CHAPTER 3

Section 3.3

1. (a)
   (b)
   (c)
   (d)

2. (a)
   (b)
   (c)

3. If $f = xy$ then ∇f is given by
$$\nabla f = y\mathbf{i} + x\mathbf{j}$$
If $f = x^2 + y^2 - z^2$ then ∇f is given by
$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$$
If $f = e^{x+y-z}$ then ∇f is given by
$$\nabla f = e^{x+y-z}\mathbf{i} + e^{x+y-z}\mathbf{j} - e^{x+y-z}\mathbf{k}$$

4. Let $f = kMm/r$, where $r = \sqrt{x^2 + y^2 + z^2}$ be the equation for the gravitational potential. Then
$$\begin{aligned}
\nabla f &= \nabla \left(\frac{kMm}{\sqrt{x^2 + y^2 + z^2}} \right) \\
&= \frac{\partial}{\partial x} \left(\frac{kMm}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{i} + \frac{\partial}{\partial y} \left(\frac{kMm}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{j} + \frac{\partial}{\partial z} \left(\frac{kMm}{\sqrt{x^2 + y^2 + z^2}} \right) \mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{x}{r}\mathbf{i} - \frac{kMm}{r^2}\frac{y}{r}\mathbf{j} - \frac{kMm}{r^2}\frac{z}{r}\mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{\mathbf{r}}{r}
\end{aligned}$$
is a vector equation for the gravitational field.

5. Let f be given by
$$f = \ln \frac{\sqrt{(x-1)^2 + y^2}}{\sqrt{(x+1)^2 + y^2}}$$

Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}} \frac{\partial}{\partial x}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{x-1}{(x-1)^2+y^2} - \frac{x+1}{(x+1)^2+y^2} \\
&= \frac{2\left(x^2-y^2-1\right)}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]} \\
\frac{\partial f}{\partial y} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}} \frac{\partial}{\partial y}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{y}{(x-1)^2+y^2} - \frac{y}{(x+1)^2+y^2} \\
&= \frac{4xy}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}
\end{aligned}$$

Hence,

$$\nabla f = \frac{1}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}\left[2\left(x^2-y^2-1\right)\mathbf{i} + 4xy\mathbf{j}\right]$$

6.

$$\begin{aligned}
\nabla(f+g) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(f+g) \\
&= \frac{\partial}{\partial x}(f+g)\mathbf{i} + \frac{\partial}{\partial y}(f+g)\mathbf{j} + \frac{\partial}{\partial z}(f+g)\mathbf{k} \\
&= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)\mathbf{k} \\
&= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) + \left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) \\
&= \nabla f + \nabla g
\end{aligned}$$

$$\begin{aligned}
\nabla(fg) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(fg) \\
&= \frac{\partial}{\partial x}(fg)\mathbf{i} + \frac{\partial}{\partial y}(fg)\mathbf{j} + \frac{\partial}{\partial z}(fg)\mathbf{k} \\
&= \left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)\mathbf{k} \\
&= \left(f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}\right) + \left(g\frac{\partial f}{\partial x}\mathbf{i} + g\frac{\partial f}{\partial y}\mathbf{j} + g\frac{\partial f}{\partial z}\mathbf{k}\right) \\
&= f\nabla g + g\nabla f
\end{aligned}$$

7. Let $f(x,y,z)$ be a composite function $F(u)$, where $u = g(x,y,z)$. Then

$$\begin{aligned}
\nabla f = \nabla F = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) F(u) &= \frac{\partial}{\partial x}F(u)\,\mathbf{i} + \frac{\partial}{\partial y}F(u)\,\mathbf{j} + \frac{\partial}{\partial z}F(u)\,\mathbf{k} \\
&= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\mathbf{k} \\
&= \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right) \\
&= F'(u)\,\nabla g
\end{aligned}$$

8.

