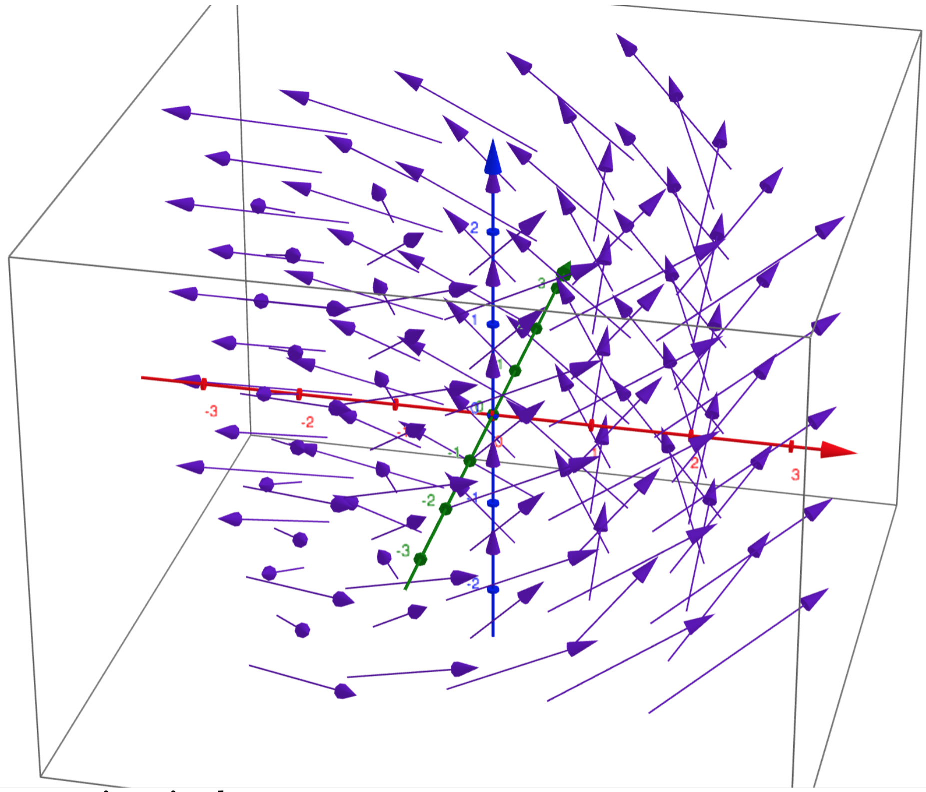
$\mathbf{v} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$

(b)



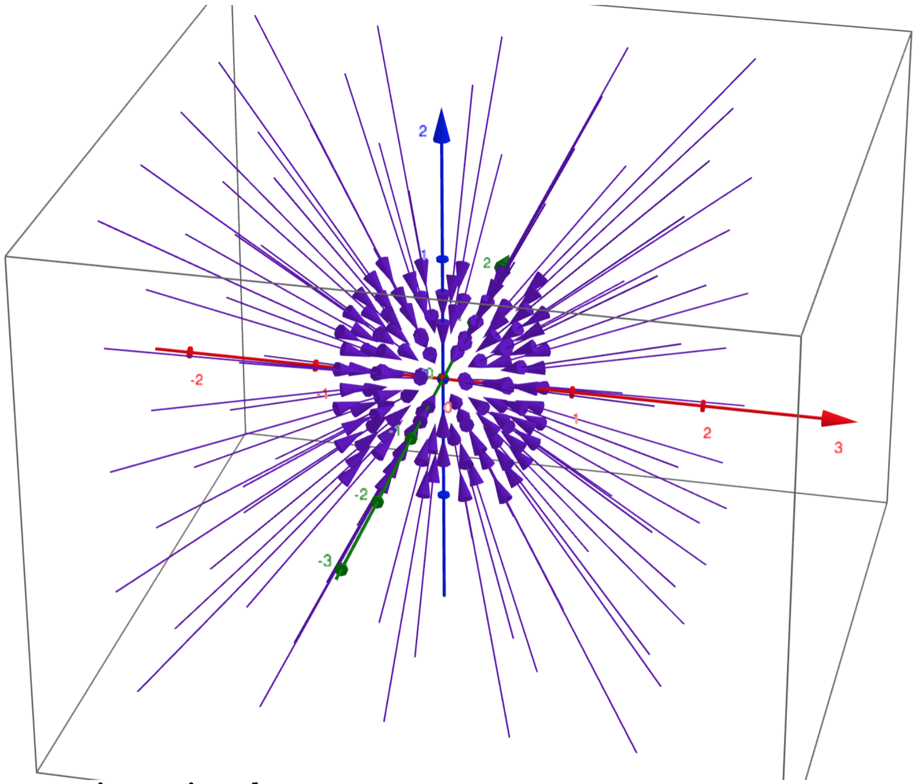
$\mathbf{u} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$

(c)



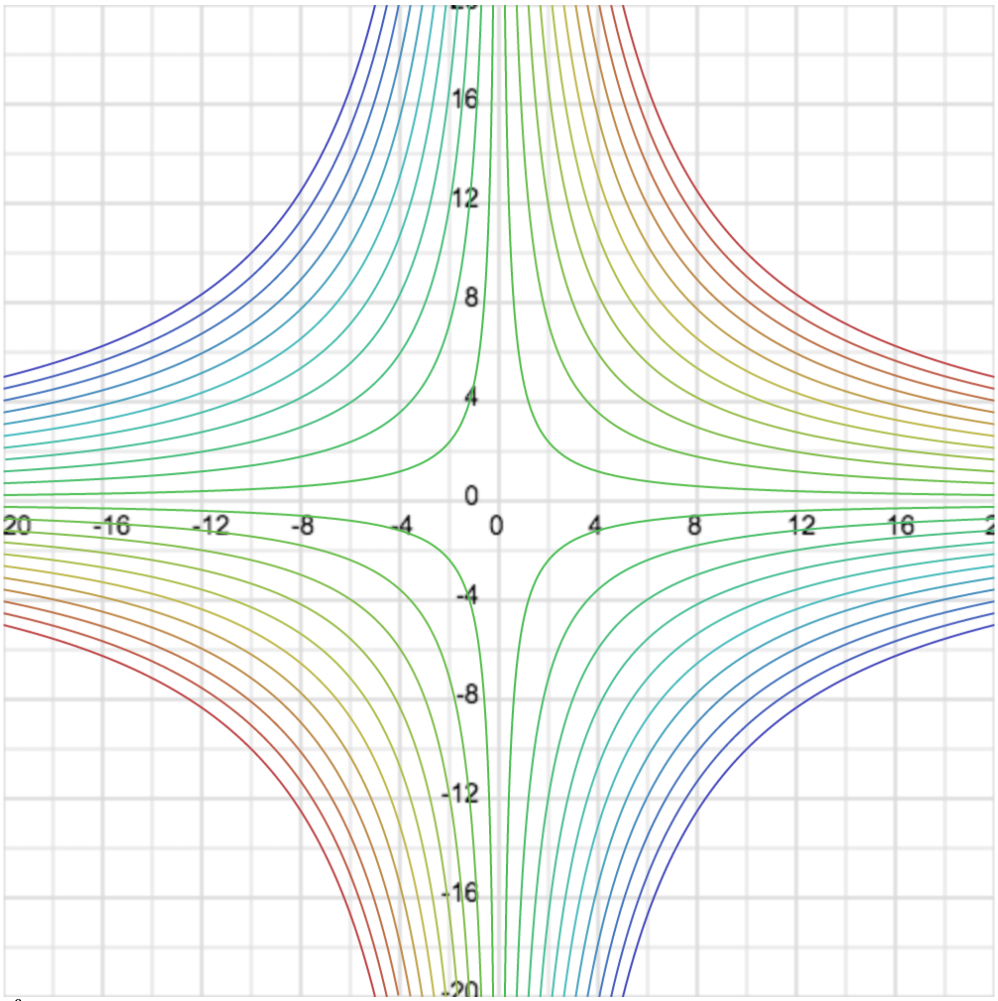
$\mathbf{v} = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$

(d)



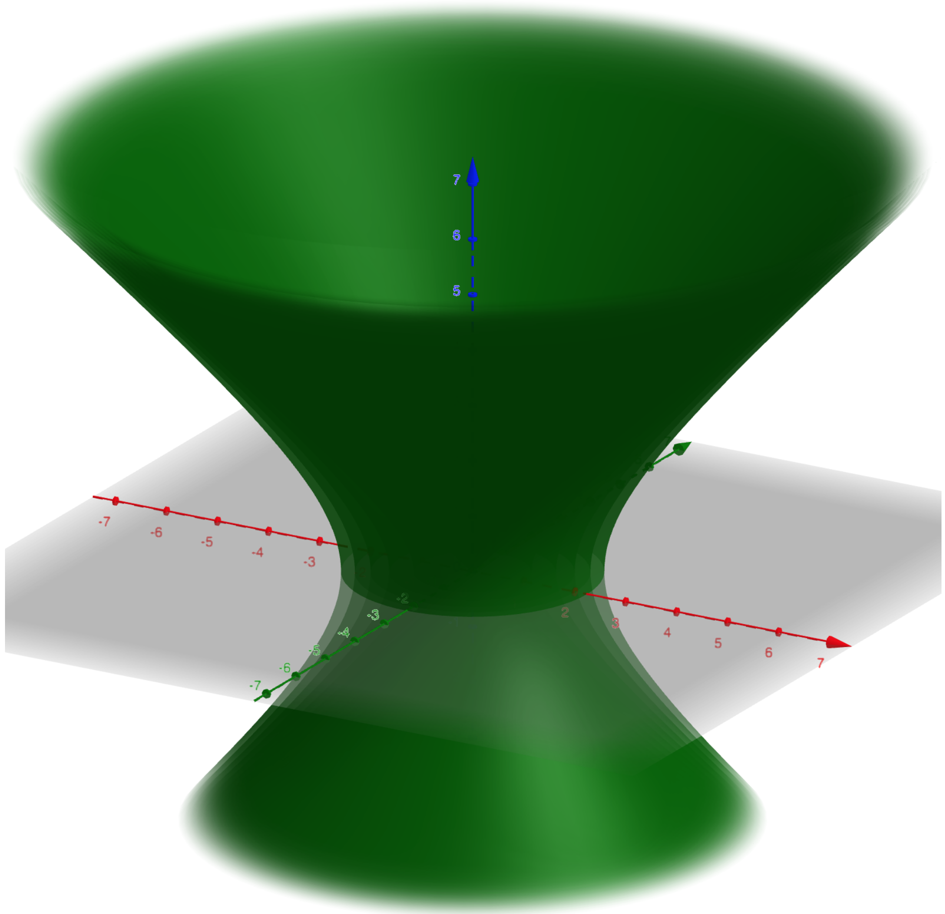
$\mathbf{v} = -x\mathbf{i} + -y\mathbf{j} - z\mathbf{k}$

2. (a)



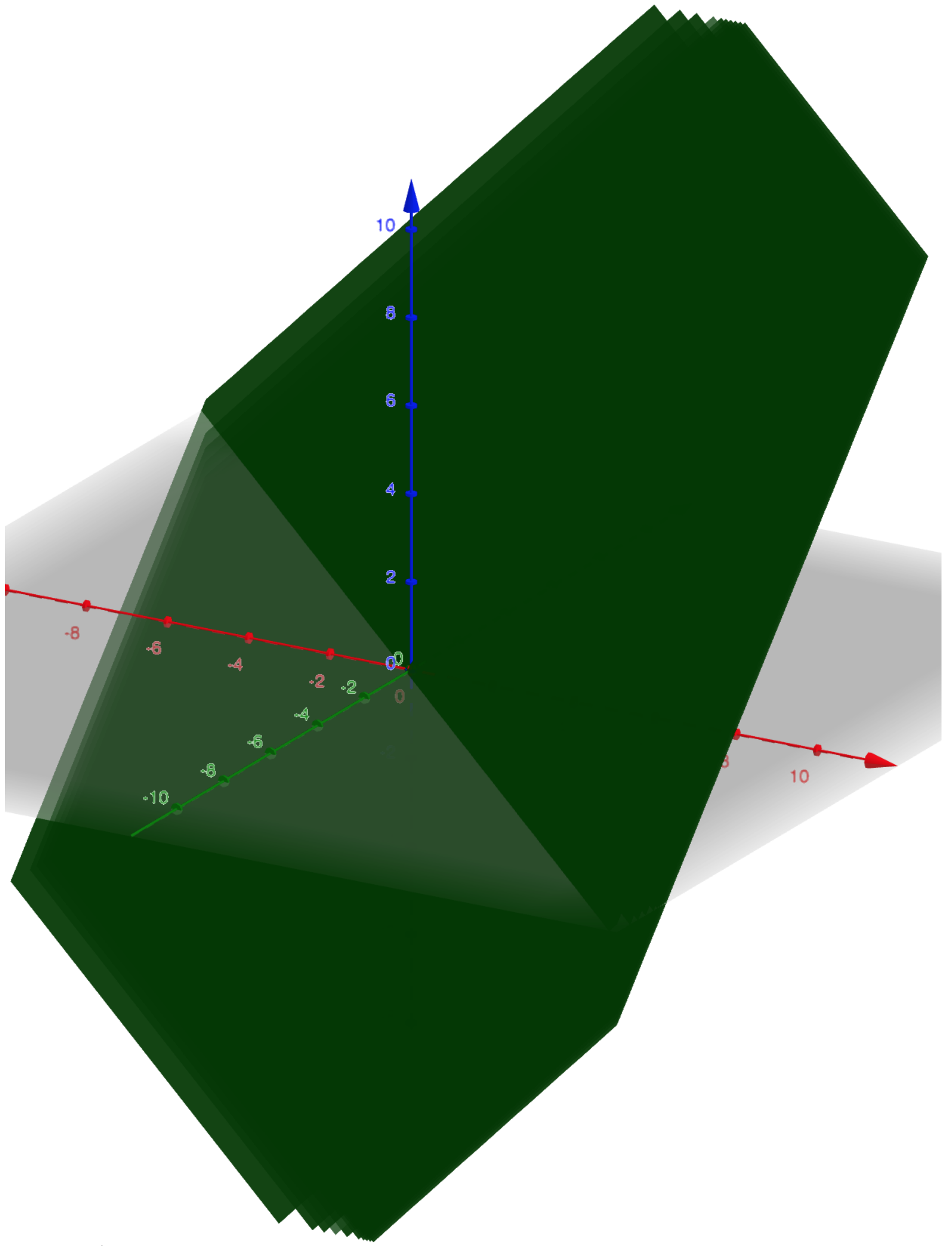
$f = xy$

(b)



$f = x^2 + y^2 - z^2$

(c)



$f = e^{x+y-z}$

3. If $f = xy$ then ∇f is given by
$$\nabla f = y\mathbf{i} + x\mathbf{j}$$
If $f = x^2 + y^2 - z^2$ then ∇f is given by
$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$$
If $f = e^{x+y-z}$ then ∇f is given by
$$\nabla f = e^{x+y-z}\mathbf{i} + e^{x+y-z}\mathbf{j} - e^{x+y-z}\mathbf{k}$$

4. Let $f = kMm/r$, where $r = \sqrt{x^2+y^2+z^2}$ be the equation for the gravitational potential. Then

$$\begin{aligned}
\nabla f &= \nabla\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{x}{r}\mathbf{i} - \frac{kMm}{r^2}\frac{y}{r}\mathbf{j} - \frac{kMm}{r^2}\frac{z}{r}\mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{\mathbf{r}}{r}
\end{aligned}$$

is a vector equation for the gravitational field.

5. Let f be given by

$$f = \ln\frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}}$$

Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}\frac{\partial}{\partial x}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{x-1}{(x-1)^2+y^2} - \frac{x+1}{(x+1)^2+y^2} \\
&= \frac{2\left(x^2-y^2-1\right)}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]} \\
\frac{\partial f}{\partial y} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}\frac{\partial}{\partial y}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{y}{(x-1)^2+y^2} - \frac{y}{(x+1)^2+y^2} \\
&= \frac{4xy}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}
\end{aligned}$$

Hence,

$$\nabla f = \frac{1}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}\left[2\left(x^2-y^2-1\right)\mathbf{i} + 4xy\mathbf{j}\right]$$

6.

$$
\begin{aligned}
\nabla(f+g) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(f+g)\\
&= \frac{\partial}{\partial x}(f+g)\,\mathbf{i}+\frac{\partial}{\partial y}(f+g)\,\mathbf{j}+\frac{\partial}{\partial z}(f+g)\,\mathbf{k}\\
&= \left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\right)\mathbf{i}+\left(\frac{\partial f}{\partial y}+\frac{\partial g}{\partial y}\right)\mathbf{j}+\left(\frac{\partial f}{\partial z}+\frac{\partial g}{\partial z}\right)\mathbf{k}\\
&= \left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)+\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}+\frac{\partial g}{\partial z}\mathbf{k}\right)\\
&= \nabla f+\nabla g
\end{aligned}
$$

$$
\begin{aligned}
\nabla(fg) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(fg)\\
&= \frac{\partial}{\partial x}(fg)\,\mathbf{i}+\frac{\partial}{\partial y}(fg)\,\mathbf{j}+\frac{\partial}{\partial z}(fg)\,\mathbf{k}\\
&= \left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)\mathbf{i}+\left(f\frac{\partial g}{\partial y}+g\frac{\partial f}{\partial y}\right)\mathbf{j}+\left(f\frac{\partial g}{\partial z}+g\frac{\partial f}{\partial z}\right)\mathbf{k}\\
&= \left(f\frac{\partial g}{\partial x}\mathbf{i}+f\frac{\partial g}{\partial y}\mathbf{j}+f\frac{\partial g}{\partial z}\mathbf{k}\right)+\left(g\frac{\partial f}{\partial x}\mathbf{i}+g\frac{\partial f}{\partial y}\mathbf{j}+g\frac{\partial f}{\partial z}\mathbf{k}\right)\\
&= f\nabla g+g\nabla f
\end{aligned}
$$

7. Let $f(x,y,z)$ be a composite function $F(u)$, where $u=g(x,y,z)$. Then

$$
\begin{aligned}
\nabla f=\nabla F=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)F(u) &= \frac{\partial}{\partial x}F(u)\,\mathbf{i}+\frac{\partial}{\partial y}F(u)\,\mathbf{j}+\frac{\partial}{\partial z}F(u)\,\mathbf{k}\\
&= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\mathbf{k}\\
&= \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial u}{\partial z}\mathbf{k}\right)\\
&= F'(u)\,\nabla g
\end{aligned}
$$

8.

$$
\begin{aligned}
\nabla \frac{f}{g} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\frac{f}{g} \\
&= \frac{\partial}{\partial x}\frac{f}{g}\mathbf{i} + \frac{\partial}{\partial y}\frac{f}{g}\mathbf{j} + \frac{\partial}{\partial z}\frac{f}{g}\mathbf{k} \\
&= \frac{gf_x - fg_x}{g^2}\mathbf{i} + \frac{gf_y - fg_y}{g^2}\mathbf{j} + \frac{gf_z - fg_z}{g^2}\mathbf{k} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) - f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)\right] \\
&= \frac{1}{g^2}\left(g\nabla f - f\nabla g\right)
\end{aligned}
$$

9. (a) If $f(x, y, z) = w = x^3y - y^3z$ then H is given by

$$
H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \begin{bmatrix} w_{xx} & w_{xy} & w_{xz} \\ w_{yx} & w_{yy} & w_{yz} \\ w_{zx} & w_{zy} & w_{zz} \end{bmatrix} = \begin{bmatrix} 6xy & 3x^2 & 0 \\ 3x^2 & -6yz & -3y^2 \\ 0 & -3y^2 & 0 \end{bmatrix}
$$

If $f(x, y, z) = w = x_1^2 + 2x_1x_2 + 5x_1x_3 + 2x_2x_1 + 4x_2^2 + x_2x_3 + 5x_3x_1 + x_3x_2 + 2x_3^2$ then H is given by

$$
H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \begin{bmatrix} w_{x_1x_1} & w_{x_1x_2} & w_{x_1x_3} \\ w_{x_2x_1} & w_{x_2x_2} & w_{x_2x_3} \\ w_{x_3x_1} & w_{x_3x_2} & w_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 2 \\ 10 & 2 & 4 \end{bmatrix}
$$

(b) As long as the function $f(x_1, \dots, x_n)$ has continuous second partial derivatives then $\partial^2 f/(\partial x_i \partial x_j) = \partial^2 f/(\partial x_j \partial x_i)$, which implies that H will be symmetric.

