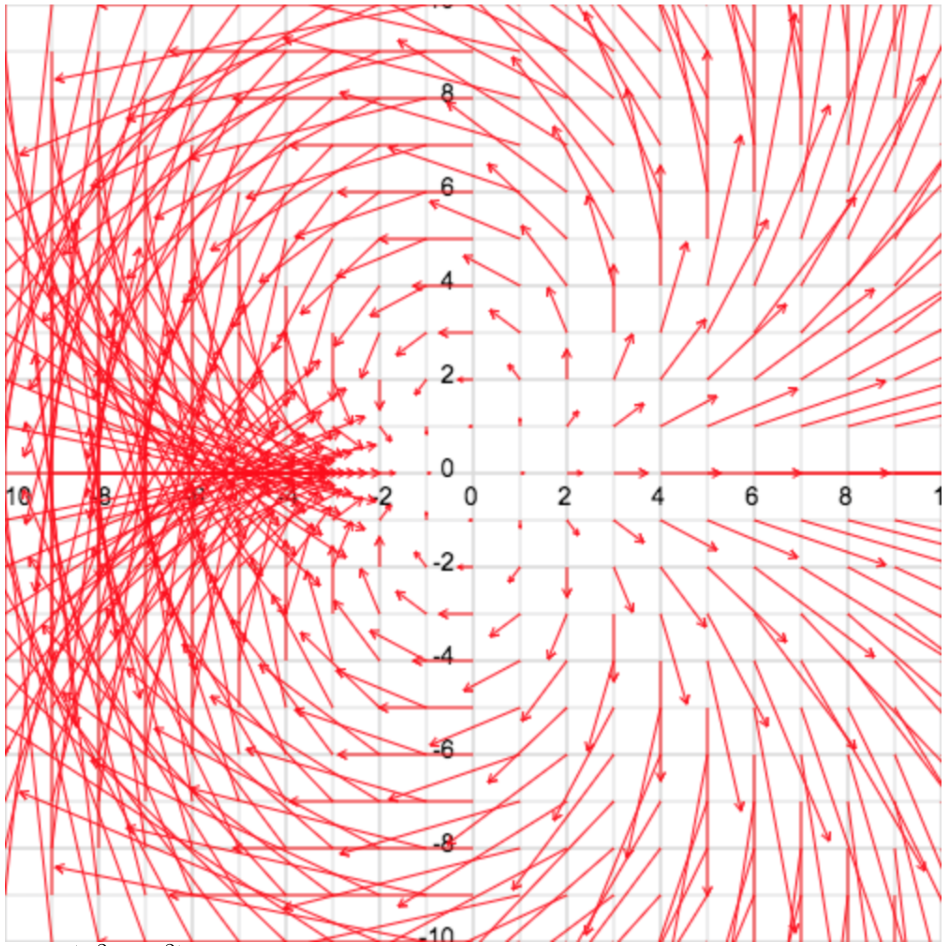
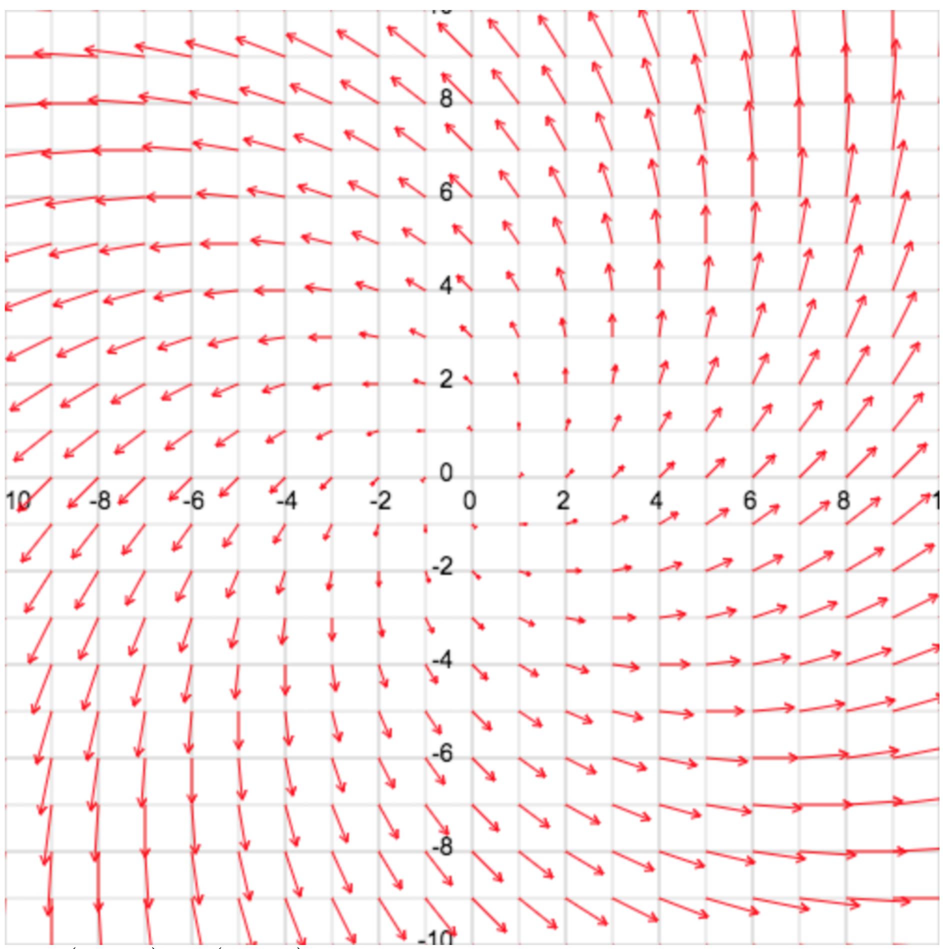
CHAPTER 3

Section 3.3

1. (a) 

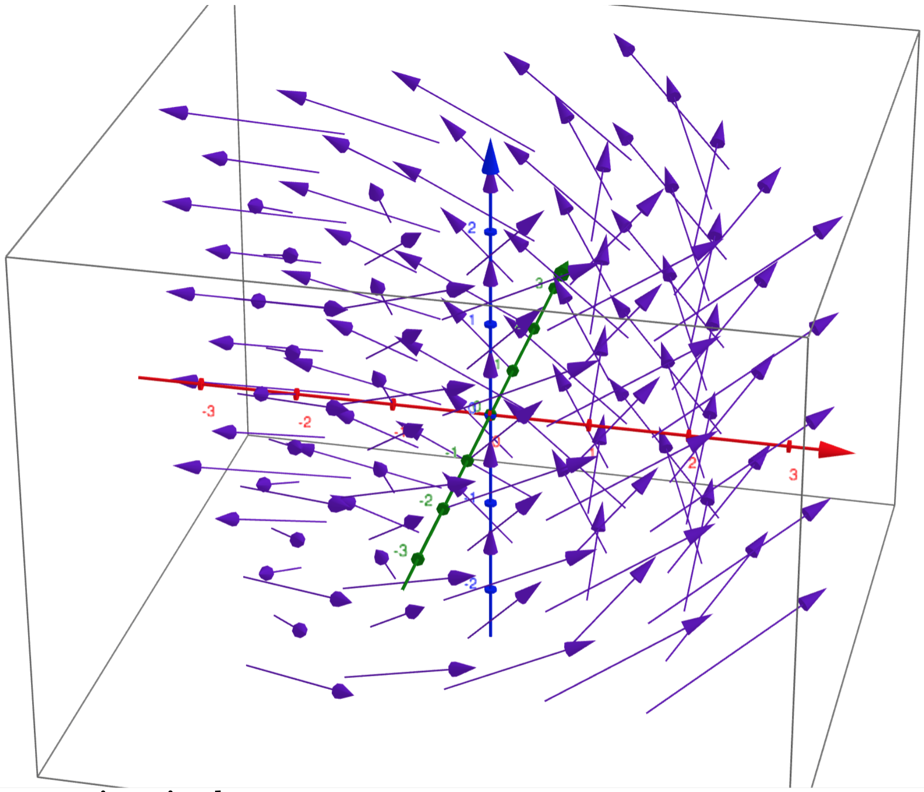
$\mathbf{v} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$

(b)



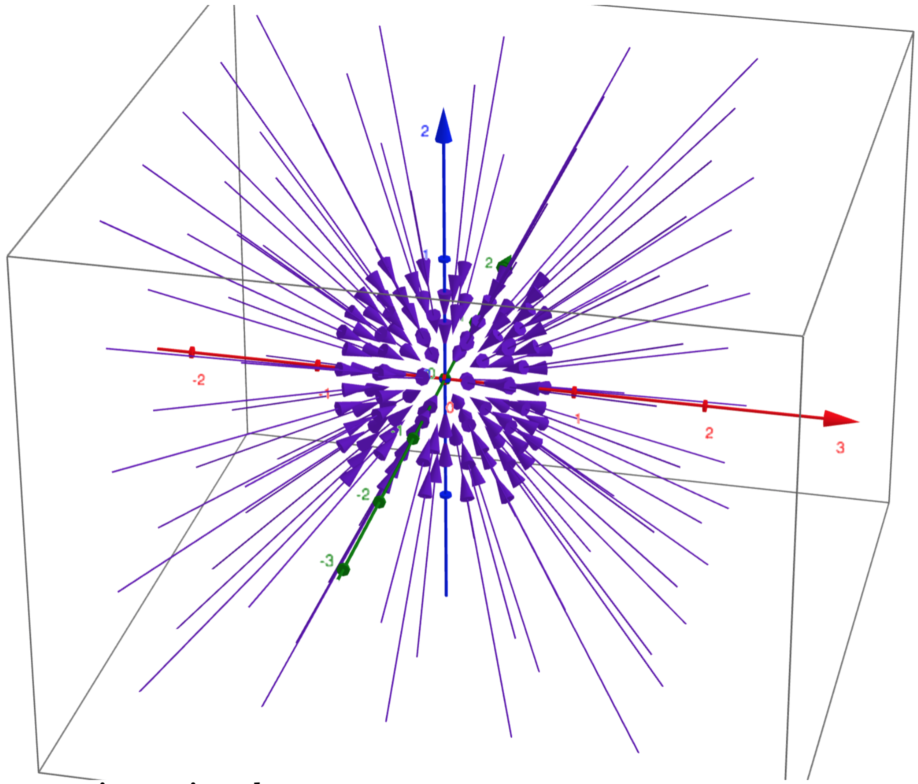
$\mathbf{u} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$

(c)



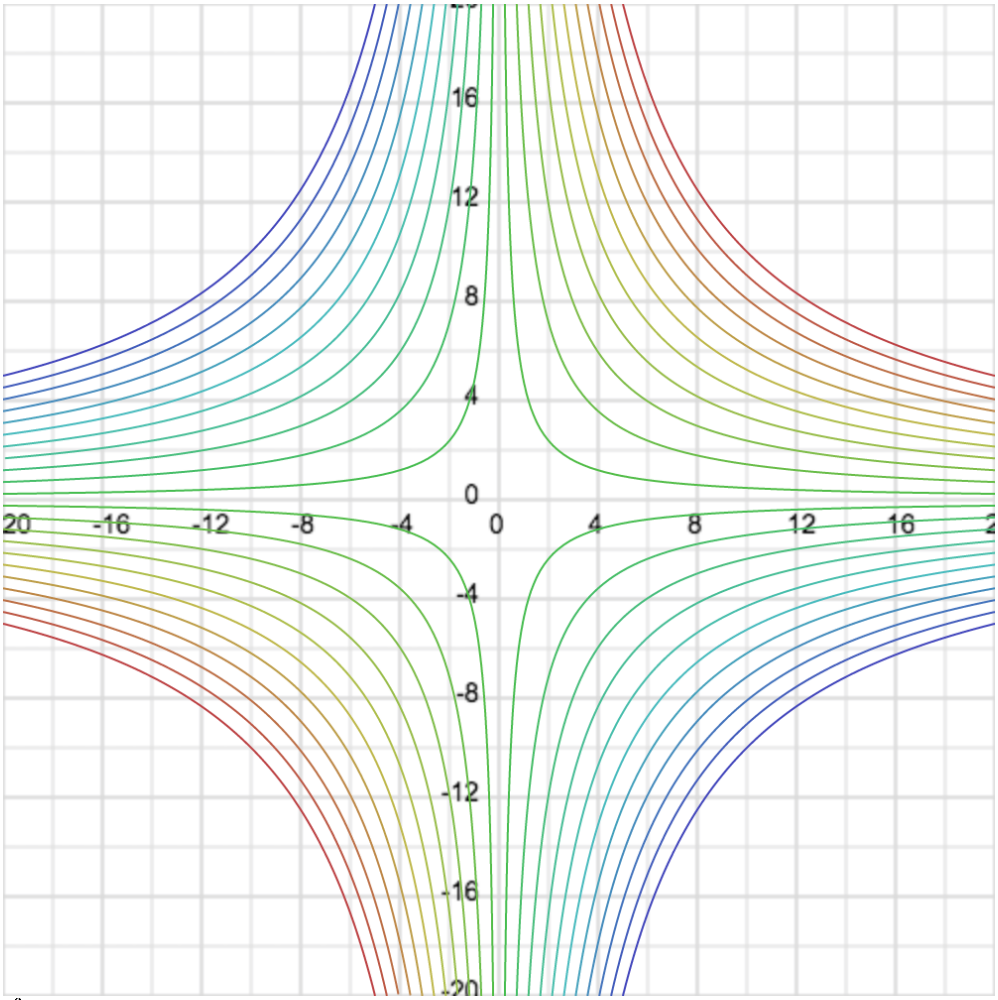
$\mathbf{v} = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$

(d)



