CHAPTER 5

Section 5.3

1. (a) From the given end points $(0,0)$, $(2,2)$ it follows that we can represent the curve C in the form $y = x$, $0 \leq x \leq 2$. Hence, by (5.6) we find

$$\int_{(0,0)}^{(2,2)} y^2\,dx = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

   (b) Given the end points $(2,1)$, $(1,2)$ we will parameterise the curve C according to: $x = 2 - t$, $y = 1 + t$, $0 \leq t \leq 1$. Then by (5.4) we find

$$\int_{(2,1)}^{(1,2)} y\,dx = -\int_0^1 (1+t)\;dt = -\left[t + \frac{t^2}{2}\right]_0^1 = -\frac{3}{2}$$

   (c) Given the end points $(1,1)$, $(2,1)$ we will parameterise the curve C according to $x = 1 + t$, $y = 1$, $0 \leq t \leq 1$. Then by (5.5) we find

$$\int_{(1,1)}^{(2,1)} x\,dy = \int_0^1 (1+t)\,(0)\;dt = 0$$

2. (a) Let us represent the curve $C : x = \sqrt{1-y^2}$ in the form $x = \cos t$, $y = \sin t$, $-\pi/2 \leq t \leq \pi/2$. Then by (5.4) and (5.5)

$$\begin{aligned}\int_{(0,-1)}^{(0,1)} y^2\,dx + x^2\,dy &= \int_{-\pi/2}^{\pi/2} -\sin^3 t\,dt + \cos^3 t\,dt \\ &= \int_{-\pi/2}^{\pi/2} -\left(1-\cos^2 t\right)\sin t + \left(1-\sin^2 t\right)\cos t\,dt \\ &= \left[\cos t - \frac{\cos^3 t}{3} + \sin t - \frac{\sin^3 t}{3}\right]_{-\pi/2}^{\pi/2} = \frac{4}{3}\end{aligned}$$

   (b) Let C be the parabola $y = x^2$. Then by (5.6) and (5.7) we find

$$\int_{(0,0)}^{(2,4)} y\,dx + x\,dy = \int_0^2 \left(x^2 + 2x^2\right)\;dx = \left[\frac{x^3}{3} + \frac{2}{3}x^3\right]_0^2 = 8$$

   (c) Let C be the curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq \pi/2$ and let us use the substitution $u = \tan^3 t$. Then by (5.4) and (5.5) we can rewrite the integral as

$$\begin{aligned}\int_{(1,0)}^{(0,1)} \frac{y\,dx - x\,dy}{x^2+y^2} &= -3\int_0^{\pi/2} \frac{\sin^4 t\cos^2 t + \sin^2 t\cos^4 t}{\cos^6 t + \sin^6 t}\,dt = \int_0^{\pi/2} \frac{-3\sin^2 t\cos^2 t}{\cos^6 t + \sin^6 t}\,dt \\ &= -\int_0^{\infty} \frac{\cos^6 t}{\cos^6 t + \sin^6 t}\,du = -\int_0^{\infty} \frac{du}{1+u^2} = \lim_{b\to\infty} -\int_0^b \frac{du}{1+u^2} \\ &= \lim_{b\to\infty} -\tan^{-1} u\Big|_0^b = \lim_{b\to\infty} -\tan^{-1} b = -\frac{\pi}{2}\end{aligned}$$

3. (a) Let C be the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$. Then the integral

$$\oint_C y^2\,dx + xy\,dy$$

can be evaluated by computing the sum of the four integrals

$$\underbrace{\int_{(1,1)}^{(-1,1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(-1,1)}^{(-1,-1)} xy\,dy}_{dx=0} \qquad \underbrace{\int_{(-1,-1)}^{(1,-1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(1,-1)}^{(1,1)} xy\,dy}_{dx=0}$$

