CHAPTER 4

Section 4.1

1. (a) Using integration by parts twice, the integral can be written as

$$\begin{aligned}\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx &= -x^2 \cos x + 2x \sin x - \int 2 \sin x\, dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C \\ &= 2x \sin x - \left(x^2 - 2\right) \cos x + C\end{aligned}$$

(b) Making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\begin{aligned}\int \frac{x}{1+x^4}\, dx = \frac{1}{2} \int \frac{2x}{1+\left(x^2\right)^2}\, dx = \frac{1}{2} \int \frac{1}{1+u^2}\, du &= \frac{1}{2} \tan^{-1} u + C \\ &= \frac{1}{2} \tan^{-1} x^2 + C\end{aligned}$$

(c) Using partial fraction expansion, we can write

$$\begin{aligned}\int \frac{1}{(x-1)(x-2)}\, dx = \int \left(-\frac{1}{x-1} + \frac{1}{x-2}\right) dx &= -\int \frac{dx}{x-1} + \int \frac{dx}{x-2} \\ &= -\ln (x-1) + \ln (x-2) + C \\ &= \ln \frac{x-2}{x-1} + C\end{aligned}$$

(d) Making the substitution $u = \sqrt{x-1}$ so that $2udu = dx$, the integral can be written as

$$\begin{aligned}\int \frac{1}{1+\sqrt{x-1}}\, dx = 2 \int \frac{u}{1+u}\, du &= 2 \int \frac{-1+1+u}{1+u}\, du \\ &= 2 \int \left(-\frac{1}{1+u} + 1\right) du \\ &= -2 \int \frac{du}{1+u} + 2 \int du \\ &= -2 \ln (1+u) + 2u + C \\ &= 2\left[\sqrt{x-1} - \ln \left(1+\sqrt{x-1}\right)\right] + C\end{aligned}$$

2. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\, dx = \int_0^1 \cos^2\theta\, d\theta = \frac{1}{2} \int_0^1 (1+\cos 2\theta)\, d\theta &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\sin 2\theta}{4}\right|_0^{\pi/2} \\ &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\cos\theta \sin\theta}{2}\right|_0^{\pi/2} = \frac{\pi}{4}\end{aligned}$$

(b) Using the identity $\sin mx \sin nx = (1/2)\cos[(m-n)x] - (1/2)\cos[(m+n)x]$, the integral can be written as

$$\int_0^\pi \sin 2x \sin 3x \, dx = \frac{1}{2}\int_0^\pi (\cos x - \cos 5x)\, dx = \frac{1}{2}\int_0^\pi \cos x \, dx - \frac{1}{2}\int_0^\pi \cos 5x \, dx$$
$$= \left.\frac{\sin x}{2}\right|_0^\pi - \left.\frac{\sin 5x}{10}\right|_0^\pi = 0$$

(c) Using integration by parts twice, the integral can be written as

$$\int_0^1 \left(2x^2 - 3x + 1\right) e^x \, dx = \left.\left(2x^2 - 3x + 1\right) e^x\right|_0^1 - \int_0^1 (4x-3)\, e^x \, dx$$
$$= \left.\left(2x^2 - 3x + 1\right) e^x\right|_0^1 - \left.(4x-3)\, e^x\right|_0^1 + \int_0^1 4e^x \, dx$$
$$= \left.\left(2x^2 - 3x + 1\right) e^x\right|_0^1 - \left.(4x-3)\, e^x\right|_0^1 + \left.4e^x\right|_0^1 = 3e - 8$$

(d) Using integration by parts, the fact that $(d/dx)\tan^{-1} x = 1/(1+x^2)$ and making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\int_0^1 \tan^{-1} x \, dx = \left. x\tan^{-1} x\right|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx = \left. x\tan^{-1} x\right|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx$$
$$= \left. x\tan^{-1} x\right|_0^1 - \frac{1}{2}\int_0^1 \frac{du}{1+u}$$
$$= \left. x\tan^{-1} x\right|_0^1 - \left.\frac{\ln(1+u)}{2}\right|_0^1$$
$$= \left. x\tan^{-1} x\right|_0^1 - \left.\frac{\ln(1+x^2)}{2}\right|_0^1 = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$$

3. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 0^+}\int_b^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 0^+}\int_b^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta = \lim_{b\to 0^+}\int_b^{\pi/2} d\theta$$
$$= \lim_{b\to 0^+} \left.\theta\right|_b^{\pi/2}$$
$$= \lim_{b\to 0^+}\left(\frac{\pi}{2} - b\right) = \frac{\pi}{2}$$