(c) As discussed in Section 2.14, the directional derivative of a function $f(x, y)$ in a given direction can be written as

$$
\nabla_\alpha f = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha = \nabla f \cdot \mathbf{u}
$$

where $\mathbf{u} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$ is a unit vector that makes an angle α with the positive x-axis. Hence,

$$
\begin{aligned}
\nabla_\alpha \nabla_\beta f &= \nabla_\alpha\left(\frac{\partial f}{\partial x}\cos\beta + \frac{\partial f}{\partial y}\sin\beta\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos\beta + \frac{\partial f}{\partial y}\sin\beta\right)\cos\alpha + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\beta + \frac{\partial f}{\partial y}\sin\beta\right)\sin\alpha \\
&= \cos\beta\left(\frac{\partial^2 f}{\partial x^2}\cos\alpha + \frac{\partial^2 f}{\partial x \partial y}\sin\alpha\right) + \sin\beta\left(\frac{\partial^2 f}{\partial y \partial x}\cos\alpha + \frac{\partial^2 f}{\partial y^2}\sin\alpha\right) \\
&= \begin{bmatrix}\cos\beta & \sin\beta\end{bmatrix}\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{bmatrix}\begin{bmatrix}\cos\alpha \\ \sin\alpha\end{bmatrix} \\
&= \begin{bmatrix}\cos\beta & \sin\beta\end{bmatrix} H \begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}^\top
\end{aligned}
$$

Section 3.6

1.

$$\begin{aligned}\nabla \cdot (\mathbf{u}+\mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left[(u_x+v_x)\,\mathbf{i}+(u_y+v_y)\,\mathbf{j}+(u_z+v_z)\,\mathbf{k}\right]\\ &= \frac{\partial}{\partial x}(u_x+v_x)+\frac{\partial}{\partial y}(u_y+v_y)+\frac{\partial}{\partial z}(u_z+v_z)\\ &= \left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)\\ &= \nabla\cdot\mathbf{u}+\nabla\cdot\mathbf{v}\end{aligned}$$

$$\begin{aligned}\nabla \cdot (f\mathbf{u}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(fu_x\mathbf{i}+fu_y\mathbf{j}+fu_z\mathbf{k})\\ &= \frac{\partial}{\partial x}(fu_x)+\frac{\partial}{\partial y}(fu_y)+\frac{\partial}{\partial z}(fu_z)\\ &= f\frac{\partial u_x}{\partial x}+u_x\frac{\partial f}{\partial x}+f\frac{\partial u_y}{\partial y}+u_y\frac{\partial f}{\partial y}+f\frac{\partial u_z}{\partial z}+u_z\frac{\partial f}{\partial z}\\ &= f\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\left(u_x\frac{\partial f}{\partial x}+u_y\frac{\partial f}{\partial y}+u_z\frac{\partial f}{\partial z}\right)\\ &= f(\nabla\cdot\mathbf{u})+(\nabla f\cdot\mathbf{u})\end{aligned}$$

2. Recognizing that $\mathbf{v}=\rho\mathbf{u}$ and using (3.22), then (3.17) can be written as

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{v}=\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=\frac{\partial\rho}{\partial t}+(\nabla\rho\cdot\mathbf{u})+\rho(\nabla\cdot\mathbf{u})=0$$

According to Problem 12 of Section 2.8, the first two terms can be written as

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{u}=\frac{\partial\rho}{\partial t}+u_x\frac{\partial\rho}{\partial x}+u_y\frac{\partial\rho}{\partial y}+u_z\frac{\partial\rho}{\partial z}=\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}\frac{dx}{dt}+\frac{\partial\rho}{\partial y}\frac{dy}{dt}+\frac{\partial\rho}{\partial z}\frac{dz}{dt}=\frac{d\rho}{dt}=\frac{D\rho}{Dt}$$

Hence, (3.17) can be written as

$$\frac{\partial\rho}{\partial t}+(\nabla\rho\cdot\mathbf{u})+\rho(\nabla\cdot\mathbf{u})=\frac{D\rho}{Dt}+\rho(\nabla\cdot\mathbf{u})=0$$

When $\rho\equiv a$, where a is some arbitrary constant, then $D\rho/dt\equiv Da/dt=0$ and the equation above reduces to

$$\rho(\nabla\cdot\mathbf{u})\equiv a(\nabla\cdot\mathbf{u})=0$$

Since $\rho\equiv a\neq 0$, the only way for this equation to make sense is if $\nabla\cdot\mathbf{u}=0$.

3.

$$
\begin{aligned}
\nabla \times (\mathbf{u}+\mathbf{v}) &= \left[\frac{\partial}{\partial y}(u_z+v_z) - \frac{\partial}{\partial z}(u_y+v_y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(u_x+v_x) - \frac{\partial}{\partial x}(u_z+v_z)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(u_y+v_y) - \frac{\partial}{\partial y}(u_x+v_x)\right]\mathbf{k} \\
&= \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&\quad + \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\
&= (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times (f\mathbf{u}) &= \left[\frac{\partial}{\partial y}(fu_z) - \frac{\partial}{\partial z}(fu_y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(fu_x) - \frac{\partial}{\partial x}(fu_z)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(fu_y) - \frac{\partial}{\partial y}(fu_x)\right]\mathbf{k} \\
&= \left[f\frac{\partial u_z}{\partial y} + u_z\frac{\partial f}{\partial y} - \left(f\frac{\partial u_y}{\partial z} + u_y\frac{\partial f}{\partial z}\right)\right]\mathbf{i} + \left[f\frac{\partial u_x}{\partial z} + u_x\frac{\partial f}{\partial z} - \left(f\frac{\partial u_z}{\partial x} + u_z\frac{\partial f}{\partial x}\right)\right]\mathbf{j} \\
&\quad + \left[f\frac{\partial u_y}{\partial x} + u_y\frac{\partial f}{\partial x} - \left(f\frac{\partial u_x}{\partial y} + u_x\frac{\partial f}{\partial y}\right)\right]\mathbf{k} \\
&= f\left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&\quad + \left[\left(u_z\frac{\partial f}{\partial y} - u_y\frac{\partial f}{\partial z}\right)\mathbf{i} + \left(u_x\frac{\partial f}{\partial z} - u_z\frac{\partial f}{\partial x}\right)\mathbf{j} + \left(u_y\frac{\partial f}{\partial x} - u_x\frac{\partial f}{\partial y}\right)\mathbf{k}\right] \\
&= (f\nabla \times \mathbf{u}) + (\nabla f \times \mathbf{u})
\end{aligned}
$$

4.

$$
\begin{aligned}
\nabla \times (\nabla f) &= \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]\mathbf{k} \\
&= \left(\frac{\partial^2 f}{\partial z\partial y} - \frac{\partial^2 f}{\partial y\partial z}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial z\partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial y\partial x} - \frac{\partial^2 f}{\partial x\partial y}\right)\mathbf{k} \\
&= \mathbf{0}
\end{aligned}
$$

5. (a) If $\mathbf{v} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$, then

$$
\begin{aligned}
\nabla \times \mathbf{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k} \\
&= \left(x^2 - x^2\right)\mathbf{i} + (2xy - 2xy)\mathbf{j} + (2xz - 2xz)\mathbf{k} \\
&= \mathbf{0}
\end{aligned}
$$

Let $f = x^2yz + a$, where a is an arbitrary constant. Then $\nabla f = \mathbf{v}$.

(b) If $\mathbf{v} = e^{xy}[(2y^2 + yz^2)\mathbf{i} + (2xy + xz^2 + 2)\mathbf{j} + 2z\mathbf{k}]$, then

$$\begin{aligned}\nabla \times \mathbf{v} &= (2xze^{xy} - 2xze^{xy})\,\mathbf{i} + (2yze^{xy} - 2yze^{xy})\,\mathbf{j} \\ &\quad + \left[ye^{xy}\left(2xy + xz^2 + 2\right) + e^{xy}\left(2y + z^2\right) - xe^{xy}\left(2y^2 + yz^2\right) - e^{xy}\left(4y + z^2\right)\right]\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

Let $f = e^{xy}(2y + z^2) + a$, where a is an arbitrary constant. Then $\nabla f = \mathbf{v}$.

6.

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\ &\quad \cdot \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\ &= \frac{\partial^2 v_z}{\partial y \partial x} - \frac{\partial^2 v_y}{\partial z \partial x} + \frac{\partial^2 v_x}{\partial z \partial y} - \frac{\partial^2 v_z}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial y \partial z} \\ &= 0\end{aligned}$$

7. (a) If $\mathbf{v} = 2x\mathbf{i} + y\mathbf{j} - 3z\mathbf{k}$, then

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 2 + 1 - 3 = 0$$

Since $\nabla \cdot \mathbf{v} = 0$, the vector $\mathbf{v} = \nabla \times \mathbf{u}$ for some vector $\mathbf{u}$. Furthermore, since by (3.32) $\nabla \times (\nabla f) = \mathbf{0}$, we can safely assume that $\mathbf{u}$ is of the form $\mathbf{u} = \mathbf{u}_0 + \nabla f$, where f is an arbitrary scalar function and $\mathbf{u}_0$ is any one vector whose curl is $\mathbf{v}$, as then $\nabla \times \mathbf{u} = \nabla \times (\mathbf{u}_0 + \nabla f) = (\nabla \times \mathbf{u}_0) + [\nabla \times (\nabla f)] = \nabla \times \mathbf{u}_0$.

Next, assume $\mathbf{u}_0 \cdot \mathbf{k} = 0$, which implies $u_{0z} = 0$. Equating the components of $\nabla \times \mathbf{u}_0$ to those of $\mathbf{v}$ then gives

$$\frac{\partial u_{0z}}{\partial y} - \frac{\partial u_{0y}}{\partial z} = -\frac{\partial u_{0y}}{\partial z} = 2x, \quad \frac{\partial u_{0x}}{\partial z} - \frac{\partial u_{0z}}{\partial x} = \frac{\partial u_{0x}}{\partial z} = y, \quad \frac{\partial u_{0y}}{\partial x} - \frac{\partial u_{0x}}{\partial y} = -3z$$

from which we may deduce that $\mathbf{u}_0 = yz\mathbf{i} - 2xz\mathbf{j}$.

(b) If $\mathbf{v} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, then going through the exact same steps as for part (a) gives $\mathbf{u}_0 = (z^2/2)\mathbf{i} + [(x^2 - 2yz)/2]\mathbf{j}$.

8.

$$
\begin{aligned}
\operatorname{div}\operatorname{grad} f = \nabla\cdot(\nabla f) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)\\
&= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z}\\
&= \nabla^2 f\\
&= \Delta f
\end{aligned}
$$

Let $f = 1/\sqrt{x^2+y^2+z^2}$. Then

$$
\nabla^2 f = \frac{2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{x^2-2y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{x^2+y^2-2z^2}{\left(x^2+y^2+z^2\right)^{5/2}} = 0
$$

9.

$$
\begin{aligned}
\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left[\left(u_y v_z-u_z v_y\right)\mathbf{i}+\left(u_z v_x-u_x v_z\right)\mathbf{j}+\left(u_x v_y-u_y v_x\right)\mathbf{k}\right]\\
&= \frac{\partial}{\partial x}\left(u_y v_z-u_z v_y\right)+\frac{\partial}{\partial y}\left(u_z v_x-u_x v_z\right)+\frac{\partial}{\partial z}\left(u_x v_y-u_y v_x\right)\\
&= u_y\frac{\partial v_z}{\partial x}+v_z\frac{\partial u_y}{\partial x}-u_z\frac{\partial v_y}{\partial x}-v_y\frac{\partial u_z}{\partial x}+u_z\frac{\partial v_x}{\partial y}+v_x\frac{\partial u_z}{\partial y}-u_x\frac{\partial v_z}{\partial y}-v_z\frac{\partial u_x}{\partial y}\\
&\quad+u_x\frac{\partial v_y}{\partial z}+v_y\frac{\partial u_x}{\partial z}-u_y\frac{\partial v_x}{\partial z}-v_x\frac{\partial u_y}{\partial z}\\
&= \left(v_x\frac{\partial u_z}{\partial y}-v_x\frac{\partial u_y}{\partial z}+v_y\frac{\partial u_x}{\partial z}-v_y\frac{\partial u_z}{\partial x}+v_z\frac{\partial u_y}{\partial x}-v_z\frac{\partial u_x}{\partial y}\right)\\
&\quad+\left(u_x\frac{\partial v_y}{\partial z}-u_x\frac{\partial v_z}{\partial y}+u_y\frac{\partial v_z}{\partial x}-u_y\frac{\partial v_x}{\partial z}+u_z\frac{\partial v_x}{\partial y}-u_z\frac{\partial v_y}{\partial x}\right)\\
&= \left(v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{k}\right)\cdot\left[\left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)\mathbf{k}\right]\\
&\quad-\left(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k}\right)\cdot\left[\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\mathbf{k}\right]\\
&= \mathbf{v}\cdot(\nabla\times\mathbf{u})-\mathbf{u}\cdot(\nabla\times\mathbf{v})
\end{aligned}
$$

10.

$$
\begin{aligned}
\nabla \times (\nabla \times \mathbf{u}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\
&\quad \times \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&= \left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\right]\mathbf{i} \\
&\quad + \left[\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\right]\mathbf{k} \\
&= \left(\frac{\partial^2 u_y}{\partial x \partial y} - \frac{\partial^2 u_x}{\partial y^2} - \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 u_z}{\partial y \partial z} - \frac{\partial^2 u_y}{\partial z^2} - \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_x}{\partial y \partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial z \partial x} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_y}{\partial z \partial y}\right)\mathbf{k} \\
&= \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} - \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial y^2} - \frac{\partial^2 u_x}{\partial z^2}\right)\mathbf{i} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial y \partial x} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z} - \frac{\partial^2 u_y}{\partial x^2} - \frac{\partial^2 u_y}{\partial y^2} - \frac{\partial^2 u_y}{\partial z^2}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial z \partial x} + \frac{\partial^2 u_y}{\partial z \partial y} + \frac{\partial^2 u_z}{\partial z^2} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2} - \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k} \\
&= \left[\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 u_x}{\partial y \partial x} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z}\right)\mathbf{j}\right. \\
&\quad \left. + \left(\frac{\partial^2 u_x}{\partial z \partial x} + \frac{\partial^2 u_y}{\partial z \partial y} + \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k}\right] \\
&\quad - \left[\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right)\mathbf{j}\right. \\
&\quad \left. + \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k}\right] \\
&= \nabla\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \left(\nabla^2 u_x \mathbf{i} + \nabla^2 u_y \mathbf{j} + \nabla^2 u_z \mathbf{k}\right) \\
&= \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}
\end{aligned}
$$

11. (a)

$$\begin{aligned}
\nabla\cdot[\mathbf{u}\times(\mathbf{v}\times\mathbf{w})] &= \nabla\cdot\underbrace{[(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}]}_{(1.19)}\\
&= \underbrace{\nabla\cdot[(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}]-\nabla\cdot[(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}]}_{(3.21)}\\
&= \underbrace{(\mathbf{u}\cdot\mathbf{w})\,(\nabla\cdot\mathbf{v})+[\nabla\,(\mathbf{u}\cdot\mathbf{w})]\cdot\mathbf{v}}_{(3.22)}-(\mathbf{u}\cdot\mathbf{v})\,(\nabla\cdot\mathbf{w})-[\nabla\,(\mathbf{u}\cdot\mathbf{v})]\cdot\mathbf{w}\\
&= (\mathbf{u}\cdot\mathbf{w})\,(\nabla\cdot\mathbf{v})-(\mathbf{u}\cdot\mathbf{v})\,(\nabla\cdot\mathbf{w})+[\nabla\,(\mathbf{u}\cdot\mathbf{w})]\cdot\mathbf{v}-[\nabla\,(\mathbf{u}\cdot\mathbf{v})]\cdot\mathbf{w}
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla\cdot[(\nabla f)\times(f\nabla g)] &= \underbrace{(f\nabla g)\cdot[\nabla\times(\nabla f)]-(\nabla f)\cdot[\nabla\times(f\nabla g)]}_{(3.35)}\\
&= (f\nabla g)\cdot\underbrace{\mathbf{0}}_{(3.31)}-(\nabla f)\cdot[\nabla\times(f\nabla g)]\\
&= -\,(\nabla f)\cdot[\nabla\times(f\nabla g)]\\
&= -\,(\nabla f)\cdot\underbrace{[f\,(\nabla\times(\nabla g))+(\nabla f)\times(\nabla g)]}_{(3.28)}\\
&= -\,(\nabla f)\cdot\left[f\underbrace{\mathbf{0}}_{(3.31)}+(\nabla f)\times(\nabla g)\right]\\
&= -\,(\nabla f)\cdot(\nabla f)\times(\nabla g)\\
&= (\nabla f)\cdot(\nabla g)\times(\nabla f)\\
&= \underbrace{(\nabla g)\cdot(\nabla f)\times(\nabla f)}_{(1.34)}\\
&= (\nabla g)\cdot\underbrace{\mathbf{0}}_{(1.19)}\\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
\nabla\times[(\nabla\times\mathbf{v})+\nabla f] = \underbrace{\nabla\times(\nabla\times\mathbf{v})+\nabla\times(\nabla f)}_{(3.27)} &= \nabla\times(\nabla\times\mathbf{v})+\underbrace{\mathbf{0}}_{(3.31)}\\
&= \nabla\times(\nabla\times\mathbf{v})
\end{aligned}$$

(d)

$$\nabla^2 f = \mathbf{0}+\nabla^2 f = \nabla\times\nabla\cdot\mathbf{v}+\nabla\cdot\nabla f = \underbrace{\nabla\cdot\nabla\times\mathbf{v}}_{(1.34)}+\nabla\cdot\nabla f = \underbrace{\nabla\cdot[(\nabla\times\mathbf{v})+\nabla f]}_{(3.21)}$$

12. (a) Let $\mathbf{u}$ be a unit vector, such that

$$\mathbf{u} = \frac{u_x}{|\mathbf{u}|}\mathbf{i} + \frac{u_y}{|\mathbf{u}|}\mathbf{j} + \frac{u_z}{|\mathbf{u}|}\mathbf{k} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

That is, $u_x/|\mathbf{u}|$, $u_y|\mathbf{u}|$, $u_z/|\mathbf{u}|$ are, by Section 1.2, simply the direction cosines of $\mathbf{u}$. Hence by (2.114),

$$(\mathbf{u}\cdot\nabla) f = \frac{u_x}{|\mathbf{u}|}\frac{\partial f}{\partial x} + \frac{u_y}{|\mathbf{u}|}\frac{\partial f}{\partial y} + \frac{u_z}{|\mathbf{u}|}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma = \nabla_u f$$

(b)

$$[(\mathbf{i}-\mathbf{j})\cdot\nabla] f = (\mathbf{i}-\mathbf{j})\cdot(\nabla f) = (\mathbf{i}-\mathbf{j})\cdot\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

(c) Let $\mathbf{v} = x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}$. Then

$$[(x\mathbf{i} - y\mathbf{j})\cdot\nabla]\left(x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}\right) = x\frac{\partial \mathbf{v}}{\partial x} - y\frac{\partial \mathbf{v}}{\partial y} = 2\left(x^2\mathbf{i} + y^2\mathbf{j}\right)$$

13.

$$\begin{aligned}
\nabla(\mathbf{u}\cdot\mathbf{v}) &= \nabla\left(u_x v_x + u_y v_y + u_z v_z\right) \\
&= \nabla\left(u_x v_x\right) + \nabla\left(u_y v_y\right) + \nabla\left(u_z v_z\right) \\
&= u_x\nabla v_x + v_x\nabla u_x + u_y\nabla v_y + v_y\nabla u_y + u_z\nabla v_z + v_z\nabla u_z \\
&= \left(u_x\nabla v_x + u_y\nabla v_y + u_z\nabla v_z\right) + \left(v_x\nabla u_x + v_y\nabla u_y + v_z\nabla u_z\right)
\end{aligned}$$

Let us for a moment focus on the first three terms $u_x\nabla v_x + u_y\nabla v_y + u_z\nabla v_z = \mathbf{a}$.