Hence,

$$\begin{aligned}\oint_C y^2\,dx + xy\,dy &= \int_1^{-1} dx - \int_1^{-1} y\,dy + \int_{-1}^{1} dx + \int_{-1}^{1} y\,dy \\ &= x\big|_1^{-1} - \frac{y^2}{2}\bigg|_1^{-1} + x\big|_{-1}^{1} + \frac{y^2}{2}\bigg|_{-1}^{1} = 0\end{aligned}$$

(b) Let C be the circle $x^2 + y^2 = 1$. Using the parameterization $x = \cos t$, $y = \sin t$ where $0 \le t \le 2\pi$, then by (5.4) and (5.5) the integral

$$\oint_C y\,dx - x\,dy$$

may be written as

$$\begin{aligned}\oint_C y\,dx - x\,dy = \int_0^{2\pi} -\sin^2 t\,dt - \cos^2 t\,dt = -\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\,dt &= -\int_0^{2\pi} dt \\ &= -2\pi\end{aligned}$$

(c) Let C be the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. Then the integral

$$\oint_C x^2y^2\,dx - xy^3\,dy$$

can be evaluated by computing the sum of the three integrals

$$\underbrace{\int_{(0,0)}^{(1,0)} x^2y^2\,dx = 0}_{dy=0} \qquad \underbrace{-\int_{(1,0)}^{(1,1)} xy^3\,dy}_{dx=0} \qquad \int_{(1,1)}^{(0,0)} x^2y^2\,dx - xy^3\,dy$$

Hence,

$$\begin{aligned}\oint_C x^2y^2\,dx - xy^3\,dy &= -\int_0^1 y^3\,dy + \int_0^1 x^4\,dx - \int_0^1 y^4\,dy \\ &= -\frac{y^4}{4}\bigg|_0^1 + \frac{x^5}{5}\bigg|_0^1 - \frac{y^5}{5}\bigg|_0^1 = -\frac{1}{4}\end{aligned}$$

4. (a) Let C be the circle $x^2 + y^2 = 4$. Then using the parametrisation $x = 4\cos t$, $y = 4\sin t$, where $0 \le t \le 2\pi$ and (5.12) the integral

$$\oint_C \left(x^2 - y^2\right)\, ds$$

may be written as

$$\oint_C \left(x^2 - y^2\right)\, ds = 64 \int_0^{2\pi} \left(\cos^2 t - \sin^2 t\right)\, dt = 64 \int_0^{2\pi} \cos 2t\, dt = 32 \sin 2t\Big|_0^{2\pi} = 0$$

(b) Let C be the line $y = x$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) the integral

$$\int_{(0,0)}^{(1,1)} x\, ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} x\, ds = \sqrt{2} \int_0^1 x\, dx = \frac{\sqrt{2}}{2} x^2 \Big|_0^1 = \frac{1}{\sqrt{2}}$$

(c) Let C be the parabola $y = x^2$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) and using the substitution $x = (1/2)\tan u$, such that $dx = (1/2)\sec^2 u\, du$ the integral

$$\int_{(0,0)}^{(1,1)} ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1 + 4x^2}\, dx = \frac{1}{2} \int_0^{\tan^{-1} 2} \sec^3 u\, du$$

In order to solve the integral on the right hand side, let us solve the indefinite integral

$$\begin{aligned}
\int \sec^3 x\, dx = \int_0 \sec^2 x \sec x\, dx &= \sec x \tan x - \int \sec x \tan^2 x\, dx + C \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right)\, dx + C \\
&= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx + C
\end{aligned}$$

Adding the term $\int \sec^3 x\, dx$ to both sides and dividing by two then gives

$$\begin{aligned}
\int \sec^3 x\, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x\, dx + C \\
&= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C
\end{aligned}$$

Substituting in the original equation then gives

$$\begin{aligned}\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1+4x^2}\,dx &= \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3 u\,du \\ &= \frac{1}{4}\sec u\tan u\Big|_0^{\tan^{-1}2} + \frac{1}{4}\ln|\sec u + \tan u|\Big|_0^{\tan^{-1}2} \\ &= \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5}+2\right)}{4}\end{aligned}$$