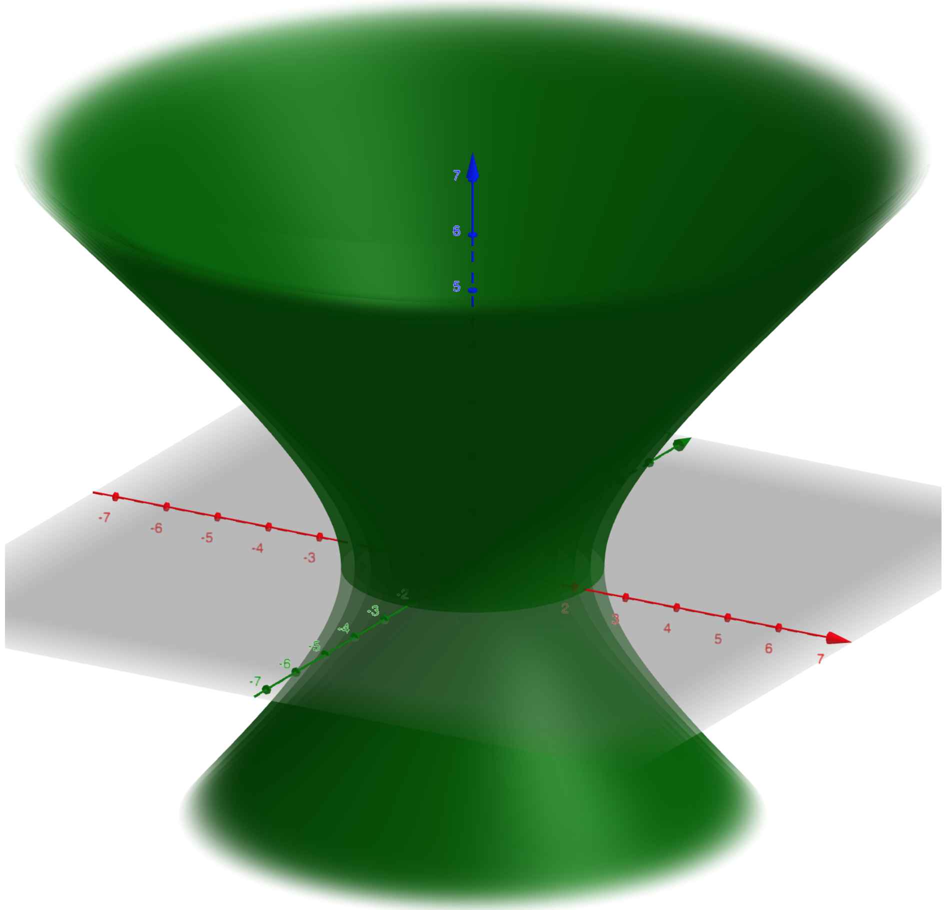
$\mathbf{v} = -x\mathbf{i} + -y\mathbf{j} - z\mathbf{k}$

2. (a)



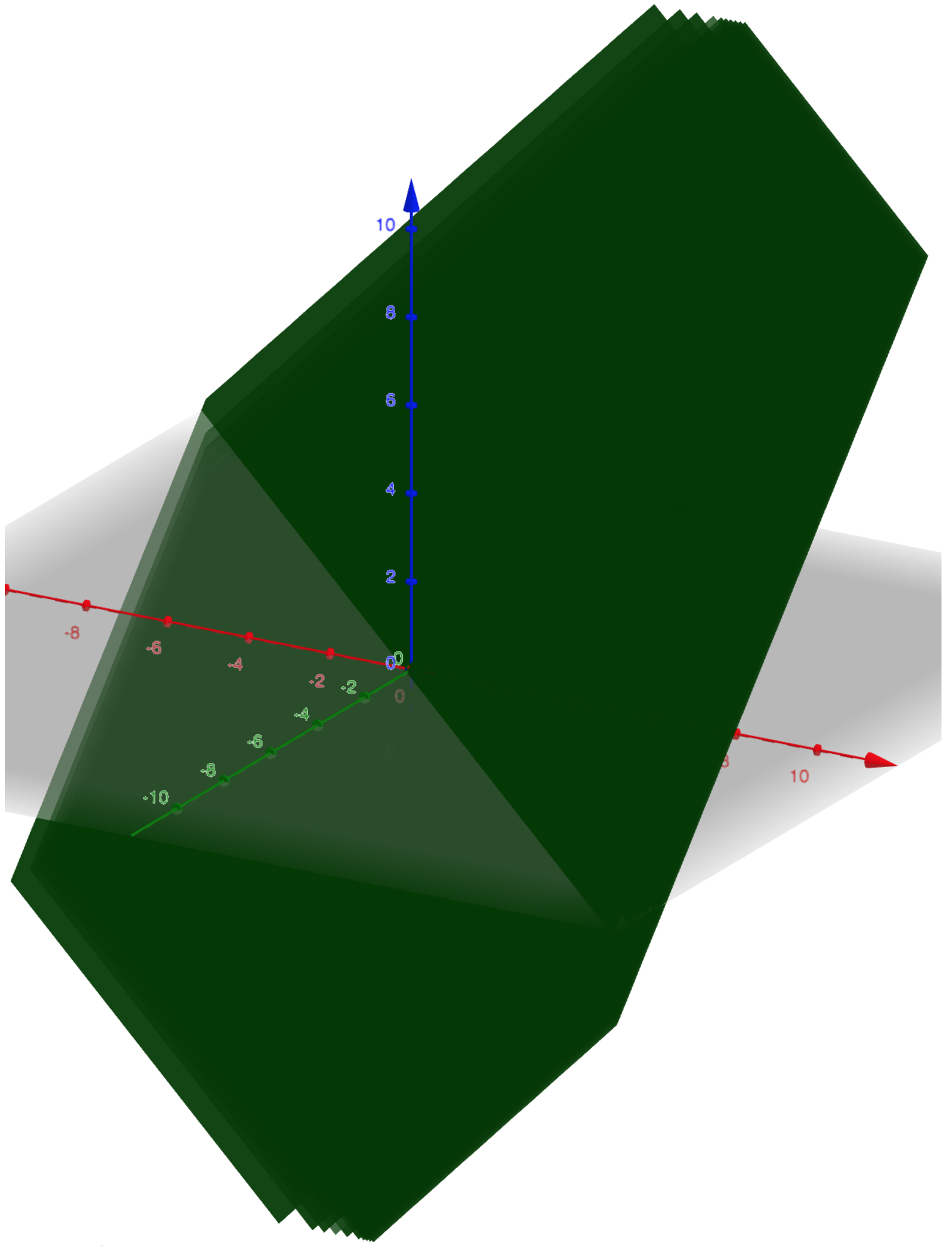
$f = xy$

(b)



$f = x^2 + y^2 - z^2$

(c)



$f = e^{x+y-z}$

3. If $f = xy$ then ∇f is given by

$$\nabla f = y\mathbf{i} + x\mathbf{j}$$

If $f = x^2 + y^2 - z^2$ then ∇f is given by

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$$

If $f = e^{x+y-z}$ then ∇f is given by

$$\nabla f = e^{x+y-z}\mathbf{i} + e^{x+y-z}\mathbf{j} - e^{x+y-z}\mathbf{k}$$

4. Let $f = kMm/r$, where $r = \sqrt{x^2+y^2+z^2}$ be the equation for the gravitational potential. Then

$$\begin{aligned}
\nabla f &= \nabla\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{x}{r}\mathbf{i} - \frac{kMm}{r^2}\frac{y}{r}\mathbf{j} - \frac{kMm}{r^2}\frac{z}{r}\mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{\mathbf{r}}{r}
\end{aligned}$$

is a vector equation for the gravitational field.

5. Let f be given by

$$f = \ln\frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}}$$

Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}\frac{\partial}{\partial x}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{x-1}{(x-1)^2+y^2} - \frac{x+1}{(x+1)^2+y^2} \\
&= \frac{2\left(x^2-y^2-1\right)}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]} \\
\frac{\partial f}{\partial y} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}\frac{\partial}{\partial y}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{y}{(x-1)^2+y^2} - \frac{y}{(x+1)^2+y^2} \\
&= \frac{4xy}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}
\end{aligned}$$

Hence,

$$\nabla f = \frac{1}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}\left[2\left(x^2-y^2-1\right)\mathbf{i} + 4xy\mathbf{j}\right]$$

6.

$$
\begin{aligned}
\nabla(f+g) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(f+g) \\
&= \frac{\partial}{\partial x}(f+g)\,\mathbf{i}+\frac{\partial}{\partial y}(f+g)\,\mathbf{j}+\frac{\partial}{\partial z}(f+g)\,\mathbf{k} \\
&= \left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\right)\mathbf{i}+\left(\frac{\partial f}{\partial y}+\frac{\partial g}{\partial y}\right)\mathbf{j}+\left(\frac{\partial f}{\partial z}+\frac{\partial g}{\partial z}\right)\mathbf{k} \\
&= \left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)+\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}+\frac{\partial g}{\partial z}\mathbf{k}\right) \\
&= \nabla f+\nabla g
\end{aligned}
$$

$$
\begin{aligned}
\nabla(fg) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(fg) \\
&= \frac{\partial}{\partial x}(fg)\,\mathbf{i}+\frac{\partial}{\partial y}(fg)\,\mathbf{j}+\frac{\partial}{\partial z}(fg)\,\mathbf{k} \\
&= \left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)\mathbf{i}+\left(f\frac{\partial g}{\partial y}+g\frac{\partial f}{\partial y}\right)\mathbf{j}+\left(f\frac{\partial g}{\partial z}+g\frac{\partial f}{\partial z}\right)\mathbf{k} \\
&= \left(f\frac{\partial g}{\partial x}\mathbf{i}+f\frac{\partial g}{\partial y}\mathbf{j}+f\frac{\partial g}{\partial z}\mathbf{k}\right)+\left(g\frac{\partial f}{\partial x}\mathbf{i}+g\frac{\partial f}{\partial y}\mathbf{j}+g\frac{\partial f}{\partial z}\mathbf{k}\right) \\
&= f\nabla g+g\nabla f
\end{aligned}
$$

7. Let $f(x,y,z)$ be a composite function $F(u)$, where $u=g(x,y,z)$. Then

$$
\begin{aligned}
\nabla f = \nabla F = \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)F(u) &= \frac{\partial}{\partial x}F(u)\,\mathbf{i}+\frac{\partial}{\partial y}F(u)\,\mathbf{j}+\frac{\partial}{\partial z}F(u)\,\mathbf{k} \\
&= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\mathbf{k} \\
&= \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial u}{\partial z}\mathbf{k}\right) \\
&= F'(u)\,\nabla g
\end{aligned}
$$

8.

$$
\begin{aligned}
\nabla \frac{f}{g} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\frac{f}{g} \\
&= \frac{\partial}{\partial x}\frac{f}{g}\mathbf{i} + \frac{\partial}{\partial y}\frac{f}{g}\mathbf{j} + \frac{\partial}{\partial z}\frac{f}{g}\mathbf{k} \\
&= \frac{gf_x - fg_x}{g^2}\mathbf{i} + \frac{gf_y - fg_y}{g^2}\mathbf{j} + \frac{gf_z - fg_z}{g^2}\mathbf{k} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) - f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)\right] \\
&= \frac{1}{g^2}\left(g\nabla f - f\nabla g\right)
\end{aligned}
$$

9. (a) If $f(x,y,z) = w = x^3y - y^3z$ then H is given by

$$
H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \begin{bmatrix} w_{xx} & w_{xy} & w_{xz} \\ w_{yx} & w_{yy} & w_{yz} \\ w_{zx} & w_{zy} & w_{zz} \end{bmatrix} = \begin{bmatrix} 6xy & 3x^2 & 0 \\ 3x^2 & -6yz & -3y^2 \\ 0 & -3y^2 & 0 \end{bmatrix}
$$

If $f(x,y,z) = w = x_1^2 + 2x_1x_2 + 5x_1x_3 + 2x_2x_1 + 4x_2^2 + x_2x_3 + 5x_3x_1 + x_3x_2 + 2x_3^2$ then H is given by

$$
H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \begin{bmatrix} w_{x_1x_1} & w_{x_1x_2} & w_{x_1x_3} \\ w_{x_2x_1} & w_{x_2x_2} & w_{x_2x_3} \\ w_{x_3x_1} & w_{x_3x_2} & w_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 2 \\ 10 & 2 & 4 \end{bmatrix}
$$

(b) As long as the function $f(x_1, \ldots, x_n)$ has continuous second partial derivatives then $\partial^2 f/(\partial x_i \partial x_j) = \partial^2 f/(\partial x_j \partial x_i)$, which implies that H will be symmetric.