Expanding these gives

$$
\begin{aligned}
\mathbf{a} &= u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i}+\frac{\partial v_x}{\partial y}\mathbf{j}+\frac{\partial v_x}{\partial z}\mathbf{k}\right)+u_y\left(\frac{\partial v_y}{\partial x}\mathbf{i}+\frac{\partial v_y}{\partial y}\mathbf{j}+\frac{\partial v_y}{\partial z}\mathbf{k}\right)+u_z\left(\frac{\partial v_z}{\partial x}\mathbf{i}+\frac{\partial v_z}{\partial y}\mathbf{j}+\frac{\partial v_z}{\partial z}\mathbf{k}\right)\\
&= u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i}+\frac{\partial v_x}{\partial y}\mathbf{j}+\frac{\partial v_x}{\partial z}\mathbf{k}\right)+u_y\left(\frac{\partial v_y}{\partial x}\mathbf{i}+\frac{\partial v_y}{\partial y}\mathbf{j}+\frac{\partial v_y}{\partial z}\mathbf{k}\right)+u_z\left(\frac{\partial v_z}{\partial x}\mathbf{i}+\frac{\partial v_z}{\partial y}\mathbf{j}+\frac{\partial v_z}{\partial z}\mathbf{k}\right)\\
&\quad+u_x\left(\frac{\partial v_y}{\partial x}\mathbf{j}-\frac{\partial v_y}{\partial x}\mathbf{j}+\frac{\partial v_z}{\partial x}\mathbf{k}-\frac{\partial v_z}{\partial x}\mathbf{k}\right)+u_y\left(\frac{\partial v_x}{\partial y}\mathbf{i}-\frac{\partial v_x}{\partial y}\mathbf{i}+\frac{\partial v_z}{\partial y}\mathbf{k}-\frac{\partial v_z}{\partial y}\mathbf{k}\right)\\
&\quad+u_z\left(\frac{\partial v_x}{\partial z}\mathbf{i}-\frac{\partial v_x}{\partial z}\mathbf{i}+\frac{\partial v_y}{\partial z}\mathbf{j}-\frac{\partial v_y}{\partial z}\mathbf{j}\right)\\
&= u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i}+\frac{\partial v_y}{\partial x}\mathbf{j}+\frac{\partial v_z}{\partial x}\mathbf{k}\right)+u_y\left(\frac{\partial v_x}{\partial y}\mathbf{i}+\frac{\partial v_y}{\partial y}\mathbf{j}+\frac{\partial v_z}{\partial y}\mathbf{k}\right)+u_z\left(\frac{\partial v_x}{\partial z}\mathbf{i}+\frac{\partial v_y}{\partial z}\mathbf{j}+\frac{\partial v_z}{\partial z}\mathbf{k}\right)\\
&\quad+\left[u_y\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)-u_z\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right]\mathbf{i}+\left[u_z\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-u_x\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right]\mathbf{j}\\
&\quad+\left[u_x\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)-u_y\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\right]\mathbf{k}\\
&= u_x\frac{\partial \mathbf{v}}{\partial x}+u_y\frac{\partial \mathbf{v}}{\partial y}+u_z\frac{\partial \mathbf{v}}{\partial z}+\left\{\mathbf{u}\times\left[\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\mathbf{k}\right]\right\}\\
&= (\mathbf{u}\cdot\nabla)\,\mathbf{v}+[\mathbf{u}\times(\nabla\times\mathbf{v})]
\end{aligned}
$$

Then, clearly

$$v_x\nabla u_x+v_y\nabla u_y+v_z\nabla u_z=(\mathbf{v}\cdot\nabla)\,\mathbf{u}+[\mathbf{v}\times(\nabla\times\mathbf{u})]$$

And so we may conclude that

$$\nabla(\mathbf{u}\cdot\mathbf{v})=(\mathbf{u}\cdot\nabla)\,\mathbf{v}+[\mathbf{u}\times(\nabla\times\mathbf{v})]+(\mathbf{v}\cdot\nabla)\,\mathbf{u}+[\mathbf{v}\times(\nabla\times\mathbf{u})]$$

14.

$$
\begin{aligned}
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla \times \left[(u_y v_z - u_z v_y)\,\mathbf{i} + (u_z v_x - u_x v_z)\,\mathbf{j} + (u_x v_y - u_y v_x)\,\mathbf{k}\right] \\
&= \left[\nabla \times (u_y v_z \mathbf{i} + u_z v_x \mathbf{j} + u_x v_y \mathbf{k})\right] - \left[\nabla \times (u_z v_y \mathbf{i} + u_x v_z \mathbf{j} + u_y v_x \mathbf{k})\right] \\
&= \left[\frac{\partial}{\partial y}(u_x v_y) - \frac{\partial}{\partial z}(u_z v_x)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(u_y v_z) - \frac{\partial}{\partial x}(u_x v_y)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(u_z v_x) - \frac{\partial}{\partial y}(u_y v_z)\right]\mathbf{k} - \left[\frac{\partial}{\partial y}(u_y v_x) - \frac{\partial}{\partial z}(u_x v_z)\right]\mathbf{i} \\
&\quad - \left[\frac{\partial}{\partial z}(u_z v_y) - \frac{\partial}{\partial x}(u_y v_x)\right]\mathbf{j} - \left[\frac{\partial}{\partial x}(u_x v_z) - \frac{\partial}{\partial y}(u_z v_y)\right]\mathbf{k} \\
&= \left(u_x\frac{\partial v_y}{\partial y} + v_y\frac{\partial u_x}{\partial y} - u_z\frac{\partial v_x}{\partial z} - v_x\frac{\partial u_z}{\partial z} - u_y\frac{\partial v_x}{\partial y} - v_x\frac{\partial u_y}{\partial y} + u_x\frac{\partial v_z}{\partial z} + v_z\frac{\partial u_x}{\partial z}\right)\mathbf{i} \\
&\quad + \left(u_y\frac{\partial v_z}{\partial z} + v_z\frac{\partial u_y}{\partial z} - u_x\frac{\partial v_y}{\partial x} - v_y\frac{\partial u_x}{\partial x} - u_z\frac{\partial v_y}{\partial z} - v_y\frac{\partial u_z}{\partial z} + u_y\frac{\partial v_x}{\partial x} + v_x\frac{\partial u_y}{\partial x}\right)\mathbf{j} \\
&\quad + \left(u_z\frac{\partial v_x}{\partial x} + v_x\frac{\partial u_z}{\partial x} - u_y\frac{\partial v_z}{\partial y} - v_z\frac{\partial u_y}{\partial y} - u_x\frac{\partial v_z}{\partial x} - v_z\frac{\partial u_x}{\partial x} + u_z\frac{\partial v_y}{\partial y} + v_y\frac{\partial u_z}{\partial y}\right)\mathbf{k} \\
&= \left(u_x\frac{\partial v_y}{\partial y} - u_z\frac{\partial v_x}{\partial z} - u_y\frac{\partial v_x}{\partial y} + u_x\frac{\partial v_z}{\partial z}\right)\mathbf{i} + \left(u_y\frac{\partial v_z}{\partial z} - u_x\frac{\partial v_y}{\partial x} - u_z\frac{\partial v_y}{\partial z} + u_y\frac{\partial v_x}{\partial x}\right)\mathbf{j} \\
&\quad + \left(u_z\frac{\partial v_x}{\partial x} - u_y\frac{\partial v_z}{\partial y} - u_x\frac{\partial v_z}{\partial x} + u_z\frac{\partial v_y}{\partial y}\right)\mathbf{k} \\
&\quad + \left(v_y\frac{\partial u_x}{\partial y} - v_x\frac{\partial u_z}{\partial z} - v_x\frac{\partial u_y}{\partial y} + v_z\frac{\partial u_x}{\partial z}\right)\mathbf{i} + \left(v_z\frac{\partial u_y}{\partial z} - v_y\frac{\partial u_x}{\partial x} - v_y\frac{\partial u_z}{\partial z} + v_x\frac{\partial u_y}{\partial x}\right)\mathbf{j} \\
&\quad + \left(v_x\frac{\partial u_z}{\partial x} - v_z\frac{\partial u_y}{\partial y} - v_z\frac{\partial u_x}{\partial x} + v_y\frac{\partial u_z}{\partial y}\right)\mathbf{k}
\end{aligned}
$$

Let us for a moment focus on the first three terms

$$
\begin{aligned}
\mathbf{a} = &\left(u_x\frac{\partial v_y}{\partial y} - u_z\frac{\partial v_x}{\partial z} - u_y\frac{\partial v_x}{\partial y} + u_x\frac{\partial v_z}{\partial z}\right)\mathbf{i} + \left(u_y\frac{\partial v_z}{\partial z} - u_x\frac{\partial v_y}{\partial x} - u_z\frac{\partial v_y}{\partial z} + u_y\frac{\partial v_x}{\partial x}\right)\mathbf{j} \\
&+ \left(u_z\frac{\partial v_x}{\partial x} - u_y\frac{\partial v_z}{\partial y} - u_x\frac{\partial v_z}{\partial x} + u_z\frac{\partial v_y}{\partial y}\right)\mathbf{k}
\end{aligned}
$$

These may be further manipulated to get

$$\begin{aligned}
\mathbf{a} &= \left(u_x\frac{\partial v_y}{\partial y} - u_z\frac{\partial v_x}{\partial z} - u_y\frac{\partial v_x}{\partial y} + u_x\frac{\partial v_z}{\partial z}\right)\mathbf{i} + \left(u_y\frac{\partial v_z}{\partial z} - u_x\frac{\partial v_y}{\partial x} - u_z\frac{\partial v_y}{\partial z} + u_y\frac{\partial v_x}{\partial x}\right)\mathbf{j} \\
&\quad + \left(u_z\frac{\partial v_x}{\partial x} - u_y\frac{\partial v_z}{\partial y} - u_x\frac{\partial v_z}{\partial x} + u_z\frac{\partial v_y}{\partial y}\right)\mathbf{k} + \left(u_x\frac{\partial v_x}{\partial x} - u_x\frac{\partial v_x}{\partial x}\right)\mathbf{i} + \left(u_y\frac{\partial v_y}{\partial y} - u_y\frac{\partial v_y}{\partial y}\right)\mathbf{j} \\
&\quad + \left(u_z\frac{\partial v_z}{\partial z} - u_z\frac{\partial v_z}{\partial z}\right)\mathbf{k} \\
&= u_x\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{i} + u_y\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{j} + u_z\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{k} \\
&\quad - u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i} + \frac{\partial v_y}{\partial x}\mathbf{j} + \frac{\partial v_z}{\partial x}\mathbf{k}\right) - u_y\left(\frac{\partial v_x}{\partial y}\mathbf{i} + \frac{\partial v_y}{\partial y}\mathbf{j} + \frac{\partial v_z}{\partial y}\mathbf{k}\right) - u_z\left(\frac{\partial v_x}{\partial z}\mathbf{i} + \frac{\partial v_y}{\partial z}\mathbf{j} + \frac{\partial v_z}{\partial z}\mathbf{k}\right) \\
&= (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) - u_z\frac{\partial \mathbf{v}}{\partial x} - u_y\frac{\partial \mathbf{v}}{\partial y} - u_z\frac{\partial \mathbf{v}}{\partial z} \\
&= \mathbf{u}\left(\nabla\cdot\mathbf{v}\right) - \left[\left(\mathbf{u}\cdot\nabla\right)\mathbf{v}\right]
\end{aligned}$$

In a similar way it may be shown that the remaining three terms can be written as $-\mathbf{v}(\nabla\cdot\mathbf{u}) + [(\mathbf{v}\cdot\nabla)\mathbf{u}]$, and hence, we may conclude that

$$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\left(\nabla\cdot\mathbf{v}\right) - \mathbf{v}\left(\nabla\cdot\mathbf{u}\right) + \left[\left(\mathbf{v}\cdot\nabla\right)\mathbf{u}\right] - \left[\left(\mathbf{u}\cdot\nabla\right)\mathbf{v}\right]$$

15. Let the sphere be given by $F(x,y,z) = x^2 + y^2 + z^2 = 9$. The unit outer normal vector to the sphere is then given by

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)$$

Next, let $\mathbf{u} = (x^2 - z^2)(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. Then, with the help of (2.117)

$$\begin{aligned}
\frac{\partial}{\partial n}\left(\nabla\cdot\mathbf{u}\right) = \nabla\left(\nabla\cdot\mathbf{u}\right)\cdot\mathbf{n} &= \nabla\left[\frac{\partial}{\partial x}\left(x^2 - z^2\right) - \frac{\partial}{\partial y}\left(x^2 - z^2\right) + 3\frac{\partial}{\partial z}\left(x^2 - z^2\right)\right]\cdot\mathbf{n} \\
&= \nabla\left(2x - 6z\right)\cdot\mathbf{n} \\
&= (2\mathbf{i} - 6\mathbf{k})\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \\
&= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(2x - 6z\right)
\end{aligned}$$

Evaluating the result at the point $(2,2,1)$ then finally gives $-2/3$.

16. If a rigid body is rotating about the z-axis with angular velocity ω, then it is moving in a circular motion in the xy-plane. Hence, a particle of the body essentially follows a path equal to that of a point restricted to lie on a cylinder. Let r be the fixed radius

of the circle the path is constrained to move on in the xy-plane and let α be the initial angle of the particle in the xy-plane relative to the positive x-axis. Then, if ω is the angular velocity, at time t the particle will have moved through angle $\omega t + \alpha$. Since the particle is constrained to lie on the circle of radius r, its x-coordinate given by $r\cos(\omega t + \alpha)$ and its $y-$coordinate by $r\sin(\omega t + \alpha)$. As the particle is free to move in the z-plane, its z-coordinate is simply given by z. As such, a vector equation for the particle is given by

$$\overrightarrow{OP} = r\cos\left(\omega t + \alpha\right)\mathbf{i} + r\sin\left(\omega t + \alpha\right)\mathbf{j} + z\mathbf{k}$$

Next, let $\boldsymbol{\omega} = \omega\mathbf{k}$ be the angular velocity vector. Now the regular velocity of the particle is given by the vector $\mathbf{v}$, which is both perpendicular to the angular velocity vector (since by definition the angular velocity vector is perpendicular to the plane of rotation and hence, $\mathbf{v}$) and the position vector $\overrightarrow{OP}$. As such, it is given by

$$\begin{aligned}
\mathbf{v} &= \frac{d}{dt}\overrightarrow{OP} \\
&= -\omega r\sin\left(\omega t + \alpha\right)\mathbf{i} + \omega r\cos\left(\omega t + \alpha\right)\mathbf{j} \\
&= \left(\omega\mathbf{k}\right) \times \left[r\cos\left(\omega t + \alpha\right)\mathbf{i} + r\sin\left(\omega t + \alpha\right)\mathbf{j} + z\mathbf{k}\right] \\
&= \boldsymbol{\omega} \times \overrightarrow{OP}
\end{aligned}$$

Knowing this, the divergence and curl of $\mathbf{v}$ are given by

$$\begin{aligned}
\nabla\cdot\mathbf{v} = \nabla\cdot\left(\boldsymbol{\omega}\times\overrightarrow{OP}\right) &= \nabla\cdot\left[-\omega r\sin\left(\omega t + \alpha\right)\mathbf{i} + \omega r\cos\left(\omega t + \alpha\right)\mathbf{j}\right] \\
&= \nabla\cdot\left(-\omega y\mathbf{i} + \omega x\mathbf{j}\right) \\
&= \frac{\partial}{\partial x}\left(-\omega y\right) + \frac{\partial}{\partial y}\left(\omega x\right) \\
&= 0 \\
\nabla\times\mathbf{v} = \nabla\times\left(\boldsymbol{\omega}\times\overrightarrow{OP}\right) &= \nabla\times\left[-\omega r\sin\left(\omega t + \alpha\right)\mathbf{i} + \omega r\cos\left(\omega t + \alpha\right)\mathbf{j}\right] \\
&= -\omega\frac{\partial}{\partial z}r\cos\left(\omega t + \alpha\right)\mathbf{i} - \omega\frac{\partial}{\partial z}r\sin\left(\omega t + \alpha\right)\mathbf{j} \\
&\quad + \left[\omega\frac{\partial}{\partial x}r\cos\left(\omega t + \alpha\right) + \omega\frac{\partial}{\partial y}r\sin\left(\omega t + \alpha\right)\right]\mathbf{k} \\
&= -\omega\frac{\partial}{\partial z}x\mathbf{i} - \omega\frac{\partial}{\partial z}y\mathbf{j} + \left(\omega\frac{\partial}{\partial x}x + \omega\frac{\partial}{\partial y}y\right)\mathbf{k} \\
&= 2\omega\mathbf{k} \\
&= 2\boldsymbol{\omega}
\end{aligned}$$

17. Let a steady fluid in motion have the velocity vector $\mathbf{u} = y\mathbf{i} = d\mathbf{r}/dt$. Since $\mathbf{u}$ has no y or z components, the position of a point in the y and z directions does not

change with time (i.e. is constant). Hence, the position of a point at time t is given by $\mathbf{r}(t) = (c_2 t + c_1)\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, where c_1, c_2 and c_3 are some arbitrary constants. As such, the path of motion for each point of the vector field is a straight line when $c_2 \neq 0$. Furthermore, $\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \nabla \cdot (y\mathbf{i}) = (\partial/\partial x)y = 0$, and hence, the flow is incompressible. The relative rate of growth of a volume occupied by the fluid is roughly proportional to $\operatorname{div}\mathbf{u}$. To be exact; $\operatorname{div}\mathbf{u} = \lim_{\Delta t \to 0} \Delta V/(V\Delta t)$. Now since $\operatorname{div}\mathbf{u} = 0$ implies that $\Delta V = 0$, the volume occupied at time $t = 1$ will be the same as that at time $t = 0$, which is simply $V(t_0) = V(t_1) = 1$ for $t_0 = 0$ and $t_1 = 1$.

18. Let a steady fluid in motion have the velocity vector $\mathbf{u} = x\mathbf{i} = d\mathbf{r}/dt$. Since $\mathbf{u}$ has no y or z components (i.e. $dy/dt = 0$, $dz/dt = 0$), the position of a point in the y and z directions does not change with time. In other words, the position of a point has coordinates $y(t) = c_2$, $z(t) = c_3$, where c_2 and c_3 are arbitrary constants. For the x-coordinate however, we find that $dx/dt = x$, so that $x(t) = c_1 e^t$, where c_1 is the initial value of x at time $t = 0$. Hence, we find $\mathbf{r}(t) = c_1 e^t\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Furthermore, $\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \nabla \cdot (x\mathbf{i}) = (\partial/\partial x)x = 1$, and as such, the flow is *not* incompressible. Since
$$\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = 1 = \frac{1}{V}\frac{dV}{dt} \qquad \Longrightarrow \qquad V(t) = V_0 e^t$$
The volume at $t = 0$ is $V(0) = V_0 = 1$. Hence, at time $t = 1$ the volume will be $V(1) = e$.

Section 3.8

1. Let $u = F(x, y, z)$, $v = G(x, y, z)$, $w = H(x, y, z)$. Then, using the result of Problem 5 following Section 2.12, we can write

$$\begin{aligned}
\nabla F &= \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \\
&= \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} \\
&= \frac{1}{J}\left(\frac{\partial(y,z)}{\partial(v,w)}\mathbf{i} + \frac{\partial(z,x)}{\partial(v,w)}\mathbf{j} + \frac{\partial(x,y)}{\partial(v,w)}\mathbf{k}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial z}{\partial v}\frac{\partial y}{\partial w}\right)\mathbf{i} + \left(\frac{\partial z}{\partial v}\frac{\partial x}{\partial w} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial w}\right)\mathbf{j} + \left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial w} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial w}\right)\mathbf{k}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)
\end{aligned}$$

$$
\begin{aligned}
\nabla G &= \frac{\partial G}{\partial x}\mathbf{i} + \frac{\partial G}{\partial y}\mathbf{j} + \frac{\partial G}{\partial z}\mathbf{k} \\
&= \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k} \\
&= \frac{1}{J}\left(\frac{\partial (y,z)}{\partial (w,u)}\mathbf{i} + \frac{\partial (z,x)}{\partial (w,u)}\mathbf{j} + \frac{\partial (x,y)}{\partial (w,u)}\mathbf{k}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial y}{\partial w}\frac{\partial z}{\partial u} - \frac{\partial z}{\partial w}\frac{\partial y}{\partial u}\right)\mathbf{i} + \left(\frac{\partial z}{\partial w}\frac{\partial x}{\partial u} - \frac{\partial x}{\partial w}\frac{\partial z}{\partial u}\right)\mathbf{j} + \left(\frac{\partial x}{\partial w}\frac{\partial y}{\partial u} - \frac{\partial y}{\partial w}\frac{\partial x}{\partial u}\right)\mathbf{k}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla H &= \frac{\partial H}{\partial x}\mathbf{i} + \frac{\partial H}{\partial y}\mathbf{j} + \frac{\partial H}{\partial z}\mathbf{k} \\
&= \frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} + \frac{\partial w}{\partial z}\mathbf{k} \\
&= \frac{1}{J}\left(\frac{\partial (y,z)}{\partial (u,v)}\mathbf{i} + \frac{\partial (z,x)}{\partial (u,v)}\mathbf{j} + \frac{\partial (x,y)}{\partial (u,v)}\mathbf{k}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\mathbf{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\mathbf{k}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)
\end{aligned}
$$

2.

$$
\begin{aligned}
\nabla F \cdot \nabla G \times \nabla H &= \underbrace{\frac{1}{J^3}\left[\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right) \times \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\right]}_{(3.48)} \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \cdot \underbrace{\left[\left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial u} - \left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial w}\right]}_{(1.19)} \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial u} \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\underbrace{\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)}_{(1.34)} = \frac{1}{J}
\end{aligned}
$$

where the last step follows from (3.44):

$$
J = \frac{\partial (x,y,z)}{\partial (u,v,w)} = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}
$$

Hence, this proves that

$$J = \frac{1}{\dfrac{\partial\left(u,v,w\right)}{\partial\left(x,y,z\right)}} = \frac{1}{\nabla F\cdot\nabla G\times\nabla H}$$

3.

$$\begin{aligned}
J\left(\nabla G\times\nabla H\right) &= \underbrace{\frac{J}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\times\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)}_{(3.48)}\\
&= \frac{1}{J}\underbrace{\left[\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\frac{\partial\mathbf{r}}{\partial u}-\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\frac{\partial\mathbf{r}}{\partial w}\right]}_{(1.19)}\\
&= \frac{1}{J}\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\frac{\partial\mathbf{r}}{\partial u}\\
&= \frac{1}{J}\underbrace{(J)}_{(3.44)}\frac{\partial\mathbf{r}}{\partial u} = \frac{\partial\mathbf{r}}{\partial u}
\end{aligned}$$

$$\begin{aligned}
J\left(\nabla H\times\nabla F\right) &= \frac{1}{J}\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\times\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\\
&= \frac{1}{J}\left[\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\frac{\partial\mathbf{r}}{\partial v}-\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\frac{\partial\mathbf{r}}{\partial u}\right]\\
&= \frac{1}{J}\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\frac{\partial\mathbf{r}}{\partial v} = \frac{\partial\mathbf{r}}{\partial v}
\end{aligned}$$

$$\begin{aligned}
J\left(\nabla F\times\nabla G\right) &= \frac{1}{J}\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\times\left(\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\\
&= \frac{1}{J}\left[\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\frac{\partial\mathbf{r}}{\partial w}-\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\frac{\partial\mathbf{r}}{\partial v}\right]\\
&= \frac{1}{J}\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\frac{\partial\mathbf{r}}{\partial w} = \frac{\partial\mathbf{r}}{\partial w}
\end{aligned}$$

4. If the vectors ∇F, ∇G, ∇H are mutually perpendicular in D, then

$$\nabla F\cdot\nabla G = 0 \qquad \nabla F\cdot\nabla H = 0 \qquad \nabla G\cdot\nabla H = 0$$

Furthermore, note that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are arbitrary vectors in D then

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= [(a_y b_z - a_z b_y)\,\mathbf{i} + (a_z b_x - a_x b_z)\,\mathbf{j} + (a_x b_y - a_y b_x)\,\mathbf{k}] \\
&\quad \cdot [(c_y d_z - c_z d_y)\,\mathbf{i} + (c_z d_x - c_x d_z)\,\mathbf{j} + (c_x d_y - c_y d_x)\,\mathbf{k}] \\
&= (a_y b_z - a_z b_y)(c_y d_z - c_z d_y) + (a_z b_x - a_x b_z)(c_z d_x - c_x d_z) \\
&\quad + (a_x b_y - a_y b_x)(c_x d_y - c_y d_x) + a_x b_x c_x d_x - a_x b_x c_x d_x + a_y b_y c_y d_y \\
&\quad - a_y b_y c_y d_y + a_z b_z c_z d_z - a_z b_z c_z d_z \\
&= a_x b_x c_x d_x + a_x b_y c_x d_y + a_x b_z c_x d_z + a_y b_x c_y d_x + a_y b_y c_y d_y + a_y b_z c_y d_z \\
&\quad + a_z b_x c_z d_x + a_z b_y c_z d_y + a_z b_z c_z d_z - a_x b_x c_x d_x - a_x b_y c_y d_x - a_x b_z c_z d_x \\
&\quad - a_y b_x c_x d_y - a_y b_y c_y d_y - a_y b_z c_z d_y - a_z b_x c_x d_z - a_z b_y c_y d_z - a_z b_z c_z d_z \\
&= (a_x c_x + a_y c_y + a_z c_z)(b_x d_x + b_y d_y + b_z d_z) \\
&\quad - (b_x c_x + b_y c_y + b_z c_z)(a_x d_x + a_y d_y + a_z d_z) \\
&= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})
\end{aligned}$$

Using (3.48) and the vector identity above we can form the three equations

$$\begin{aligned}
\nabla F\cdot\nabla G &= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right) \\
&= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial w}\right)\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\right) - \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial w}\right)\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}\right) = 0
\end{aligned}$$

$$\begin{aligned}
\nabla F\cdot\nabla H &= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right) \\
&= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial v}\right) - \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial v}\right) = 0
\end{aligned}$$

$$\begin{aligned}
\nabla G\cdot\nabla H &= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right) \\
&= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}\right) - \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial v}\right) = 0
\end{aligned}$$

For these equations to make sense it is sufficient to require that

$$\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u} = 0 \qquad \frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u} = 0 \qquad \frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial v} = 0$$

In other words, the tangent vectors $\partial\mathbf{r}/\partial u$, $\partial\mathbf{r}/\partial v$, $\partial\mathbf{r}/\partial w$ form a triple of mutually perpendicular vectors at each point of D and hence, the coordinates are orthogonal.