(b) Making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\int_0^\infty e^{-x}\, dx = \lim_{b\to\infty}\int_0^b e^{-x}\, dx = \lim_{b\to-\infty} -\int_0^b e^u \, du = \lim_{b\to-\infty} -\left.e^u\right|_0^b = \lim_{b\to-\infty}\left(-e^b + 1\right) = 1$$

(c) Using integration by parts, the integral can be written as

$$\begin{aligned}\int_0^1 \ln x\,dx = \lim_{b\to 0^+}\int_b^1 \ln x\,dx = \lim_{b\to 0^+} x\ln x\big|_b^1 - \lim_{b\to 0^+}\int_b^1 dx &= \lim_{b\to 0^+}\left(x\ln x - x\right)\big|_b^1\\ &= \lim_{b\to 0^+}\left(-1 - b\ln b + b\right)\\ &= -1 - \lim_{b\to 0^+} b\ln b = -1\end{aligned}$$

where the last step follows from the fact that

$$\lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{1/b} \overset{LH}{=} \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = \lim_{b\to 0^+} -b = 0$$

using L'Hopital's rule.

(d) Making the substitutions $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, $2u = \theta$ so that $2du = d\theta$, $v = \cos u$ so that $dv = -\sin u du$ and $w = \sin u$ so that $dw = \cos u du$ and the identities $1 + \tan^2\theta = \sec^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, the integral can be written as

$$\begin{aligned}\int_1^\infty \frac{dx}{x\sqrt{1+x^2}} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x\sqrt{1+x^2}} = \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}}\,d\theta\\ &= \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{d\theta}{\sin\theta} = \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{2du}{\sin 2u} = \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{du}{\sin u\cos u}\\ &= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin^2 u + \cos^2 u}{\sin u\cos u}\,du\\ &= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin u}{\cos u}\,du + \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\cos u}{\sin u}\,du\\ &= \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2+\sqrt{2}}/2}^b -\frac{dv}{v} + \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2-\sqrt{2}}/2}^b \frac{dw}{w}\\ &= \lim_{b\to\sqrt{2}/2} -\ln v\big|_{\sqrt{2+\sqrt{2}}/2}^b + \lim_{b\to\sqrt{2}/2}\ln w\big|_{\sqrt{2-\sqrt{2}}/2}^b\\ &= \lim_{b\to\sqrt{2}/2} -\ln b + \ln\frac{\sqrt{2+\sqrt{2}}}{2} + \lim_{b\to\sqrt{2}/2}\ln b - \ln\frac{\sqrt{2-\sqrt{2}}}{2}\\ &= \ln\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \frac{1}{2}\ln\left(3+2\sqrt{2}\right) = \frac{1}{2}\ln\left(1+\sqrt{2}\right)^2 = \ln\left(1+\sqrt{2}\right)\end{aligned}$$

(e) Using integration by parts twice and making the substitution $u = -x$ so that

$du = -dx$, the integral can be written as

$$\begin{aligned}\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx &= \lim_{b\to-\infty}\int_0^b -u^2 e^u\,du \\ &= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty}\int_0^b 2ue^u\,du \\ &= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty}\int_0^b 2e^u\,du \\ &= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty} 2e^u\Big|_0^b \\ &= \lim_{b\to-\infty}\left(-b^2e^b + 2be^b - 2e^b + 2\right) = 2\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to-\infty} -b^2e^b = \lim_{b\to-\infty} -\frac{b^2}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty}\frac{2b}{e^{-b}} \overset{LH}{=} \lim_{b\to-\infty} -\frac{2}{e^{-b}} = 0$$

(f) Using integration by parts, the integral can be written as

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} -\frac{\ln x}{x}\bigg|_1^b + \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} \\ &= \lim_{b\to\infty} -\frac{\ln x}{x}\bigg|_1^b - \lim_{b\to\infty}\frac{1}{x}\bigg|_1^b = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right) = 1\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to\infty} -\frac{\ln b}{b} \overset{LH}{=} \lim_{b\to\infty} -\frac{1/b}{1} = \lim_{b\to\infty} -\frac{1}{b} = 0$$