5. Let a path $x = \phi(t)$, $y = \psi(t)$, $h \le t \le k$, where x and y are continuous and have continuous derivatives for $h \le t \le k$ like (5.1) be given. Next, let us make a change of parameter by the equation $t = g(\tau)$, $\alpha \le \tau \le \beta$, where $g'(\tau)$ is continuous and positive in the interval and $g(\alpha) = h$, $g(\beta) = k$. Then by (5.4) the line integral $\int_C f(x,y)\,dx$ on the path $x = \phi(g(\tau))$, $y = \psi(g(\tau))$, such that $dx = (d/d\tau)\phi(g(\tau))\,d\tau$, is given by

$$\begin{aligned}\int\limits_C f(x,y)\,dx &= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right),\psi\left(g\left(\tau\right)\right)\right]\frac{d}{d\tau}\phi\left(g\left(\tau\right)\right)\,d\tau \\ &= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right),\psi\left(g\left(\tau\right)\right)\right]\frac{d\phi}{dt}\frac{d}{d\tau}g\left(\tau\right)\,d\tau \\ &= \int_h^k f\left[\phi\left(t\right),\psi\left(t\right)\right]\frac{d\phi}{dt}\frac{dt}{d\tau}\,d\tau = \int_h^k f\left[\phi\left(t\right),\psi\left(t\right)\right]\phi'\left(t\right)\,dt\end{aligned}$$

6. (a) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFG$ may be approximated as

$$\begin{aligned}\int\limits_C P\,dx + Q\,dy &\sim \left[\frac{1}{2}\left(0+3\right)\cdot 1 + \frac{1}{2}\left(1+2\right)\cdot 0\right] + \left[\frac{1}{2}\left(3+0\right)\cdot 0 + \frac{1}{2}\left(2+4\right)\cdot 1\right] \\ &\quad + \left[\frac{1}{2}\left(0+5\right)\cdot 1 + \frac{1}{2}\left(4+6\right)\cdot 0\right] = 7\end{aligned}$$

(b) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFGKH$ may be approximated as

$$\begin{aligned}\int\limits_C P\,dx + Q\,dy &\sim \left[\frac{1}{2}\left(0+0\right)\cdot 1 + \frac{1}{2}\left(1+4\right)\cdot 1\right] + \left[\frac{1}{2}\left(0+5\right)\cdot 1 + \frac{1}{2}\left(4+6\right)\cdot 0\right] \\ &\quad + \left[\frac{1}{2}\left(5+0\right)\cdot 0 + \frac{1}{2}\left(6+9\right)\cdot 1\right] + \left[\frac{1}{2}\left(0+2\right)\cdot 1 + \frac{1}{2}\left(9+8\right)\cdot -1\right] \\ &= 5\end{aligned}$$

(c) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABCDHLSONMIEA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+8)\cdot 1 + \frac{1}{2}(2+3)\cdot 0\right] \\
& + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+4)\cdot 0\right] + \left[\frac{1}{2}(5+2)\cdot 0 + \frac{1}{2}(4+8)\cdot 1\right] \\
& + \left[\frac{1}{2}(2+1)\cdot 0 + \frac{1}{2}(8+2)\cdot 1\right] + \left[\frac{1}{2}(1+4)\cdot 0 + \frac{1}{2}(2+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(4+3)\cdot -1 + \frac{1}{2}(6+2)\cdot 0\right] + \left[\frac{1}{2}(3+7)\cdot -1 + \frac{1}{2}(2+8)\cdot 0\right] \\
& + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] \\
& + \left[\frac{1}{2}(8+3)\cdot 0 + \frac{1}{2}(3+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\
= & \, 8
\end{aligned}$$

(d) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFJNMIJFA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(4+1)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 0 + \frac{1}{2}(4+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(5+7)\cdot 0 + \frac{1}{2}(6+8)\cdot 1\right] + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] \\
& + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+6)\cdot 0\right] \\
& + \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+4)\cdot -1\right] + \left[\frac{1}{2}(0+0)\cdot -1 + \frac{1}{2}(4+1)\cdot -1\right] \\
= & \, \frac{11}{2}
\end{aligned}$$

