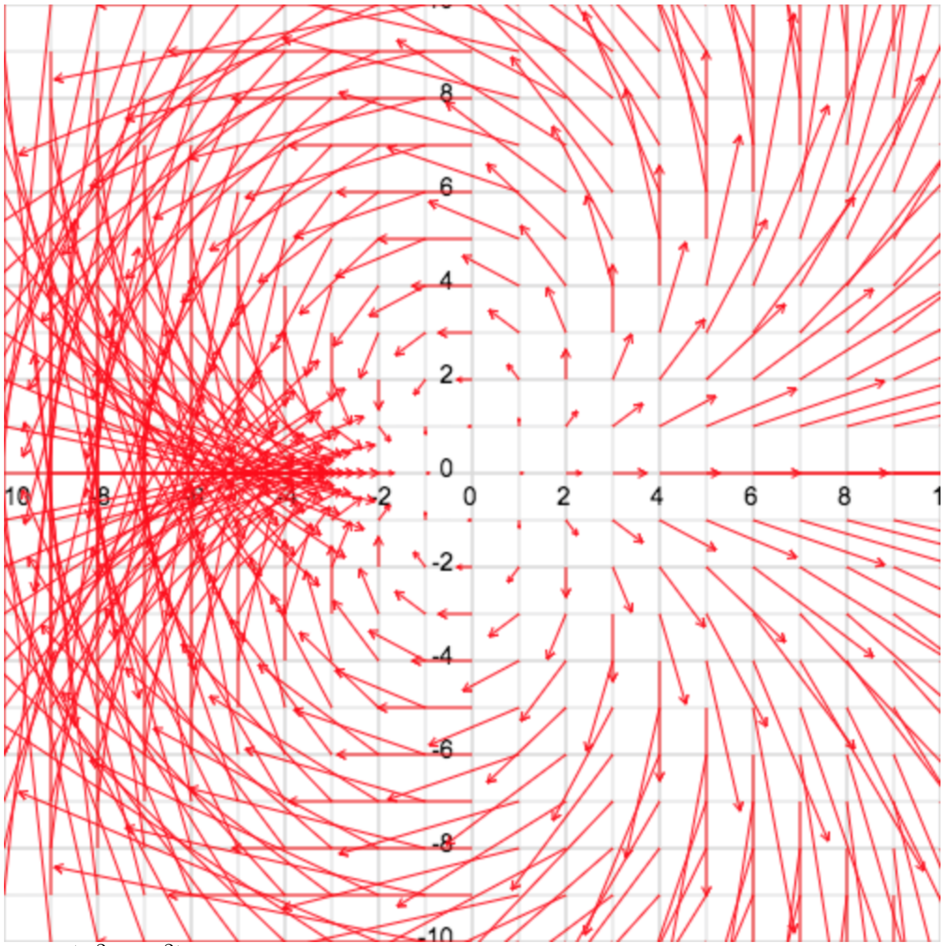
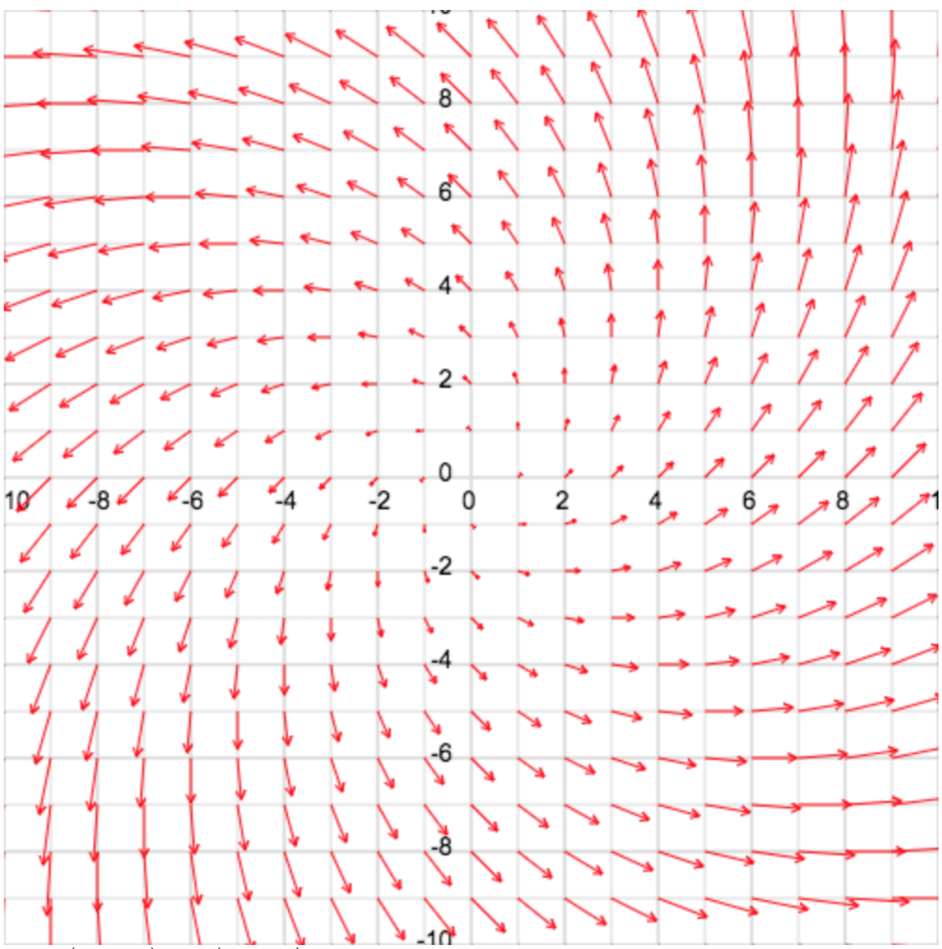
CHAPTER 3