4. (a)

$$\begin{aligned}\int_{-1}^1 \frac{dx}{x^{1/3}} = \int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} &= \lim_{b\to0^-}\int_{-1}^b \frac{dx}{x^{1/3}} + \lim_{b\to0^+}\int_b^1 \frac{dx}{x^{1/3}} \\ &= \lim_{b\to0^-}\frac{3}{2}x^{2/3}\bigg|_{-1}^b + \lim_{b\to0^+}\frac{3}{2}x^{2/3}\bigg|_b^1 \\ &= \lim_{b\to0^-}\frac{3}{2}\left(b^{2/3}-1\right) + \lim_{b\to0^+}\frac{3}{2}\left(1-b^{2/3}\right) = 0\end{aligned}$$

(b)

$$\begin{aligned}\int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} &= \lim_{b\to0^-}\int_{-1}^b \frac{dx}{x^3} + \lim_{b\to0^+}\int_b^1 \frac{dx}{x^3} \\ &= \lim_{b\to0^-} -\frac{1}{4x^4}\bigg|_{-1}^b + \lim_{b\to0^+}\frac{1}{4x^4}\bigg|_b^1 \\ &= \lim_{b\to0^-}\frac{1}{4}\left(-b^{-4}+1\right) + \lim_{b\to0^+}\frac{1}{4}\left(1-b^{-4}\right) = -\infty\end{aligned}$$

Hence, the integral is divergent.

(c) Making the substitution $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, the integral can be written as

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\pi/2}\int_0^b \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \lim_{b\to\pi/2}\int_0^b d\theta = \lim_{b\to\pi/2}\theta\Big|_0^b$$
$$= \lim_{b\to\pi/2} b = \frac{\pi}{2}$$

(d) Using a partial fraction expansion, the integral can be written as

$$\begin{aligned}\int_0^\infty \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx &= \lim_{b\to 0^+}\int_b^1 \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx + \lim_{b\to\infty}\int_1^b \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx\\ &= \lim_{b\to 0^+}\int_b^1 \left[\frac{4x-2}{3\left(x^2-x+1\right)} - \frac{1}{3\left(x+1\right)} - \frac{1}{x}\right]dx + \ldots\\ &= \lim_{b\to 0^+}\frac{1}{3}\int_b^1 \frac{4x-2}{x^2-x+1}\,dx - \lim_{b\to 0^+}\frac{1}{3}\int_b^1 \frac{dx}{x+1} - \lim_{b\to 0^+}\int_b^1 \frac{dx}{x} + \ldots\end{aligned}$$

It is clear to see that the first two integrals (obtained by a partial fraction expansion) belonging to the first partial integral converge. However, the third diverges:

$$\lim_{b\to 0^+}\int_b^1 \frac{dx}{x} = \lim_{b\to 0^+}\ln x\Big|_b^1 = -\lim_{b\to 0^+}\ln b = \infty$$

Hence, since the first partial integral diverges, we conclude that the original integral is divergent.

(e)

$$\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty} -\cos x\Big|_0^b = \lim_{b\to\infty}\left(-\cos b + 1\right)$$

Since $\lim_{b\to\infty}\cos b$ does not exist the integral is divergent.

(f) Making the substitution $u = \cosh x$ so that $du = \sinh x dx$, the integral can be written as

$$\begin{aligned}\int_0^\infty \left(1-\tanh x\right)\,dx = \lim_{b\to\infty}\int_0^b \left(1-\tanh x\right)\,dx &= \lim_{b\to\infty}\int_0^b dx - \lim_{b\to\infty}\int_0^b \tanh x\,dx\\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_0^b \frac{\sinh x}{\cosh x}\,dx\\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_1^{\cosh b}\frac{du}{u}\\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\ln u\Big|_1^{\cosh b}\\ &= \lim_{b\to\infty}\left(b - \ln\cosh b\right) = \ln 2\end{aligned}$$

where the last step follows from the fact that

$$\lim_{b\to\infty}(b - \ln\cosh b) = \lim_{b\to\infty}\left[\ln e^b - \ln\left(e^b + e^{-b}\right) + \ln 2\right] = \lim_{b\to\infty}\left[\ln 2 - \ln\left(1 + e^{-2b}\right)\right]$$
$$= \ln 2$$

5. (a) The curves $y = 0$, $y = 1 - x^2$ intersect at the point $(-1, 0)$, $(1, 0)$. Hence, the area between the curves is given by

$$A = \int_{-1}^{1}\left(1 - x^2\right)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$

(b) The curves $y = x^3$, $y = x^{1/3}$ intersect at the points $(-1, -1)$, $(0, 0)$, $(1, 1)$. Hence the area between the curves is given by

$$A = 2\int_{0}^{1}\left(x^{1/3} - x^3\right)\,dx = \left[\frac{3x^{4/3}}{2} - \frac{x^4}{2}\right]_{0}^{1} = 1$$

Note that we have used the fact that the intersection of the two curves is anti-symmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x = 0$ to $x = 1$ and multiply the result by two.