$$\begin{aligned}
\nabla\frac{f}{g} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\frac{f}{g} \\
&= \frac{\partial}{\partial x}\frac{f}{g}\mathbf{i} + \frac{\partial}{\partial y}\frac{f}{g}\mathbf{j} + \frac{\partial}{\partial z}\frac{f}{g}\mathbf{k} \\
&= \frac{gf_x - fg_x}{g^2}\mathbf{i} + \frac{gf_y - fg_y}{g^2}\mathbf{j} + \frac{gf_z - fg_z}{g^2}\mathbf{k} \\
&= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) - f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)\right] \\
&= \frac{1}{g^2}\left(g\nabla f - f\nabla g\right)
\end{aligned}$$

9. (a) If $f(x,y,z) = w = x^3y - y^3z$ then H is given by

$$H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \begin{bmatrix} w_{xx} & w_{xy} & w_{xz} \\ w_{yx} & w_{yy} & w_{yz} \\ w_{zx} & w_{zy} & w_{zz} \end{bmatrix} = \begin{bmatrix} 6xy & 3x^2 & 0 \\ 3x^2 & -6yz & -3y^2 \\ 0 & -3y^2 & 0 \end{bmatrix}$$

If $f(x,y,z) = w = x_1^2 + 2x_1x_2 + 5x_1x_3 + 2x_2x_1 + 4x_2^2 + x_2x_3 + 5x_3x_1 + x_3x_2 + 2x_3^2$ then H is given by

$$H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \begin{bmatrix} w_{x_1x_1} & w_{x_1x_2} & w_{x_1x_3} \\ w_{x_2x_1} & w_{x_2x_2} & w_{x_2x_3} \\ w_{x_3x_1} & w_{x_3x_2} & w_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 2 \\ 10 & 2 & 4 \end{bmatrix}$$

(b) As long as the function $f(x_1, \ldots, x_n)$ has continuous second partial derivatives then $\partial^2 f/(\partial x_i \partial x_j) = \partial^2 f/(\partial x_j \partial x_i)$, which implies that H will be symmetric.

(c) As discussed in Section 2.14, the directional derivative of a function $f(x,y)$ in a given direction can be written as

$$\nabla_\alpha f = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha = \nabla f \cdot \mathbf{u}$$

where $\mathbf{u} = \cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}$ is a unit vector that makes an angle α with the positive x-axis. Hence,

$$\begin{aligned}
\nabla_\alpha \nabla_\beta f &= \nabla_\alpha \left(\frac{\partial f}{\partial x} \cos\beta + \frac{\partial f}{\partial y} \sin\beta \right) \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \cos\beta + \frac{\partial f}{\partial y} \sin\beta \right) \cos\alpha + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \cos\beta + \frac{\partial f}{\partial y} \sin\beta \right) \sin\alpha \\
&= \cos\beta \left(\frac{\partial^2 f}{\partial x^2} \cos\alpha + \frac{\partial^2 f}{\partial x \partial y} \sin\alpha \right) + \sin\beta \left(\frac{\partial^2 f}{\partial y \partial x} \cos\alpha + \frac{\partial^2 f}{\partial y^2} \sin\alpha \right) \\
&= \begin{bmatrix} \cos\beta & \sin\beta \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos\beta & \sin\beta \end{bmatrix} H \begin{bmatrix} \cos\alpha & \sin\alpha \end{bmatrix}^\top
\end{aligned}$$

Section 3.6

1.