(c) As discussed in Section 2.14, the directional derivative of a function $f(x,y)$ in a given direction can be written as

$$
\nabla_\alpha f = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha = \nabla f \cdot \mathbf{u}
$$

where $\mathbf{u} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$ is a unit vector that makes an angle α with the positive x-axis. Hence,

$$
\begin{aligned}
\nabla_\alpha \nabla_\beta f &= \nabla_\alpha \left(\frac{\partial f}{\partial x}\cos\beta + \frac{\partial f}{\partial y}\sin\beta\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos\beta + \frac{\partial f}{\partial y}\sin\beta\right)\cos\alpha + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\beta + \frac{\partial f}{\partial y}\sin\beta\right)\sin\alpha \\
&= \cos\beta\left(\frac{\partial^2 f}{\partial x^2}\cos\alpha + \frac{\partial^2 f}{\partial x \partial y}\sin\alpha\right) + \sin\beta\left(\frac{\partial^2 f}{\partial y \partial x}\cos\alpha + \frac{\partial^2 f}{\partial y^2}\sin\alpha\right) \\
&= \begin{bmatrix} \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos\beta & \sin\beta \end{bmatrix} H \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix}^\top
\end{aligned}
$$

Section 3.6

1.

$$\begin{aligned}
\nabla \cdot (\mathbf{u}+\mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left[(u_x+v_x)\,\mathbf{i}+(u_y+v_y)\,\mathbf{j}+(u_z+v_z)\,\mathbf{k}\right] \\
&= \frac{\partial}{\partial x}(u_x+v_x)+\frac{\partial}{\partial y}(u_y+v_y)+\frac{\partial}{\partial z}(u_z+v_z) \\
&= \left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right) \\
&= \nabla\cdot\mathbf{u}+\nabla\cdot\mathbf{v}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot (f\mathbf{u}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(fu_x\mathbf{i}+fu_y\mathbf{j}+fu_z\mathbf{k}) \\
&= \frac{\partial}{\partial x}(fu_x)+\frac{\partial}{\partial y}(fu_y)+\frac{\partial}{\partial z}(fu_z) \\
&= f\frac{\partial u_x}{\partial x}+u_x\frac{\partial f}{\partial x}+f\frac{\partial u_y}{\partial y}+u_y\frac{\partial f}{\partial y}+f\frac{\partial u_z}{\partial z}+u_z\frac{\partial f}{\partial z} \\
&= f\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\left(u_x\frac{\partial f}{\partial x}+u_y\frac{\partial f}{\partial y}+u_z\frac{\partial f}{\partial z}\right) \\
&= f\,(\nabla\cdot\mathbf{u})+(\nabla f\cdot\mathbf{u})
\end{aligned}$$

2. Recognizing that $\mathbf{v}=\rho\mathbf{u}$ and using (3.22), then (3.17) can be written as

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{v}=\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=\frac{\partial\rho}{\partial t}+(\nabla\rho\cdot\mathbf{u})+\rho\,(\nabla\cdot\mathbf{u})=0$$

According to Problem 12 of Section 2.8, the first two terms can be written as

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{u}=\frac{\partial\rho}{\partial t}+u_x\frac{\partial\rho}{\partial x}+u_y\frac{\partial\rho}{\partial y}+u_z\frac{\partial\rho}{\partial z}=\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}\frac{dx}{dt}+\frac{\partial\rho}{\partial y}\frac{dy}{dt}+\frac{\partial\rho}{\partial z}\frac{dz}{dt}=\frac{d\rho}{dt}=\frac{D\rho}{Dt}$$

Hence, (3.17) can be written as

$$\frac{\partial\rho}{\partial t}+(\nabla\rho\cdot\mathbf{u})+\rho\,(\nabla\cdot\mathbf{u})=\frac{D\rho}{Dt}+\rho\,(\nabla\cdot\mathbf{u})=0$$

When $\rho\equiv a$, where a is some arbitrary constant, then $D\rho/dt\equiv Da/dt=0$ and the equation above reduces to

$$\rho\,(\nabla\cdot\mathbf{u})\equiv a\,(\nabla\cdot\mathbf{u})=0$$

Since $\rho\equiv a\neq 0$, the only way for this equation to make sense is if $\nabla\cdot\mathbf{u}=0$.

3.

$$
\begin{aligned}
\nabla \times (\mathbf{u}+\mathbf{v}) &= \left[\frac{\partial}{\partial y}(u_z+v_z) - \frac{\partial}{\partial z}(u_y+v_y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(u_x+v_x) - \frac{\partial}{\partial x}(u_z+v_z)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(u_y+v_y) - \frac{\partial}{\partial y}(u_x+v_x)\right]\mathbf{k} \\
&= \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&\quad + \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\
&= (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times (f\mathbf{u}) &= \left[\frac{\partial}{\partial y}(fu_z) - \frac{\partial}{\partial z}(fu_y)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(fu_x) - \frac{\partial}{\partial x}(fu_z)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(fu_y) - \frac{\partial}{\partial y}(fu_x)\right]\mathbf{k} \\
&= \left[f\frac{\partial u_z}{\partial y} + u_z\frac{\partial f}{\partial y} - \left(f\frac{\partial u_y}{\partial z} + u_y\frac{\partial f}{\partial z}\right)\right]\mathbf{i} + \left[f\frac{\partial u_x}{\partial z} + u_x\frac{\partial f}{\partial z} - \left(f\frac{\partial u_z}{\partial x} + u_z\frac{\partial f}{\partial x}\right)\right]\mathbf{j} \\
&\quad + \left[f\frac{\partial u_y}{\partial x} + u_y\frac{\partial f}{\partial x} - \left(f\frac{\partial u_x}{\partial y} + u_x\frac{\partial f}{\partial y}\right)\right]\mathbf{k} \\
&= f\left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&\quad + \left[\left(u_z\frac{\partial f}{\partial y} - u_y\frac{\partial f}{\partial z}\right)\mathbf{i} + \left(u_x\frac{\partial f}{\partial z} - u_z\frac{\partial f}{\partial x}\right)\mathbf{j} + \left(u_y\frac{\partial f}{\partial x} - u_x\frac{\partial f}{\partial y}\right)\mathbf{k}\right] \\
&= (f\nabla \times \mathbf{u}) + (\nabla f \times \mathbf{u})
\end{aligned}
$$

4.

$$
\begin{aligned}
\nabla \times (\nabla f) &= \left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]\mathbf{k} \\
&= \left(\frac{\partial^2 f}{\partial z\partial y} - \frac{\partial^2 f}{\partial y\partial z}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial z\partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial y\partial x} - \frac{\partial^2 f}{\partial x\partial y}\right)\mathbf{k} \\
&= \mathbf{0}
\end{aligned}
$$

5. (a) If $\mathbf{v} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$, then

$$
\begin{aligned}
\nabla \times \mathbf{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k} \\
&= \left(x^2 - x^2\right)\mathbf{i} + (2xy - 2xy)\mathbf{j} + (2xz - 2xz)\mathbf{k} \\
&= \mathbf{0}
\end{aligned}
$$

Let $f = x^2yz + a$, where a is an arbitrary constant. Then $\nabla f = \mathbf{v}$.

(b) If $\mathbf{v} = e^{xy}[(2y^2 + yz^2)\mathbf{i} + (2xy + xz^2 + 2)\mathbf{j} + 2z\mathbf{k}]$, then

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left(2xze^{xy} - 2xze^{xy}\right)\mathbf{i} + \left(2yze^{xy} - 2yze^{xy}\right)\mathbf{j} \\
&\quad + \left[ye^{xy}\left(2xy + xz^2 + 2\right) + e^{xy}\left(2y + z^2\right) - xe^{xy}\left(2y^2 + yz^2\right) - e^{xy}\left(4y + z^2\right)\right]\mathbf{k} \\
&= \mathbf{0}
\end{aligned}$$

Let $f = e^{xy}(2y + z^2) + a$, where a is an arbitrary constant. Then $\nabla f = \mathbf{v}$.

6.

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\
&\quad \cdot \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\
&= \frac{\partial^2 v_z}{\partial y \partial x} - \frac{\partial^2 v_y}{\partial z \partial x} + \frac{\partial^2 v_x}{\partial z \partial y} - \frac{\partial^2 v_z}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial y \partial z} \\
&= 0
\end{aligned}$$

7. (a) If $\mathbf{v} = 2x\mathbf{i} + y\mathbf{j} - 3z\mathbf{k}$, then

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 2 + 1 - 3 = 0$$

Since $\nabla \cdot \mathbf{v} = 0$, the vector $\mathbf{v} = \nabla \times \mathbf{u}$ for some vector $\mathbf{u}$. Furthermore, since by (3.32) $\nabla \times (\nabla f) = \mathbf{0}$, we can safely assume that $\mathbf{u}$ is of the form $\mathbf{u} = \mathbf{u}_0 + \nabla f$, where f is an arbitrary scalar function and $\mathbf{u}_0$ is any one vector whose curl is $\mathbf{v}$, as then $\nabla \times \mathbf{u} = \nabla \times (\mathbf{u}_0 + \nabla f) = (\nabla \times \mathbf{u}_0) + [\nabla \times (\nabla f)] = \nabla \times \mathbf{u}_0$.
Next, assume $\mathbf{u}_0 \cdot \mathbf{k} = 0$, which implies $u_{0z} = 0$. Equating the components of $\nabla \times \mathbf{u}_0$ to those of $\mathbf{v}$ then gives

$$\frac{\partial u_{0z}}{\partial y} - \frac{\partial u_{0y}}{\partial z} = -\frac{\partial u_{0y}}{\partial z} = 2x, \quad \frac{\partial u_{0x}}{\partial z} - \frac{\partial u_{0z}}{\partial x} = \frac{\partial u_{0x}}{\partial z} = y, \quad \frac{\partial u_{0y}}{\partial x} - \frac{\partial u_{0x}}{\partial y} = -3z$$

from which we may deduce that $\mathbf{u}_0 = yz\mathbf{i} - 2xz\mathbf{j}$.

(b) If $\mathbf{v} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, then going through the exact same steps as for part (a) gives $\mathbf{u}_0 = (z^2/2)\mathbf{i} + [(x^2 - 2yz)/2]\mathbf{j}$.

8.

$$\begin{aligned}
\operatorname{div}\operatorname{grad} f = \nabla\cdot(\nabla f) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z} \\
&= \nabla^2 f \\
&= \Delta f
\end{aligned}$$

Let $f = 1/\sqrt{x^2+y^2+z^2}$. Then

$$\nabla^2 f = \frac{2x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{x^2-2y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}} - \frac{x^2+y^2-2z^2}{\left(x^2+y^2+z^2\right)^{5/2}} = 0$$

9.

$$\begin{aligned}
\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left[\left(u_y v_z - u_z v_y\right)\mathbf{i} + \left(u_z v_x - u_x v_z\right)\mathbf{j} + \left(u_x v_y - u_y v_x\right)\mathbf{k}\right] \\
&= \frac{\partial}{\partial x}\left(u_y v_z - u_z v_y\right) + \frac{\partial}{\partial y}\left(u_z v_x - u_x v_z\right) + \frac{\partial}{\partial z}\left(u_x v_y - u_y v_x\right) \\
&= u_y\frac{\partial v_z}{\partial x} + v_z\frac{\partial u_y}{\partial x} - u_z\frac{\partial v_y}{\partial x} - v_y\frac{\partial u_z}{\partial x} + u_z\frac{\partial v_x}{\partial y} + v_x\frac{\partial u_z}{\partial y} - u_x\frac{\partial v_z}{\partial y} - v_z\frac{\partial u_x}{\partial y} \\
&\quad + u_x\frac{\partial v_y}{\partial z} + v_y\frac{\partial u_x}{\partial z} - u_y\frac{\partial v_x}{\partial z} - v_x\frac{\partial u_y}{\partial z} \\
&= \left(v_x\frac{\partial u_z}{\partial y} - v_x\frac{\partial u_y}{\partial z} + v_y\frac{\partial u_x}{\partial z} - v_y\frac{\partial u_z}{\partial x} + v_z\frac{\partial u_y}{\partial x} - v_z\frac{\partial u_x}{\partial y}\right) \\
&\quad + \left(u_x\frac{\partial v_y}{\partial z} - u_x\frac{\partial v_z}{\partial y} + u_y\frac{\partial v_z}{\partial x} - u_y\frac{\partial v_x}{\partial z} + u_z\frac{\partial v_x}{\partial y} - u_z\frac{\partial v_y}{\partial x}\right) \\
&= \left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right)\cdot\left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&\quad - \left(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}\right)\cdot\left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\
&= \mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v})
\end{aligned}$$

10.

$$
\begin{aligned}
\nabla \times (\nabla \times \mathbf{u}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\
&\quad \times \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&= \left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\right]\mathbf{i} \\
&\quad + \left[\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\right]\mathbf{k} \\
&= \left(\frac{\partial^2 u_y}{\partial x \partial y} - \frac{\partial^2 u_x}{\partial y^2} - \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 u_z}{\partial y \partial z} - \frac{\partial^2 u_y}{\partial z^2} - \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_x}{\partial y \partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial z \partial x} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_y}{\partial z \partial y}\right)\mathbf{k} \\
&= \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} - \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial y^2} - \frac{\partial^2 u_x}{\partial z^2}\right)\mathbf{i} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial y \partial x} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z} - \frac{\partial^2 u_y}{\partial x^2} - \frac{\partial^2 u_y}{\partial y^2} - \frac{\partial^2 u_y}{\partial z^2}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial z \partial x} + \frac{\partial^2 u_y}{\partial z \partial y} + \frac{\partial^2 u_z}{\partial z^2} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2} - \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k} \\
&= \left[\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 u_x}{\partial y \partial x} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z}\right)\mathbf{j}\right. \\
&\quad \left. + \left(\frac{\partial^2 u_x}{\partial z \partial x} + \frac{\partial^2 u_y}{\partial z \partial y} + \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k}\right] \\
&\quad - \left[\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right)\mathbf{j}\right. \\
&\quad \left. + \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k}\right] \\
&= \nabla\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \left(\nabla^2 u_x \mathbf{i} + \nabla^2 u_y \mathbf{j} + \nabla^2 u_z \mathbf{k}\right) \\
&= \nabla\left(\nabla \cdot \mathbf{u}\right) - \nabla^2 \mathbf{u}
\end{aligned}
$$