(c) The curves $y = 6\sin^{-1}x$, $y = \pi\sin\pi x$ intersect at the points $(-1/2, -\pi)$, $(0, 0)$, $(1/2, \pi)$. Hence, the area between the curves is given by

$$\begin{aligned}
A &= 2\int_{0}^{1/2}\left(\pi\sin\pi x - 6\sin^{-1}x\right)\,dx \\
&= 2\pi\int_{0}^{1/2}\sin\pi x\,dx - 12\int_{0}^{1/2}\sin^{-1}x\,dx \\
&= -2\cos\pi x\Big|_{0}^{1/2} - 12x\sin^{-1}x\Big|_{0}^{1/2} + \int_{0}^{1/2}\frac{12x}{\sqrt{1-x^2}}\,dx \\
&= 2 - \pi - 6\int_{1}^{3/4}\frac{du}{\sqrt{u}} \\
&= 2 - \pi - 6\int_{1}^{3/4}u^{-1/2}\,du \\
&= 2 - \pi - 12\sqrt{u}\Big|_{1}^{3/4} = 14 - \pi - 6\sqrt{3}
\end{aligned}$$

where we have used integration by parts and the substitution $u = 1 - x^2$ so that $du = -2xdx$ in order to solve the second integral. Note that we have used the fact that the intersection of the two curves is anti-symmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x = 0$ to $x = 1/2$ and multiply the result by two.

6. (a)
$$\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \sin x\, dx = -\frac{2\cos x}{\pi}\bigg|_0^{\pi/2} = \frac{2}{\pi}$$

(b)
$$\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_{-\pi/2}^{0} \sin x\, dx = -\frac{2\cos x}{\pi}\bigg|_{-\pi/2}^{0} = \frac{2\cos x}{\pi}\bigg|_0^{-\pi/2} = -\frac{2}{\pi}$$

(c) Using the identities $\sin^2 x + \cos^2 x = 1$, $\cos 2x = 2\cos^2 x - 1$, we find
$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^2 x\, dx &= \frac{2}{\pi}\int_0^{\pi/2} \left(1-\cos^2 x\right)\, dx \\ &= \frac{1}{\pi}\int_0^{\pi/2} \left(1-\cos 2x\right)\, dx \\ &= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\end{aligned}$$

(d)
$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\, dx = \frac{1}{x_2-x_1}\int_{x_1}^{x_2} (ax+b)\, dx &= \frac{1}{x_2-x_1}\left[\frac{ax^2}{2} + bx\right]_{x_1}^{x_2} \\ &= b + \frac{a}{2}(x_1+x_2)\end{aligned}$$

7. Let $f(x)$ and $g(x)$ be continuous for $a \le x \le b$ and $|g(x) - f(x)| \le \epsilon$ for $a \le x \le b$. Defining $h(x) = g(x) - f(x)$ so that $|h(x)| \le \epsilon$ and using (4.6) we then find
$$\left|\int_a^b h(x)\, dx\right| = \left|\int_a^b [g(x)-f(x)]\, dx\right| = \left|\int_a^b g(x)\, dx - \int_a^b f(x)\, dx\right| \le \epsilon\,(b-a)$$

8. (a)
$$\int_0^1 \sin x^2\, dx \approxeq \int_0^1 \left(x^2 - \frac{x^6}{6}\right)\, dx = \left[\frac{x^3}{3} - \frac{x^7}{42}\right]_0^1 = \frac{13}{42} \approxeq 0.3095$$
The worst error is approximately 0.0081 at the point $x = 1$.

(b)
$$\int_0^1 e^{-x^2}\, dx \approxeq \int_0^1 \left(1 - x^2 + \frac{x^4}{2}\right)\, dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1 = \frac{23}{30} \approxeq 0.7667$$
The worst error is approximately 0.1321 at the point $x = 1$.