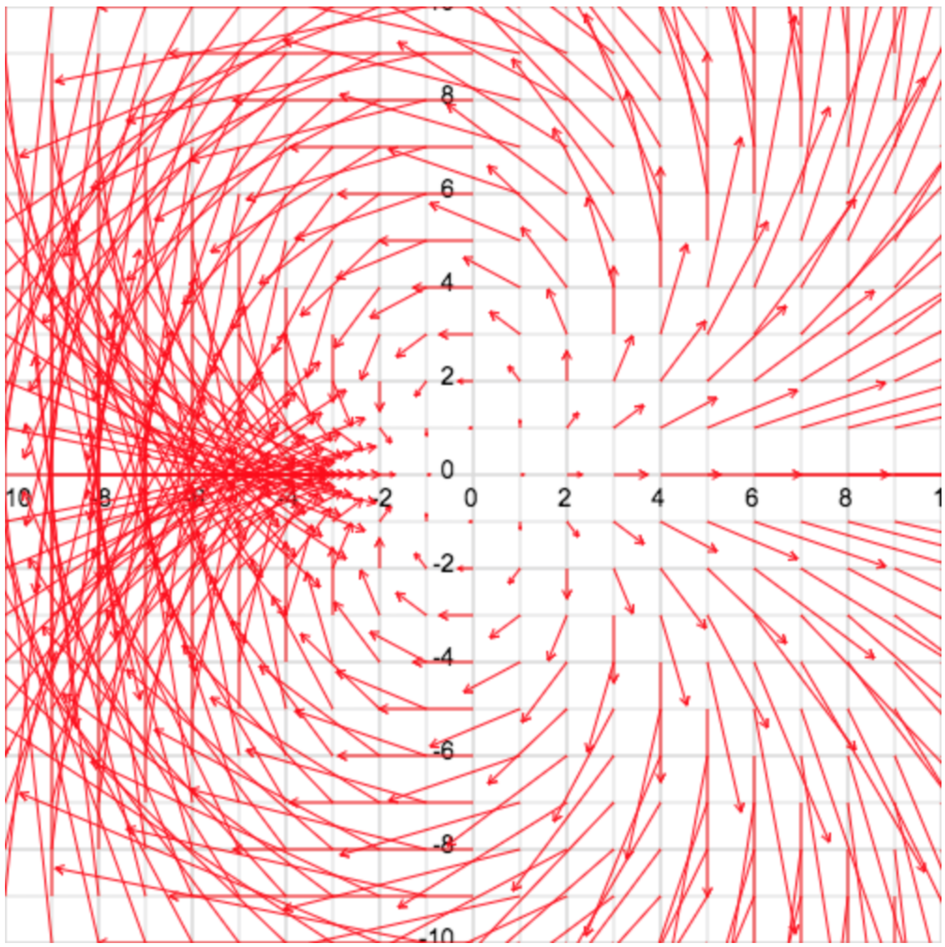

Figure 1: $\mathbf{v} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$