11. (a)

$$\begin{aligned}
\nabla \cdot [\mathbf{u} \times (\mathbf{v} \times \mathbf{w})] &= \nabla \cdot \underbrace{[(\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}]}_{(1.19)} \\
&= \underbrace{\nabla \cdot [(\mathbf{u} \cdot \mathbf{w})\,\mathbf{v}] - \nabla \cdot [(\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}]}_{(3.21)} \\
&= \underbrace{(\mathbf{u} \cdot \mathbf{w})\,(\nabla \cdot \mathbf{v}) + [\nabla\,(\mathbf{u} \cdot \mathbf{w})] \cdot \mathbf{v}}_{(3.22)} - (\mathbf{u} \cdot \mathbf{v})\,(\nabla \cdot \mathbf{w}) - [\nabla\,(\mathbf{u} \cdot \mathbf{v})] \cdot \mathbf{w} \\
&= (\mathbf{u} \cdot \mathbf{w})\,(\nabla \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})\,(\nabla \cdot \mathbf{w}) + [\nabla\,(\mathbf{u} \cdot \mathbf{w})] \cdot \mathbf{v} - [\nabla\,(\mathbf{u} \cdot \mathbf{v})] \cdot \mathbf{w}
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla \cdot [(\nabla f) \times (f \nabla g)] &= \underbrace{(f \nabla g) \cdot [\nabla \times (\nabla f)] - (\nabla f) \cdot [\nabla \times (f \nabla g)]}_{(3.35)} \\
&= (f \nabla g) \cdot \underbrace{\mathbf{0}}_{(3.31)} - (\nabla f) \cdot [\nabla \times (f \nabla g)] \\
&= -\,(\nabla f) \cdot [\nabla \times (f \nabla g)] \\
&= -\,(\nabla f) \cdot \underbrace{[f\,(\nabla \times (\nabla g)) + (\nabla f) \times (\nabla g)]}_{(3.28)} \\
&= -\,(\nabla f) \cdot \left[f \underbrace{\mathbf{0}}_{(3.31)} + (\nabla f) \times (\nabla g) \right] \\
&= -\,(\nabla f) \cdot (\nabla f) \times (\nabla g) \\
&= (\nabla f) \cdot (\nabla g) \times (\nabla f) \\
&= \underbrace{(\nabla g) \cdot (\nabla f) \times (\nabla f)}_{(1.34)} \\
&= (\nabla g) \cdot \underbrace{\mathbf{0}}_{(1.19)} \\
&= 0
\end{aligned}$$

(c)

$$\begin{aligned}
\nabla \times [(\nabla \times \mathbf{v}) + \nabla f] = \underbrace{\nabla \times (\nabla \times \mathbf{v}) + \nabla \times (\nabla f)}_{(3.27)} &= \nabla \times (\nabla \times \mathbf{v}) + \underbrace{\mathbf{0}}_{(3.31)} \\
&= \nabla \times (\nabla \times \mathbf{v})
\end{aligned}$$

(d)

$$\nabla^2 f = \mathbf{0} + \nabla^2 f = \nabla \times \nabla \cdot \mathbf{v} + \nabla \cdot \nabla f = \underbrace{\nabla \cdot \nabla \times \mathbf{v}}_{(1.34)} + \nabla \cdot \nabla f = \underbrace{\nabla \cdot [(\nabla \times \mathbf{v}) + \nabla f]}_{(3.21)}$$

12. (a) Let $\mathbf{u}$ be a unit vector, such that

$$\mathbf{u} = \frac{u_x}{|\mathbf{u}|}\mathbf{i} + \frac{u_y}{|\mathbf{u}|}\mathbf{j} + \frac{u_z}{|\mathbf{u}|}\mathbf{k} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

That is, $u_x/|\mathbf{u}|$, $u_y|\mathbf{u}|$, $u_z/|\mathbf{u}|$ are, by Section 1.2, simply the direction cosines of $\mathbf{u}$. Hence by (2.114),

$$(\mathbf{u}\cdot\nabla) f = \frac{u_x}{|\mathbf{u}|}\frac{\partial f}{\partial x} + \frac{u_y}{|\mathbf{u}|}\frac{\partial f}{\partial y} + \frac{u_z}{|\mathbf{u}|}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma = \nabla_u f$$

(b)

$$[(\mathbf{i}-\mathbf{j})\cdot\nabla] f = (\mathbf{i}-\mathbf{j})\cdot(\nabla f) = (\mathbf{i}-\mathbf{j})\cdot\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

(c) Let $\mathbf{v} = x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}$. Then

$$[(x\mathbf{i} - y\mathbf{j})\cdot\nabla]\left(x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}\right) = x\frac{\partial \mathbf{v}}{\partial x} - y\frac{\partial \mathbf{v}}{\partial y} = 2\left(x^2\mathbf{i} + y^2\mathbf{j}\right)$$

13.

$$\begin{aligned}\nabla(\mathbf{u}\cdot\mathbf{v}) &= \nabla(u_x v_x + u_y v_y + u_z v_z)\\ &= \nabla(u_x v_x) + \nabla(u_y v_y) + \nabla(u_z v_z)\\ &= u_x\nabla v_x + v_x\nabla u_x + u_y\nabla v_y + v_y\nabla u_y + u_z\nabla v_z + v_z\nabla u_z\\ &= (u_x\nabla v_x + u_y\nabla v_y + u_z\nabla v_z) + (v_x\nabla u_x + v_y\nabla u_y + v_z\nabla u_z)\end{aligned}$$

Let us for a moment focus on the first three terms $u_x\nabla v_x + u_y\nabla v_y + u_z\nabla v_z = \mathbf{a}$.

Expanding these gives

$$
\begin{aligned}
\mathbf{a} &= u_x \left(\frac{\partial v_x}{\partial x}\mathbf{i} + \frac{\partial v_x}{\partial y}\mathbf{j} + \frac{\partial v_x}{\partial z}\mathbf{k} \right) + u_y \left(\frac{\partial v_y}{\partial x}\mathbf{i} + \frac{\partial v_y}{\partial y}\mathbf{j} + \frac{\partial v_y}{\partial z}\mathbf{k} \right) + u_z \left(\frac{\partial v_z}{\partial x}\mathbf{i} + \frac{\partial v_z}{\partial y}\mathbf{j} + \frac{\partial v_z}{\partial z}\mathbf{k} \right) \\
&= u_x \left(\frac{\partial v_x}{\partial x}\mathbf{i} + \frac{\partial v_x}{\partial y}\mathbf{j} + \frac{\partial v_x}{\partial z}\mathbf{k} \right) + u_y \left(\frac{\partial v_y}{\partial x}\mathbf{i} + \frac{\partial v_y}{\partial y}\mathbf{j} + \frac{\partial v_y}{\partial z}\mathbf{k} \right) + u_z \left(\frac{\partial v_z}{\partial x}\mathbf{i} + \frac{\partial v_z}{\partial y}\mathbf{j} + \frac{\partial v_z}{\partial z}\mathbf{k} \right) \\
&\quad + u_x \left(\frac{\partial v_y}{\partial x}\mathbf{j} - \frac{\partial v_y}{\partial x}\mathbf{j} + \frac{\partial v_z}{\partial x}\mathbf{k} - \frac{\partial v_z}{\partial x}\mathbf{k} \right) + u_y \left(\frac{\partial v_x}{\partial y}\mathbf{i} - \frac{\partial v_x}{\partial y}\mathbf{i} + \frac{\partial v_z}{\partial y}\mathbf{k} - \frac{\partial v_z}{\partial y}\mathbf{k} \right) \\
&\quad + u_z \left(\frac{\partial v_x}{\partial z}\mathbf{i} - \frac{\partial v_x}{\partial z}\mathbf{i} + \frac{\partial v_y}{\partial z}\mathbf{j} - \frac{\partial v_y}{\partial z}\mathbf{j} \right) \\
&= u_x \left(\frac{\partial v_x}{\partial x}\mathbf{i} + \frac{\partial v_y}{\partial x}\mathbf{j} + \frac{\partial v_z}{\partial x}\mathbf{k} \right) + u_y \left(\frac{\partial v_x}{\partial y}\mathbf{i} + \frac{\partial v_y}{\partial y}\mathbf{j} + \frac{\partial v_z}{\partial y}\mathbf{k} \right) + u_z \left(\frac{\partial v_x}{\partial z}\mathbf{i} + \frac{\partial v_y}{\partial z}\mathbf{j} + \frac{\partial v_z}{\partial z}\mathbf{k} \right) \\
&\quad + \left[u_y \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) - u_z \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \right] \mathbf{i} + \left[u_z \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) - u_x \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \right] \mathbf{j} \\
&\quad + \left[u_x \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) - u_y \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \right] \mathbf{k} \\
&= u_x \frac{\partial \mathbf{v}}{\partial x} + u_y \frac{\partial \mathbf{v}}{\partial y} + u_z \frac{\partial \mathbf{v}}{\partial z} + \left\{ \mathbf{u} \times \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k} \right] \right\} \\
&= \left(\mathbf{u} \cdot \nabla \right) \mathbf{v} + \left[\mathbf{u} \times \left(\nabla \times \mathbf{v} \right) \right]
\end{aligned}
$$

Then, clearly

$$
v_x \nabla u_x + v_y \nabla u_y + v_z \nabla u_z = \left(\mathbf{v} \cdot \nabla \right) \mathbf{u} + \left[\mathbf{v} \times \left(\nabla \times \mathbf{u} \right) \right]
$$

And so we may conclude that

$$
\nabla \left(\mathbf{u} \cdot \mathbf{v} \right) = \left(\mathbf{u} \cdot \nabla \right) \mathbf{v} + \left[\mathbf{u} \times \left(\nabla \times \mathbf{v} \right) \right] + \left(\mathbf{v} \cdot \nabla \right) \mathbf{u} + \left[\mathbf{v} \times \left(\nabla \times \mathbf{u} \right) \right]
$$

14.

$$
\begin{aligned}
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla \times \left[(u_y v_z - u_z v_y)\,\mathbf{i} + (u_z v_x - u_x v_z)\,\mathbf{j} + (u_x v_y - u_y v_x)\,\mathbf{k} \right] \\
&= \left[\nabla \times (u_y v_z \mathbf{i} + u_z v_x \mathbf{j} + u_x v_y \mathbf{k}) \right] - \left[\nabla \times (u_z v_y \mathbf{i} + u_x v_z \mathbf{j} + u_y v_x \mathbf{k}) \right] \\
&= \left[\frac{\partial}{\partial y}(u_x v_y) - \frac{\partial}{\partial z}(u_z v_x) \right] \mathbf{i} + \left[\frac{\partial}{\partial z}(u_y v_z) - \frac{\partial}{\partial x}(u_x v_y) \right] \mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(u_z v_x) - \frac{\partial}{\partial y}(u_y v_z) \right] \mathbf{k} - \left[\frac{\partial}{\partial y}(u_y v_x) - \frac{\partial}{\partial z}(u_x v_z) \right] \mathbf{i} \\
&\quad - \left[\frac{\partial}{\partial z}(u_z v_y) - \frac{\partial}{\partial x}(u_y v_x) \right] \mathbf{j} - \left[\frac{\partial}{\partial x}(u_x v_z) - \frac{\partial}{\partial y}(u_z v_y) \right] \mathbf{k} \\
&= \left(u_x \frac{\partial v_y}{\partial y} + v_y \frac{\partial u_x}{\partial y} - u_z \frac{\partial v_x}{\partial z} - v_x \frac{\partial u_z}{\partial z} - u_y \frac{\partial v_x}{\partial y} - v_x \frac{\partial u_y}{\partial y} + u_x \frac{\partial v_z}{\partial z} + v_z \frac{\partial u_x}{\partial z} \right) \mathbf{i} \\
&\quad + \left(u_y \frac{\partial v_z}{\partial z} + v_z \frac{\partial u_y}{\partial z} - u_x \frac{\partial v_y}{\partial x} - v_y \frac{\partial u_x}{\partial x} - u_z \frac{\partial v_y}{\partial z} - v_y \frac{\partial u_z}{\partial z} + u_y \frac{\partial v_x}{\partial x} + v_x \frac{\partial u_y}{\partial x} \right) \mathbf{j} \\
&\quad + \left(u_z \frac{\partial v_x}{\partial x} + v_x \frac{\partial u_z}{\partial x} - u_y \frac{\partial v_z}{\partial y} - v_z \frac{\partial u_y}{\partial y} - u_x \frac{\partial v_z}{\partial x} - v_z \frac{\partial u_x}{\partial x} + u_z \frac{\partial v_y}{\partial y} + v_y \frac{\partial u_z}{\partial y} \right) \mathbf{k} \\
&= \left(u_x \frac{\partial v_y}{\partial y} - u_z \frac{\partial v_x}{\partial z} - u_y \frac{\partial v_x}{\partial y} + u_x \frac{\partial v_z}{\partial z} \right) \mathbf{i} + \left(u_y \frac{\partial v_z}{\partial z} - u_x \frac{\partial v_y}{\partial x} - u_z \frac{\partial v_y}{\partial z} + u_y \frac{\partial v_x}{\partial x} \right) \mathbf{j} \\
&\quad + \left(u_z \frac{\partial v_x}{\partial x} - u_y \frac{\partial v_z}{\partial y} - u_x \frac{\partial v_z}{\partial x} + u_z \frac{\partial v_y}{\partial y} \right) \mathbf{k} \\
&\quad + \left(v_y \frac{\partial u_x}{\partial y} - v_x \frac{\partial u_z}{\partial z} - v_x \frac{\partial u_y}{\partial y} + v_z \frac{\partial u_x}{\partial z} \right) \mathbf{i} + \left(v_z \frac{\partial u_y}{\partial z} - v_y \frac{\partial u_x}{\partial x} - v_y \frac{\partial u_z}{\partial z} + v_x \frac{\partial u_y}{\partial x} \right) \mathbf{j} \\
&\quad + \left(v_x \frac{\partial u_z}{\partial x} - v_z \frac{\partial u_y}{\partial y} - v_z \frac{\partial u_x}{\partial x} + v_y \frac{\partial u_z}{\partial y} \right) \mathbf{k}
\end{aligned}
$$

Let us for a moment focus on the first three terms

$$
\begin{aligned}
\mathbf{a} = &\left(u_x \frac{\partial v_y}{\partial y} - u_z \frac{\partial v_x}{\partial z} - u_y \frac{\partial v_x}{\partial y} + u_x \frac{\partial v_z}{\partial z} \right) \mathbf{i} + \left(u_y \frac{\partial v_z}{\partial z} - u_x \frac{\partial v_y}{\partial x} - u_z \frac{\partial v_y}{\partial z} + u_y \frac{\partial v_x}{\partial x} \right) \mathbf{j} \\
&+ \left(u_z \frac{\partial v_x}{\partial x} - u_y \frac{\partial v_z}{\partial y} - u_x \frac{\partial v_z}{\partial x} + u_z \frac{\partial v_y}{\partial y} \right) \mathbf{k}
\end{aligned}
$$