9. Let $f(x)$ be continuous for $0 \leq x \leq 1$. Then (4.20) may be used to approximate the integral of $f(x)$ numerically:

$$\int_0^1 f(x)\, dx \sim \frac{1}{2n}\left[f(0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(1)\right]$$

where $0 < x_1 < x_2 \cdots < x_{n-1} < 1$. If we then let $n \to \infty$ and choose $x_1 = 1/n$, $x_2 = 2/n, \ldots, x_{n-1} = (n-1)/n$, $x_n = n/n$ such that the endpoints converge to $x = 0$ and $x = 1$ respectively, while at the same time choosing an infinite number of equally spaced, but infinitely close interior points $x_1, x_2, \ldots, x_{n-1}$ the finite sum converges to:

$$\lim_{n\to\infty} \frac{1}{2n}\left[f(0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(1)\right] =$$

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\, dx$$

Note that the end points of the first and second limits differ by a factor of $1/2$. However, since $2\infty = \infty$ this difference is of no importance.

10. (a)

$$\lim_{n\to\infty} \frac{1 + 2 + \cdots + n}{n^2} = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n} + \frac{n}{n}\right) = \int_0^1 x\, dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

(b)

$$\lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2 + \left(\frac{n}{n}\right)^2\right]$$

$$= \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

(c) Provided that $P \geq 0$

$$\lim_{n\to\infty} \frac{1^P + 2^P + \cdots + n^P}{n^{P+1}} = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^P + \left(\frac{2}{n}\right)^P + \cdots + \left(\frac{n-1}{n}\right)^P + \left(\frac{n}{n}\right)^P\right]$$

$$= \int_0^1 x^P\, dx = \left.\frac{x^{P+1}}{P+1}\right|_0^1 = \frac{1}{P+1}$$

(d) Taking the natural log of both sides of the equation gives $\ln(4/e) = \ln 4 - 1$ and

$$\ln\left\{\lim_{n\to\infty} \frac{1}{n}\left[(n+1)(n+2)\ldots(2n)\right]^{1/n}\right\} = \mathfrak{L}$$

Then, manipulating the left-hand side further, we find

$$\begin{aligned}
\mathfrak{L} &= \lim_{n\to\infty} \ln \left\{ \frac{1}{n} \left[(n+1)(n+2)\dots(2n) \right]^{1/n} \right\} = \lim_{n\to\infty} \frac{1}{n} \ln \left[\frac{1}{n^n} (n+1)(n+2)\dots(2n) \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \ln \left(\frac{n+1}{n} \frac{n+2}{n} + \dots + \frac{2n}{n} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left(\ln \frac{n+1}{n} + \ln \frac{n+2}{n} + \dots + \ln \frac{n+n}{n} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \left[\ln \left(1 + \frac{1}{n} \right) + \ln \left(1 + \frac{2}{n} \right) + \dots + \ln \left(1 + \frac{n-1}{n} \right) + \ln \left(1 + \frac{n}{n} \right) \right] \\
&= \int_0^1 \ln(1+x)\ dx = x \ln(1+x) \Big|_0^1 - \int_0^1 \frac{x}{1+x}\, dx = \ln 2 - \int_1^2 \frac{u-1}{u}\, du \\
&= \ln 2 - \int_1^2 \left(1 - \frac{1}{u} \right)\ du = \ln 2 - \left[u - \ln u \right]_1^2 = \ln 4 - 1 = \ln \frac{4}{e}
\end{aligned}$$

where we have used integration by parts to solve the first integral and the substitution $u = 1 + x$ so that $du = dx$ to solve the second integral.

11. Let $f(x)$ be a continuous function for $a \le x \le b$ and let it be a given fact that

$$\int_{a_1}^{b_1} f(x)\ dx = 0$$

for every interval $a_1 \le x \le b_1$ contained in the interval $a \le x \le b$. Next, let us choose a fixed point x_0 such that $a_1 \le x_0$, $x_0 + \delta \le b_1$, where $\delta > 0$. Then by (4.13) we find

$$\int_{x_0}^{x_0+\delta} f(x)\ dx = f(x^*)\,\delta = 0 \qquad \text{for } x_0 \le x^* \le x_0 + \delta$$

Now if we let $\delta \to 0$ we get

$$\lim_{\delta\to 0} \frac{1}{\delta} \int_{x_0}^{x_0+\delta} f(x)\ dx = f(x_0) = 0$$

As x_0 was chosen arbitrarily within $a_1 \le x \le b_1$, we conclude that $f(x) \equiv 0$.