5. Using (3.56) we can write

$$\mathbf{p} = (\alpha p_u)\left(\frac{1}{\alpha^2}\frac{\partial\mathbf{r}}{\partial u}\right) + (\beta p_v)\left(\frac{1}{\beta^2}\frac{\partial\mathbf{r}}{\partial v}\right) + (\gamma p_w)\left(\frac{1}{\gamma^2}\frac{\partial\mathbf{r}}{\partial w}\right)$$

The curl of $\mathbf{p}$ is the sum of the curls of the terms on the right-hand side. By (3.27), (3.28) and (3.55) we can thus write

$$\begin{aligned}
\nabla \times \mathbf{p} &= \nabla \times \left[(\alpha p_u)\left(\frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right) + (\beta p_v)\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right) + (\gamma p_w)\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right)\right] \\
&= \left[\nabla \times (\alpha p_u)\left(\frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right)\right] + \left[\nabla \times (\beta p_v)\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right)\right] + \left[\nabla \times (\gamma p_w)\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right)\right] \\
&= (\alpha p_u)\left(\nabla \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right) + \left[(\nabla \alpha p_u) \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right] + (\beta p_v)\left(\nabla \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right) \\
&\quad + \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right] + (\gamma p_w)\left(\nabla \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right) + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right] \\
&= \left[(\nabla \alpha p_u) \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right] + \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right] + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]
\end{aligned}$$

The u component of $\nabla \times \mathbf{p}$ can be obtained by taking the scalar product of both sides of the equation above with $(1/\alpha)(\partial \mathbf{r}/\partial u)$, giving

$$\begin{aligned}
[\nabla \times \mathbf{p}]_u &= \nabla \times \mathbf{p} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u} \\
&= \underbrace{\left[(\nabla \alpha p_u) \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u}\right]}_{0} + \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u}\right] + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u}\right] \\
&= \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right]_u + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]_u \\
&= \underbrace{[\nabla \beta p_v]_v \left[\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right]_w - [\nabla \beta p_v]_w \left[\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right]_v}_{(3.59)} + [\nabla \gamma p_w]_v \left[\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]_w - [\nabla \gamma p_w]_w \left[\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]_v \\
&= \underbrace{\left(\frac{\beta}{\beta}\frac{\partial p_v}{\partial v}\right)}_{(3.60)} \underbrace{\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial w}\right)}_{0} - \left(\frac{\beta}{\gamma}\frac{\partial p_v}{\partial w}\right) \underbrace{\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial v}\right)}_{1/\beta} \\
&\quad + \left(\frac{\gamma}{\beta}\frac{\partial p_w}{\partial v}\right) \underbrace{\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial w}\right)}_{1/\gamma} - \left(\frac{\gamma}{\gamma}\frac{\partial p_w}{\partial w}\right) \underbrace{\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial v}\right)}_{0} \\
&= \frac{1}{\beta}\frac{\partial p_w}{\partial v} - \frac{1}{\gamma}\frac{\partial p_v}{\partial w} = \frac{1}{\beta\gamma}\left[\frac{\partial}{\partial v}(\gamma p_w) - \frac{\partial}{\partial w}(\beta p_v)\right]
\end{aligned}$$

The remaining components can be found in exactly the same way.

6. (a) Cylindrical coordinates are given by the relations

$$x = f(r, \theta, z) = r\cos\theta \qquad y = g(r, \theta, z) = r\sin\theta \qquad z = h(r, \theta, z) = z$$

and

$$r = F(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \theta = G(x,y,z) = \tan^{-1}\frac{y}{x} \quad z = H(x,y,z) = z$$

Now since

$$(\alpha\nabla F)\cdot(\beta\nabla G)\times(\gamma\nabla H) = \underbrace{\left(\frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial r}\right)\cdot\left(\frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\times\left(\frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial z}\right)}_{(3.52)}$$

$$= (\cos\theta\mathbf{i}+\sin\theta\mathbf{j})\cdot\left[\frac{1}{r}(-r\sin\theta\mathbf{i}+r\cos\theta\mathbf{j})\times\mathbf{k}\right] = 1$$

where $\alpha = 1$, $\beta = r$, $\gamma = 1$, the vectors $\alpha\nabla F$, $\beta\nabla G$, $\gamma\nabla H$ are mutually perpendicular unit vectors. Hence, the surfaces $F = r = \text{const}$, $G = \theta = \text{const}$, $H = z = \text{const}$ must meet at right angles and thus form a triply orthogonal family of surfaces. Furthermore, by (3.51)

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

(b) By (3.54) we conclude

$$ds^2 = \alpha^2 dr^2 + \beta^2 d\theta^2 + \gamma^2 dz^2 = dr^2 + r^2 d\theta^2 + dz^2$$

(c) Using (3.57), we find

$$p_r = \mathbf{p}\cdot\frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial r} = \frac{1}{\alpha}\left(p_x\frac{\partial x}{\partial r}+p_y\frac{\partial y}{\partial r}+p_z\frac{\partial z}{\partial r}\right) = p_x\cos\theta + p_y\sin\theta$$
$$p_\theta = \mathbf{p}\cdot\frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\beta}\left(p_x\frac{\partial x}{\partial \theta}+p_y\frac{\partial y}{\partial \theta}+p_z\frac{\partial z}{\partial \theta}\right) = -p_x\sin\theta + p_y\cos\theta$$
$$p_z = \mathbf{p}\cdot\frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial z} = \frac{1}{\gamma}\left(p_x\frac{\partial x}{\partial z}+p_y\frac{\partial y}{\partial z}+p_z\frac{\partial z}{\partial z}\right) = p_z$$

(d) By (3.60) we find

$$[\nabla U]_r = \frac{1}{\alpha}\frac{\partial U}{\partial r} = \frac{\partial U}{\partial r} \qquad [\nabla U]_\theta = \frac{1}{\beta}\frac{\partial U}{\partial \theta} = \frac{1}{r}\frac{\partial U}{\partial \theta} \qquad [\nabla U]_z = \frac{1}{\gamma}\frac{\partial U}{\partial z} = \frac{\partial U}{\partial z}$$

(e) By (3.61) we find

$$\nabla\cdot\mathbf{p} = \frac{1}{\alpha\beta\gamma}\left[\frac{\partial}{\partial r}(\beta\gamma p_r)+\frac{\partial}{\partial \theta}(\alpha\gamma p_\theta)+\frac{\partial}{\partial z}(\alpha\beta p_z)\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(rp_r)+\frac{\partial p_\theta}{\partial \theta}+r\frac{\partial p_z}{\partial z}\right]$$

(f) By (3.62) we find

$$\begin{aligned}
[\nabla\times\mathbf{p}]_r &= \frac{1}{\beta\gamma}\left[\frac{\partial}{\partial\theta}(\gamma p_z)-\frac{\partial}{\partial z}(\beta p_\theta)\right] = \frac{1}{r}\left[\frac{\partial p_z}{\partial\theta}-r\frac{\partial p_\theta}{\partial z}\right]\\
[\nabla\times\mathbf{p}]_\theta &= \frac{1}{\alpha\gamma}\left[\frac{\partial}{\partial z}(\alpha p_r)-\frac{\partial}{\partial r}(\gamma p_z)\right] = \frac{\partial p_r}{\partial z}-\frac{\partial p_z}{\partial r}\\
[\nabla\times\mathbf{p}]_z &= \frac{1}{\alpha\beta}\left[\frac{\partial}{\partial r}(\beta p_\theta)-\frac{\partial}{\partial\theta}(\alpha p_r)\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(r p_\theta)-\frac{\partial p_r}{\partial\theta}\right]
\end{aligned}$$

(g) From (3.56) and (3.60) it follows that

$$\nabla U = [\nabla U]_r\,\frac{1}{\alpha}\frac{\partial\mathbf{r}}{\partial r}+[\nabla U]_\theta\,\frac{1}{\beta}\frac{\partial\mathbf{r}}{\partial\theta}+[\nabla U]_z\,\frac{1}{\gamma}\frac{\partial\mathbf{r}}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial\mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{\partial U}{\partial z}\frac{\partial\mathbf{r}}{\partial z}$$

Furthermore, from part (a) we known that

$$\frac{\partial\mathbf{r}}{\partial r}=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}\qquad\qquad \frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}=-\sin\theta\mathbf{i}+\cos\theta\mathbf{j}\qquad\qquad \frac{\partial\mathbf{r}}{\partial z}=\mathbf{k}$$

And so

$$\begin{aligned}
\nabla^2 U &= \nabla\cdot(\nabla U)\\
&= \left(\frac{\partial}{\partial r}\frac{\partial\mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{\partial}{\partial z}\frac{\partial\mathbf{r}}{\partial z}\right)\cdot\left(\frac{\partial U}{\partial r}\frac{\partial\mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{\partial U}{\partial z}\frac{\partial\mathbf{r}}{\partial z}\right)\\
&= \frac{\partial\mathbf{r}}{\partial r}\cdot\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}\frac{\partial\mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{\partial U}{\partial z}\frac{\partial\mathbf{r}}{\partial z}\right)+\frac{1}{r^2}\frac{\partial\mathbf{r}}{\partial\theta}\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial U}{\partial r}\frac{\partial\mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{\partial U}{\partial z}\frac{\partial\mathbf{r}}{\partial z}\right)\\
&\quad+\frac{\partial\mathbf{r}}{\partial z}\cdot\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial r}\frac{\partial\mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{\partial U}{\partial z}\frac{\partial\mathbf{r}}{\partial z}\right)\\
&= \frac{\partial\mathbf{r}}{\partial r}\cdot\left[\frac{\partial^2 U}{\partial r^2}\frac{\partial\mathbf{r}}{\partial r}+\frac{\partial U}{\partial r}\underbrace{\frac{\partial^2\mathbf{r}}{\partial r^2}}_{0}-\frac{1}{r^2}\frac{\partial U}{\partial\theta}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2 U}{\partial\theta\partial r}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial U}{\partial\theta}\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)}_{0}\right]\\
&\quad+\frac{\partial\mathbf{r}}{\partial r}\cdot\left[\frac{\partial^2 U}{\partial z\partial r}\frac{\partial\mathbf{r}}{\partial z}+\frac{\partial U}{\partial z}\underbrace{\frac{\partial^2\mathbf{r}}{\partial z^2}}_{0}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{r^2}\frac{\partial\mathbf{r}}{\partial\theta}\cdot\left[\frac{\partial^2U}{\partial r\partial\theta}\frac{\partial\mathbf{r}}{\partial r}+\frac{\partial U}{\partial r}\underbrace{\frac{\partial^2\mathbf{r}}{\partial r\partial\theta}}_{(1/r)(\partial\mathbf{r}/\partial\theta)}+\frac{1}{r}\frac{\partial^2U}{\partial\theta^2}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial U}{\partial\theta}\underbrace{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)}_{-\partial\mathbf{r}/\partial r}\right]\\
&+\frac{1}{r^2}\frac{\partial\mathbf{r}}{\partial\theta}\cdot\left[\frac{\partial^2U}{\partial z\partial\theta}\frac{\partial\mathbf{r}}{\partial z}+\frac{\partial U}{\partial z}\underbrace{\frac{\partial^2\mathbf{r}}{\partial z\partial\theta}}_{0}\right]\\
&+\frac{\partial\mathbf{r}}{\partial z}\cdot\left[\frac{\partial^2U}{\partial r\partial z}\frac{\partial\mathbf{r}}{\partial r}+\frac{\partial U}{\partial r}\underbrace{\frac{\partial^2\mathbf{r}}{\partial r\partial z}}_{0}+\frac{1}{r^2}\frac{\partial^2U}{\partial\theta\partial z}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{1}{r}\frac{\partial U}{\partial\theta}\underbrace{\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)}_{0}+\frac{\partial^2U}{\partial z^2}\frac{\partial\mathbf{r}}{\partial z}+\frac{\partial U}{\partial z}\underbrace{\frac{\partial^2\mathbf{r}}{\partial z^2}}_{0}\right]\\
&=\frac{\partial\mathbf{r}}{\partial r}\cdot\left[\frac{\partial^2U}{\partial r^2}\frac{\partial\mathbf{r}}{\partial r}-\frac{1}{r^2}\frac{\partial U}{\partial\theta}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2U}{\partial\theta\partial r}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)+\frac{\partial^2U}{\partial z\partial r}\frac{\partial\mathbf{r}}{\partial z}\right]\\
&\quad+\frac{1}{r^2}\frac{\partial\mathbf{r}}{\partial\theta}\cdot\left[\frac{\partial^2U}{\partial r\partial\theta}\frac{\partial\mathbf{r}}{\partial r}+\frac{\partial U}{\partial r}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2U}{\partial\theta^2}\left(\frac{1}{r}\frac{\partial\mathbf{r}}{\partial\theta}\right)-\frac{1}{r}\frac{\partial U}{\partial\theta}\frac{\partial\mathbf{r}}{\partial r}+\frac{\partial^2U}{\partial z\partial\theta}\frac{\partial\mathbf{r}}{\partial z}\right]\\
&\quad+\frac{\partial\mathbf{r}}{\partial z}\cdot\left[\frac{\partial^2U}{\partial r\partial z}\frac{\partial\mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial^2U}{\partial\theta\partial z}\frac{\partial\mathbf{r}}{\partial\theta}+\frac{\partial^2U}{\partial z^2}\frac{\partial\mathbf{r}}{\partial z}\right]\\
&=\frac{\partial^2U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2U}{\partial\theta^2}+\frac{\partial^2U}{\partial z^2}\\
&=\frac{1}{r^2}\left[r^2\frac{\partial^2U}{\partial r^2}+r\frac{\partial U}{\partial r}+\frac{\partial^2U}{\partial\theta^2}+r^2\frac{\partial^2U}{\partial z^2}\right]=\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2U}{\partial\theta^2}+r^2\frac{\partial^2U}{\partial z^2}\right]
\end{aligned}
$$

7. (a) Spherical coordinates are given by the relations

$$x=f(\rho,\phi,\theta)=\rho\sin\phi\cos\theta\quad y=g(\rho,\phi,\theta)=\rho\sin\phi\sin\theta\quad z=h(\rho,\phi,\theta)=\rho\cos\phi$$

and

$$\rho=F(x,y,z)=\sqrt{x^2+y^2+z^2}\quad\phi=G(x,y,z)=\cos^{-1}\frac{z}{\rho}\quad\theta=H(x,y,z)=\tan^{-1}\frac{y}{x}$$

Now since

$$
\begin{aligned}
(\alpha\nabla F)\cdot(\beta\nabla G)\times(\gamma\nabla H)&=\underbrace{\left(\frac{1}{\alpha}\frac{\partial\mathbf{r}}{\partial\rho}\right)\cdot\left(\frac{1}{\beta}\frac{\partial\mathbf{r}}{\partial\phi}\right)\times\left(\frac{1}{\gamma}\frac{\partial\mathbf{r}}{\partial\theta}\right)}_{(3.52)}\\
&=(\sin\phi\cos\theta\mathbf{i}+\sin\phi\sin\theta\mathbf{j}+\cos\phi\mathbf{k})\\
&\quad\cdot\frac{1}{\rho}(\rho\cos\phi\cos\theta\mathbf{i}+\rho\cos\phi\sin\theta\mathbf{j}-\rho\sin\phi\mathbf{k})\\
&\quad\times\frac{1}{\rho\sin\phi}(-\rho\sin\phi\sin\theta\mathbf{i}+\rho\sin\phi\cos\theta\mathbf{j})=1
\end{aligned}
$$

where $\alpha = 1$, $\beta = \rho$, $\gamma = \rho \sin\phi$, the vectors $\alpha\nabla F$, $\beta\nabla G$, $\gamma\nabla H$ are mutually perpendicular unit vectors. Hence, the surfaces $F = \rho = \text{const}$, $G = \phi = \text{const}$, $H = \theta = \text{const}$ must meet at right angles and thus form a triply orthogonal family of surfaces. Furthermore, by (3.51)

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin\phi$$

(b) By (3.54) we conclude

$$ds^2 = \alpha^2 d\rho^2 + \beta^2 d\phi^2 + \gamma^2 d\theta^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi d\theta^2$$

(c) Using (3.57) we find

$$\begin{aligned}
p_\rho &= \mathbf{p} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial \rho} = \frac{1}{\alpha}\left(p_x \frac{\partial x}{\partial \rho} + p_y \frac{\partial y}{\partial \rho} + p_z \frac{\partial z}{\partial \rho}\right) = p_x \sin\phi\cos\theta + p_y \sin\phi\sin\theta + p_z\cos\phi \\
p_\phi &= \mathbf{p} \cdot \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial \phi} = \frac{1}{\beta}\left(p_x \frac{\partial x}{\partial \phi} + p_y \frac{\partial y}{\partial \phi} + p_z \frac{\partial z}{\partial \phi}\right) = p_x \cos\phi\cos\theta + p_y \cos\phi\sin\theta - p_z\sin\phi \\
p_\theta &= \mathbf{p} \cdot \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\gamma}\left(p_x \frac{\partial x}{\partial \theta} + p_y \frac{\partial y}{\partial \theta} + p_z \frac{\partial z}{\partial \theta}\right) = -p_x \sin\theta + p_y\cos\theta
\end{aligned}$$

(d) By (3.60) we find

$$[\nabla U]_\rho = \frac{1}{\alpha}\frac{\partial U}{\partial \rho} = \frac{\partial U}{\partial \rho} \quad [\nabla U]_\phi = \frac{1}{\beta}\frac{\partial U}{\partial \phi} = \frac{1}{\rho}\frac{\partial U}{\partial \phi} \quad [\nabla U]_\theta = \frac{1}{\gamma}\frac{\partial U}{\partial \theta} = \frac{1}{\rho\sin\phi}\frac{\partial U}{\partial \theta}$$

(e) By (3.61) we find

$$\begin{aligned}
\nabla \cdot \mathbf{p} &= \frac{1}{\alpha\beta\gamma}\left[\frac{\partial}{\partial \rho}(\beta\gamma p_\rho) + \frac{\partial}{\partial \phi}(\alpha\gamma p_\phi) + \frac{\partial}{\partial \theta}(\alpha\beta p_\theta)\right] \\
&= \frac{1}{\rho^2\sin\phi}\left[\sin\phi\frac{\partial}{\partial \rho}(\rho^2 p_\rho) + \rho\frac{\partial}{\partial \phi}(p_\phi \sin\phi) + \rho\frac{\partial p_\theta}{\partial \theta}\right]
\end{aligned}$$

(f) By (3.62) we find

$$\begin{aligned}
[\nabla \times \mathbf{p}]_\rho &= \frac{1}{\beta\gamma}\left[\frac{\partial}{\partial \phi}(\gamma p_\theta) - \frac{\partial}{\partial \theta}(\beta p_\phi)\right] = \frac{1}{\rho\sin\phi}\left[\frac{\partial}{\partial \phi}(p_\theta \sin\phi) - \frac{\partial p_\phi}{\partial \theta}\right] \\
[\nabla \times \mathbf{p}]_\phi &= \frac{1}{\alpha\gamma}\left[\frac{\partial}{\partial \theta}(\alpha p_\rho) - \frac{\partial}{\partial \rho}(\gamma p_\theta)\right] = \frac{1}{\rho\sin\phi}\left[\frac{\partial p_\rho}{\partial \theta} - \sin\phi\frac{\partial}{\partial \rho}(\rho p_\theta)\right] \\
[\nabla \times \mathbf{p}]_\theta &= \frac{1}{\alpha\beta}\left[\frac{\partial}{\partial \rho}(\beta p_\phi) - \frac{\partial}{\partial \phi}(\alpha p_\rho)\right] = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho p_\phi) - \frac{\partial p_\rho}{\partial \phi}\right]
\end{aligned}$$

Note that there is a typo in the book for the second term of the first component, i.e. $\partial p_\phi/\partial\phi$ should be $\partial p_\phi/\partial\theta$.