These may be further manipulated to get

$$\begin{aligned}
\mathbf{a} &= \left(u_x\frac{\partial v_y}{\partial y} - u_z\frac{\partial v_x}{\partial z} - u_y\frac{\partial v_x}{\partial y} + u_x\frac{\partial v_z}{\partial z}\right)\mathbf{i} + \left(u_y\frac{\partial v_z}{\partial z} - u_x\frac{\partial v_y}{\partial x} - u_z\frac{\partial v_y}{\partial z} + u_y\frac{\partial v_x}{\partial x}\right)\mathbf{j} \\
&\quad + \left(u_z\frac{\partial v_x}{\partial x} - u_y\frac{\partial v_z}{\partial y} - u_x\frac{\partial v_z}{\partial x} + u_z\frac{\partial v_y}{\partial y}\right)\mathbf{k} + \left(u_x\frac{\partial v_x}{\partial x} - u_x\frac{\partial v_x}{\partial x}\right)\mathbf{i} + \left(u_y\frac{\partial v_y}{\partial y} - u_y\frac{\partial v_y}{\partial y}\right)\mathbf{j} \\
&\quad + \left(u_z\frac{\partial v_z}{\partial z} - u_z\frac{\partial v_z}{\partial z}\right)\mathbf{k} \\
&= u_x\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{i} + u_y\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{j} + u_z\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{k} \\
&\quad - u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i} + \frac{\partial v_y}{\partial x}\mathbf{j} + \frac{\partial v_z}{\partial x}\mathbf{k}\right) - u_y\left(\frac{\partial v_x}{\partial y}\mathbf{i} + \frac{\partial v_y}{\partial y}\mathbf{j} + \frac{\partial v_z}{\partial y}\mathbf{k}\right) - u_z\left(\frac{\partial v_x}{\partial z}\mathbf{i} + \frac{\partial v_y}{\partial z}\mathbf{j} + \frac{\partial v_z}{\partial z}\mathbf{k}\right) \\
&= (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) - u_z\frac{\partial \mathbf{v}}{\partial x} - u_y\frac{\partial \mathbf{v}}{\partial y} - u_z\frac{\partial \mathbf{v}}{\partial z} \\
&= \mathbf{u}\left(\nabla\cdot\mathbf{v}\right) - \left[\left(\mathbf{u}\cdot\nabla\right)\mathbf{v}\right]
\end{aligned}$$

In a similar way it may be shown that the remaining three terms can be written as $-\mathbf{v}(\nabla\cdot\mathbf{u}) + [(\mathbf{v}\cdot\nabla)\mathbf{u}]$, and hence, we may conclude that

$$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}\left(\nabla\cdot\mathbf{v}\right) - \mathbf{v}\left(\nabla\cdot\mathbf{u}\right) + \left[\left(\mathbf{v}\cdot\nabla\right)\mathbf{u}\right] - \left[\left(\mathbf{u}\cdot\nabla\right)\mathbf{v}\right]$$

15. Let the sphere be given by $F(x,y,z) = x^2 + y^2 + z^2 = 9$. The unit outer normal vector to the sphere is then given by

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)$$

Next, let $\mathbf{u} = (x^2 - z^2)(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. Then, with the help of (2.117)

$$\begin{aligned}
\frac{\partial}{\partial n}\left(\nabla\cdot\mathbf{u}\right) = \nabla\left(\nabla\cdot\mathbf{u}\right)\cdot\mathbf{n} &= \nabla\left[\frac{\partial}{\partial x}\left(x^2 - z^2\right) - \frac{\partial}{\partial y}\left(x^2 - z^2\right) + 3\frac{\partial}{\partial z}\left(x^2 - z^2\right)\right]\cdot\mathbf{n} \\
&= \nabla\left(2x - 6z\right)\cdot\mathbf{n} \\
&= (2\mathbf{i} - 6\mathbf{k})\cdot\frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \\
&= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(2x - 6z\right)
\end{aligned}$$

Evaluating the result at the point $(2,2,1)$ then finally gives $-2/3$.

16. If a rigid body is rotating about the z-axis with angular velocity ω, then it is moving in a circular motion in the xy-plane. Hence, a particle of the body essentially follows a path equal to that of a point restricted to lie on a cylinder. Let r be the fixed radius

of the circle the path is constrained to move on in the xy-plane and let α be the initial angle of the particle in the xy-plane relative to the positive x-axis. Then, if ω is the angular velocity, at time t the particle will have moved through angle $\omega t + \alpha$. Since the particle is constrained to lie on the circle of radius r, its x-coordinate given by $r\cos(\omega t+\alpha)$ and its $y-$coordinate by $r\sin(\omega t+\alpha)$. As the particle is free to move in the z-plane, its z-coordinate is simply given by z. As such, a vector equation for the particle is given by

$$\overrightarrow{OP} = r\cos\left(\omega t+\alpha\right)\mathbf{i} + r\sin\left(\omega t+\alpha\right)\mathbf{j} + z\mathbf{k}$$

Next, let $\boldsymbol{\omega} = \omega\mathbf{k}$ be the angular velocity vector. Now the regular velocity of the particle is given by the vector $\mathbf{v}$, which is both perpendicular to the angular velocity vector (since by definition the angular velocity vector is perpendicular to the plane of rotation and hence, $\mathbf{v}$) and the position vector $\overrightarrow{OP}$. As such, it is given by

$$\begin{aligned}
\mathbf{v} &= \frac{d}{dt}\overrightarrow{OP} \\
&= -\omega r\sin\left(\omega t+\alpha\right)\mathbf{i} + \omega r\cos\left(\omega t+\alpha\right)\mathbf{j} \\
&= (\omega\mathbf{k}) \times \left[r\cos\left(\omega t+\alpha\right)\mathbf{i} + r\sin\left(\omega t+\alpha\right)\mathbf{j} + z\mathbf{k}\right] \\
&= \boldsymbol{\omega} \times \overrightarrow{OP}
\end{aligned}$$

Knowing this, the divergence and curl of $\mathbf{v}$ are given by

$$\begin{aligned}
\nabla\cdot\mathbf{v} = \nabla\cdot\left(\boldsymbol{\omega}\times\overrightarrow{OP}\right) &= \nabla\cdot\left[-\omega r\sin\left(\omega t+\alpha\right)\mathbf{i} + \omega r\cos\left(\omega t+\alpha\right)\mathbf{j}\right] \\
&= \nabla\cdot\left(-\omega y\mathbf{i} + \omega x\mathbf{j}\right) \\
&= \frac{\partial}{\partial x}\left(-\omega y\right) + \frac{\partial}{\partial y}\left(\omega x\right) \\
&= 0 \\
\nabla\times\mathbf{v} = \nabla\times\left(\boldsymbol{\omega}\times\overrightarrow{OP}\right) &= \nabla\times\left[-\omega r\sin\left(\omega t+\alpha\right)\mathbf{i} + \omega r\cos\left(\omega t+\alpha\right)\mathbf{j}\right] \\
&= -\omega\frac{\partial}{\partial z}r\cos\left(\omega t+\alpha\right)\mathbf{i} - \omega\frac{\partial}{\partial z}r\sin\left(\omega t+\alpha\right)\mathbf{j} \\
&\quad + \left[\omega\frac{\partial}{\partial x}r\cos\left(\omega t+\alpha\right) + \omega\frac{\partial}{\partial y}r\sin\left(\omega t+\alpha\right)\right]\mathbf{k} \\
&= -\omega\frac{\partial}{\partial z}x\mathbf{i} - \omega\frac{\partial}{\partial z}y\mathbf{j} + \left(\omega\frac{\partial}{\partial x}x + \omega\frac{\partial}{\partial y}y\right)\mathbf{k} \\
&= 2\omega\mathbf{k} \\
&= 2\boldsymbol{\omega}
\end{aligned}$$

17. Let a steady fluid in motion have the velocity vector $\mathbf{u} = y\mathbf{i} = d\mathbf{r}/dt$. Since $\mathbf{u}$ has no y or z components, the position of a point in the y and z directions does not

change with time (i.e. is constant). Hence, the position of a point at time t is given by $\mathbf{r}(t) = (c_2 t + c_1)\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, where c_1, c_2 and c_3 are some arbitrary constants. As such, the path of motion for each point of the vector field is a straight line when $c_2 \neq 0$. Furthermore, $\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \nabla \cdot (y\mathbf{i}) = (\partial/\partial x)y = 0$, and hence, the flow is incompressible. The relative rate of growth of a volume occupied by the fluid is roughly proportional to $\operatorname{div}\mathbf{u}$. To be exact; $\operatorname{div}\mathbf{u} = \lim_{\Delta t \to 0} \Delta V/(V\Delta t)$. Now since $\operatorname{div}\mathbf{u} = 0$ implies that $\Delta V = 0$, the volume occupied at time $t = 1$ will be the same as that at time $t = 0$, which is simply $V(t_0) = V(t_1) = 1$ for $t_0 = 0$ and $t_1 = 1$.

18. Let a steady fluid in motion have the velocity vector $\mathbf{u} = x\mathbf{i} = d\mathbf{r}/dt$. Since $\mathbf{u}$ has no y or z components (i.e. $dy/dt = 0$, $dz/dt = 0$), the position of a point in the y and z directions does not change with time. In other words, the position of a point has coordinates $y(t) = c_2$, $z(t) = c_3$, where c_2 and c_3 are arbitrary constants. For the x-coordinate however, we find that $dx/dt = x$, so that $x(t) = c_1 e^t$, where c_1 is the initial value of x at time $t = 0$. Hence, we find $\mathbf{r}(t) = c_1 e^t\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Furthermore, $\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \nabla \cdot (x\mathbf{i}) = (\partial/\partial x)x = 1$, and as such, the flow is *not* incompressible. Since
$$\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = 1 = \frac{1}{V}\frac{dV}{dt} \qquad \Longrightarrow \qquad V(t) = V_0 e^t$$
The volume at $t = 0$ is $V(0) = V_0 = 1$. Hence, at time $t = 1$ the volume will be $V(1) = e$.

Section 3.8

1. Let $u = F(x, y, z)$, $v = G(x, y, z)$, $w = H(x, y, z)$. Then, using the result of Problem 5 following Section 2.12, we can write

$$\begin{aligned}
\nabla F &= \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} \\
&= \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k} \\
&= \frac{1}{J}\left(\frac{\partial(y,z)}{\partial(v,w)}\mathbf{i} + \frac{\partial(z,x)}{\partial(v,w)}\mathbf{j} + \frac{\partial(x,y)}{\partial(v,w)}\mathbf{k}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial z}{\partial v}\frac{\partial y}{\partial w}\right)\mathbf{i} + \left(\frac{\partial z}{\partial v}\frac{\partial x}{\partial w} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial w}\right)\mathbf{j} + \left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial w} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial w}\right)\mathbf{k}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)
\end{aligned}$$

$$
\begin{aligned}
\nabla G &= \frac{\partial G}{\partial x}\mathbf{i} + \frac{\partial G}{\partial y}\mathbf{j} + \frac{\partial G}{\partial z}\mathbf{k} \\
&= \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k} \\
&= \frac{1}{J}\left(\frac{\partial (y,z)}{\partial (w,u)}\mathbf{i} + \frac{\partial (z,x)}{\partial (w,u)}\mathbf{j} + \frac{\partial (x,y)}{\partial (w,u)}\mathbf{k}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial y}{\partial w}\frac{\partial z}{\partial u} - \frac{\partial z}{\partial w}\frac{\partial y}{\partial u}\right)\mathbf{i} + \left(\frac{\partial z}{\partial w}\frac{\partial x}{\partial u} - \frac{\partial x}{\partial w}\frac{\partial z}{\partial u}\right)\mathbf{j} + \left(\frac{\partial x}{\partial w}\frac{\partial y}{\partial u} - \frac{\partial y}{\partial w}\frac{\partial x}{\partial u}\right)\mathbf{k}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla H &= \frac{\partial H}{\partial x}\mathbf{i} + \frac{\partial H}{\partial y}\mathbf{j} + \frac{\partial H}{\partial z}\mathbf{k} \\
&= \frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} + \frac{\partial w}{\partial z}\mathbf{k} \\
&= \frac{1}{J}\left(\frac{\partial (y,z)}{\partial (u,v)}\mathbf{i} + \frac{\partial (z,x)}{\partial (u,v)}\mathbf{j} + \frac{\partial (x,y)}{\partial (u,v)}\mathbf{k}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\mathbf{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{j} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\mathbf{k}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)
\end{aligned}
$$

2.

$$
\begin{aligned}
\nabla F \cdot \nabla G \times \nabla H &= \underbrace{\frac{1}{J^3}\left[\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right) \times \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\right]}_{(3.48)} \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \cdot \underbrace{\left[\left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial u} - \left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial w}\right]}_{(1.19)} \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial u} \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \\
&= \frac{1}{J^3}\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\underbrace{\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)}_{(1.34)} = \frac{1}{J}
\end{aligned}
$$

where the last step follows from (3.44):

$$
J = \frac{\partial (x,y,z)}{\partial (u,v,w)} = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}
$$

Hence, this proves that

$$J = \frac{1}{\dfrac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{1}{\nabla F \cdot \nabla G \times \nabla H}$$

3.