12. Let $f(x)$ be a continuous function for $a \le x \le b$, $f(x) \ge 0$ on the interval and

$$\int_a^b f(x)\ dx = 0$$

Next, let c be such that $a < c < b$. Then the integrals $\int_a^c f(x)\,dx$, $\int_c^b f(x)\,dx$ are either positive or zero. However, since their sum must be zero, the only option is that in fact they both are zero. The interval of each partial integral thus obtained can in turn be subdivided into smaller intervals over which to individually integrate $f(x)$ and since again $f(x)$ is either positive or zero on this new sub interval, but the total integral over $a \le x \le b$ must be zero, we conclude that each partial integral must be zero over the relevant sub interval. We can continue to apply this argument indefinitely for every smaller sub interval obtained from a larger sub interval and so we conclude that $\int_{a_1}^{b_1} f(x)\,dx = 0$ for every choice a_1, b_1 on the interval $a \le x \le b$. Hence, by Problem 11, $f(x) \equiv 0$.

Section 4.2

1. (a) Let $f(x) = x$ and $F(x) = \int_0^x x\,dx$. Then

x	x	$\int_x^{x+1} t\,dt$	$\int_0^x t\,dt = F(x)$
0	0	0.5	0.0
1	1	1.5	0.5
2	2	2.5	2.0
3	3	3.5	4.5
4	4	4.5	8.0
5	5	5.5	12.5
6	6	6.5	18.0
7	7	7.5	24.5
8	8	8.5	32.0
9	9	9.5	40.5
10	10		50.0

(b) Let $f(x) = e^{-x^2}$ and $F(x) = \int_0^1 e^{-x^2}\,dx$. Then

x	e^{-x^2}	$\int_x^{x+0.1} e^{-t^2}\,dt$	$\int_0^x e^{-t^2}\,dt = F(x)$
0	1.0	0.100	0.00
0.1	0.99	0.098	0.100
0.2	0.96	0.094	0.197
0.3	0.91	0.088	0.291
0.4	0.85	0.082	0.379
0.5	0.78	0.074	0.460
0.6	0.70	0.066	0.534
0.7	0.61	0.057	0.600
0.8	0.53	0.049	0.657
0.9	0.44	0.041	0.705
1.0	0.37		0.746

(c) Let $f(x) = \cos x$ and $F(x) = \int_0^1 \cos x\,dx$. Then

x	$\cos x$	$\int_x^{x+0.1} \cos t\,dt$	$\int_0^x \cos t\,dt = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.099	0.100
0.2	0.98	0.097	0.199
0.3	0.95	0.094	0.295
0.4	0.92	0.090	0.389
0.5	0.88	0.085	0.479
0.6	0.83	0.080	0.564
0.7	0.76	0.073	0.644
0.8	0.70	0.066	0.717
0.9	0.62	0.058	0.783
1.0	0.54		0.841

(d) Let $f(x) = 1/(1+x^3)$ and $F(x) = \int_0^1 dx/(1+x^3)$. Then

x	$1/(1+x^3)$	$\int_x^{x+0.1} dt/(1+t^3)$	$\int_0^x dt/(1+t^3) = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.100	0.100
0.2	0.99	0.098	0.200
0.3	0.97	0.096	0.298
0.4	0.94	0.091	0.393
0.5	0.89	0.086	0.485
0.6	0.82	0.078	0.570
0.7	0.75	0.070	0.649
0.8	0.66	0.062	0.719
0.9	0.58	0.054	0.781
1.0	0.50		0.835

(e) Let $f(x) = \sqrt{1-x^3}$ and $F(x) = \int_0^0 .5\sqrt{1-x^3}\,dx$. Then

x	$\sqrt{1-x^3}$	$\int_x^{x+0.1} \sqrt{1-t^3}\,dt$	$\int_0^x \sqrt{1-t^3}\,dt = F(x)$
0	1.00	0.100	0.00
0.1	1.00	0.100	0.100
0.2	1.00	0.099	0.200
0.3	0.99	0.098	0.299
0.4	0.97	0.095	0.397
0.5	0.04		0.492

2. Let $f(x)$ be continuous for $a \le x \le b$

(a) By (4.15) and (4.9) we find

$$\frac{d}{dx}\int_x^b f(t)\,dt = \frac{d}{dx}\left[-\int_b^x f(t)\,dt\right] = \frac{d}{dx}\left[-F(x)\right] = -\frac{dF}{dx} = -f(x)$$