(g) From (3.56) and (3.60) it follows that

$$\nabla U = [\nabla U]_\rho \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial \rho} + [\nabla U]_\phi \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial \phi} + [\nabla U]_\theta \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}$$

Furthermore, from part (a) we know that

$$\frac{\partial \mathbf{r}}{\partial \rho} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k} \quad \frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi} = \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k}$$
$$\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

And so

$$\begin{aligned}
\nabla^2 U &= \nabla\cdot(\nabla U)\\
&= \left(\frac{\partial}{\partial\rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial}{\partial\phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial}{\partial\theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\cdot\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&= \frac{\partial \mathbf{r}}{\partial \rho}\cdot\frac{\partial}{\partial\rho}\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&\quad + \frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&\quad + \frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&= \frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{\partial^2 U}{\partial\rho^2}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{\partial U}{\partial \rho}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial \rho^2}}_{0} - \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right) + \frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\partial\rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right) + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\underbrace{\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)}_{0}\right]\\
&\quad + \frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[-\frac{1}{\rho^2\sin\phi}\frac{\partial U}{\partial \theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right) + \frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta\partial\rho}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&\quad + \frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial \theta}\underbrace{\frac{\partial}{\partial\rho}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)}_{0}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\phi}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial\rho\partial\phi}}_{(1/\rho)(\partial\mathbf{r}/\partial\phi)}+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi^2}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial U}{\partial \phi}\underbrace{\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)}_{-\partial\mathbf{r}/\partial\rho}\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[-\frac{\cos\phi}{\rho\sin^2\phi}\frac{\partial U}{\partial \theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta\partial\phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial \phi}\underbrace{\frac{\partial}{\partial\phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)}_{0}\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\theta}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial\rho\partial\theta}}_{(1/\rho)(\partial\mathbf{r}/\partial\theta)}+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\partial\theta}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial U}{\partial \phi}\underbrace{\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)}_{(\cot(\phi)/\rho)(\partial\mathbf{r}/\partial\theta)}\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta^2}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial \theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
=&\frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{\partial^2 U}{\partial\rho^2}\frac{\partial \mathbf{r}}{\partial \rho}-\frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\partial\rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)-\frac{1}{\rho^2\sin\phi}\frac{\partial U}{\partial \theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta\partial\rho}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\phi}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi^2}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)-\frac{1}{\rho}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \rho}\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[-\frac{\cos\phi}{\rho\sin^2\phi}\frac{\partial U}{\partial \theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta\partial\phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\theta}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\partial\theta}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{\cos\phi}{\rho}\frac{\partial U}{\partial \phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta^2}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial \theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
=&\frac{\partial^2 U}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial U}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2}+\frac{\cot\phi}{\rho^2}\frac{\partial U}{\partial \phi}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 U}{\partial\theta^2}\\
=&\frac{1}{\rho^2\sin^2\phi}\left[\rho^2\sin^2\phi\frac{\partial^2 U}{\partial\rho^2}+2\rho\sin^2\phi\frac{\partial U}{\partial \rho}+\sin^2\phi\frac{\partial^2 U}{\partial\phi^2}+\sin\phi\cos\phi\frac{\partial U}{\partial \phi}+\frac{\partial^2 U}{\partial\theta^2}\right]\\
=&\frac{1}{\rho^2\sin^2\phi}\left[\sin^2\phi\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial U}{\partial \rho}\right)+\sin\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial U}{\partial \phi}\right)+\frac{\partial^2 U}{\partial\theta^2}\right]
\end{aligned}
$$

9. Assuming that for each surface of Problem 8 the functions f, g, h have continuous first

partial derivatives in D and that the Jacobian matrix

$$\begin{pmatrix} f_u & g_u & h_u \\ f_v & g_v & h_v \end{pmatrix}^\top$$

has rank 2 in D, then we can apply the Implicit Function Theorem of Section 2.10, as in Section 2.12 to show that the inverse functions $u = \phi(x, y)$, $v = \psi(x, y)$ of $x = f(u, v)$, $y = g(u, v)$ is well defined in a neighborhood D_0 of a point (u_0, v_0) in D, under the condition that the Jacobian of the mapping $\partial(f, g)/\partial(u, v) \neq 0$ at the point (u_0, v_0).

For the sphere: $x = f(u, v) = \sin u \cos v$, $y = g(u, v) = \sin u \sin v$, $z = h(u, v) = \cos u$ we can define the implicit equations

$$F(x, y, u, v) = f(u, v) - x = 0 \qquad G(x, y, u, v) = g(u, v) - y = 0$$

Then by (2.61) we find

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\cos u} \qquad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\sin v}{\cos u}$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & -1 \\ g_u & 0 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = -\frac{\sin v}{\sin u} \qquad \frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & 0 \\ g_u & -1 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\sin u}$$

Hence, as long as $u \neq n\pi/2$, where $n = 0, \pm1, \pm2 \ldots$, the inverse mapping will be well defined.

For the cylinder: $x = \cos u$, $y = \sin u$, $z = v$ we find

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,H)}{\partial(x,v)}}{\dfrac{\partial(F,H)}{\partial(u,v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & h_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = -\frac{1}{\sin u} \qquad \frac{\partial u}{\partial z} = -\frac{\dfrac{\partial(F,H)}{\partial(z,v)}}{\dfrac{\partial(F,H)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & h_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = 0$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F,H)}{\partial(u,x)}}{\dfrac{\partial(F,H)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & -1 \\ h_u & 0 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = 0 \qquad \frac{\partial v}{\partial z} = -\frac{\dfrac{\partial(F,H)}{\partial(u,z)}}{\dfrac{\partial(F,H)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & 0 \\ h_u & -1 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = 1$$

Hence, as long as $u \neq n\pi$, where $n = 0, \pm1, \pm2, \ldots$, the inverse mapping will be well defined and is given by $u = \tan^{-1} y/x$, $v = z$.

For the cone: $x = \sinh u \sin v$, $y = \sinh u \cos v$, $z = \sinh u$ we find

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\sin v}{\cosh u} \quad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\cosh u}$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & -1 \\ g_u & 0 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\sinh u} \quad \frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & 0 \\ g_u & -1 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = -\frac{\sin v}{\sinh u}$$

Hence, as long as $u \neq 0$, the inverse mapping will be well defined.

10. (a) Let a surface S be given as in Problems 8 and 9 and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point (x, y, z). The equations $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ can then be interpreted as defining a vector function $\mathbf{r} = \mathbf{r}(u, v)$. When $v = v_0 = \text{const}$, this is the vector representation $\mathbf{r} = \mathbf{r}(u, v_0)$ of one of a family curves obtained by varying u for different values of $v = v_0 = \text{const}$. The tangent vector to this curve is defined as in Section 2.13 to be the derivative of $\mathbf{r}$ with respect to the parameter u: $\partial\mathbf{r}/\partial u$. Similarly, fixing $u = u_0 = \text{const}$ while allowing v to vary results in one of a family of curves obtained by varying v for different values of $u = u_0 = \text{const}$, and the tangent vector to this curve is $\partial\mathbf{r}/\partial v$.

(b) If the curves $v = \text{const}$, $u = \text{const}$ intersect at right angles, then this implies that the corresponding tangent vectors to these curves, $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ respectively, are perpendicular at the point of intersection, i.e.:

$$\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right)\cdot\left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right) = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0$$

(c) The element of arc on a curve $u = u(t)$, $v = v(t)$ on S is given by

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)^2 + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)^2 + \left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right)^2 \\
&= \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]du^2 + \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]dv^2 \\
&\quad + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}dudv + 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}dudv + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}dudv \\
&= \left|\frac{\partial\mathbf{r}}{\partial u}\right|^2 du^2 + \left|\frac{\partial\mathbf{r}}{\partial v}\right|^2 dv^2 + 2\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}dudv \\
&= E du^2 + G dv^2 + 2F dudv
\end{aligned}$$

(d) For part (b) it was shown that the coordinates are orthogonal if and only if $(\partial\mathbf{r}/\partial u)\cdot(\partial\mathbf{r}/\partial v)=0$. Hence, for the element of arc ds^2 this implies

$$ds^2=\left|\frac{\partial\mathbf{r}}{\partial u}\right|^2du^2+\left|\frac{\partial\mathbf{r}}{\partial v}\right|^2dv^2+2\underbrace{\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}}_{0}dudv=\left|\frac{\partial\mathbf{r}}{\partial u}\right|^2du^2+\left|\frac{\partial\mathbf{r}}{\partial v}\right|^2dv^2$$

(e) Let $u=u(t)$, $v=v(t)$ and $u=U(\tau)$, $v=V(\tau)$ be two curves on S meeting at a point P_0 of S for $t=t_0$, $\tau=\tau_0$, so that $u(t_0)=u_0=U(\tau_0)$, $v(t_0)=v_0=V(\tau_0)$. Then, using (1.9), the angle θ between the corresponding velocity vectors $\partial\mathbf{r}/dt$ at t_0 and $\partial\mathbf{r}/d\tau$ at τ_0 (assumed both to be non-zero) is given by

$$\begin{aligned}
\cos\theta&=\frac{\dfrac{d\mathbf{r}}{dt}\cdot\dfrac{d\mathbf{r}}{d\tau}}{\left|\dfrac{d\mathbf{r}}{dt}\right|\left|\dfrac{d\mathbf{r}}{d\tau}\right|}\\
&=\frac{\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right)\cdot\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right)}{\left|\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right|\left|\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right|}\\
&=\frac{\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right)\cdot\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right)}{\left[\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right)^2\right]^{1/2}\left[\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau}+\dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right)^2\right]^{1/2}}\\
&=\frac{E\dfrac{du}{dt}\dfrac{du}{d\tau}+G\dfrac{dv}{dt}\dfrac{dv}{d\tau}+F\left(\dfrac{du}{dt}\dfrac{dv}{d\tau}+\dfrac{dv}{dt}\dfrac{du}{d\tau}\right)}{\left[E\left(\dfrac{du}{dt}\right)^2+G\left(\dfrac{dv}{dt}\right)^2+2F\dfrac{du}{dt}\dfrac{dv}{dt}\right]^{1/2}\left[E\left(\dfrac{du}{d\tau}\right)^2+G\left(\dfrac{dv}{d\tau}\right)^2+2F\dfrac{du}{d\tau}\dfrac{dv}{d\tau}\right]^{1/2}}\\
&=\frac{Eu'U'+Gv'V'+F\left(u'V'+v'U'\right)}{\left(Eu'^2+gv'^2+2Fu'v'\right)^{1/2}\left(EU'^2+GV'^2+2FU'V'\right)^{1/2}}
\end{aligned}$$

where E, F, G are evaluated at (u_0,v_0) and $u'=u'(t_0)$, $v'=v'(t_0)$, $U'=U'(\tau_0)$, $V'=V'(\tau_0)$.

(f) If the paths of part (e) are the coordinate lines

$$u\left(t\right)=u_0+t-t_0\qquad v\left(t\right)=v_0\qquad U\left(\tau\right)=u_0\qquad V\left(\tau\right)=v_0+\tau-\tau_0$$

such that

$$u'=\frac{d}{dt}\left(u_0+t-t_0\right)=1\quad v'=\frac{dv_0}{dt}=0\quad U'=\frac{du_0}{d\tau}=0\quad V'=\frac{d}{d\tau}\left(v_0+\tau-\tau_0\right)=1$$

then $\cos\theta=F(EG)^{-1/2}$ at the point (u_0,v_0).

(g) In order to apply the Implicit Function Theorem of Section 2.10, it is assumed that at least one of $\partial(g,h)/\partial(u,v) \neq 0$, $\partial(f,h)/\partial(u,v) \neq 0$ or $\partial(f,g)/\partial(u,v) \neq 0$, or equivalently, that $(\partial\mathbf{r}/\partial u) \times (\partial\mathbf{r}/\partial v) > \mathbf{0}$. Hence, the two vectors $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are not parallel and thus linearly independent in D.

(h) To show that $E > 0$, $G > 0$ follows from the fact that since $(\partial\mathbf{r}/\partial u)\times(\partial\mathbf{r}/\partial v) > \mathbf{0}$, both $(\partial\mathbf{r}/\partial u) > \mathbf{0}$ and $(\partial\mathbf{r}/\partial v) > \mathbf{0}$, which in turn implies

$$E = \left|\frac{\partial\mathbf{r}}{\partial u}\right|^2 > 0 \qquad\qquad G = \left|\frac{\partial\mathbf{r}}{\partial v}\right|^2 > 0$$

Furthermore, recalling the identity

$$|\mathbf{u}\times\mathbf{v}|^2 = \begin{vmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v} \end{vmatrix}$$

from Problem 12 (a) following Section 1.5 we find that

$$\left|\frac{\partial\mathbf{r}}{\partial u}\right|^2 \left|\frac{\partial\mathbf{r}}{\partial v}\right|^2 - \left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}\right)^2 = EG - F^2 = \left|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right|^2 > 0$$

where the last inequality again follows from (g).

(i) As stated in Section 2.21 a quadratic form is called positive definite if it is positive for all non-zero values of its argument. As such, the expression for the element of arc $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is a positive definite quadratic form, since $ds^2 \geq 0$. Furthermore, a quadratic form is positive definite if and only if all eigenvalues of the $n \times n$ symmetric coefficient matrix $\mathbf{A}$ of the quadratic form are positive. In the case of ds^2 the coefficient matrix $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Then the eigenvalues of $\mathbf{A}$ are the solutions of

$$\begin{vmatrix} E-\lambda & F \\ F & G-\lambda \end{vmatrix} = \lambda^2 - (E+G)\,\lambda + EG - F^2 = 0$$

Hence,

$$\lambda = \frac{E + G \pm \sqrt{(E+G)^2 - 4\,(EG - F^2)}}{2} = \frac{E + G \pm \sqrt{(E-G)^2 + 4F^2}}{2}$$

Now since we want both roots to be positive, it is sufficient to require that $EG - F^2 > 0$ and $E + G > 0$. The last condition certainly is satisfied when $E > 0$, $G > 0$.

Section 3.11

1. In E^2 let (ξ^2, ξ^2) be standard coordinates and let (x^1, x^2) be new coordinates given by $x^1 = 3\xi^1 + 2\xi^2$, $x^2 = 4\xi^1 + 3\xi^2$ with inverses $\xi^1 = 3x^1 - 2x^2$, $\xi^2 = -4x^1 + 3x^2$.

   (a) Let $U_1 = f_1(\xi^1, \xi^2) = \xi^1\xi^2$, $U_2 = f_1(\xi^1, \xi^2) = \xi^1 - \xi^2$. Then using the rule

   $$u_i = \sum_{j=1}^{n} \frac{\partial \xi^j}{\partial x^i} U_j$$

   we find

   $$\begin{aligned} u_1 = 3\xi^1\xi^2 - 4\left(\xi^1 - \xi^2\right) &= 3\left(3x^1 - 2x^2\right)\left(-4x^1 + 3x^2\right) - 4\left(7x^1 - 5x^2\right) \\ &= -36x^1x^1 + 51x^1x^2 - 18x^2x^2 - 28x^1 + 20x^2 \\ u_2 = -2\xi^1\xi^2 + 3\left(\xi^1 - \xi^2\right) &= -2\left(3x^1 - 2x^2\right)\left(-4x^1 + 3x^2\right) + 3\left(7x^1 - 5x^2\right) \\ &= 24x^1x^1 - 34x^1x^2 + 12x^2x^2 + 21x^1 - 15x^2 \end{aligned}$$

   (b) Let $V^1 = f_1(\xi^2, \xi^2) = \xi^1 \cos\xi^2$, $V^2 = f_2(\xi^1, \xi^2) = \xi^1 \sin\xi^2$. Then using the rule

   $$v^i = \sum_{j=1}^{n} \frac{\partial x^i}{\partial \xi^j} V^j$$

   we find

   $$\begin{aligned} v^1 &= 3\xi^1\cos\xi^2 + 2\xi^1\sin\xi^2 = \left(3x^1 - 2x^2\right)\left[3\cos\left(-4x^1 + 3x^2\right) + 2\sin\left(-4x^1 + 3x^2\right)\right] \\ v^2 &= 4\xi^1\cos\xi^2 + 3\xi^1\sin\xi^2 = \left(3x^1 - 2x^2\right)\left[4\cos\left(-4x^1 + 3x^2\right) + 3\sin\left(-4x^1 + 3x^2\right)\right] \end{aligned}$$

   (c) Let $W_{11} = f_{11}(\xi^1, \xi^2) = 0$, $W_{12} = f_{12}(\xi^1, \xi^2) = \xi^1\xi^2$, $W_{21} = f_{21}(\xi^1, \xi^2) = -\xi^1\xi^2$, $W_{22} = f_{22}(\xi^1, \xi^2) = 0$. Then using the rule

   $$w_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial \xi^k}{\partial x^i}\frac{\partial \xi^l}{\partial x^j} W_{kl}$$

   we find

   $$\begin{aligned} w_{11} &= (3)(3)(0) + (3)(-4)\left(\xi^1\xi^2\right) + (-4)(3)\left(-\xi^1\xi^2\right) + (-4)(-4)(0) = 0 \\ w_{12} &= (3)(-2)(0) + (3)(3)\left(\xi^1\xi^2\right) + (-4)(-2)\left(-\xi^1\xi^2\right) + (-4)(3)(0) \\ &= \xi^1\xi^2 = \left(3x^1 - 2x^2\right)\left(-4x^1 + 3x^2\right) = -12x^1x^1 + 17x^1x^2 - 6x^2x^2 \\ w_{21} &= (-2)(3)(0) + (-2)(-4)\left(\xi^1\xi^2\right) + (3)(3)\left(-\xi^1\xi^2\right) + (3)(-4)(0) \\ &= -\xi^1\xi^2 = -w_{12} \\ w_{22} &= (-2)(-2)(0) + (-2)(3)\left(\xi^1\xi^2\right) + (3)(-2)\left(-\xi^1\xi^2\right) + (3)(3)(0) = 0 \end{aligned}$$

(d) Let $Z_1^1 = f_{11}(\xi^1, \xi^2) = \xi^1 + \xi^2$, $Z_2^1 = f_{12}(\xi^1, \xi^2) = Z_1^2 = f_{21}(\xi^1, \xi^2) = 3\xi^1 + 2\xi^2$, $Z_2^2 = f_{22}(\xi^1, \xi^2) = \xi^1 - \xi^2$. Then using the rule

$$z_j^i = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial x^i}{\partial \xi^k} \frac{\partial \xi^l}{\partial x^j} Z_j^i$$

we find

$$\begin{aligned}
z_1^1 &= (3)(3)\left(\xi^1 + \xi^2\right) + (3)(-4)\left(3\xi^1 + 2\xi^2\right) + (2)(3)\left(3\xi^1 + 2\xi^2\right) + (2)(-4)\left(\xi^1 - \xi^2\right) \\
&= -17\xi^1 + 5\xi^2 = -17\left(3x^1 - 2x^2\right) + 5\left(-4x^1 + 3x^2\right) = -71x^1 + 49x^2 \\
z_2^1 &= (3)(-2)\left(\xi^1 + \xi^2\right) + (3)(3)\left(3\xi^1 + 2\xi^2\right) + (2)(-2)\left(3\xi^1 + 2\xi^2\right) + (2)(3)\left(\xi^1 - \xi^2\right) \\
&= 15\xi^1 - 2\xi^2 = 15\left(3x^1 - 2x^2\right) - 2\left(-4x^1 + 3x^2\right) = 53x^1 - 36x^2 \\
z_1^2 &= (4)(3)\left(\xi^1 + \xi^2\right) + (4)(-4)\left(3\xi^1 + 2\xi^2\right) + (3)(3)\left(3\xi^1 + 2\xi^2\right) + (3)(-4)\left(\xi^1 - \xi^2\right) \\
&= -21\xi^1 + 10\xi^2 = -21\left(3x^1 - 2x^2\right) + 10\left(-4x^1 + 3x^2\right) = -103x^1 + 72x^2 \\
z_2^2 &= (4)(-2)\left(\xi^1 + \xi^2\right) + (4)(3)\left(3\xi^1 + 2\xi^2\right) + (3)(-2)\left(3\xi^1 + 2\xi^2\right) + (3)(3)\left(\xi^1 - \xi^2\right) \\
&= 19\xi^1 - 5\xi^2 = 19\left(3x^1 - 2x^2\right) - 5\left(-4x^1 + 3x^2\right) = 77x^1 - 53x^2
\end{aligned}$$