(e) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFEAEFBA$ may be approximated as

$$\begin{aligned}\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right] \\ & + \left[\frac{1}{2}(0+3)\cdot -1 + \frac{1}{2}(4+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\ & + \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(1+2)\cdot 1\right] + \left[\frac{1}{2}(3+0)\cdot 1 + \frac{1}{2}(2+4)\cdot 0\right] \\ & + \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(4+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot -1 + \frac{1}{2}(2+1)\cdot 0\right] \\ = & \; 0\end{aligned}$$

7. Let C be a smooth curve in the xy-plane and let $f(x,y) > 0$ be a continuous function defined over a region of the xy-plane containing the curve C. The equation $z = f(x,y)$ then is the equation of a surface that lies above the region of the xy-plane containing the curve C. Next, we imagine moving a straight line along C perpendicular to the xy-plane, effectively tracing out a "wall" standing on C, orthogonal to the xy-plane. This "wall" cuts the surface $z = f(x,y)$, forming a curve on it that lies above the curve C. In fact, the curve C may be interpreted as the projection of the surface curve onto the xy-plane. Using (5.11), the line integral

$$\int\limits_C f(x,y)\,ds = \lim_{\substack{n\to\infty \\ \max \Delta_i s \to 0}} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta_i s$$

then may be interpreted as an infinite sum of the length of each straight line directed from C to the surface curve lying above it in the limit where the distance Δs between each subsequent line becomes infinitely small, effectively tracing out a "wall" with height at each point (x,y) given by $f(x,y)$. This may be interpreted the as the area of the cylindrical surface $0 \le z \le f(x,y)$, (x,y) on C.

Section 5.5

1. Let the vector $v = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ be given. Then by (5.25) and (5.29)

   (a) The integral $\int_C v_T\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int\limits_C v_T\,ds = \int\limits_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6),(5.9)}{=} \int_0^1 2x^2\,dx + \int_0^1 2y^2\,dy = \frac{4}{3}$$

   (b) The integral $\int_C v_T\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int\limits_C v_T\,ds = \int\limits_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6),(5.7)}{=} \int_0^1 \left(x^2+5x^4\right)\,dx = \frac{4}{3}$$

(c) The integral $\int_C v_T\, ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_T\, ds &= \int\limits_C \left(x^2+y^2\right)\, dx + 2xy\, dy \\ &= \int_{(0,0)}^{(1,0)} \left(x^2+y^2\right)\, dx + 2xy\, dy + \int_{(1,0)}^{(1,1)} \left(x^2+y^2\right)\, dx + 2xy\, dy \\ &= \int_0^1 x^2\, dx + \int_0^1 2y\, dy = \frac{4}{3}\end{aligned}$$

2. Let $\mathbf{v} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be the same vector as given in Problem 1, and let $\mathbf{n}$ be the unit normal vector 90° behind the tangent vector $\mathbf{T}$ as given by (5.37). Then the normal component of $\mathbf{v}$ is given by $v_n = \mathbf{v}\cdot\mathbf{n} = (P\mathbf{i}+Q\mathbf{j})\cdot(y_s\mathbf{i} - x_s\mathbf{j}) = -Qx_s + Py_s$. Then by (5.25) and (5.29)

(a) The integral $\int_C v_n\, ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\, ds = \int\limits_C -2xy\, dx + \left(x^2+y^2\right)\, dy \overset{(5.6)\,(5.9)}{=} \int_0^1 -2x^2\, dx + \int_0^1 2y^2\, dy = 0$$

(b) The integral $\int_C v_n\, ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\, ds = \int\limits_C -2xy\, dx + \left(x^2+y^2\right)\, dy \overset{(5.6)\,(5.7)}{=} \int_0^1 2x^5\, dx = \frac{1}{3}$$