$$\begin{aligned}
J(\nabla G \times \nabla H) &= \underbrace{\frac{J}{J^2}\left(\frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right) \times \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)}_{(3.48)} \\
&= \frac{1}{J}\underbrace{\left[\left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial u} - \left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial w}\right]}_{(1.19)} \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\frac{\partial \mathbf{r}}{\partial u} \\
&= \frac{1}{J}\underbrace{(J)}_{(3.44)}\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{r}}{\partial u}
\end{aligned}$$

$$\begin{aligned}
J(\nabla H \times \nabla F) &= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \times \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\frac{\partial \mathbf{r}}{\partial v} - \left(\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\frac{\partial \mathbf{r}}{\partial u}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{r}}{\partial v}
\end{aligned}$$

$$\begin{aligned}
J(\nabla F \times \nabla G) &= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) \times \left(\frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right) \\
&= \frac{1}{J}\left[\left(\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right)\frac{\partial \mathbf{r}}{\partial w} - \left(\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right)\frac{\partial \mathbf{r}}{\partial v}\right] \\
&= \frac{1}{J}\left(\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial w} \times \frac{\partial \mathbf{r}}{\partial u}\right)\frac{\partial \mathbf{r}}{\partial w} = \frac{\partial \mathbf{r}}{\partial w}
\end{aligned}$$

4. If the vectors ∇F, ∇G, ∇H are mutually perpendicular in D, then

$$\nabla F \cdot \nabla G = 0 \qquad \nabla F \cdot \nabla H = 0 \qquad \nabla G \cdot \nabla H = 0$$

Furthermore, note that if **a**, **b**, **c**, **d** are arbitrary vectors in D then

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= [(a_yb_z-a_zb_y)\,\mathbf{i}+(a_zb_x-a_xb_z)\,\mathbf{j}+(a_xb_y-a_yb_x)\,\mathbf{k}]\\
&\quad\cdot[(c_yd_z-c_zd_y)\,\mathbf{i}+(c_zd_x-c_xd_z)\,\mathbf{j}+(c_xd_y-c_yd_x)\,\mathbf{k}]\\
&=(a_yb_z-a_zb_y)\,(c_yd_z-c_zd_y)+(a_zb_x-a_xb_z)\,(c_zd_x-c_xd_z)\\
&\quad+(a_xb_y-a_yb_x)\,(c_xd_y-c_yd_x)+a_xb_xc_xd_x-a_xb_xc_xd_x+a_yb_yc_yd_y\\
&\quad-a_yb_yc_yd_y+a_zb_zc_zd_z-a_zb_zc_zd_z\\
&=a_xb_xc_xd_x+a_xb_yc_xd_y+a_xb_zc_xd_z+a_yb_xc_yd_x+a_yb_yc_yd_y+a_yb_zc_yd_z\\
&\quad+a_zb_xc_zd_x+a_zb_yc_zd_y+a_zb_zc_zd_z-a_xb_xc_xd_x-a_xb_yc_yd_x-a_xb_zc_zd_x\\
&\quad-a_yb_xc_xd_y-a_yb_yc_yd_y-a_yb_zc_zd_y-a_zb_xc_xd_z-a_zb_yc_yd_z-a_zb_zc_zd_z\\
&=(a_xc_x+a_yc_y+a_zc_z)\,(b_xd_x+b_yd_y+b_zd_z)\\
&\quad-(b_xc_x+b_yc_y+b_zc_z)\,(a_xd_x+a_yd_y+a_zd_z)\\
&=(\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d})
\end{aligned}$$

Using (3.48) and the vector identity above we can form the three equations

$$\begin{aligned}
\nabla F\cdot\nabla G &= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\\
&=\frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial w}\right)\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)-\frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial w}\right)\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)=0
\end{aligned}$$

$$\begin{aligned}
\nabla F\cdot\nabla H &= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\times\frac{\partial\mathbf{r}}{\partial w}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\\
&=\frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial v}\right)-\frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial v}\right)=0
\end{aligned}$$

$$\begin{aligned}
\nabla G\cdot\nabla H &= \frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\times\frac{\partial\mathbf{r}}{\partial u}\right)\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)\\
&=\frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}\right)-\frac{1}{J^2}\left(\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial u}\right)\left(\frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial v}\right)=0
\end{aligned}$$

For these equations to make sense it is sufficient to require that

$$\frac{\partial\mathbf{r}}{\partial v}\cdot\frac{\partial\mathbf{r}}{\partial u}=0 \qquad \frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial u}=0 \qquad \frac{\partial\mathbf{r}}{\partial w}\cdot\frac{\partial\mathbf{r}}{\partial v}=0$$

In other words, the tangent vectors $\partial\mathbf{r}/\partial u$, $\partial\mathbf{r}/\partial v$, $\partial\mathbf{r}/\partial w$ form a triple of mutually perpendicular vectors at each point of D and hence, the coordinates are orthogonal.

5. Using (3.56) we can write

$$\mathbf{p}=(\alpha p_u)\left(\frac{1}{\alpha^2}\frac{\partial\mathbf{r}}{\partial u}\right)+(\beta p_v)\left(\frac{1}{\beta^2}\frac{\partial\mathbf{r}}{\partial v}\right)+(\gamma p_w)\left(\frac{1}{\gamma^2}\frac{\partial\mathbf{r}}{\partial w}\right)$$

The curl of $\mathbf{p}$ is the sum of the curls of the terms on the right-hand side. By (3.27), (3.28) and (3.55) we can thus write

$$\begin{aligned}
\nabla \times \mathbf{p} &= \nabla \times \left[(\alpha p_u)\left(\frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right) + (\beta p_v)\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right) + (\gamma p_w)\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right)\right] \\
&= \left[\nabla \times (\alpha p_u)\left(\frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right)\right] + \left[\nabla \times (\beta p_v)\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right)\right] + \left[\nabla \times (\gamma p_w)\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right)\right] \\
&= (\alpha p_u)\left(\nabla \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right) + \left[(\nabla \alpha p_u) \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right] + (\beta p_v)\left(\nabla \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right) \\
&\quad + \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right] + (\gamma p_w)\left(\nabla \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right) + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right] \\
&= \left[(\nabla \alpha p_u) \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u}\right] + \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right] + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]
\end{aligned}$$

The u component of $\nabla \times \mathbf{p}$ can be obtained by taking the scalar product of both sides of the equation above with $(1/\alpha)(\partial \mathbf{r}/\partial u)$, giving

$$\begin{aligned}
[\nabla \times \mathbf{p}]_u &= \nabla \times \mathbf{p} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u} \\
&= \underbrace{\left[(\nabla \alpha p_u) \times \frac{1}{\alpha^2}\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u}\right]}_{0} + \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u}\right] + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial u}\right] \\
&= \left[(\nabla \beta p_v) \times \frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right]_u + \left[(\nabla \gamma p_w) \times \frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]_u \\
&= \underbrace{[\nabla \beta p_v]_v \left[\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right]_w - [\nabla \beta p_v]_w \left[\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v}\right]_v}_{(3.59)} + [\nabla \gamma p_w]_v \left[\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]_w - [\nabla \gamma p_w]_w \left[\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w}\right]_v \\
&= \underbrace{\left(\frac{\beta}{\beta}\frac{\partial p_v}{\partial v}\right)}_{(3.60)} \underbrace{\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial w}\right)}_{0} - \left(\frac{\beta}{\gamma}\frac{\partial p_v}{\partial w}\right) \underbrace{\left(\frac{1}{\beta^2}\frac{\partial \mathbf{r}}{\partial v} \cdot \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial v}\right)}_{1/\beta} \\
&\quad + \left(\frac{\gamma}{\beta}\frac{\partial p_w}{\partial v}\right) \underbrace{\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial w}\right)}_{1/\gamma} - \left(\frac{\gamma}{\gamma}\frac{\partial p_w}{\partial w}\right) \underbrace{\left(\frac{1}{\gamma^2}\frac{\partial \mathbf{r}}{\partial w} \cdot \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial v}\right)}_{0} \\
&= \frac{1}{\beta}\frac{\partial p_w}{\partial v} - \frac{1}{\gamma}\frac{\partial p_v}{\partial w} = \frac{1}{\beta\gamma}\left[\frac{\partial}{\partial v}(\gamma p_w) - \frac{\partial}{\partial w}(\beta p_v)\right]
\end{aligned}$$

The remaining components can be found in exactly the same way.

6. (a) Cylindrical coordinates are given by the relations

$$x = f(r, \theta, z) = r\cos\theta \qquad y = g(r, \theta, z) = r\sin\theta \qquad z = h(r, \theta, z) = z$$

and

$$r = F\left(x,y,z\right) = \sqrt{x^2+y^2+z^2} \quad \theta = G\left(x,y,z\right) = \tan^{-1}\frac{y}{x} \quad z = H\left(x,y,z\right) = z$$

Now since

$$\begin{aligned}(\alpha\nabla F)\cdot(\beta\nabla G)\times(\gamma\nabla H) &= \underbrace{\left(\frac{1}{\alpha}\frac{\partial\mathbf{r}}{\partial r}\right)\cdot\left(\frac{1}{\beta}\frac{\partial\mathbf{r}}{\partial\theta}\right)\times\left(\frac{1}{\gamma}\frac{\partial\mathbf{r}}{\partial z}\right)}_{(3.52)} \\ &= \left(\cos\theta\mathbf{i}+\sin\theta\mathbf{j}\right)\cdot\left[\frac{1}{r}\left(-r\sin\theta\mathbf{i}+r\cos\theta\mathbf{j}\right)\times\mathbf{k}\right] = 1\end{aligned}$$

where $\alpha = 1$, $\beta = r$, $\gamma = 1$, the vectors $\alpha\nabla F$, $\beta\nabla G$, $\gamma\nabla H$ are mutually perpendicular unit vectors. Hence, the surfaces $F = r = \text{const}$, $G = \theta = \text{const}$, $H = z = \text{const}$ must meet at right angles and thus form a triply orthogonal family of surfaces. Furthermore, by (3.51)

$$J = \frac{\partial\left(x,y,z\right)}{\partial\left(r,\theta,z\right)} = r$$

(b) By (3.54) we conclude

$$ds^2 = \alpha^2 dr^2 + \beta^2 d\theta^2 + \gamma^2 dz^2 = dr^2 + r^2 d\theta^2 + dz^2$$

(c) Using (3.57), we find

$$\begin{aligned} p_r &= \mathbf{p}\cdot\frac{1}{\alpha}\frac{\partial\mathbf{r}}{\partial r} = \frac{1}{\alpha}\left(p_x\frac{\partial x}{\partial r}+p_y\frac{\partial y}{\partial r}+p_z\frac{\partial z}{\partial r}\right) = p_x\cos\theta + p_y\sin\theta \\ p_\theta &= \mathbf{p}\cdot\frac{1}{\beta}\frac{\partial\mathbf{r}}{\partial\theta} = \frac{1}{\beta}\left(p_x\frac{\partial x}{\partial\theta}+p_y\frac{\partial y}{\partial\theta}+p_z\frac{\partial z}{\partial\theta}\right) = -p_x\sin\theta + p_y\cos\theta \\ p_z &= \mathbf{p}\cdot\frac{1}{\gamma}\frac{\partial\mathbf{r}}{\partial z} = \frac{1}{\gamma}\left(p_x\frac{\partial x}{\partial z}+p_y\frac{\partial y}{\partial z}+p_z\frac{\partial z}{\partial z}\right) = p_z \end{aligned}$$

(d) By (3.60) we find

$$[\nabla U]_r = \frac{1}{\alpha}\frac{\partial U}{\partial r} = \frac{\partial U}{\partial r} \qquad [\nabla U]_\theta = \frac{1}{\beta}\frac{\partial U}{\partial\theta} = \frac{1}{r}\frac{\partial U}{\partial\theta} \qquad [\nabla U]_z = \frac{1}{\gamma}\frac{\partial U}{\partial z} = \frac{\partial U}{\partial z}$$

(e) By (3.61) we find

$$\nabla\cdot\mathbf{p} = \frac{1}{\alpha\beta\gamma}\left[\frac{\partial}{\partial r}\left(\beta\gamma p_r\right)+\frac{\partial}{\partial\theta}\left(\alpha\gamma p_\theta\right)+\frac{\partial}{\partial z}\left(\alpha\beta p_z\right)\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rp_r\right)+\frac{\partial p_\theta}{\partial\theta}+r\frac{\partial p_z}{\partial z}\right]$$

(f) By (3.62) we find

$$
\begin{aligned}
[\nabla \times \mathbf{p}]_r &= \frac{1}{\beta\gamma}\left[\frac{\partial}{\partial\theta}(\gamma p_z) - \frac{\partial}{\partial z}(\beta p_\theta)\right] = \frac{1}{r}\left[\frac{\partial p_z}{\partial\theta} - r\frac{\partial p_\theta}{\partial z}\right] \\
[\nabla \times \mathbf{p}]_\theta &= \frac{1}{\alpha\gamma}\left[\frac{\partial}{\partial z}(\alpha p_r) - \frac{\partial}{\partial r}(\gamma p_z)\right] = \frac{\partial p_r}{\partial z} - \frac{\partial p_z}{\partial r} \\
[\nabla \times \mathbf{p}]_z &= \frac{1}{\alpha\beta}\left[\frac{\partial}{\partial r}(\beta p_\theta) - \frac{\partial}{\partial\theta}(\alpha p_r)\right] = \frac{1}{r}\left[\frac{\partial}{\partial r}(r p_\theta) - \frac{\partial p_r}{\partial\theta}\right]
\end{aligned}
$$

(g) From (3.56) and (3.60) it follows that

$$
\nabla U = [\nabla U]_r \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial r} + [\nabla U]_\theta \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial\theta} + [\nabla U]_z \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial z} = \frac{\partial U}{\partial r}\frac{\partial \mathbf{r}}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial \mathbf{r}}{\partial\theta} + \frac{\partial U}{\partial z}\frac{\partial \mathbf{r}}{\partial z}
$$

Furthermore, from part (a) we known that

$$
\frac{\partial \mathbf{r}}{\partial r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \qquad \frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \qquad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}
$$