2. (a) In standard coordinates, the fundamental metric tensor G_{ij} reduces to δ_{ij} (Kronecker delta), since in the (ξ^i) coordinates $ds^2 = d\xi^i d\xi^i = \delta_{ij} d\xi^i d\xi^j$. Hence, we can regard g_{ij} as the covariant tensor obtained from δ_{ij} (constant functions) in standard coordinates. In the (x^i) coordinates we find, using (3.84):

$$\begin{aligned}
g_{11} &= \frac{\partial \xi^1}{\partial x^1} \frac{\partial \xi^1}{\partial x^1} + \frac{\partial \xi^2}{\partial x^1} \frac{\partial \xi^2}{\partial x^1} = (3)(3) + (-4)(-4) = 25 \\
g_{12} &= \frac{\partial \xi^1}{\partial x^1} \frac{\partial \xi^1}{\partial x^2} + \frac{\partial \xi^2}{\partial x^1} \frac{\partial \xi^2}{\partial x^2} = (3)(-2) + (-4)(3) = -18 \\
g_{21} &= g_{12} \text{ (since according to (3.84) } g_{ij} \text{ is symmetric)} \\
g_{22} &= \frac{\partial \xi^1}{\partial x^2} \frac{\partial \xi^1}{\partial x^2} + \frac{\partial \xi^2}{\partial x^2} \frac{\partial \xi^2}{\partial x^2} = (-2)(-2) + (3)(3) = 13
\end{aligned}$$

(b) By setting $G_j^i = \delta_{ij}$ in standard coordinates (ξ^i), we obtain a mixed second-order tensor g_j^i. In the (x^i) coordinates we obtain

$$g_j^i = \frac{\partial x^i}{\partial \xi^k} \frac{\partial \xi^l}{\partial x^j} \delta_{kl} = \frac{\partial x^i}{\partial \xi^k} \frac{\partial \xi^k}{\partial x^j} = \delta_{ij}$$

since the matrices $(\partial x^i / \partial \xi^k)$ and $(\partial \xi^k / \partial x^j)$ are inverses of each other. Hence, $g_j^i = g_i^j = \delta_{ij}$ in all coordinates. Next, by setting $G^{ij} = \delta_{ij}$ in standard (ξ^i) coordinates we obtain a contravariant second-order tensor g^{ij}:

$$g^{ij} = \frac{\partial x^i}{\partial \xi^k} \frac{\partial x^j}{\partial \xi^l} \delta_{kl} = \frac{\partial x^i}{\partial \xi^k} \frac{\partial x^j}{\partial \xi^k} = g^{ji}$$

Comparing this expression with (3.84) then suggests that (g^{ij}) is the inverse matrix of (g_{ij}), so that $g_{i\alpha}g^{\alpha j} = \delta_{ij}$. Hence,

$$g^{11} = g_{22} = 13 \qquad g^{12} = g^{21} = -g_{21} = 18$$
$$g^{22} = g_{11} = 25$$

3. As the text states, we can raise or lower indices of a given tensor $u^{ij\cdots}_{kl\ldots}$ by multiplying by the metric tensor g_{pq} or g^{pg} to first increase the order by two, and then using contraction, based on q and one selected index of $u^{ij\cdots}_{kl\ldots}$ to lower the order by two.

   (a) In order to find the contravariant vector u^i associated with the covariant vector u_i from Problem 1 (a) in the (x^i) coordinates we compute $u^i = g^{ij}u_j$, where g^{ij} is the metric tensor in (x^i) from Problem 2 (b):

   $$u^1 = g^{11}u_1 + g^{12}u_2 = 13u_1 + 18u_2 \qquad u^2 = g^{21}u_1 + g^{22}u_2 = 18u_1 + 25u_2$$

   where u_1 and u_2 are as in the answer to Problem 1 (a).

   (b) In order to find the covariant vector v_i associated with the contravariant vector v^i from Problem 1 (b) in the (x^i) coordinates we compute $v_i = g_{ij}u^j$, where g_{ij} is the metric tensor in (x^i) from Problem 2 (a):

   $$v_1 = g_{11}v^1 + g_{12}v^2 = 25v^1 - 18v^2 \qquad v_2 = g_{21}v^1 + g_{22}v^2 = -18v^1 + 13v^2$$

   where v^1 and v^2 are as in the answer to Problem 1 (b).

   (c) In order to find the second-order tensor w^{ij} contravariant in both indices associated with the tensor w_{ij} from Problem 1 (c) in the (x^i) coordinates we compute $w^{ij} = g^{ikjl}w_{kl}$, where $g^{ikjl} = g^{ik}g^{jl}$ is the tensor product of two second-order metric tensors in (x^i) identical to the second-order metric tensor contravariant in both indices from Problem 2 (b):

   $$\begin{aligned}
   w^{11} &= g^{1k11}w_{k1} + g^{1k12}w_{k2} \\
   &= g^{1111}w_{11} + g^{1112}w_{12} + g^{1211}w_{21} + g^{1212}w_{22} \\
   &= g^{1111}(0) + g^{1112}w_{12} - g^{1211}w_{12} + g^{1212}(0) = 0 \\
   w^{12} &= g^{1k21}w_{k1} + g^{1k22}w_{k2} \\
   &= g^{1121}w_{11} + g^{1122}w_{12} + g^{1221}w_{21} + g^{1222}w_{22} \\
   &= g^{1121}(0) + g^{1122}w_{12} - g^{1221}w_{12} + g^{1222}(0) = w_{12} \\
   w^{21} &= g^{2k11}w_{k1} + g^{2k12}w_{k2} \\
   &= g^{2111}w_{11} + g^{2112}w_{12} + g^{2211}w_{21} + g^{2212}w_{22} \\
   &= g^{2111}(0) + g^{2112}w_{12} - g^{2211}w_{12} + g^{2212}(0) = -w_{12} \\
   w^{22} &= g^{2k21}w_{k1} + g^{2k22}w_{k2} \\
   &= g^{2121}w_{11} + g^{2122}w_{12} + g^{2221}w_{21} + g^{2222}w_{22} \\
   &= g^{2121}(0) + g^{2122}w_{12} - g^{2221}w_{12} + g^{2222}(0) = 0
   \end{aligned}$$

   where w_{11}, w_{12}, w_{21}, w_{22} are as in the answer to Problem 1 (c).

4. (a) Let a tensor $u^{ij\ldots}_{kl\ldots}$ be such that all of its components are equal to zero at a given point in (x^i) coordinates. Changing to $(\bar{x}^i)$ coordinates,

$$
\begin{aligned}
\bar{u}^{ij\ldots}_{kl\ldots} &= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\cdots u^{pq\ldots}_{rs\ldots} \\
&= \frac{\partial \bar{x}^i}{\partial x^1}\frac{\partial \bar{x}^j}{\partial x^1}\cdots\frac{\partial x^1}{\partial \bar{x}^k}\frac{\partial x^1}{\partial \bar{x}^l}\cdots \underbrace{u^{11\ldots}_{11\ldots}}_{0} + \cdots + \frac{\partial \bar{x}^i}{\partial x^n}\frac{\partial \bar{x}^j}{\partial x^n}\cdots\frac{\partial x^n}{\partial \bar{x}^k}\frac{\partial x^n}{\partial \bar{x}^l}\cdots \underbrace{u^{nn\ldots}_{nn\ldots}}_{0}
\end{aligned}
$$

then implies that the tensor $\bar{u}^{ij\ldots}_{kl\ldots}$ will have all of its components equal to zero in $(\bar{x}^i)$ coordinates as well, since we are summing over products of derivatives of coordinates and the values of tensor components, all of which are equal to zero.

(b) Let two tensors $u^{ij\ldots}_{kl\ldots}$ and $v^{ij\ldots}_{kl\ldots}$ of the same type be such that their corresponding components are equal in (x^i) coordinates, i.e. $u^{ij\ldots}_{kl\ldots} = v^{ij\ldots}_{kl\ldots}$. Changing to $(\bar{x}^i)$ coordinates,

$$
\bar{u}^{ij\ldots}_{kl\ldots} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\cdots u^{pq\ldots}_{rs\ldots} = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\cdots v^{pq\ldots}_{rs\ldots} = \bar{v}^{ij\ldots}_{kl\ldots}
$$

then indicates that corresponding components are equal in every allowed coordinate system.

(c) Let a second-order tensor u_{ij} be such that $u_{ij} = u_{ji}$ in (x^i) coordinates. Changing to $(\bar{x}^i)$ coordinates,

$$
\bar{u}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}u_{kl} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}u_{lk} = \frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^k}{\partial \bar{x}^i}u_{lk} = \bar{u}_{ji}
$$

then indicates that $\bar{u}_{ij} = \bar{u}_{ji}$ in every other coordinate system $(\bar{x}^i)$, and hence, that the second-order tensor u_{ij} is symmetric.

(d) Let a second-order tensor u_{ij} be such that $u_{ij} = -u_{ji}$ in (x^i) coordinates. Changing to $(\bar{x}^i)$ coordinates,

$$
\bar{u}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}u_{kl} = -\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}u_{lk} = -\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial x^k}{\partial \bar{x}^i}u_{lk} = -\bar{u}_{ji}
$$

then indicates that $\bar{u}_{ij} = -\bar{u}_{ji}$ in every other coordinate system $(\bar{x}^i)$, and hence, that the second-order tensor u_{ij} is alternating.

5. Let us define the components U_i of a covariant vector field defined in (ξ^i) coordinates, that is n functions $U_1(\xi^1,\ldots,\xi^n),\ldots,U_n(\xi^1,\ldots,\xi^n)$ in D. Then to each other coordinate system, say (x^i), we can assign corresponding components u_i by the equation $u_i = (\partial\xi^j/\partial x^i)U_j$. Next, let $(\bar{x}^i)$ be some other coordinate system such that we have similarly $\bar{u}_i = (\partial\xi^j/\partial\bar{x}^i)U_j$. Now since $(\partial\xi^j/\partial x^i)$ has the matrix $(\partial x^j/\partial\xi^i)$ as its inverse, we can solve the equation for u_i in terms of U_j: $U_j = (\partial x^k/\partial\xi^j)u_k$. If we substitute for U_j in the equation for $\bar{u}_i$ we obtain

$$
\bar{u}_i = \frac{\partial \xi^j}{\partial \bar{x}^i}U_j = \frac{\partial \xi^j}{\partial \bar{x}^i}\frac{\partial x^k}{\partial \xi^j}u_k = \frac{\partial x^k}{\partial \bar{x}^i}u_k
$$

which is none other than (3.75). Accordingly, once we have assigned components in the standard coordinates (ξ^i), we automatically obtain components in all other coordinate systems, related by (3.75), and a covariant vector is obtained.

6. To show that (g^{ij}) is the inverse matrix of (g_{ij}) we compute

$$g_{i\alpha}g^{\alpha j} = \underbrace{\frac{\partial \xi^k}{\partial x^i}\frac{\partial \xi^k}{\partial x^\alpha}}_{(3.84)}\underbrace{\frac{\partial x^\alpha}{\partial \xi^l}\frac{\partial x^j}{\partial \xi^l}}_{(3.86)} = \frac{\partial \xi^k}{\partial x^i}\frac{\partial x^j}{\partial \xi^l}\frac{\partial \xi^k}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \xi^l} = \frac{\partial \xi^k}{\partial x^i}\frac{\partial x^j}{\partial \xi^l}\delta_{kl} = \frac{\partial \xi^k}{\partial x^i}\frac{\partial x^j}{\partial \xi^k} = \delta_{ij}$$

7. To prove that $g = \det(g_{ij})$ is positive note that by (3.84) the matrix (g_{ij}) is a product of two matrices:

$$g_{ij} = \frac{\partial \xi^r}{\partial x^i}\frac{\partial \xi^r}{\partial x^j}$$

Then taking determinants of both sides of this equation gives

$$\det(g_{ij}) = \det\left(\frac{\partial \xi^r}{\partial x^i}\frac{\partial \xi^r}{\partial x^j}\right) = \underbrace{\det\left(\frac{\partial \xi^r}{\partial x^i}\right)\det\left(\frac{\partial \xi^r}{\partial x^j}\right)}_{(1.60)} > 0$$

since the individual determinants will produce the same scalar result and hence, their product will be positive.

8. (a) Let v^i_j and w^i_j be two tensors of the same type in (x^i) coordinates, given in the neighborhood D of a point. Next, let $v^i_j + w^i_j = u^i_j$ define the sum of the two tensors, producing another tensor of the same type. In order to check that this indeed is the case, consider changing to $(\bar{x}^i)$ coordinates:

$$\bar{v}^i_j + \bar{w}^i_j = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}v^k_l + \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}w^k_l = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}\left(v^k_l + w^k_l\right) = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}u^k_l = \bar{u}^k_l$$

which shows that the sum of two tensors of the same type produces another tensor of the same type under a coordinate transformation, and hence, that the addition of two tensors of the same type is indeed defined by (3.90).

(b) Let $v^{ij\dots}_{kl\dots}$ and $w^{ij\dots}_{kl\dots}$ be two tensors of the same, but arbitrary type in (x^i) coordinates, given in the neighborhood D of a point. Next, let $v^{ij\dots}_{kl\dots} + w^{ij\dots}_{kl\dots} = u^{ij\dots}_{kl\dots}$ define the sum of the two tensors, producing another tensor of the same type. As for part (a), in order to check that this is indeed the case, consider changing to $(\bar{x}^i)$ coordinates:

$$\begin{aligned}\bar{v}^{ij\dots}_{kl\dots} + \bar{w}^{ij\dots}_{kl\dots} &= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\cdots v^{pq\dots}_{rs\dots} + \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\cdots w^{pq\dots}_{rs\dots} \\ &= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\cdots(v^{pq\dots}_{rs\dots} + w^{pq\dots}_{rs\dots}) \\ &= \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial \bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^s}{\partial \bar{x}^l}\cdots u^{pq\dots}_{rs\dots} = \bar{u}^{ij\dots}_{kl\dots}\end{aligned}$$

which shows that the sum of two tensors of the same, but arbitrary type produces another tensor of the same type under a coordinate transformation, and hence, that the addition of two tensors of the same, but arbitrary type is indeed defined by $v^{ij\dots}_{kl\dots} + w^{ij\dots}_{kl\dots} = u^{ij\dots}_{kl\dots}$.

9. Let u_{ij} be a tensor covariant in both indices and let f be an arbitrary scalar invariant. We can then define multiplication of a tensor by an invariant as $fu_{ij} = v_{ij}$, producing another tensor of the same type. In order to check that this is indeed the case, consider changing to $(\bar{x}^i)$ coordinates:

$$f\bar{u}_{ij} = f\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}u_{kl} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}fu_{kl} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}v_{kl} = \bar{v}_{ij}$$

which shows that the product of a tensor with a scalar invariant produces another tensor of the same type under a coordinate transformation, and hence, that (3.91) holds.

10. Let u_i, v_j, w^k, z^l, p_{mh} be tensors. Then the given tensor products are a tensor since

(a)

$$\bar{u}_i\bar{v}_j = \left(\frac{\partial x^k}{\partial \bar{x}^i}u_k\right)\frac{\partial x^l}{\partial \bar{x}^j}v_l = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}u_kv_l = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}s_{kl} = \bar{s}_{ij}$$

(b)

$$\bar{u}_i\bar{w}^k = \left(\frac{\partial x^l}{\partial \bar{x}^i}u_l\right)\frac{\partial \bar{x}^k}{\partial x^j}w^j = \frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^j}u_lw^j = \frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial \bar{x}^k}{\partial x^j}t^j_l = \bar{t}^k_i$$

(c)

$$\bar{w}^k\bar{z}^l = \left(\frac{\partial \bar{x}^k}{\partial x^i}w^i\right)\frac{\partial \bar{x}^l}{\partial x^j}z^j = \frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^l}{\partial x^j}w^iz^j = \frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^l}{\partial x^j}q^{ij} = \bar{q}^{kl}$$

(d)

$$\bar{w}^k\bar{p}_{mh} = \left(\frac{\partial \bar{x}^k}{\partial x^l}w^l\right)\frac{\partial x^n}{\partial \bar{x}^m}\frac{\partial x^i}{\partial \bar{x}^h}p_{ni} = \frac{\partial \bar{x}^k}{\partial x^l}\frac{\partial x^n}{\partial \bar{x}^m}\frac{\partial x^i}{\partial \bar{x}^h}w^lp_{ni} = \frac{\partial \bar{x}^k}{\partial x^l}\frac{\partial x^n}{\partial \bar{x}^m}\frac{\partial x^i}{\partial \bar{x}^h}d^l_{ni} = \bar{d}^k_{mh}$$

when changing to $(\bar{x}^i)$ coordinates.

11. (a) Let u^{ij}_{kl} be a tensor contravariant in two indices and covariant in two indices in (x^i) coordinates. Then in order to prove that contracting the aforementioned tensor will produce a second-order tensor, we will show that the same procedure can be carried out while transforming from (x^i) to $(\bar{x}^i)$ coordinates:

$$\begin{aligned}\bar{w}^j_k = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}w^q_r = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}u^{lq}_{rl} = \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\delta_{lm}u^{lq}_{rm} &= \frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^i}u^{lq}_{rm} \\ &= \frac{\partial \bar{x}^i}{\partial x^l}\frac{\partial \bar{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^m}{\partial \bar{x}^i}u^{lq}_{rm} = \bar{u}^{ij}_{ki}\end{aligned}$$

where the matrices $(\partial \bar{x}^i/\partial x^l)$ and $(\partial x^m/\partial \bar{x}^i)$ are inverses of each other. Similarly, let u^{ij}_{kj} be a tensor contravariant in two indices and covariant in two indices in (x^i) coordinates. Transforming to $(\bar{x}^i)$ then gives

$$\begin{aligned}\bar{v}^i_k = \frac{\partial \bar{x}^i}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}v^q_r = \frac{\partial \bar{x}^i}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}u^{ql}_{rl} = \frac{\partial \bar{x}^i}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\delta_{lm}u^{ql}_{rm} &= \frac{\partial \bar{x}^i}{\partial x^q}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial \bar{x}^j}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^j}u^{ql}_{rm} \\ &= \frac{\partial \bar{x}^i}{\partial x^q}\frac{\partial \bar{x}^j}{\partial x^l}\frac{\partial x^r}{\partial \bar{x}^k}\frac{\partial x^m}{\partial \bar{x}^j}u^{ql}_{rm} = \bar{u}^{ij}_{kj}\end{aligned}$$

(b) Let $u^{ij\ldots}_{kl\ldots}$ be a tensor of arbitrary type in (x^i) coordinates. Then in order to prove that contracting the aforementioned tensor will produce a tensor with an order of two lower than the original tensor, we will show that the same procedure can be carried out while transforming from (x^i) to $(\bar{x}^i)$ coordinates:

$$\begin{aligned}\bar{w}^{jk\ldots}_{lm\ldots} = \frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^s}{\partial \bar{x}^m}\cdots w^{jk\ldots}_{lm\ldots} &= \frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^s}{\partial \bar{x}^m}\cdots u^{tpq\ldots}_{rs\ldots t} \\ &= \frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^s}{\partial \bar{x}^m}\cdots u^{pq\ldots t}_{rs\ldots t} \\ &= \frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^s}{\partial \bar{x}^m}\cdots \delta_{tu}u^{pq\ldots t}_{rs\ldots u} \\ &= \frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\cdots\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^s}{\partial \bar{x}^m}\cdots\frac{\partial \bar{x}^i}{\partial x^t}\frac{\partial x^u}{\partial \bar{x}^i}u^{pq\ldots t}_{rs\ldots u} \\ &= \frac{\partial \bar{x}^j}{\partial x^p}\frac{\partial \bar{x}^k}{\partial x^q}\frac{\partial \bar{x}^i}{\partial x^t}\cdots\frac{\partial x^r}{\partial \bar{x}^l}\frac{\partial x^s}{\partial \bar{x}^m}\frac{\partial x^u}{\partial \bar{x}^i}\cdots u^{pq\ldots t}_{rs\ldots u} \\ &= \bar{u}^{jk\ldots i}_{lm\ldots i} = \bar{u}^{ijk\ldots}_{lm\ldots i}\end{aligned}$$

where the matrices $(\partial \bar{x}^i/\partial x^t)$ and $(\partial x^u/\partial \bar{x}^i)$ are inverses of each other.