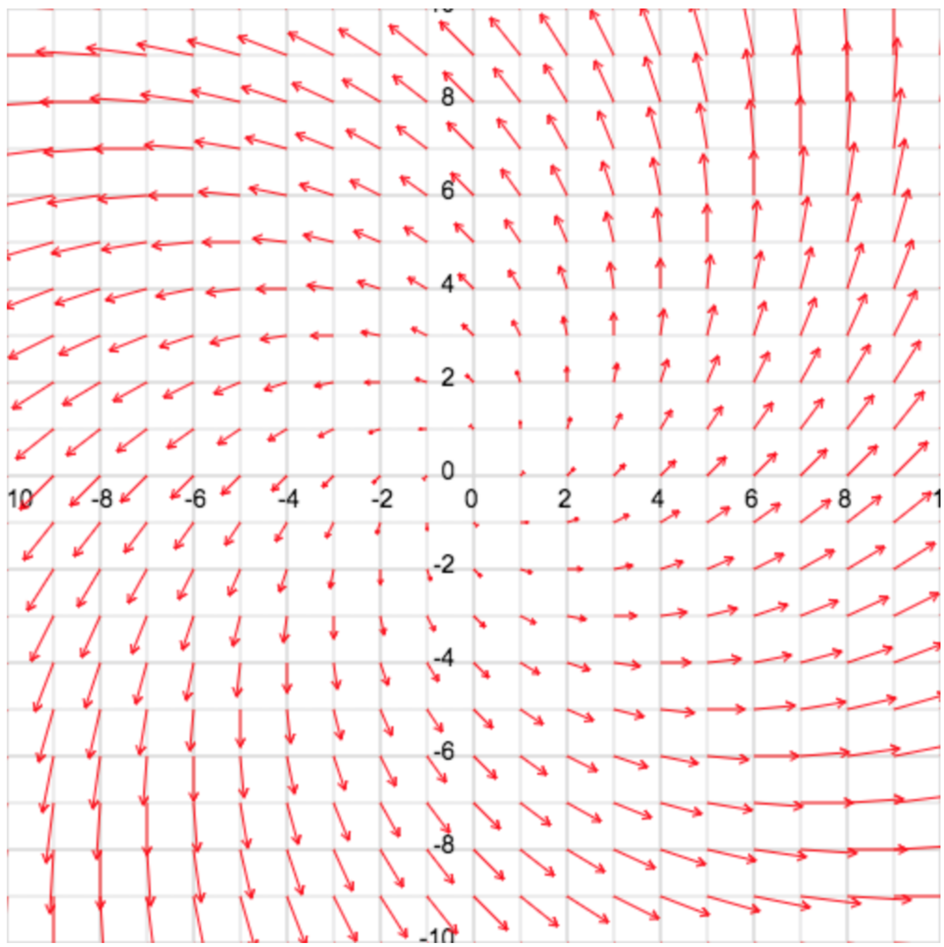

Figure 2: $\mathbf{u} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$

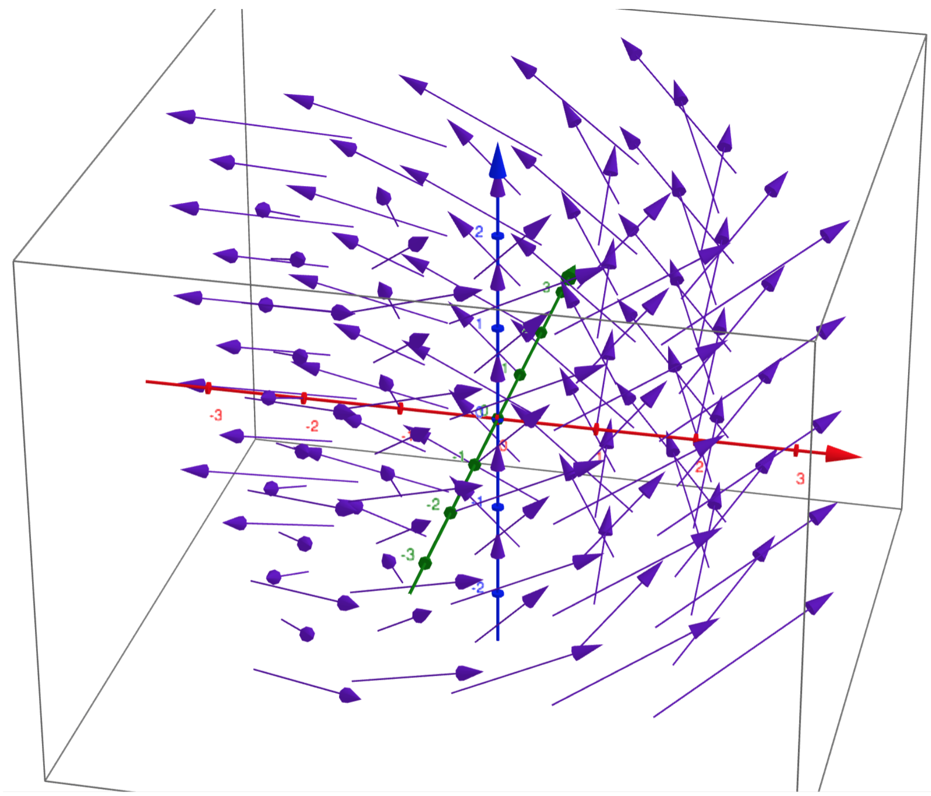