(c) The integral $\int_C v_n\, ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\begin{aligned}\int\limits_C v_n\, ds &= \int\limits_C -2xy\, dx + \left(x^2+y^2\right)\, dy \\ &= \int_{(0,0)}^{(1,0)} -2xy\, dx + \left(x^2+y^2\right)\, dy + \int_{(1,0)}^{(1,1)} -2xy\, dx + \left(x^2+y^2\right)\, dy \\ &= \int_0^1 \left(1+y^2\right)\, dy = \frac{4}{3}\end{aligned}$$

3. Let the gravitational force near a point on the earth's surface be represented approximately by the vector $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = -mg\mathbf{j}$, where the y-axis points upwards. Then by (5.29) and the fact that $P(x,y) = 0$ the work done by the force $\mathbf{F}$ on a body moving in a vertical plane from height h_1 to height h_2 along any path is equal to

$$\int\limits_C F_T\, ds = \int\limits_C \left(P\cos\alpha + Q\sin\alpha\right)\, ds = \int\limits_C Q\, dy = -\int_{h_1}^{h_2} mg\, dy = -mgy\Big|_{h_1}^{h_2} = mg\left(h_1 - h_2\right)$$

4. Let the gravitational force $\mathbf{F}$ be given by $\mathbf{F} = -(kMm/r^2)(\mathbf{r}/r)$. Then in order to compute the work by the gravitational force $\mathbf{F}$ in bringing a particle to its present position r from infinite distance along the ray through the earth's center, we will represents the curve C in terms of parameter t and then use (5.34) to get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_\infty^r \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt = \int_\infty^r \left(-\frac{kMm}{t^2}\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}\right) dt = \int_\infty^r -\frac{kMm}{t^2}\, dt = \left.\frac{kMm}{t}\right|_\infty^r$$
$$= kMm\left(\frac{1}{r}-\frac{1}{\infty}\right) = \frac{kMm}{r}$$
$$= -U$$

where $(d\mathbf{r}/dr)\cdot(d\mathbf{r}/dr) = 1$ follows from the fact that $d\mathbf{r}/dr$ is a unit vector.

5. (a) By (5.40) the integral $\oint_C ay\,dx + bx\,dy$ may be written as

$$\oint_C ay\,dx + bx\,dy = \iint_R (b-a)\; dx\,dy = (b-a)\,A$$

where A is the area enclosed by the curve C.

(b) By (5.40) the integral $\oint e^x \sin y\,dx + e^x\cos y\,dy$ around the rectangle with vertices $(0,0), (1,0), (1,\pi/2), (0,\pi/2)$ may be written as

$$\oint e^x\sin y\,dx + e^x\cos y\,dy = \int_0^{\pi/2}\int_0^1 (e^x\cos y - e^x\cos y)\; dx\,dy = 0$$

(c) By (5.40) and (4.61) the integral $\oint (2x^3-y^3)\,dx + (x^3+y^3)\,dy$ around the circle $x^2+y^2=1$ may be written as

$$\oint \left(2x^3-y^3\right)\; dx + \left(x^3+y^3\right)\; dy = 3\int_0^1\int_0^{2\pi} r^3\,d\theta\,dr = 6\pi\int_0^1 r^3\,dr = \frac{3\pi}{2}$$

(d) By (5.43) and (3.31) the integral $\oint_C u_T\,ds$, where $\mathbf{u} = \operatorname{grad}(x^2y)$ and C is the circle $x^2+y^2=1$ may be written as

$$\oint_C u_T\,ds = \iint_R \operatorname{curl}_z \mathbf{u}\,dx\,dy = \iint_R \operatorname{curl}_z \operatorname{grad}\left(x^2y\right)\; dx\,dy = 0$$

(e) By (5.44) the integral $\oint_C v_n\,ds$, where $\mathbf{v} = (x^2+y^2)\mathbf{i} - 2xy\mathbf{j}$ and C is the circle $x^2+y^2=1$ ($\mathbf{n}$ being the outer normal) may be written as

$$\oint_C v_n\,ds = \iint_R \operatorname{div}\mathbf{v}\,dx\,dy = \iint_R \operatorname{div}\;\left[\left(x^2+y^2\right)\mathbf{i} - 2xy\mathbf{j}\right]\; dx\,dy = \iint_R (2x-2x)\; dx\,dy$$
$$= 0$$