12. To show that the Γ^i_{jl} can be expressed as in (3.94), firstly note that

$$\begin{aligned}\Gamma^i_{jl}g_{is} &= \frac{\partial^2 \xi^\alpha}{\partial x^j \partial x^l}\frac{\partial x^i}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^i}\frac{\partial \xi^\alpha}{\partial x^s} \\ &= \frac{\partial^2 \xi^\alpha}{\partial x^j \partial x^l}\frac{\partial \xi^\alpha}{\partial x^s} \\ &= \frac{1}{2}\left(2\frac{\partial^2 \xi^\alpha}{\partial x^l \partial x^j}\frac{\partial \xi^\alpha}{\partial x^s} + \frac{\partial \xi^\alpha}{\partial x^j}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^l} - \frac{\partial \xi^\alpha}{\partial x^j}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^l} + \frac{\partial \xi^\alpha}{\partial x^l}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^j} - \frac{\partial \xi^\alpha}{\partial x^l}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^j}\right) \\ &= \frac{1}{2}\left(\frac{\partial \xi^\alpha}{\partial x^j}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^l} + \frac{\partial^2 \xi^\alpha}{\partial x^l \partial x^j}\frac{\partial \xi^\alpha}{\partial x^s} + \frac{\partial \xi^\alpha}{\partial x^l}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^j} + \frac{\partial^2 \xi^\alpha}{\partial x^l \partial x^j}\frac{\partial \xi^\alpha}{\partial x^s}\right) \\ &\quad - \frac{1}{2}\left(\frac{\partial \xi^\alpha}{\partial x^j}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^l} + \frac{\partial \xi^\alpha}{\partial x^l}\frac{\partial^2 \xi^\alpha}{\partial x^s \partial x^j}\right) \\ &= \frac{1}{2}\left[\frac{\partial}{\partial x^l}\left(\frac{\partial \xi^\alpha}{\partial x^j}\frac{\partial \xi^\alpha}{\partial x^s}\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial \xi^\alpha}{\partial x^l}\frac{\partial \xi^\alpha}{\partial x^s}\right) - \frac{\partial}{\partial x^s}\left(\frac{\partial \xi^\alpha}{\partial x^j}\frac{\partial \xi^\alpha}{\partial x^l}\right)\right] \\ &= \frac{1}{2}\left(\frac{\partial g_{js}}{\partial x^l} + \frac{\partial g_{ls}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^s}\right)\end{aligned}$$

Multiplying both sides by g^{st} then gives

$$\begin{aligned}
\Gamma^i_{jl} g_{is} g^{st} &= \frac{1}{2}\left(\frac{\partial g_{js}}{\partial x^l} + \frac{\partial g_{ls}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^s}\right) g^{st} \\
\Gamma^i_{jl} \delta_{it} &= \frac{1}{2} g^{st}\left(\frac{\partial g_{js}}{\partial x^l} + \frac{\partial g_{ls}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^s}\right) \\
\Gamma^t_{jl} &= \frac{1}{2} g^{st}\left(\frac{\partial g_{js}}{\partial x^l} + \frac{\partial g_{ls}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^s}\right)
\end{aligned}$$

where we can move the matrix (g^{st}) to before the brackets on the right hand side, since it is real and symmetric.

13. From (3.101) and (3.92) it follows that

$$\operatorname{div}\mathbf{u} = \operatorname{div} u^i = \Delta_\alpha u^\alpha = \frac{\partial u^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\alpha l} u^l$$

Next, setting $i = j = \alpha$ in (3.94) gives

$$\begin{aligned}
\Gamma^\alpha_{\alpha l} = \frac{1}{2} g^{\alpha s}\left(\frac{\partial g_{\alpha s}}{\partial x^l} + \frac{\partial g_{ls}}{\partial x^\alpha} - \frac{\partial g_{\alpha l}}{\partial x^s}\right) &= \frac{1}{2}\left(g^{\alpha s}\frac{\partial g_{\alpha s}}{\partial x^l} + g^{\alpha s}\frac{\partial g_{ls}}{\partial x^\alpha} - g^{\alpha s}\frac{\partial g_{\alpha l}}{\partial x^s}\right) \\
&= \frac{1}{2}\left(g^{\alpha s}\frac{\partial g_{\alpha s}}{\partial x^l} + \frac{\partial g_{ls}}{\partial x^s} - \frac{\partial g_{sl}}{\partial x^s}\right) \\
&= \frac{1}{2}\left(g^{\alpha s}\frac{\partial g_{\alpha s}}{\partial x^l} + \frac{\partial g_{ls}}{\partial x^s} - \frac{\partial g_{ls}}{\partial x^s}\right) = \frac{1}{2} g^{\alpha s}\frac{\partial g_{\alpha s}}{\partial x^l}
\end{aligned}$$

Furthermore, by (1.31) and (1.32), and more generally, using Laplace expansion, expanding along the α^{th} row of the matrix $(g_{\alpha s})$, note that the determinant $g = \det(g_{\alpha s})$ can be written as

$$g = \det(g_{\alpha s}) = g_{\alpha 1} A_{\alpha 1} + g_{\alpha,2} A_{\alpha 2} + \cdots + g_{\alpha n} A_{\alpha n} = \sum_{s=1}^{n} g_{\alpha s} A_{\alpha s}$$

where for the moment we do not use the summation convention (i.e. there is no sum over α implied here) and where $A_{\alpha s}$ denotes the cofactor of the matrix $(g_{\alpha,s})$ when expanding along the α^{th} row: $A_{\alpha s} = (-1)^{\alpha+s} M_{\alpha s}$. Here $M_{\alpha s}$ denotes the minor of the entry in the α^{th} row and s^{th} column of $(g_{\alpha s})$, i.e. $M_{\alpha s}$ is the determinant of the sub-matrix formed by deleting the α^{th} row and s^{th} column of $(g_{\alpha s})$. Now the determinant of $(g_{\alpha s})$ can be considered to be a function of the elements of $(g_{\alpha s})$: $\det(g_{\alpha s}) = F(g_{11}, g_{12}, \ldots, g_{21}, g_{22}, \ldots, g_{nn})$ so that by the general chain rule (2.43)

$$\frac{\partial}{\partial x^l}\det(g_{\alpha s}) = \sum_{\alpha=1}^{n}\sum_{s=1}^{n}\frac{\partial F}{\partial g_{\alpha s}}\frac{\partial g_{\alpha s}}{\partial x^l}$$

where the summation is performed over all $n \times n$ elements of the matrix $(g_{\alpha s})$. To find $\partial F/\partial g_{\alpha s}$, note that the index α can be chosen at will when expanding the determinant according to Laplace's formula. In particular, it can be chosen so as to coincide with the first index of $\partial/\partial g_{\alpha s}$. Hence,

$$\begin{aligned}
\frac{\partial}{\partial g_{\alpha s}} \det(g_{\alpha s}) = \frac{\partial}{\partial g_{\alpha s}} \sum_{k=1}^{n} g_{\alpha k} A_{\alpha k} = \sum_{k=1}^{n} \frac{\partial}{\partial g_{\alpha s}} (g_{\alpha k} A_{\alpha k}) &= \sum_{k=1}^{n} \frac{\partial g_{\alpha k}}{\partial g_{\alpha s}} A_{\alpha k} + \sum_{k=1}^{n} g_{\alpha k} \underbrace{\frac{\partial A_{\alpha k}}{\partial g_{\alpha s}}}_{0} \\
&= \sum_{k=1}^{n} \frac{\partial g_{\alpha k}}{\partial g_{\alpha s}} A_{\alpha k} \\
&= \sum_{k=1}^{n} \delta_{ks} A_{\alpha k} = A_{\alpha s}
\end{aligned}$$

The right-most term $\partial A_{\alpha k}/\partial g_{\alpha s}$ is zero because if an element of the matrix $(g_{\alpha s})$ and the cofactor $A_{\alpha k}$ associated with the matrix element $g_{\alpha k}$ lie on the same row, then the cofactor will not be a function of $(g_{\alpha s})$, since the cofactor of $g_{\alpha k}$ is expressed in terms of elements not in its own row. As such

$$\frac{\partial}{\partial x^l} \det(g_{\alpha s}) = \sum_{\alpha=1}^{n} \sum_{s=1}^{n} \frac{\partial F}{\partial g_{\alpha s}} \frac{\partial g_{\alpha s}}{\partial x^l} = \sum_{\alpha=1}^{n} \sum_{s=1}^{n} A_{\alpha s} \frac{\partial g_{\alpha s}}{\partial x^l} = A_{\alpha s} \frac{\partial g_{\alpha s}}{\partial x^l}$$

where in the last step the summation convention is assumed again. Now since $g_{\alpha s} = g_{s\alpha}$ (i.e. the matrix $(g_{\alpha s})$ is symmetric) we have $A^{\top}_{\alpha s} = \operatorname{adj}(g_{\alpha s}) = A_{\alpha s}$. Then by Problem 7 (c) following Section 1.9 we find $A_{\alpha s} = g g^{\alpha s}$, where by (3.87) the matrix $(g^{\alpha s}) = (g^{s\alpha})$ is the inverse of the matrix $(g_{\alpha s})$. Hence, we conclude that

$$\frac{\partial}{\partial x^l} \det(g_{\alpha s}) = g g^{\alpha s} \frac{\partial g_{\alpha s}}{\partial x^l} \qquad \Longrightarrow \qquad g^{\alpha s} \frac{\partial g_{\alpha s}}{\partial x^l} = \frac{1}{g} \frac{\partial}{\partial x^l} \det(g_{\alpha s}) = \frac{1}{g} \frac{\partial g}{\partial x^l}$$

and so $\Gamma^{\alpha}_{\alpha l} = (2g)^{-1}(\partial g/\partial x^l)$, thus allowing us to write $\operatorname{div} \mathbf{u}$ as

$$\begin{aligned}
\operatorname{div} \mathbf{u} = \operatorname{div} u^i = \Delta_\alpha u^\alpha = \frac{\partial u^\alpha}{\partial x^\alpha} + \Gamma^{\alpha}_{\alpha l} u^l &= \frac{\partial u^\alpha}{\partial x^\alpha} + \frac{1}{2g} \frac{\partial g}{\partial x^l} u^l \\
&= \frac{\partial u^\alpha}{\partial x^\alpha} + \frac{1}{2g} \frac{\partial g}{\partial x^\alpha} u^\alpha \qquad \text{(rename dummy index } l\text{)} \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{g} g^{\alpha l} u_l \right)
\end{aligned}$$

14. By (3.108) and (3.95) we find

$$\begin{aligned}
\operatorname{curl} \mathbf{u} = b_{ij} = \Delta_i u_j - \Delta_j u_i = \underbrace{\frac{\partial u_j}{\partial x^i} - \Gamma^l_{ji} u_l}_{(3.95)} - \frac{\partial u_i}{\partial x^j} + \Gamma^l_{ij} u_l &= \frac{\partial u_j}{\partial x^i} - \Gamma^l_{ji} u_l - \frac{\partial u_i}{\partial x^j} + \underbrace{\Gamma^l_{ji}}_{(3.93)} u_l \\
&= \frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j}
\end{aligned}$$

15. Let the norm of a vector $\mathbf{u}$ be its norm or length in standard coordinates (ξ^i). Thus for components U^i or U_i we have $|\mathbf{u}| = (U^i U^i)^{1/2} = (U_i U_i)^{1/2}$. Using the relations $U^i = (\partial \xi^i/\partial x^j)u^j$ and $U_i = (\partial x^j/\partial \xi^i)U_j)$, the fact that $U^i(\xi^1, \ldots, \xi^n) = f_i(\xi^1, \ldots, \xi^n) = U_i(\xi^1, \ldots, \xi^n)$ and (3.84) and (3.86) we then find additionally

$$|\mathbf{u}| = \left(U^k U^k\right)^{1/2} = \left(U_k U^k\right)^{1/2} = \left(\frac{\partial x^i}{\partial \xi^k} u_i \frac{\partial \xi^k}{\partial x^j} u^j\right)^{1/2} = \left(\frac{\partial x^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j} u_i u^j\right)^{1/2} = \left(\delta_{ij} u_i u^j\right)^{1/2}$$
$$= \left(u_i u^i\right)^{1/2}$$
$$|\mathbf{u}| = \left(U^k U^k\right)^{1/2} = \left(\frac{\partial \xi^k}{\partial x^i} u^i \frac{\partial \xi^k}{\partial x^j} u^j\right)^{1/2} = \left(\frac{\partial \xi^k}{\partial x^i}\frac{\partial \xi^k}{\partial x^j} u^i u^j\right)^{1/2} = \left(g_{ij} u^i u^j\right)^{1/2}$$
$$|\mathbf{u}| = \left(U_k U_k\right)^{1/2} = \left(\frac{\partial x^i}{\partial \xi^k} u_i \frac{\partial x^j}{\partial \xi^k} u_j\right)^{1/2} = \left(\frac{\partial x^i}{\partial \xi^k}\frac{\partial x^j}{\partial \xi^k} u_i u_j\right)^{1/2} = \left(g^{ij} u_i u_j\right)^{1/2}$$

16. Let the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ be defined as the invariant $\mathbf{u} = u_i v^i$ in some arbitrary coordinate system (x^i). Hence, in standard coordinates (ξ^i) we find

$$(\mathbf{u}, \mathbf{v}) = u_i v^i = \delta_{ij} u_i v^j = \frac{\partial x^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j} u_i v^j = \frac{\partial x^i}{\partial \xi^k} u_i \frac{\partial \xi^k}{\partial x^j} v^j = U_k V^k = U^k V^k = U_k V_k$$

where the last two equalities hold since in standard coordinates (ξ^i) it is true that $U^i(\xi^1, \ldots, \xi^n) = f_i(\xi^1, \ldots, \xi^n) = U_i(\xi^1, \ldots, \xi^n)$ and $V^i(\xi^1, \ldots, \xi^n) = g_i(\xi^1, \ldots, \xi^n) = V_i(\xi^1, \ldots, \xi^n)$. Additionally, we then find

$$(\mathbf{u}, \mathbf{v}) = U^k V^k = \frac{\partial \xi^k}{\partial x^i} u^i \frac{\partial \xi^k}{\partial x^j} v^j = \frac{\partial \xi^k}{\partial x^i}\frac{\partial \xi^k}{\partial x^j} u^i v^j = g_{ij} u^i v^j$$
$$(\mathbf{u}, \mathbf{v}) = U_k V_k = \frac{\partial x^i}{\partial \xi^k} u_i \frac{\partial x^j}{\partial \xi^k} v_j = \frac{\partial x^i}{\partial \xi^k}\frac{\partial x^j}{\partial \xi^k} u_i v_j = g^{ij} u_i v_j$$
$$(\mathbf{u}, \mathbf{u}) = U_k U^k = \frac{\partial x^i}{\partial \xi^k} u_i \frac{\partial \xi^k}{\partial x^j} u^j = \frac{\partial x^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j} u_i u^i = \delta_{ij} u_i u^i = u_i u^i = |\mathbf{u}|^2$$

17. (a) Let u^i and v^j be two contravariant tensors in (x^i) coordinates and let the tensor product $b_{ij} u^i v^j$ be an invariant. In other words, the scalar quantity obtained from the tensor product $b_{ij} u^i v^j$ should be the same in every other coordinate system $(\bar{x}^i)$, i.e. it should hold that $b_{ij} u^i v^j = \bar{b}_{ij} \bar{u}^i \bar{v}^j$. To this end, let us assume that b_{ij} is a covariant tensor so that $\bar{b}_{ij} = (\partial x^k/\partial \bar{x}^i)(\partial x^l/\partial \bar{x}^j) b_{kl}$. Then

$$\bar{b}_{ij} \bar{u}^i \bar{v}^j = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} b_{kl} \frac{\partial \bar{x}^i}{\partial x^r} u^r \frac{\partial \bar{x}^j}{\partial x^s} u^s = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial x^l}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^s} b_{kl} u^r u^s = \delta_{kr}\delta_{ls} b_{kl} u^r u^s$$
$$= b_{kl} \delta_{kr} u^r \delta_{ls} u^s$$
$$= b_{kl} u^k u^l$$

which confirms that the tensor product $b_{ij} u^i v^j$ indeed produces an invariant when b_{ij} is a second-order covariant tensor.

(b) Let u_i, $v_j, \ldots$, w^k, $z^l, \ldots$ be covariant and contravariant vectors in (x^i) coordinates and let the tensor product $b^{ij\cdots}_{kl\ldots}u_iv_j\ldots w^kz^l\ldots$ be an invariant. In other words, the scalar quantity obtained from the tensor product $b^{ij\cdots}_{kl\ldots}u_iv_j\ldots w^kz^l\ldots$ should be the same in every other coordinate system $(\bar{x}^i)$, i.e. it should hold that $b^{ij\cdots}_{kl\ldots}u_iv_j\ldots w^kz^l\cdots = \bar{b}^{ij\cdots}_{kl\ldots}\bar{u}_i\bar{v}_j\ldots\bar{w}^k\bar{z}^l\ldots$. To this end, let us assume that $b^{ij\cdots}_{kl\ldots}$ is a tensor of the type indicated so that

$$\bar{b}^{ij\cdots}_{kl\ldots} = \frac{\partial\bar{x}^i}{\partial x^p}\frac{\partial\bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial\bar{x}^k}\frac{\partial x^s}{\partial\bar{x}^l}\cdots b^{pq}_{rs}$$

Then

$$\begin{aligned}
\bar{b}^{ij\cdots}_{kl\ldots}\bar{u}_i\bar{v}_j\ldots\bar{w}^k\bar{z}^l &= \frac{\partial\bar{x}^i}{\partial x^p}\frac{\partial\bar{x}^j}{\partial x^q}\cdots\frac{\partial x^r}{\partial\bar{x}^k}\frac{\partial x^s}{\partial\bar{x}^l}\cdots b^{pq\ldots}_{rs\ldots}\frac{\partial x^\alpha}{\partial\bar{x}^i}u_\alpha\frac{\partial x^\beta}{\partial\bar{x}^j}v_\beta\cdots\frac{\partial\bar{x}^k}{\partial x^\kappa}w^\kappa\frac{\partial\bar{x}^l}{\partial x^\lambda}z^\lambda\ldots \\
&= \frac{\partial\bar{x}^i}{\partial x^p}\frac{\partial x^\alpha}{\partial\bar{x}^i}\frac{\partial\bar{x}^j}{\partial x^q}\frac{\partial x^\beta}{\partial\bar{x}^j}\cdots\frac{\partial x^r}{\partial\bar{x}^k}\frac{\partial\bar{x}^k}{\partial x^\kappa}\frac{\partial x^s}{\partial\bar{x}^l}\frac{\partial\bar{x}^l}{\partial x^\lambda}\cdots b^{pq\cdots}_{rs\ldots}u_\alpha v_\beta\ldots w^\kappa z^\lambda\ldots \\
&= \delta_{p\alpha}\delta_{q\beta}\ldots\delta_{r\kappa}\delta_{s\lambda}\ldots b^{pq\cdots}_{rs\ldots}u_\alpha v_\beta\ldots w^\kappa z^\lambda\ldots \\
&= b^{pq\cdots}_{rs\ldots}\delta_{p\alpha}\delta_{q\beta}u_\alpha v_\beta\ldots\delta_{r\kappa}\delta_{s\lambda}w^\kappa z^\lambda\cdots = b^{pq\cdots}_{rs\ldots}u_pv_q\ldots w^rz^s\ldots
\end{aligned}$$

which confirms that the tensor product $b^{ij\cdots}_{kl\ldots}u_iv_j\ldots w^kz^l\ldots$ indeed produces an invariant when $b^{ij\cdots}_{kl\ldots}$ is a tensor of the type indicated.

18. (a) In order to obtain the contravariant tensor b^{ij} associated with the covariant tensor b_{ij} we can simply multiply b_{ij} by the contravariant metric tensor g^{pq} twice to raise both indices of b_{ij} as is discussed in Section 3.9. Hence,

$$b^{ij} = g^{ik}g^{jl}b_{kl} = g^{ik}g^{jl}\underbrace{\left(\frac{\partial u_l}{\partial x^k} - \frac{\partial u_k}{\partial x^l}\right)}_{(3.109)}$$

(b) By (3.83) and (3.84), and using the fact that the (x^i) coordinates are assumed to be orthogonal, we find for $n=3$: $ds^2 = g_{11}dx^1dx^2 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3$. That is, the metric tensor (g_{ij}) is diagonal, since for an arbitrary off-diagonal element g_{kl}, $k \neq l$ we have[1]

$$g_{kl} = \frac{\partial\xi^i}{\partial x^k}\frac{\partial\xi^i}{\partial x^l} = \frac{\partial\boldsymbol{\Xi}}{\partial x^k}\cdot\frac{\partial\boldsymbol{\Xi}}{\partial x^l} = 0 \qquad \text{for } k \neq l$$

Comparing this result with (3.54) and recognizing that $x^1 = u$, $x^2 = v$, $x^3 = w$ we thus conclude that $g_{11} = \alpha^2$, $g_{22} = \beta^2$, $g_{33} = \gamma^2$, and by (3.87) we thus find that $g^{11} = 1/\alpha^2$, $g^{22} = 1/\beta^2$, $g^{33} = 1/\gamma^2$. Now as (3.58') shows, the components p_u, p_v, p_w of the vector $\mathbf{p}$ from Section 3.8 are neither contravariant nor covariant. Instead these components are obtained geometrically by perpendicular projection.

[1]Also see the discussion surrounding (3.53) and (3.54).