Figure 3: $\mathbf{v} = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$

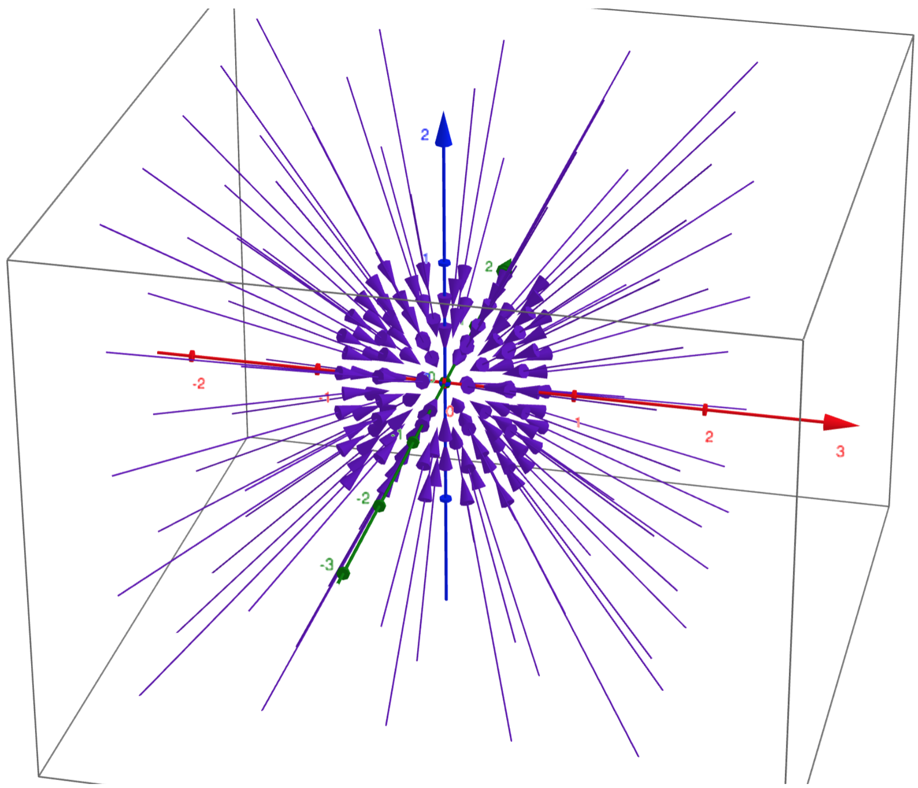