(f) Let $F = (x-2)^2 + y^2$. Then by (2.117) $\partial F/\partial n = \nabla F \cdot \mathbf{n} = \mathbf{v}\cdot\mathbf{n}$ and since $\oint_C \mathbf{v}\cdot\mathbf{n}\,ds = \oint_C v_n\,ds$ we find by (5.44) and (4.64)

$$\oint_C v_n\,ds = \iint_R \operatorname{div}\,(\nabla F)\;dx\,dy = \iint_R \nabla\cdot\nabla F\,dx\,dy = \iint_R \nabla^2 F\,dx\,dy$$

$$= \iint_R \nabla^2\left[(x-2)^2 + y^2\right]\,dx\,dy$$

$$= 4\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = 4\pi$$

(g) Let $F = \ln[(x-2)^2 + y^2]^{-1}$. Then by (2.117) $\partial F/\partial n = \nabla F\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n}$ and since $\oint_C \mathbf{v}\cdot\mathbf{n}\,ds = \oint_C v_n\,ds$ we find by (5.44)

$$\oint_C v_n\,ds = \iint_R \operatorname{div}\,(\nabla F)\;dx\,dy = \iint_R \nabla\cdot\nabla F\,dx\,dy = \iint \nabla^2 F\,dx\,dy$$

$$= \iint_R \nabla\ln\frac{1}{(x-2)^2 + y^2}\,dx\,dy$$

$$= 2\iint_R \frac{x^2 - 4x + 4 - y^2 - (x-2)^2 + y^2}{\left[(x-2)^2 + y^2\right]^2}\,dx\,dxy = 0$$

(h) By (5.40) the integral $\oint_C f(x)\,dx + g(y)\,dy$ may be written as

$$\oint_C f(x)\;dx + g(y)\;dy = \iint_R \left[\frac{\partial}{\partial x}g(y) - \frac{\partial}{\partial y}f(x)\right]\,dx\,dy = 0$$

6. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector of an arbitrary point (x,y) and let $\mathbf{n}$ be the outer normal to some arbitrary closed curve C. Then by (5.44)

$$\frac{1}{2}\oint_C r_n\,ds = \frac{1}{2}\oint_C \mathbf{r}\cdot\mathbf{n}\,ds = \frac{1}{2}\iint_R \operatorname{div}\mathbf{r}\,dx\,dy = \frac{1}{2}\iint_R \nabla\cdot(x\mathbf{i} + y\mathbf{j})\;dx\,dy$$

$$= \frac{1}{2}\iint_R \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right)\cdot(x\mathbf{i} + y\mathbf{j})\;dx\,dy$$

$$= \iint_R dx\,dy = A$$

7. As for Problem 2(a), let the line integral $\int_{(0,-1)}^{(0,1)} y^2\,dx + x^2\,dy$, where C is the semi-circle

$x = \sqrt{1-y^2}$ be given. Then by (5.40) and (4.64)

$$\oint\limits_C y^2\,dx + x^2\,dy = \iint\limits_R \left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right)\,dx\,dy = 2\iint\limits_R (x-y)\,dx\,dy$$
$$= 2\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} (x-y)\,dx\,dy$$
$$= 2\int_{-\pi/2}^{\pi/2}\int_0^1 (\cos\theta - \sin\theta)\,r^2\,dr\,d\theta$$
$$= \frac{2}{3}\int_{\pi/2}^{\pi/2} (\cos\theta - \sin\theta)\,d\theta = \frac{4}{3}$$