From (3.58') we see that αp_u, βp_v, γp_w are covariant components of $\mathbf{p}$. Hence, substituting in the equation of part (a) to find the three non-zero components b^{23}, b^{31}, b^{12} of the tensor b^{ij} gives

$$\begin{aligned}
b^{23} &= \underbrace{g^{21}g^{31}}_{0}(\ldots) + g^{22}\underbrace{g^{32}}_{0}(\ldots) + g^{23}\underbrace{g^{33}}_{0}(\ldots) + g^{22}\underbrace{g^{31}}_{0}(\ldots) + \underbrace{g^{21}g^{32}}_{0}(\ldots) \\
&\quad + \underbrace{g^{23}g^{31}}_{0}(\ldots) + \underbrace{g^{21}}_{0}g^{33}(\ldots) + \underbrace{g^{23}g^{32}}_{0}(\ldots) + g^{22}g^{33}\left(\frac{\partial u_3}{\partial x^2} - \frac{\partial u_2}{\partial x^3}\right) \\
&= \frac{1}{\beta^2\gamma^2}\left[\frac{\partial}{\partial v}(\gamma p_w) - \frac{\partial}{\partial w}(\beta p_v)\right] \\
b^{31} &= \underbrace{g^{31}}_{0}g^{11}(\ldots) + \underbrace{g^{32}g^{12}}_{0}(\ldots) + g^{33}\underbrace{g^{13}}_{0}(\ldots) + g^{32}\underbrace{g^{11}}_{0}(\ldots) + \underbrace{g^{31}g^{12}}_{0}(\ldots) \\
&\quad + g^{33}g^{11}\left(\frac{\partial u_1}{\partial x^3} - \frac{\partial u_3}{\partial x^1}\right) + \underbrace{g^{31}g^{13}}_{0}(\ldots) + g^{33}\underbrace{g^{12}}_{0}(\ldots) + \underbrace{g^{32}g^{13}}_{0}(\ldots) \\
&= \frac{1}{\gamma^2\alpha^2}\left[\frac{\partial}{\partial w}(\alpha p_u) - \frac{\partial}{\partial u}(\gamma p_w)\right] \\
b^{12} &= g^{11}\underbrace{g^{21}}_{0}(\ldots) + g^{12}\underbrace{g^{22}}_{0}(\ldots) + \underbrace{g^{13}g^{23}}_{0}(\ldots) + \underbrace{g^{12}g^{21}}_{0}(\ldots) + g^{11}g^{22}\left(\frac{\partial u_2}{\partial x^1} - \frac{\partial u_1}{\partial x^2}\right) \\
&\quad + \underbrace{g^{13}g^{21}}_{0}(\ldots) + \underbrace{g^{11}}_{0}g^{23}(\ldots) + g^{13}\underbrace{g^{22}}_{0}(\ldots) + \underbrace{g^{12}g^{23}}_{0}(\ldots) \\
&= \frac{1}{\alpha^2\beta^2}\left[\frac{\partial}{\partial u}(\beta p_v) - \frac{\partial}{\partial v}(\alpha p_u)\right]
\end{aligned}$$

Comparing this result with (3.62), we see that each components differs by a factor of $1/\beta\gamma$, $1/\gamma\alpha$, $1/\alpha\beta$ respectively. Again, this difference can be explained due to the fact that the components given by (3.62) are neither contravariant nor covariant. Let us focus on component b^{23} for now. Taking a hint from the solution to Problem 5 following Section 3.8 and (3.46), we can obtain the first of (3.62) by accounting for the extra scaling by $\beta\gamma$ due to dotting $\nabla \times \mathbf{p}$ with $(1/\alpha)(\partial \mathbf{r}/\partial u) = \beta\gamma\,(\nabla G \times \nabla H)$ in order to obtain the u component of $\operatorname{curl}\mathbf{p}$. In the same way we can account for the difference in scaling between respectively the second and third of (3.62) and the components b^{31} and b^{12} of the contravariant tensor b^{ij}.

(c) A covariant tensor is called alternating if interchanging two subscripts changes the sign. Now since

$$b_{ij} = \frac{\partial u_j}{\partial x^i} - \frac{\partial u_i}{\partial x^j} = -\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} = -\left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\right) = -b_{ji}$$

the second-order covariant tensor b_{ij} is alternating.

19. For $n = 3$, let u_i, v_j be two covariant tensors and let their exterior (or wedge) product be given by
$$w_{ij} = u_i \wedge v_j = 2\mathcal{A}\,(u_i v_j) = \sum \eta_{ij} u_i v_j =$$
where the sum is over all permutations of i and j and η_{ij} is $+1$ for an even permutation, -1 for an odd permutation. Hence, we find
$$w_{23} = u_2 v_3 - u_3 v_2 = -w_{32}, \quad w_{31} = u_3 v_1 - u_1 v_3 = -w_{13}, \quad w_{12} = u_1 v_2 - u_2 v_1 = -w_{21}$$
so that in standard coordinates x, y, z, with usual orientation and $u_1 = u_x, \ldots, v_1 = v_x, \ldots$, the three components of the vector product $(u_x\mathbf{i} + \ldots) \times (v_x\mathbf{i} + \ldots)$ are none other than the components w_{23}, w_{31}, w_{12} respectively.

20. Let $u_{i_1 \ldots i_r}$ be a covariant tensor in $E^n\,(r \geq 2)$ and let it be alternating in some coordinate system (x^i). Transforming to $(\bar{x}^i)$ coordinates and interchanging the first two indices then gives
$$\begin{aligned}\bar{u}_{i_1 i_2 \ldots i_r} = \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{i_r}} u_{j_1 j_2 \ldots j_r} &= \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{i_r}} \left(-u_{j_2 j_1 \ldots j_r}\right) \\ &= -\frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{i_r}} u_{j_2 j_1 \ldots j_r} = -\bar{u}_{i_2 i_1 \ldots i_r}\end{aligned}$$
which shows that interchanging two indices does nothing more than reversing the sign of the original alternating tensor after an odd number of interchanges or nothing after an even number of interchanges and reordering the summation operations (which has no effect on the end result). Hence, we may conclude that if a tensor $u_{i_1 \ldots i_r}$ is alternating in one coordinate system (x^i) it is alternating in every other allowed coordinate system $(\bar{x}^i)$.

21. Let u_i, v_i, w_{ij} be alternating covariant tensors in E^3.

    (a) Let the tensor product $u_i v_j = p_{ij}$ be a second-order covariant tensor. Then p_{ij} is alternating if $p_{ij} = -p_{ji}$ $(i = 1, \ldots, n,\ j = 1, \ldots, n)$, which in turn implies that $p_{ij} = 0$ if $i = j$, and hence, $p_{11} = u_1 v_1 = 0$. As such, if $p_{11} = u_1 v_1 \neq 0$, then $p_{ij} = u_i v_j$ fails to be an alternating second-order covariant tensor.

    (b) Let the tensor product $u_i w_{jk} = p_{ijk}$ be a third-order covariant tensor. Then p_{ijk} is alternating if for $1 \leq i,\ j,\ k \leq n$: $p_{ijk} = -p_{jik} = -p_{ikj} = -p_{kji}$, which in turn implies that when each components of p_{ijk} has a subscript equal it will be zero, i.e. $u_{ijk} = 0$ if $i = j$ or $i = k$ or $j = k$. Hence, $p_{121} = u_1 w_{21} = 0$. As such, if $p_{121} = u_1 w_{21} \neq 0$, then $p_{ijk} = u_i w_{jk}$ fails to be an alternating third-order covariant tensor.

    (c) In some coordinate system (x^i) let $u_i = 1$ for $i = 1, 2, 3$ and let the tensor product $p_{ij} = u_i v_j$ be a second-order covariant tensor. Then p_{ij} is alternating if for $i = 1, 2, 3,\ j = 1, 2, 3$ it holds that $p_{ij} = u_i v_j = -p_{ji} = -u_j v_i$. Since $u_i = 1$ for all $i = 1, 2, 3$ this condition reduces to $v_j = -v_i$, which obviously can only be satisfied if $v_j = 0$ for $j = 1, 2, 3$.

22. Let $u_{i_1 \ldots i_m}$ be a covariant tensor in coordinate system (x^i). Then equation (3.114) defines a new covariant tensor $v_{i_1 \ldots i_m}$ which is alternating since in $(\bar{x}^i)$ coordinates

$$\begin{aligned}
\bar{v}_{i_1 \ldots i_m} = \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_m}}{\partial \bar{x}^{i_m}} v_{j_1 \ldots j_m} &= \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_m}}{\partial \bar{x}^{i_m}} \mathcal{A}\left(u_{j_1 \ldots j_m}\right) \\
&= \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_m}}{\partial \bar{x}^{i_m}} \frac{1}{m!} \sum \eta_{j_1 j_2 \ldots j_m} u_{j_1 j_2 \ldots j_m} \\
&= \frac{1}{m!} \sum \eta_{j_1 j_2 \ldots j_m} \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{j_m}}{\partial \bar{x}^{i_m}} u_{j_1 j_2 \ldots j_m} \\
&= \frac{1}{m!} \sum \eta_{j_1 j_2 \ldots j_m} \bar{u}_{i_1 i_2 \ldots i_m} = \mathcal{A}\left(\bar{u}_{i_1 i_2 \ldots i_m}\right)
\end{aligned}$$

23. (a) Using the fact $\mathbf{r} = \mathbf{r}(u(t), v(t))$, such that $\partial \mathbf{r}/\partial u = \mathbf{r}_u$ and $\partial \mathbf{r}/\partial v = \mathbf{r}_v$, that the parameter t is arc length s on C (so that $|\mathbf{v}| = ds/dt = 1$), and (2.40) we can write the equation for the unit tangent vector $\mathbf{T} = d\mathbf{r}/ds$ as

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{ds} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{ds} = \frac{du}{ds}\mathbf{r}_u + \frac{dv}{ds}\mathbf{r}_v$$

(b) The acceleration vector $\mathbf{a} = d\mathbf{T}/ds$ is given by

$$\begin{aligned}
\mathbf{a} = \frac{d\mathbf{T}}{ds} &= \frac{d}{ds}\left(\frac{\partial \mathbf{r}}{\partial u}\frac{du}{ds} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{ds}\right) \\
&= \frac{\partial \mathbf{r}}{\partial u}\underbrace{\frac{d^2 u}{ds^2}}_{0} + \frac{du}{ds}\frac{d}{ds}\left(\frac{\partial \mathbf{r}}{\partial u}\right) + \frac{\partial \mathbf{r}}{\partial v}\underbrace{\frac{d^2 u}{ds^2}}_{0} + \frac{dv}{ds}\frac{d}{ds}\left(\frac{\partial \mathbf{r}}{\partial v}\right) \\
&= \frac{du}{ds}\left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\frac{du}{ds} + \frac{\partial^2 \mathbf{r}}{\partial u \partial v}\frac{dv}{ds}\right) + \frac{dv}{ds}\left(\frac{\partial^2 \mathbf{r}}{\partial v^2}\frac{dv}{ds} + \frac{\partial^2 \mathbf{r}}{\partial v \partial u}\frac{du}{ds}\right) \\
&= \left(\frac{du}{ds}\right)^2 \frac{\partial^2 \mathbf{r}}{\partial u^2} + 2\frac{du}{ds}\frac{dv}{ds}\frac{\partial^2 \mathbf{r}}{\partial u \partial v} + \left(\frac{dv}{ds}\right)^2 \frac{\partial^2 \mathbf{r}}{\partial v^2} \\
&= \left(\frac{du}{ds}\right)^2 \mathbf{r}_{uu} + 2\frac{du}{ds}\frac{dv}{ds}\mathbf{r}_{uv} + \left(\frac{dv}{ds}\right)^2 \mathbf{r}_{vv}
\end{aligned}$$

Note that we are assuming that $d^2u/ds^2 = d^2v/ds^2 = 0$ because of Problem 10 (f) following Section 3.8.

(c) Using the first Frenet formula from Problem 6 (d) following Section 2.13 $\kappa \mathbf{N} = d\mathbf{T}/ds$ we can write

$$\begin{aligned}
\kappa \mathbf{N} \cdot \mathbf{n} = \frac{d\mathbf{T}}{ds} \cdot \mathbf{n} &= \left[\left(\frac{du}{ds}\right)^2 \mathbf{r}_{uu} + 2\frac{du}{ds}\frac{dv}{ds}\mathbf{r}_{uv} + \left(\frac{dv}{ds}\right)^2 \mathbf{r}_{vv}\right] \cdot \mathbf{n} \\
&= \left(\frac{du}{ds}\right)^2 \mathbf{r}_{uu} \cdot \mathbf{n} + 2\frac{du}{ds}\frac{dv}{ds}\mathbf{r}_{uv} \cdot \mathbf{n} + \left(\frac{dv}{ds}\right)^2 \mathbf{r}_{vv} \cdot \mathbf{n} \\
&= L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2
\end{aligned}$$

(d) Let us consider a fixed point $P_0\,:\,(u_0, v_0)$ on the surface in space S and let $\mathbf{i}_1$, $\mathbf{j}_1$, $\mathbf{k}_1$ be a special basis for vectors in space with $\mathbf{k}_1 = \mathbf{n}$ at point P_0. Then we write $\mathbf{r} = \mathbf{r}_0 + \xi\mathbf{i}_1 + \eta\mathbf{j}_1 + \zeta\mathbf{k}_1$, with $(\xi, \eta\,\zeta) = (0, 0, 0)$ at P_0. The unit normal $\mathbf{n} = \mathbf{w}/|\mathbf{w}|$, where $\mathbf{w} = \mathbf{r}_u \times \mathbf{r}_v$. Hence, $\mathbf{r}_u \perp \mathbf{n}$, $\mathbf{r}_v \perp \mathbf{n}$ (i.e. both $\mathbf{r}_u$ and $\mathbf{r}_v$ are orthogonal to $\mathbf{n}$), and so $\mathbf{r}_u \cdot \mathbf{n} = \mathbf{r}_v \cdot \mathbf{n} = 0$ at P_0. Now since at P_0 we also have $\mathbf{r}_u \cdot \mathbf{n} = (\xi_u\mathbf{i}_1 + \eta_u\mathbf{j}_1 + \zeta_u\mathbf{k}_1) \cdot \mathbf{k}_1 = \zeta_u$, we conclude that $\zeta_u = 0$ at P_0. A similar reasoning shows that $\zeta_v = 0$ at P_0. Furthermore, we find $L = \mathbf{r}_{uu} \cdot \mathbf{n} = (\xi_{uu}\mathbf{i}_1 + \eta_{uu}\mathbf{j}_1 + \zeta_{uu}\mathbf{k}_1) \cdot \mathbf{k}_1 = \zeta_{uu}$, $M = \mathbf{r}_{uv} \cdot \mathbf{n} = (\xi_{uv}\mathbf{i}_1 + \eta_{uv}\mathbf{j}_1 + \zeta_{uv}\mathbf{k}_1) \cdot \mathbf{k}_1 = \zeta_{uv}$, $N = \mathbf{r}_{vv} \cdot \mathbf{n} = (\xi_{vv}\mathbf{i}_1 + \eta_{vv}\mathbf{j}_1 + \zeta_{vv}\mathbf{k}_1) \cdot \mathbf{k}_1 = \zeta_{vv}$ at P_0. These conditions must hold at P_0 for each allowable coordinate system (u, v). Hence, for new coordinates $(\bar{u}, \bar{v})$ with $\bar{u} = \bar{u}_0$, $\bar{v} = \bar{v}_0$ at P_0 we find, using (2.133) and Problem 6 following that section, that at P_0

$$\begin{aligned}
\zeta_{uu} &= \frac{\partial^2\zeta}{\partial u^2} = \underbrace{\frac{\partial\zeta}{\partial\bar{u}}}_{0}\frac{\partial^2\bar{u}}{\partial u^2} + \frac{\partial^2\zeta}{\partial\bar{u}^2}\left(\frac{\partial\bar{u}}{\partial u}\right)^2 + 2\frac{\partial^2\zeta}{\partial\bar{u}\partial\bar{v}}\frac{\partial\bar{u}}{\partial u}\frac{\partial\bar{v}}{\partial u} + \frac{\partial^2\zeta}{\partial\bar{v}^2}\left(\frac{\partial\bar{v}}{\partial u}\right)^2 + \underbrace{\frac{\partial\zeta}{\partial\bar{v}}}_{0}\frac{\partial^2\bar{v}}{\partial u^2} \\
&= \zeta_{\bar{u}\bar{u}}\left(\bar{u}_u\right)^2 + 2\zeta_{\bar{u}\bar{v}}\bar{u}_u\bar{v}_u + \zeta_{\bar{v}\bar{v}}\left(\bar{v}_u\right)^2 \\
\zeta_{uv} &= \frac{\partial^2\zeta}{\partial u\partial v} = \underbrace{\frac{\partial\zeta}{\partial\bar{u}}}_{0}\frac{\partial^2\bar{u}}{\partial u\partial v} + \frac{\partial^2\zeta}{\partial\bar{u}^2}\frac{\partial\bar{u}}{\partial u}\frac{\partial\bar{u}}{\partial v} + \frac{\partial^2\zeta}{\partial\bar{u}\partial\bar{v}}\frac{\partial\bar{u}}{\partial v}\frac{\partial\bar{v}}{\partial u} + \frac{\partial^2\zeta}{\partial\bar{u}\partial\bar{v}}\frac{\partial\bar{u}}{\partial u}\frac{\partial\bar{v}}{\partial v} + \frac{\partial^2\zeta}{\partial\bar{v}^2}\frac{\partial\bar{v}}{\partial u}\frac{\partial\bar{v}}{\partial v} \\
&\qquad + \underbrace{\frac{\partial\zeta}{\partial\bar{v}}}_{0}\frac{\partial^2\bar{v}}{\partial u\partial v} = \zeta_{\bar{u}\bar{u}}\bar{u}_u\bar{u}_v + \zeta_{\bar{u}\bar{v}}\bar{u}_v\bar{v}_u + \zeta_{\bar{u}\bar{v}}\bar{u}_u\bar{v}_v + \zeta_{\bar{v}\bar{v}}\bar{v}_u\bar{v}_v \\
\zeta_{vv} &= \frac{\partial^2\zeta}{\partial v^2} = \underbrace{\frac{\partial\zeta}{\partial\bar{u}}}_{0}\frac{\partial^2\bar{u}}{\partial v^2} + \frac{\partial^2\zeta}{\partial\bar{u}^2}\left(\frac{\partial\bar{u}}{\partial v}\right)^2 + 2\frac{\partial^2\zeta}{\partial\bar{u}\partial\bar{v}}\frac{\partial\bar{u}}{\partial v}\frac{\partial\bar{v}}{\partial v} + \frac{\partial^2\zeta}{\partial\bar{v}^2}\left(\frac{\partial\bar{v}}{\partial v}\right)^2 + \underbrace{\frac{\partial\zeta}{\partial\bar{v}}}_{0}\frac{\partial^2\bar{v}}{\partial v^2} \\
&= \zeta_{\bar{u}\bar{u}}\left(\bar{u}_v\right)^2 + 2\zeta_{\bar{u}\bar{v}}\bar{u}_v\bar{v}_v + \zeta_{\bar{v}\bar{v}}\left(\bar{v}_v\right)^2
\end{aligned}$$

Let $i \le i,\, j,\, k,\, l \le 2$. Then it is trivial to see that the four equations above can be written as the tensor equation

$$L_{ij} = \bar{L}_{kl}\frac{\partial\bar{u}^k}{\partial u^i}\frac{\partial\bar{u}^l}{\partial u^j}$$

24. (a) The result of Problem 23 (c) shows that the curvature κ satisfies the equation

$$\kappa\mathbf{N} \cdot \mathbf{n} = L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2$$

Using (1.9) and the fact that the principal normal vector $\mathbf{N}$ and the normal vector $\mathbf{n}$ for the surface S are both unit vectors, this equation can be rewritten as

$$\kappa\cos\theta = L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2$$

where $\theta = \measuredangle(\mathbf{N}, \mathbf{n})$. Since $\mathbf{T}$ is assumed to be kept fixed (such that $du/ds =$ const, $dv/ds =$ const), while $\mathbf{N}$ is varied, this reduces to $\kappa \cos\theta = \text{const} = \pm\kappa_n$, $\kappa_n \geq 0$, where the $\pm$ sign follows from the fact that $\cos\theta > 0$ for $0 \leq \theta < \pi/2$ and $\cos\theta < 0$ for $\pi/2 < \theta \leq \pi$.

(b) From part (a) and the fact that $\kappa_n \geq 0$ it follows immediately that

$$\kappa_n = \left| L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2 \right|$$

25. Let the tangent vector $\mathbf{T}$ vary at a fixed point P_0 and let $\xi = du/ds$, $\eta = dv/ds$. Then by Problem 24 (b) $\kappa_n = |L\xi^2 + 2M\xi\eta + N\eta^2|$. Then on order to find the maximum and minimum of the normal curvature κ_n for the tangent vector $\mathbf{T}$ at the fixed point P_0 we are led to seek the critical points of the function $f(\xi, \eta) = L\xi^2 + 2M\xi\eta + N\eta^2$ subject to the side condition $g(\xi, \eta) = E\xi^2 + 2F\xi\eta + G\eta^2 = 1$ from Problem 23 (a). Next, taking a hint from Problem 14 and 15 following Section 2.21, we can write $f(\xi, \eta) = \begin{bmatrix}\xi & \eta\end{bmatrix} \mathbf{A} \begin{bmatrix}\xi & \eta\end{bmatrix}^\top$, $g(\xi, \eta) = \begin{bmatrix}\xi & \eta\end{bmatrix} \mathbf{B} \begin{bmatrix}\xi & \eta\end{bmatrix}^\top$, where

$$\mathbf{A} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

from which we can obtain the quadratic equation

$$\det(\mathbf{A} - \lambda\mathbf{B}) = \begin{vmatrix} L - \lambda E & M - \lambda F \\ M - \lambda F & N - \lambda G \end{vmatrix} = 0$$

from which we may determine the two real eigenvalues $\lambda_1 \leq \lambda_2$, corresponding to the eigenvectors $\mathbf{v}_1 = (\xi_1, \eta_1)$, $\mathbf{v}_2 = (\xi_2, \eta_2)$ respectively. Hence, according to Section 2.21, f has an absolute minimum λ_1 and absolute maximum λ_2 under the given side condition.