(b) Let us make the substitution $u = x^2$ and note that $a \le u \le b$. Then

$$\frac{d}{dx}\int_a^{x^2} f(t)\ dt = \frac{d}{dx}\int_a^u f(t)\ dt = \frac{d}{dx}F(u) = \frac{dF}{du}\frac{du}{dx} = 2xf(u) = 2xf\left(x^2\right)$$

(c) Let us make the substitution $u = x^2$ and note that $a \le u \le b$. Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{b} f(t)\ dt = \frac{d}{dx}\left[-\int_b^{x^2} f(t)\ dt\right] = \frac{d}{dx}\left[-\int_b^{u} f(t)\ dt\right] &= \frac{d}{dx}\left[-F(u)\right] \\ &= -\frac{dF}{du}\frac{du}{dx} \\ &= -2xf(u) = -2xf\left(x^2\right)\end{aligned}$$

(d) Let us make the substitutions $u = x^2$, $v = x^3$ and note that $a \le u, v \le b$. Furthermore, let c be a fixed point $x^2 \le c \le x^3$. Then

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{x^3} f(t)\ dt &= \frac{d}{dx}\left(\int_c^{x^3} f(t)\ dt + \int_{x^2}^{c} f(t)\ dt\right) \\ &= \frac{d}{dx}\int_c^{v} f(t)\ dt + \frac{d}{dx}\int_u^{c} f(t)\ dt \\ &= \frac{d}{dx}\int_c^{v} f(t)\ dt - \frac{d}{dx}\int_c^{u} f(t)\ dt \\ &= \frac{dF}{dv}\frac{dv}{dx} - \frac{dF}{du}\frac{du}{dx} = 3x^2 f\left(x^3\right) - 2xf\left(x^2\right)\end{aligned}$$

3. (a) By (4.20) we find approximately

$$\begin{aligned}\ln 1 &= \int_1^1 \frac{dt}{t} \sim \frac{1-1}{4}\left[1+1\right] = 0 \\ \ln 2 &= \int_1^2 \frac{dt}{t} \sim \frac{1}{20}\left(1 + 1.818 + 1.667 + 1.538 + \cdots + 1.053 + 0.5\right) \approxeq 0.694 \\ \ln 0.5 &= \int_1^{1/2} \frac{dt}{t} = -\int_{1/2}^{1} \frac{dt}{t} \sim -\frac{1}{40}\left(2 + 3.636 + 3.333 + \cdots + 2.105 + 1\right) \approxeq -0.694\end{aligned}$$

(b) Using the definition

$$\ln x = \int_1^x \frac{dt}{t} \qquad x > 0$$

let $F(x) = \ln x$. Then by (4.10) we find

$$\frac{dF}{dx} = \frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x} \qquad x > 0$$

Hence, the first derivative of $\ln x$, $x > 0$ exists and as such, $\ln x$ is defined and continuous for $0 < x < \infty$.

(c) Let $F(x) = \ln ax - \ln x$ for $a > 0$ and $x > 0$. Next, let us make the substitution $u = ax$ so that $du = adx$. Then by (b) we find

$$\begin{aligned} \frac{dF}{dx} = \frac{d}{dx}(\ln ax - \ln x) = a\frac{d}{du}\ln u - \frac{d}{dx}\ln x &= a\frac{d}{du}\int_1^u \frac{dt}{t} - \frac{d}{dx}\int_1^x \frac{dt}{t} \\ &= \frac{a}{u} - \frac{1}{x} = \frac{a}{ax} - \frac{1}{x} = 0 \end{aligned}$$

Hence, $F'(x) \equiv 0$ so that $F(x) \equiv \text{const} = \ln a$. And so

$$F(x) = \ln a = \ln ax - \ln x \qquad \Longrightarrow \qquad \ln ax = \ln a + \ln x \quad \text{for } a, x > 0$$

4. Let an ellipse be given by the parametric equations: $x = a\cos\phi$, $y = b\sin\phi$, $b > a > 0$. Then by (3.53) the element of arc ds on the curve traced out by the ellipse is defined as $ds^2 = dx^2 + dy^2$. Hence, the length of arc from $\phi = 0$ to $\phi = \alpha$ is given by

$$\begin{aligned} s = \int_0^\alpha ds = \int_0^\alpha \sqrt{dx^2 + dy^2} &= \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\, d\phi \\ &= \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2\left(1 - \sin^2\phi\right)}\, d\phi \\ &= \int_0^\alpha \sqrt{b^2 - \left(b^2 - a^2\right)\sin^2\phi}\, d\phi \\ &= \int_0^\alpha b\sqrt{1 - \frac{b^2 - a^2}{b^2}\sin^2\phi}\, d\phi = b\int_0^\alpha \sqrt{1 - k^2\sin^2\phi}\, d\phi \end{aligned}$$