And so

$$
\begin{aligned}
\nabla^2 U &= \nabla\cdot(\nabla U) \\
&= \left(\frac{\partial}{\partial r}\frac{\partial \mathbf{r}}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{\partial \mathbf{r}}{\partial\theta} + \frac{\partial}{\partial z}\frac{\partial \mathbf{r}}{\partial z}\right)\cdot\left(\frac{\partial U}{\partial r}\frac{\partial \mathbf{r}}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial \mathbf{r}}{\partial\theta} + \frac{\partial U}{\partial z}\frac{\partial \mathbf{r}}{\partial z}\right) \\
&= \frac{\partial \mathbf{r}}{\partial r}\cdot\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}\frac{\partial \mathbf{r}}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial \mathbf{r}}{\partial\theta} + \frac{\partial U}{\partial z}\frac{\partial \mathbf{r}}{\partial z}\right) + \frac{1}{r^2}\frac{\partial \mathbf{r}}{\partial\theta}\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial U}{\partial r}\frac{\partial \mathbf{r}}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial \mathbf{r}}{\partial\theta} + \frac{\partial U}{\partial z}\frac{\partial \mathbf{r}}{\partial z}\right) \\
&\quad + \frac{\partial \mathbf{r}}{\partial z}\cdot\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial r}\frac{\partial \mathbf{r}}{\partial r} + \frac{1}{r^2}\frac{\partial U}{\partial\theta}\frac{\partial \mathbf{r}}{\partial\theta} + \frac{\partial U}{\partial z}\frac{\partial \mathbf{r}}{\partial z}\right) \\
&= \frac{\partial \mathbf{r}}{\partial r}\cdot\left[\frac{\partial^2 U}{\partial r^2}\frac{\partial \mathbf{r}}{\partial r} + \frac{\partial U}{\partial r}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial r^2}}_{0} - \frac{1}{r^2}\frac{\partial U}{\partial\theta}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right) + \frac{1}{r}\frac{\partial^2 U}{\partial\theta\partial r}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right) + \frac{1}{r}\frac{\partial U}{\partial\theta}\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)}_{0}\right] \\
&\quad + \frac{\partial \mathbf{r}}{\partial r}\cdot\left[\frac{\partial^2 U}{\partial z\partial r}\frac{\partial \mathbf{r}}{\partial z} + \frac{\partial U}{\partial z}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial z^2}}_{0}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{r^2}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{\partial^2 U}{\partial r\partial\theta}\frac{\partial \mathbf{r}}{\partial r}+\frac{\partial U}{\partial r}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial r\partial\theta}}_{(1/r)(\partial\mathbf{r}/\partial\theta)}+\frac{1}{r}\frac{\partial^2 U}{\partial\theta^2}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial U}{\partial\theta}\underbrace{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)}_{-\partial\mathbf{r}/\partial r}\right]\\
&+\frac{1}{r^2}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{\partial^2 U}{\partial z\partial\theta}\frac{\partial \mathbf{r}}{\partial z}+\frac{\partial U}{\partial z}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial z\partial\theta}}_{0}\right]\\
&+\frac{\partial \mathbf{r}}{\partial z}\cdot\left[\frac{\partial^2 U}{\partial r\partial z}\frac{\partial \mathbf{r}}{\partial r}+\frac{\partial U}{\partial r}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial r\partial z}}_{0}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta\partial z}\frac{\partial \mathbf{r}}{\partial\theta}+\frac{1}{r}\frac{\partial U}{\partial\theta}\underbrace{\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)}_{0}+\frac{\partial^2 U}{\partial z^2}\frac{\partial \mathbf{r}}{\partial z}+\frac{\partial U}{\partial z}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial z^2}}_{0}\right]\\
=&\ \frac{\partial \mathbf{r}}{\partial r}\cdot\left[\frac{\partial^2 U}{\partial r^2}\frac{\partial \mathbf{r}}{\partial r}-\frac{1}{r^2}\frac{\partial U}{\partial\theta}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2 U}{\partial\theta\partial r}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)+\frac{\partial^2 U}{\partial z\partial r}\frac{\partial \mathbf{r}}{\partial z}\right]\\
&+\frac{1}{r^2}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{\partial^2 U}{\partial r\partial\theta}\frac{\partial \mathbf{r}}{\partial r}+\frac{\partial U}{\partial r}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2 U}{\partial\theta^2}\left(\frac{1}{r}\frac{\partial \mathbf{r}}{\partial\theta}\right)-\frac{1}{r}\frac{\partial U}{\partial\theta}\frac{\partial \mathbf{r}}{\partial r}+\frac{\partial^2 U}{\partial z\partial\theta}\frac{\partial \mathbf{r}}{\partial z}\right]\\
&+\frac{\partial \mathbf{r}}{\partial z}\cdot\left[\frac{\partial^2 U}{\partial r\partial z}\frac{\partial \mathbf{r}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta\partial z}\frac{\partial \mathbf{r}}{\partial\theta}+\frac{\partial^2 U}{\partial z^2}\frac{\partial \mathbf{r}}{\partial z}\right]\\
=&\ \frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2}+\frac{\partial^2 U}{\partial z^2}\\
=&\ \frac{1}{r^2}\left[r^2\frac{\partial^2 U}{\partial r^2}+r\frac{\partial U}{\partial r}+\frac{\partial^2 U}{\partial\theta^2}+r^2\frac{\partial^2 U}{\partial z^2}\right]=\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{\partial^2 U}{\partial\theta^2}+r^2\frac{\partial^2 U}{\partial z^2}\right]
\end{aligned}
$$

7. (a) Spherical coordinates are given by the relations

$$x=f(\rho,\phi,\theta)=\rho\sin\phi\cos\theta \quad y=g(\rho,\phi,\theta)=\rho\sin\phi\sin\theta \quad z=h(\rho,\phi,\theta)=\rho\cos\phi$$

and

$$\rho=F(x,y,z)=\sqrt{x^2+y^2+z^2} \quad \phi=G(x,y,z)=\cos^{-1}\frac{z}{\rho} \quad \theta=H(x,y,z)=\tan^{-1}\frac{y}{x}$$

Now since

$$
\begin{aligned}
(\alpha\nabla F)\cdot(\beta\nabla G)\times(\gamma\nabla H)&=\underbrace{\left(\frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial\rho}\right)\cdot\left(\frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial\phi}\right)\times\left(\frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial\theta}\right)}_{(3.52)}\\
&=(\sin\phi\cos\theta\mathbf{i}+\sin\phi\sin\theta\mathbf{j}+\cos\phi\mathbf{k})\\
&\quad\cdot\frac{1}{\rho}(\rho\cos\phi\cos\theta\mathbf{i}+\rho\cos\phi\sin\theta\mathbf{j}-\rho\sin\phi\mathbf{k})\\
&\quad\times\frac{1}{\rho\sin\phi}(-\rho\sin\phi\sin\theta\mathbf{i}+\rho\sin\phi\cos\theta\mathbf{j})=1
\end{aligned}
$$

where $\alpha = 1$, $\beta = \rho$, $\gamma = \rho \sin\phi$, the vectors $\alpha\nabla F$, $\beta\nabla G$, $\gamma\nabla H$ are mutually perpendicular unit vectors. Hence, the surfaces $F = \rho = \text{const}$, $G = \phi = \text{const}$, $H = \theta = \text{const}$ must meet at right angles and thus form a triply orthogonal family of surfaces. Furthermore, by (3.51)

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin\phi$$

(b) By (3.54) we conclude

$$ds^2 = \alpha^2 d\rho^2 + \beta^2 d\phi^2 + \gamma^2 d\theta^2 = d\rho^2 + \rho^2 d\phi^2 + \rho^2 \sin^2\phi d\theta^2$$

(c) Using (3.57) we find

$$\begin{aligned}
p_\rho &= \mathbf{p} \cdot \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial \rho} = \frac{1}{\alpha}\left(p_x \frac{\partial x}{\partial \rho} + p_y \frac{\partial y}{\partial \rho} + p_z \frac{\partial z}{\partial \rho}\right) = p_x \sin\phi\cos\theta + p_y \sin\phi\sin\theta + p_z\cos\phi \\
p_\phi &= \mathbf{p} \cdot \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial \phi} = \frac{1}{\beta}\left(p_x \frac{\partial x}{\partial \phi} + p_y \frac{\partial y}{\partial \phi} + p_z \frac{\partial z}{\partial \phi}\right) = p_x \cos\phi\cos\theta + p_y \cos\phi\sin\theta - p_z\sin\phi \\
p_\theta &= \mathbf{p} \cdot \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{1}{\gamma}\left(p_x \frac{\partial x}{\partial \theta} + p_y \frac{\partial y}{\partial \theta} + p_z \frac{\partial z}{\partial \theta}\right) = -p_x \sin\theta + p_y\cos\theta
\end{aligned}$$

(d) By (3.60) we find

$$[\nabla U]_\rho = \frac{1}{\alpha}\frac{\partial U}{\partial \rho} = \frac{\partial U}{\partial \rho} \quad [\nabla U]_\phi = \frac{1}{\beta}\frac{\partial U}{\partial \phi} = \frac{1}{\rho}\frac{\partial U}{\partial \phi} \quad [\nabla U]_\theta = \frac{1}{\gamma}\frac{\partial U}{\partial \theta} = \frac{1}{\rho\sin\phi}\frac{\partial U}{\partial \theta}$$

(e) By (3.61) we find

$$\begin{aligned}
\nabla \cdot \mathbf{p} &= \frac{1}{\alpha\beta\gamma}\left[\frac{\partial}{\partial \rho}(\beta\gamma p_\rho) + \frac{\partial}{\partial \phi}(\alpha\gamma p_\phi) + \frac{\partial}{\partial \theta}(\alpha\beta p_\theta)\right] \\
&= \frac{1}{\rho^2\sin\phi}\left[\sin\phi\frac{\partial}{\partial \rho}(\rho^2 p_\rho) + \rho\frac{\partial}{\partial \phi}(p_\phi\sin\phi) + \rho\frac{\partial p_\theta}{\partial \theta}\right]
\end{aligned}$$

(f) By (3.62) we find

$$\begin{aligned}
[\nabla \times \mathbf{p}]_\rho &= \frac{1}{\beta\gamma}\left[\frac{\partial}{\partial \phi}(\gamma p_\theta) - \frac{\partial}{\partial \theta}(\beta p_\phi)\right] = \frac{1}{\rho\sin\phi}\left[\frac{\partial}{\partial \phi}(p_\theta\sin\phi) - \frac{\partial p_\phi}{\partial \theta}\right] \\
[\nabla \times \mathbf{p}]_\phi &= \frac{1}{\alpha\gamma}\left[\frac{\partial}{\partial \theta}(\alpha p_\rho) - \frac{\partial}{\partial \rho}(\gamma p_\theta)\right] = \frac{1}{\rho\sin\phi}\left[\frac{\partial p_\rho}{\partial \theta} - \sin\phi\frac{\partial}{\partial \rho}(\rho p_\theta)\right] \\
[\nabla \times \mathbf{p}]_\theta &= \frac{1}{\alpha\beta}\left[\frac{\partial}{\partial \rho}(\beta p_\phi) - \frac{\partial}{\partial \phi}(\alpha p_\rho)\right] = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho p_\phi) - \frac{\partial p_\rho}{\partial \phi}\right]
\end{aligned}$$

Note that there is a typo in the book for the second term of the first component, i.e. $\partial p_\phi/\partial\phi$ should be $\partial p_\phi/\partial\theta$.

(g) From (3.56) and (3.60) it follows that

$$\nabla U = [\nabla U]_\rho \frac{1}{\alpha}\frac{\partial \mathbf{r}}{\partial \rho} + [\nabla U]_\phi \frac{1}{\beta}\frac{\partial \mathbf{r}}{\partial \phi} + [\nabla U]_\theta \frac{1}{\gamma}\frac{\partial \mathbf{r}}{\partial \theta} = \frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}$$

Furthermore, from part (a) we know that

$$\frac{\partial \mathbf{r}}{\partial \rho} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k} \quad \frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi} = \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k}$$