Section 3.3

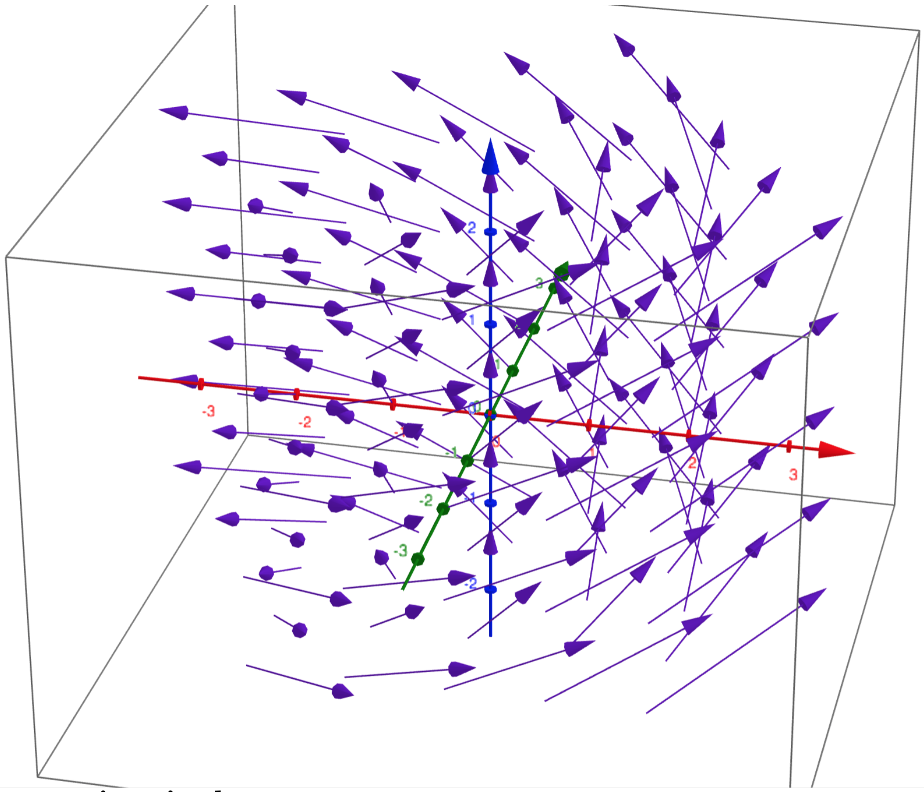
1. (a) 

$\mathbf{v} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$

(b) 