As for Problem 3(a), let the line integral $\oint_C y^2\,dx + xy\,dy$, where C is the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$ be given. Then by (5.40)

$$\oint\limits_C y^2\,dx + xy\,dy = \iint\limits_R \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}y^2\right]\,dx\,dy = -\iint\limits_R y\,dx\,dy = -\int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy$$
$$= -\int_{-1}^{1} xy\Big|_{-1}^{1}\,dy = -2\int_{-1}^{1} y\,dy$$
$$= -y^2\Big|_{-1}^{1} = 0$$

As for Problem 3(b), let the line integral $\oint_C y\,dx - x\,dy$, where C is the circle $x^2+y^2=1$ be given. Then by (5.40) and (4.64)

$$\oint\limits_C y\,dx - x\,dy = -\iint\limits_R \left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y\right)\,dx\,dy = -2\iint\limits_R dx\,dy = -2\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$
$$= -\int_0^{2\pi} d\theta = -2\pi$$

As for Problem 3(c), let the line integral $\oint_C x^2y^2\,dx - xy^3\,dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$ be given. Then by (5.40)

$$\oint\limits_C x^2y^2\,dx - xy^3\,dy = -\iint\limits_R \left[\frac{\partial}{\partial x}\left(xy^3\right) + \frac{\partial}{\partial y}\left(x^2y^2\right)\right]\,dx\,dy$$
$$= -\iint\limits_R \left(y^3 + 2x^2y\right)\,dx\,dy = -\int_0^1\int_0^x \left(y^3 + 2x^2y\right)\,dy\,dx$$
$$= -\int_0^1 \left[\frac{y^4}{4} + x^2y^2\right]_0^x\,dx = -\frac{5}{4}\int_0^1 x^4\,dx = -\frac{1}{4}$$

As for Problem 4(a), let the line integral $\oint_C (x^2-y^2)\,ds$, where C is the circle $x^2+y^2=4$ be given. Then by (5.44) and the fact that $\mathbf{n}$ may be written as $\mathbf{n}=(x\mathbf{i}+y\mathbf{j})/|x+y|$

$$\oint\limits_C (x^2-y^2)\,ds = a\iint\limits_R \text{div }(x\mathbf{i}-y\mathbf{j})\;dx\,dy = a\iint\limits_R \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}\right)\cdot(x\mathbf{i}-y\mathbf{j})\;dx\,dy = 0$$

Section 5.7

1. (a) Let

$$dF = 2xy\,dx + x^2\,dy \qquad\qquad \int_C\limits{}_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$$

where C is the curve $y=x^{3/2}$. To determine the function $F(x,y)$ we firstly note that

$$dF = 2xy\,dx + x^2\,dy = P\,dx + Q\,dy$$

where the functions $P(x,y)$ and $Q(x,y)$ are defined and continuous in the domain D given by $\{x,y\in\mathbb{R}\mid -\infty<x,y<\infty\}$. From inspection it then follows that

$$F\,(x,y) = x^2y + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = 2xy \qquad\qquad \frac{\partial F}{\partial y} = x^2$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 2xy\,dx + x^2\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$$

is independent of path (and hence curve C given by $y=x^{3/2}$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy = F\,(1,1) - F\,(0,0) = 1$$

(b) Let

$$dF = ye^{xy}\,dx + xe^{xy}\,dy \qquad\qquad \int_C \int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

where C is the curve $y = \sin^3 x$. From inspection it follows that

$$F\left(x,y\right) = e^{xy} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = ye^{xy} \qquad\qquad \frac{\partial F}{\partial y} = xe^{xy}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = ye^{xy}\,dx + xe^{xy}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy$$

is independent of path (and hence curve C given by $y = \sin^3 x$) and can easily be evaluated by (5.48) to have the value

$$\int_{(0,0)}^{(\pi,0)} ye^{xy}\,dx + xe^{xy}\,dy = F\left(\pi,0\right) - F\left(0,0\right) = 0$$