5. (a) Let $F(x)$ be as in (4.24). Then by (4.10) we find

$$\frac{dF}{dx} = \frac{d}{dx}\int_0^x \frac{dt}{\sqrt{1 - k^2\sin^2 t}} = \frac{1}{\sqrt{1 - k^2\sin^2 x}}$$

Hence, the first derivative of $F(x)$, $0 < k^2 < 1$ exists for all x and as such, $F(x)$ is defined and continuous for all x.

(b) Let $x_2 > x_1$. Then since $F'(x) > 0$ for $0 < k^2 < 1$ it follows from the very definition of the derivative that $F(x_2) > F(x_1)$. Hence, we conclude that as x increases, $F(x)$ increases.

(c) Let $F(x)$ be as in (4.24). Then to show that $F(x + \pi) - F(x) = \text{const}$ we use (4.10) to find

$$\begin{aligned} \frac{d}{dx}\left[F(x + \pi) - F(x)\right] &= \frac{d}{dx}\int_0^{x+\pi} \frac{dt}{\sqrt{1 - k^2\sin^2 t}} - \frac{d}{dx}\int_0^x \frac{dt}{\sqrt{1 - k^2\sin^2 t}} \\ &= \frac{1}{\sqrt{1 - k^2\sin^2(x + \pi)}} - \frac{1}{\sqrt{1 - k^2\sin^2 x}} \\ &= \frac{1}{\sqrt{1 - k^2(-\sin x)^2}} - \frac{1}{\sqrt{1 - k^2\sin^2 x}} = 0 \end{aligned}$$

Hence, since $F'(x+\pi)-F'(x) \equiv 0$, we conclude that the quantity $F(x+\pi)-F(x) = 2K$, where $K > 0$ is some positive constant. The fact that K must be positive and non-zero follows from (b).

(d) We know from (b) that as x increases, $F(x)$ increases. Furthermore, since $F(x) \geq 0$ for $0 < k^2 < 1$ it then follows that $\lim_{x\to\infty} F(x) = \infty$. Next, to show that $\lim_{x\to-\infty} F(x) = -\infty$ we write

$$\begin{aligned}
\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \int_0^x \frac{dt}{\sqrt{1-k^2\sin^2 t}} &= \lim_{x\to-\infty} \int_x^0 -\frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= \lim_{x\to\infty} \int_0^x -\frac{dt}{\sqrt{1-k^2\sin^2 t}} \\
&= -\lim_{x\to\infty} F(x) = -\infty
\end{aligned}$$

6. Let $x = am(y)$ be the inverse of the function $y = F(x)$ of (4.24).

(a) Let $y_2 > y_1$. Furthermore, from (a) and (b) of Problem 5 we know that $F'(x) > 0$ for $0 < k^2 < 1$ so that $F(x_2) > F(x_1)$ for $x_2 > x_1$. Since $y = F(x)$ this implies that $y_2 = F(x_2)$, $y_1 = F(x_1)$. Then noting that $x = am(y)$ is defined as the inverse of $y = F(x)$ we can write $am(y_2) = x_2$, $am(y_1) = x_1$. Now since $x_2 > x_1$ we conclude that $am(y_2) > am(y_1)$ for $y_2 > y_1$.

(b) From (c) of Problem 5 we know that $F(x+\pi) = F(x) + 2K = y + 2K$. Using the fact that $x = am(y)$ is defined as the inverse of $y = F(x)$ we then find

$$am(y+2K) = am[F(x+\pi)] = x + \pi = am(y) + \pi$$

(c) From (a) of Problem 5 we know that $dF/dx = dy/dx = 1/\sqrt{1-k^2\sin^2 x}$. Since $(dy/dx)(dx/dy) \equiv 1$ we thus conclude that $am'(y) = dx/dy = \sqrt{1-k^2\sin^2 x}$. Hence, the first derivative of $am(y)$, $0 < k^2 < 1$ exists for all y and as such, $am(y)$ is defined and continuous for all y.

7. (a)