Figure 4: $\mathbf{v} = -x\mathbf{i} + -y\mathbf{j} - z\mathbf{k}$

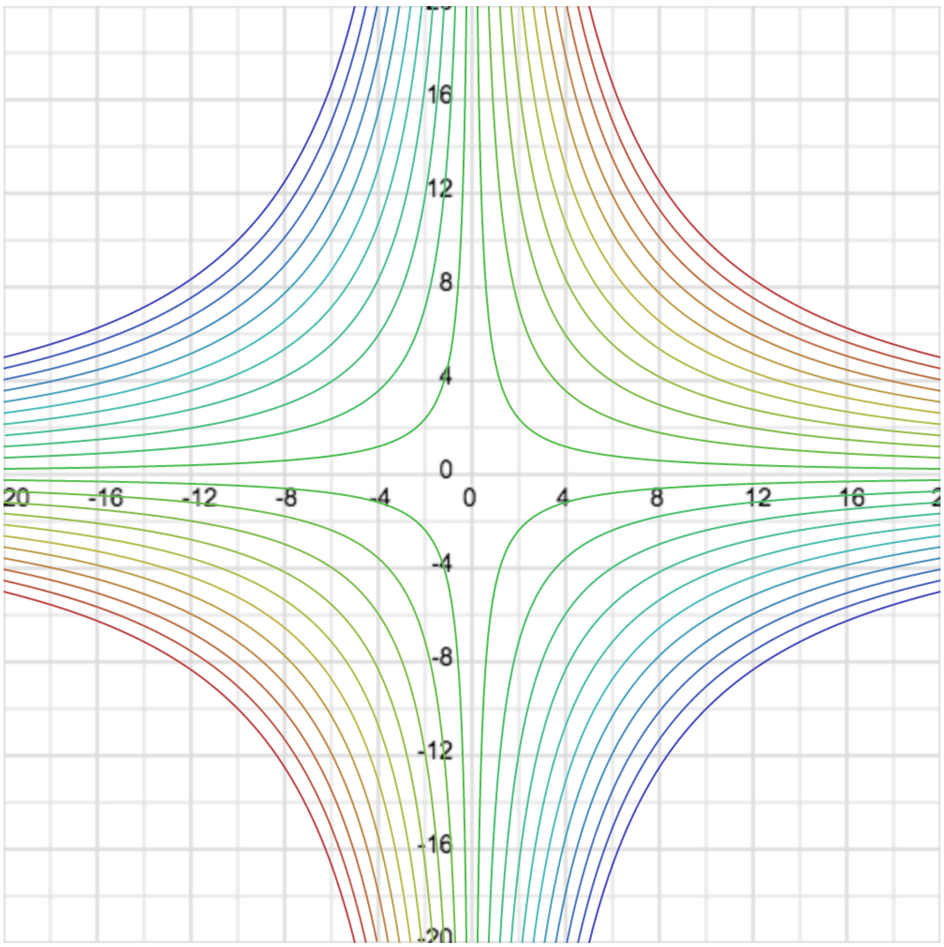

Figure 5: $f = xy$