(c) Let

$$dF = \frac{x\,dx + y\,dy}{\left(x^2+y^2\right)^{3/2}} \qquad\qquad \int_C \int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x\,dx + y\,dy}{\left(x^2+y^2\right)^{3/2}}$$

where C is the curve $x = e^t \cos t\, y = e^t \sin t$. From inspection it follows that

$$F\left(x,y\right) = -\frac{1}{\sqrt{x^2+y^2}} + C$$

where C is some arbitrary constant, since

$$\frac{\partial F}{\partial x} = \frac{x}{\left(x^2+y^2\right)^{3/2}} \qquad\qquad \frac{\partial F}{\partial y} = \frac{y}{\left(x^2+y^2\right)^{3/2}}$$

from which follows

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy = P\,dx + Q\,dy$$

Hence, by (5.46) we may conclude that the integral

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x}{\left(x^2+y^2\right)^{3/2}}\,dx + \frac{y}{\left(x^2+y^2\right)^{3/2}}\,dy$$

is independent of path (and hence curve C given by $x = e^t \cos t\, y = e^t \sin t$) and can easily be evaluated by (5.48) to have the value

$$\int_{(1,0)}^{\left(e^{2\pi},0\right)} \frac{x\,dx + y\,dy}{\left(x^2+y^2\right)^{3/2}} = F\left(e^{2\pi},0\right) - F\left(1,0\right) = 1 - e^{-2\pi}$$

2. (a) Let

$$\int_{C\ (1,-2)}^{(3,4)} \frac{y\,dx - x\,dy}{x^2} = \int_{C\ (1,-2)}^{(3,4)} P\,dx + Q\,dy$$

where C is the line $y = 3x - 5$ be given. From inspection we can define the function $F(x,y) = -(y/x) + D$, where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{y}{x^2} = P\left(x,y\right) \qquad\qquad \frac{\partial F}{\partial y} = -\frac{1}{x} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $x = 0$. And so by (5.48) the integral has the value $F(3,4) - F(1,-2) = -10/3$.

(b) Let

$$\int_{C\ (0,2)}^{(1,3)} \frac{3x^2}{y}\,dx - \frac{x^3}{y^2}\,dy$$

where C is the parabola $y = 2 + x^2$ be given. From inspection we can define the function $F(x,y) = x^3/y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \frac{3x^2}{y} = P\left(x,y\right) \qquad\qquad \frac{\partial F}{\partial y} = -\frac{x^3}{y^2} = Q\left(x,y\right)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\mathbb{R}$ excluding the line $y = 0$. And so by (5.48) the integral has the value $F(1,3) - F(0,2) = 1/3$.

(c) Let

$$\int_C^{(-1,0)}_{(1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy$$

where C is the circular arc $y=\sqrt{1-x^2}$, $-1 \le x \le 1$ be given. From inspection it follows that we cannot define a function $F(x,y)$ such that (5.46) holds. Hence, the given integral is not independent of path. Instead we use (5.6) and (5.7) to find

$$\begin{aligned}\int_C^{(-1,0)}_{(1,0)} (2xy-1)\ dx + \left(x^2+6y\right)\ dy &= \int_1^{-1} \left(2x\sqrt{1-x^2} - 1 - \frac{x^3}{\sqrt{1-x^2}} - 3x\right)\ dx \\ &= 2\end{aligned}$$

where we have made use of the fact the first, third and fourth term in the integral on the right hand side are odd and hence, will be zero when integrated from $x=1$ to $x=-1$.

(d) Let

$$\int_C^{(\pi/4,\pi/4)}_{(0,0)} \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$$

where C is the curve $y = 16x^3/\pi^2$. From inspection we can define the function $F(x,y) = \tan x \tan y + D$ where D is some arbitrary constant, such that

$$\frac{\partial F}{\partial x} = \sec^2 x \tan y = P\,(x,y) \qquad\qquad \frac{\partial F}{\partial y} = \sec^2 y \tan x = Q\,(x,y)$$

Hence, by Theorem I the integral is independent of path in D, where D is $\{x, y \mid x, y \neq k\pi/2\}$ for $k = \pm 1, 3, 5, \ldots$. And so by (5.48) the integral has the value $F(\pi/4, \pi/4) - F(0,0) = 1$.

3. (a)