$$\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$

And so

$$\begin{aligned}
\nabla^2 U &= \nabla\cdot(\nabla U)\\
&= \left(\frac{\partial}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\cdot\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&= \frac{\partial \mathbf{r}}{\partial \rho}\cdot\frac{\partial}{\partial \rho}\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&\quad + \frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\frac{\partial}{\partial \phi}\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&\quad + \frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial \rho}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial U}{\partial \theta}\frac{\partial \mathbf{r}}{\partial \theta}\right)\\
&= \frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{\partial^2 U}{\partial \rho^2}\frac{\partial \mathbf{r}}{\partial \rho} + \frac{\partial U}{\partial \rho}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial \rho^2}}_{0} - \frac{1}{\rho^2}\frac{\partial U}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right) + \frac{1}{\rho}\frac{\partial^2 U}{\partial \phi\partial \rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right) + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\underbrace{\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)}_{0}\right]\\
&\quad + \frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[-\frac{1}{\rho^2\sin\phi}\frac{\partial U}{\partial \theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right) + \frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial \theta\partial \rho}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&\quad + \frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial \theta}\underbrace{\frac{\partial}{\partial \rho}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)}_{0}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\phi}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial\rho\partial\phi}}_{(1/\rho)(\partial\mathbf{r}/\partial\phi)}+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi^2}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial U}{\partial \phi}\underbrace{\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)}_{-\partial\mathbf{r}/\partial\rho}\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[-\frac{\cos\phi}{\rho\sin^2\phi}\frac{\partial U}{\partial\theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta\partial\phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial\phi}\underbrace{\frac{\partial}{\partial\phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)}_{0}\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\theta}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\underbrace{\frac{\partial^2 \mathbf{r}}{\partial\rho\partial\theta}}_{(1/\rho)(\partial\mathbf{r}/\partial\theta)}+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\partial\theta}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial U}{\partial \phi}\underbrace{\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)}_{(\cot(\phi)/\rho)(\partial\mathbf{r}/\partial\theta)}\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta^2}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
=&\frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{\partial^2 U}{\partial\rho^2}\frac{\partial \mathbf{r}}{\partial \rho}-\frac{1}{\rho^2}\frac{\partial U}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\partial\rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)-\frac{1}{\rho^2\sin\phi}\frac{\partial U}{\partial\theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{\partial \mathbf{r}}{\partial \rho}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta\partial\rho}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\phi}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi^2}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)-\frac{1}{\rho}\frac{\partial U}{\partial \phi}\frac{\partial \mathbf{r}}{\partial \rho}\right]\\
&+\frac{1}{\rho^2}\frac{\partial \mathbf{r}}{\partial \phi}\cdot\left[-\frac{\cos\phi}{\rho\sin^2\phi}\frac{\partial U}{\partial\theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta\partial\phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{\partial^2 U}{\partial\rho\partial\theta}\frac{\partial \mathbf{r}}{\partial \rho}+\frac{\partial U}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho}\frac{\partial^2 U}{\partial\phi\partial\theta}\left(\frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi}\right)+\frac{\cos\phi}{\rho}\frac{\partial U}{\partial \phi}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
&+\frac{1}{\rho^2\sin^2\phi}\frac{\partial \mathbf{r}}{\partial \theta}\cdot\left[\frac{1}{\rho\sin\phi}\frac{\partial^2 U}{\partial\theta^2}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)+\frac{1}{\rho\sin\phi}\frac{\partial U}{\partial\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\rho\sin\phi}\frac{\partial \mathbf{r}}{\partial \theta}\right)\right]\\
=&\frac{\partial^2 U}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial U}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2}+\frac{\cot\phi}{\rho^2}\frac{\partial U}{\partial\phi}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 U}{\partial\theta^2}\\
=&\frac{1}{\rho^2\sin^2\phi}\left[\rho^2\sin^2\phi\frac{\partial^2 U}{\partial\rho^2}+2\rho\sin^2\phi\frac{\partial U}{\partial\rho}+\sin^2\phi\frac{\partial^2 U}{\partial\phi^2}+\sin\phi\cos\phi\frac{\partial U}{\partial\phi}+\frac{\partial^2 U}{\partial\theta^2}\right]\\
=&\frac{1}{\rho^2\sin^2\phi}\left[\sin^2\phi\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial U}{\partial\rho}\right)+\sin\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial U}{\partial\phi}\right)+\frac{\partial^2 U}{\partial\theta^2}\right]
\end{aligned}
$$

9. Assuming that for each surface of Problem 8 the functions f, g, h have continuous first

partial derivatives in D and that the Jacobian matrix

$$\begin{pmatrix} f_u & g_u & h_u \\ f_v & g_v & h_v \end{pmatrix}^\top$$

has rank 2 in D, then we can apply the Implicit Function Theorem of Section 2.10, as in Section 2.12 to show that the inverse functions $u = \phi(x, y)$, $v = \psi(x, y)$ of $x = f(u, v)$, $y = g(u, v)$ is well defined in a neighborhood D_0 of a point (u_0, v_0) in D, under the condition that the Jacobian of the mapping $\partial(f, g)/\partial(u, v) \neq 0$ at the point (u_0, v_0).

For the sphere: $x = f(u, v) = \sin u \cos v$, $y = g(u, v) = \sin u \sin v$, $z = h(u, v) = \cos u$ we can define the implicit equations

$$F(x, y, u, v) = f(u, v) - x = 0 \qquad G(x, y, u, v) = g(u, v) - y = 0$$

Then by (2.61) we find

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F, G)}{\partial(x, v)}}{\dfrac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\cos u} \qquad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F, G)}{\partial(y, v)}}{\dfrac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\sin v}{\cos u}$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F, G)}{\partial(u, x)}}{\dfrac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} f_u & -1 \\ g_u & 0 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = -\frac{\sin v}{\sin u} \qquad \frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(F, G)}{\partial(u, y)}}{\dfrac{\partial(F, G)}{\partial(u, v)}} = -\frac{\begin{vmatrix} f_u & 0 \\ g_u & -1 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\sin u}$$

Hence, as long as $u \neq n\pi/2$, where $n = 0, \pm 1, \pm 2 \ldots$, the inverse mapping will be well defined.

For the cylinder: $x = \cos u$, $y = \sin u$, $z = v$ we find

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F, H)}{\partial(x, v)}}{\dfrac{\partial(F, H)}{\partial(u, v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & h_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = -\frac{1}{\sin u} \qquad \frac{\partial u}{\partial z} = -\frac{\dfrac{\partial(F, H)}{\partial(z, v)}}{\dfrac{\partial(F, H)}{\partial(u, v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & h_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = 0$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F, H)}{\partial(u, x)}}{\dfrac{\partial(F, H)}{\partial(u, v)}} = -\frac{\begin{vmatrix} f_u & -1 \\ h_u & 0 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = 0 \qquad \frac{\partial v}{\partial z} = -\frac{\dfrac{\partial(F, H)}{\partial(u, z)}}{\dfrac{\partial(F, H)}{\partial(u, v)}} = -\frac{\begin{vmatrix} f_u & 0 \\ h_u & -1 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ h_u & h_v \end{vmatrix}} = 1$$

Hence, as long as $u \neq n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$, the inverse mapping will be well defined and is given by $u = \tan^{-1} y/x$, $v = z$.

For the cone: $x = \sinh u \sin v$, $y = \sinh u \cos v$, $z = \sinh u$ we find

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\sin v}{\cosh u} \quad \frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 0 & f_v \\ -1 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\cosh u}$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & -1 \\ g_u & 0 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{\cos v}{\sinh u} \quad \frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} f_u & 0 \\ g_u & -1 \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = -\frac{\sin v}{\sinh u}$$

Hence, as long as $u \neq 0$, the inverse mapping will be well defined.

10. (a) Let a surface S be given as in Problems 8 and 9 and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point (x, y, z). The equations $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ can then be interpreted as defining a vector function $\mathbf{r} = \mathbf{r}(u, v)$. When $v = v_0 = \text{const}$, this is the vector representation $\mathbf{r} = \mathbf{r}(u, v_0)$ of one of a family curves obtained by varying u for different values of $v = v_0 = \text{const}$. The tangent vector to this curve is defined as in Section 2.13 to be the derivative of $\mathbf{r}$ with respect to the parameter u: $\partial\mathbf{r}/\partial u$. Similarly, fixing $u = u_0 = \text{const}$ while allowing v to vary results in one of a family of curves obtained by varying v for different values of $u = u_0 = \text{const}$, and the tangent vector to this curve is $\partial\mathbf{r}/\partial v$.

(b) If the curves $v = \text{const}$, $u = \text{const}$ intersect at right angles, then this implies that the corresponding tangent vectors to these curves, $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ respectively, are perpendicular at the point of intersection, i.e.:

$$\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right)\cdot\left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right) = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0$$

(c) The element of arc on a curve $u = u(t)$, $v = v(t)$ on S is given by

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)^2 + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)^2 + \left(\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right)^2 \\
&= \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]du^2 + \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]dv^2 \\
&\quad + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}dudv + 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}dudv + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}dudv \\
&= \left|\frac{\partial\mathbf{r}}{\partial u}\right|^2 du^2 + \left|\frac{\partial\mathbf{r}}{\partial v}\right|^2 dv^2 + 2\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}dudv \\
&= Edu^2 + Gdv^2 + 2Fdudv
\end{aligned}$$

(d) For part (b) it was shown that the coordinates are orthogonal if and only if $(\partial\mathbf{r}/\partial u)\cdot(\partial\mathbf{r}/\partial v) = 0$. Hence, for the element of arc ds^2 this implies

$$ds^2 = \left|\frac{\partial\mathbf{r}}{\partial u}\right|^2 du^2 + \left|\frac{\partial\mathbf{r}}{\partial v}\right|^2 dv^2 + 2\underbrace{\frac{\partial\mathbf{r}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial v}}_{0} dudv = \left|\frac{\partial\mathbf{r}}{\partial u}\right|^2 du^2 + \left|\frac{\partial\mathbf{r}}{\partial v}\right|^2 dv^2$$

(e) Let $u = u(t)$, $v = v(t)$ and $u = U(\tau)$, $v = V(\tau)$ be two curves on S meeting at a point P_0 of S for $t = t_0$, $\tau = \tau_0$, so that $u(t_0) = u_0 = U(\tau_0)$, $v(t_0) = v_0 = V(\tau_0)$. Then, using (1.9), the angle θ between the corresponding velocity vectors $\partial\mathbf{r}/dt$ at t_0 and $\partial\mathbf{r}/d\tau$ at τ_0 (assumed both to be non-zero) is given by

$$\begin{aligned}
\cos\theta &= \frac{\dfrac{d\mathbf{r}}{dt}\cdot\dfrac{d\mathbf{r}}{d\tau}}{\left|\dfrac{d\mathbf{r}}{dt}\right|\left|\dfrac{d\mathbf{r}}{d\tau}\right|} \\
&= \frac{\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right)\cdot\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right)}{\left|\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right|\left|\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right|} \\
&= \frac{\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right)\cdot\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right)}{\left[\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{dt} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{dt}\right)^2\right]^{1/2}\left[\left(\dfrac{\partial\mathbf{r}}{\partial u}\dfrac{du}{d\tau} + \dfrac{\partial\mathbf{r}}{\partial v}\dfrac{dv}{d\tau}\right)^2\right]^{1/2}} \\
&= \frac{E\dfrac{du}{dt}\dfrac{du}{d\tau} + G\dfrac{dv}{dt}\dfrac{dv}{d\tau} + F\left(\dfrac{du}{dt}\dfrac{dv}{d\tau} + \dfrac{dv}{dt}\dfrac{du}{d\tau}\right)}{\left[E\left(\dfrac{du}{dt}\right)^2 + G\left(\dfrac{dv}{dt}\right)^2 + 2F\dfrac{du}{dt}\dfrac{dv}{dt}\right]^{1/2}\left[E\left(\dfrac{du}{d\tau}\right)^2 + G\left(\dfrac{dv}{d\tau}\right)^2 + 2F\dfrac{du}{d\tau}\dfrac{dv}{d\tau}\right]^{1/2}} \\
&= \frac{Eu'U' + Gv'V' + F\left(u'V' + v'U'\right)}{\left(Eu'^2 + gv'^2 + 2Fu'v'\right)^{1/2}\left(EU'^2 + GV'^2 + 2FU'V'\right)^{1/2}}
\end{aligned}$$

where E, F, G are evaluated at (u_0, v_0) and $u' = u'(t_0)$, $v' = v'(t_0)$, $U' = U'(\tau_0)$, $V' = V'(\tau_0)$.

(f) If the paths of part (e) are the coordinate lines

$$u\left(t\right) = u_0 + t - t_0 \qquad v\left(t\right) = v_0 \qquad U\left(\tau\right) = u_0 \qquad V\left(\tau\right) = v_0 + \tau - \tau_0$$

such that

$$u' = \frac{d}{dt}\left(u_0 + t - t_0\right) = 1 \quad v' = \frac{dv_0}{dt} = 0 \quad U' = \frac{du_0}{d\tau} = 0 \quad V' = \frac{d}{d\tau}\left(v_0 + \tau - \tau_0\right) = 1$$

then $\cos\theta = F(EG)^{-1/2}$ at the point (u_0, v_0).

(g) In order to apply the Implicit Function Theorem of Section 2.10, it is assumed that at least one of $\partial(g,h)/\partial(u,v) \neq 0$, $\partial(f,h)/\partial(u,v) \neq 0$ or $\partial(f,g)/\partial(u,v) \neq 0$, or equivalently, that $(\partial \mathbf{r}/\partial u) \times (\partial \mathbf{r}/\partial v) > \mathbf{0}$. Hence, the two vectors $\partial \mathbf{r}/\partial u$ and $\partial \mathbf{r}/\partial v$ are not parallel and thus linearly independent in D.

(h) To show that $E > 0$, $G > 0$ follows from the fact that since $(\partial \mathbf{r}/\partial u) \times (\partial \mathbf{r}/\partial v) > \mathbf{0}$, both $(\partial \mathbf{r}/\partial u) > \mathbf{0}$ and $(\partial \mathbf{r}/\partial v) > \mathbf{0}$, which in turn implies

$$E = \left|\frac{\partial \mathbf{r}}{\partial u}\right|^2 > 0 \qquad\qquad G = \left|\frac{\partial \mathbf{r}}{\partial v}\right|^2 > 0$$

Furthermore, recalling the identity

$$|\mathbf{u} \times \mathbf{v}|^2 = \begin{vmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{vmatrix}$$

from Problem 12 (a) following Section 1.5 we find that

$$\left|\frac{\partial \mathbf{r}}{\partial u}\right|^2 \left|\frac{\partial \mathbf{r}}{\partial v}\right|^2 - \left(\frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}\right)^2 = EG - F^2 = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|^2 > 0$$

where the last inequality again follows from (g).

(i) As stated in Section 2.21 a quadratic form is called positive definite if it is positive for all non-zero values of its argument. As such, the expression for the element of arc $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ is a positive definite quadratic form, since $ds^2 \geq 0$. Furthermore, a quadratic form is positive definite if and only if all eigenvalues of the $n \times n$ symmetric coefficient matrix $\mathbf{A}$ of the quadratic form are positive. In the case of ds^2 the coefficient matrix $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Then the eigenvalues of $\mathbf{A}$ are the solutions of

$$\begin{vmatrix} E - \lambda & F \\ F & G - \lambda \end{vmatrix} = \lambda^2 - (E + G)\,\lambda + EG - F^2 = 0$$

Hence,

$$\lambda = \frac{E + G \pm \sqrt{(E+G)^2 - 4\,(EG - F^2)}}{2} = \frac{E + G \pm \sqrt{(E-G)^2 + 4F^2}}{2}$$

Now since we want both roots to be positive, it is sufficient to require that $EG - F^2 > 0$ and $E + G > 0$. The last condition certainly is satisfied when $E > 0$, $G > 0$.

Section 3.11

1.