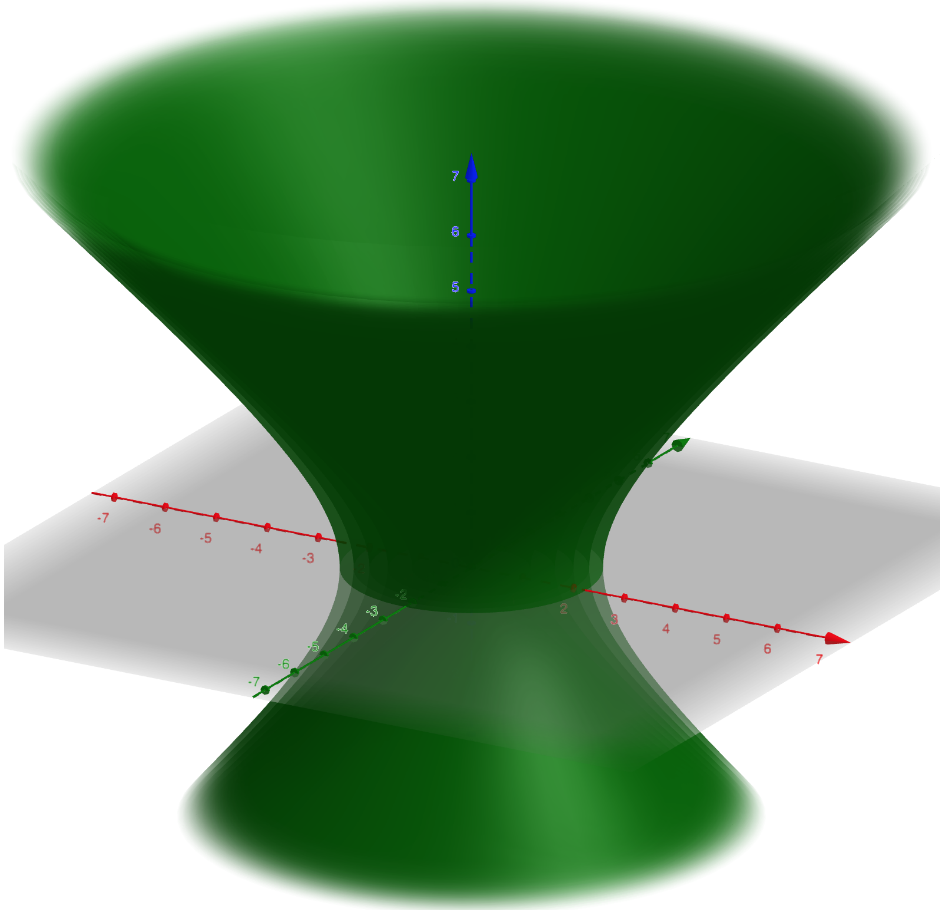

Figure 6: $f = x^2 + y^2 - z^2$

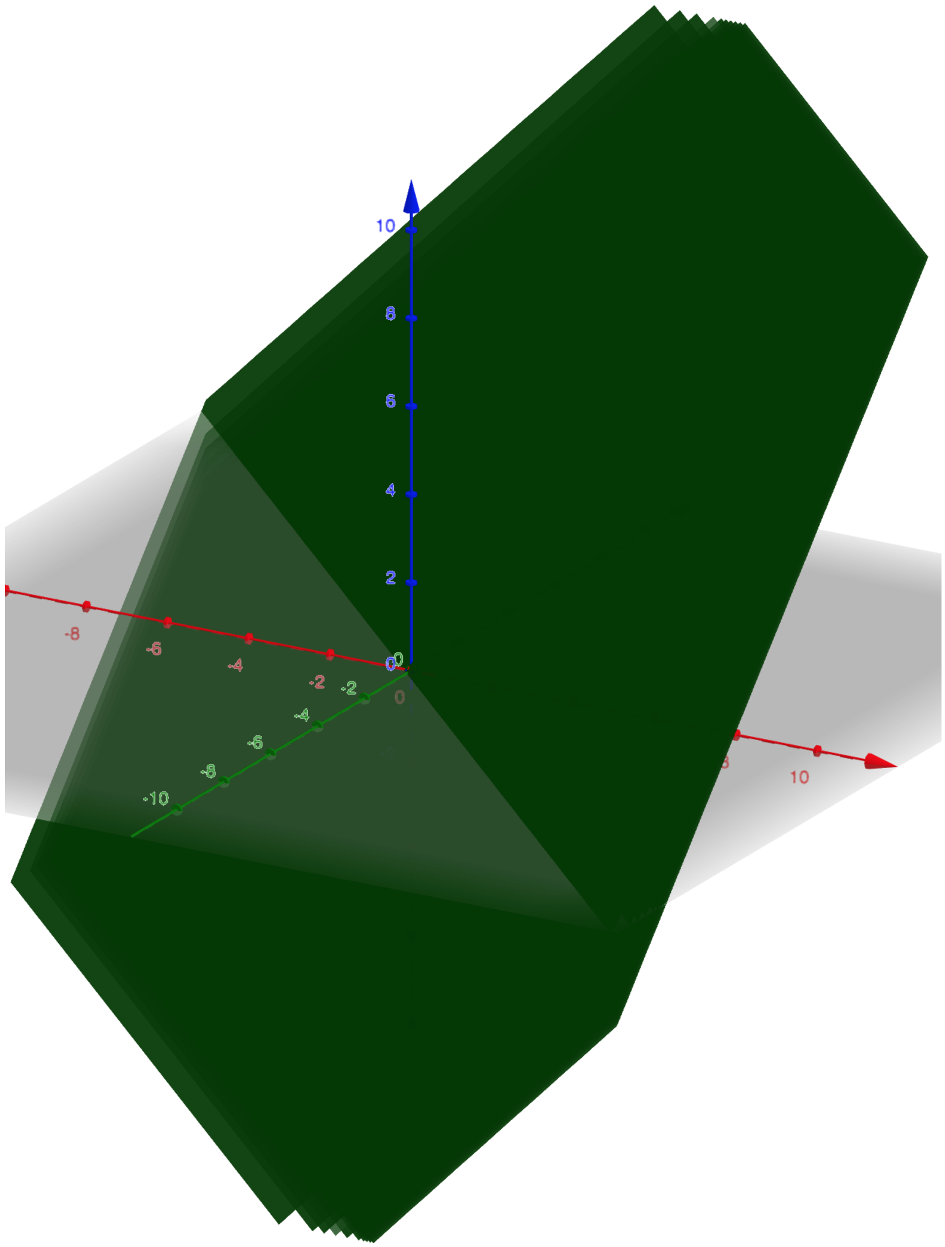

Figure 7: $f = e^{x+y-z}$