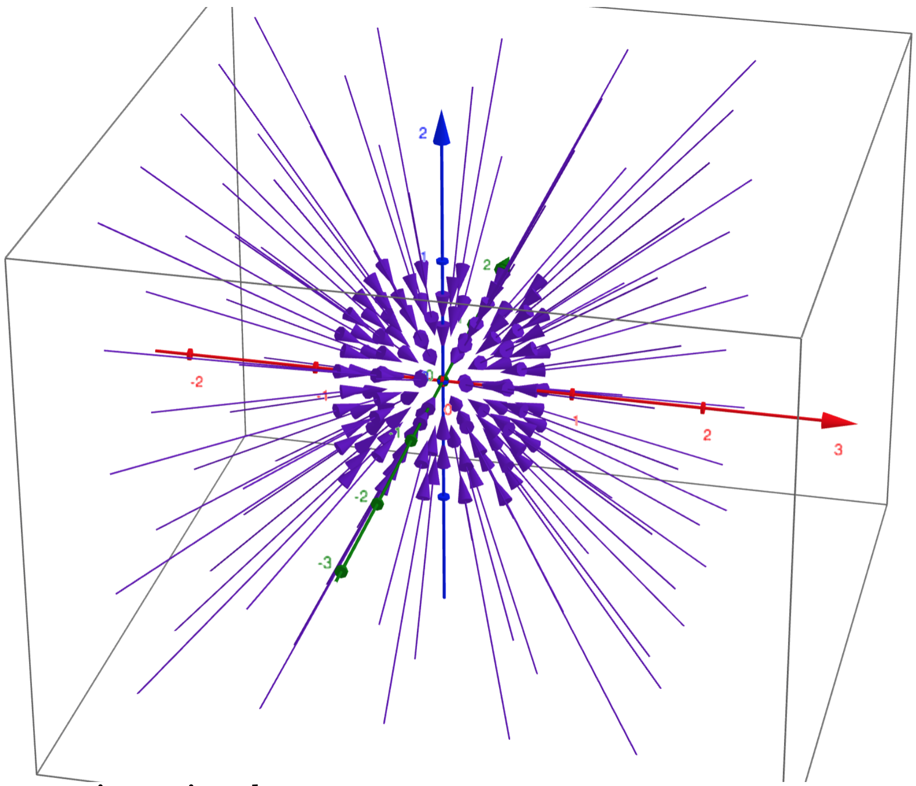
$\mathbf{u} = (x - y)\mathbf{i} + (x + y)\mathbf{j}$

(c)

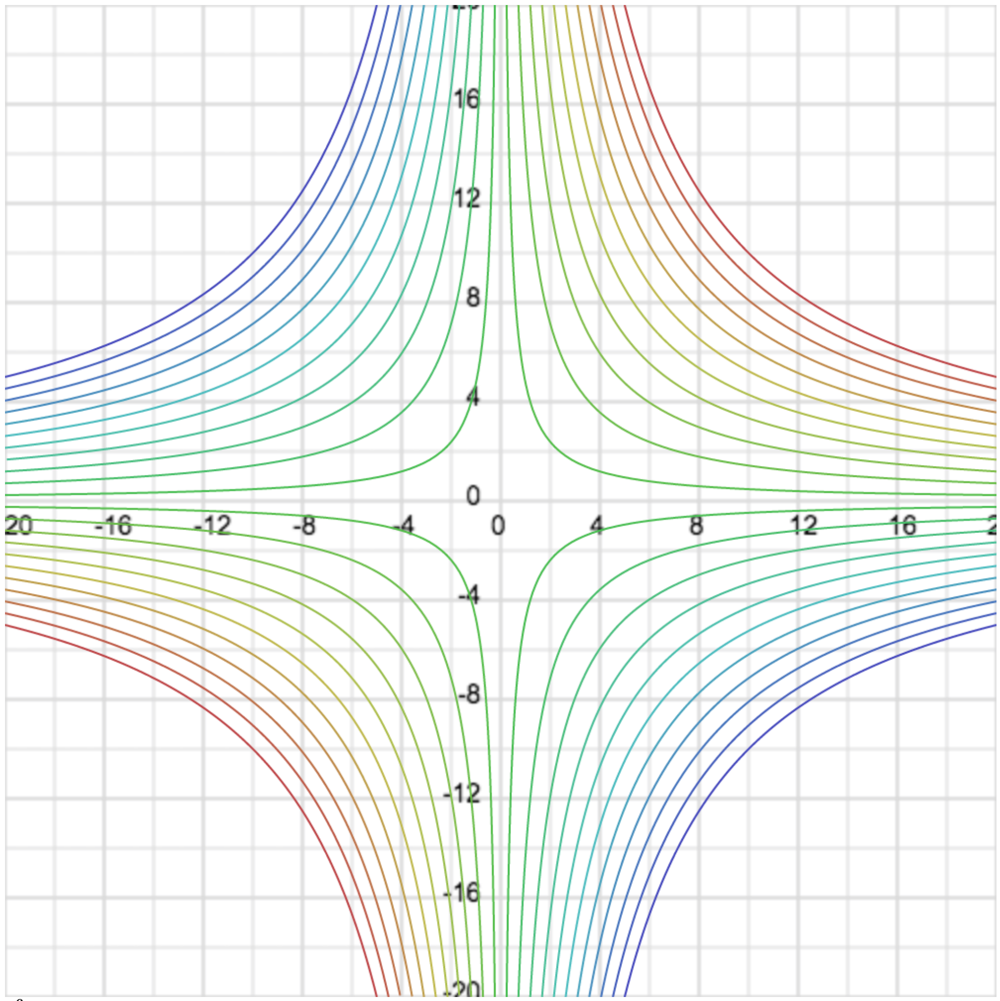


$\mathbf{v} = -y\mathbf{i} + x\mathbf{j} + \mathbf{k}$

(d)

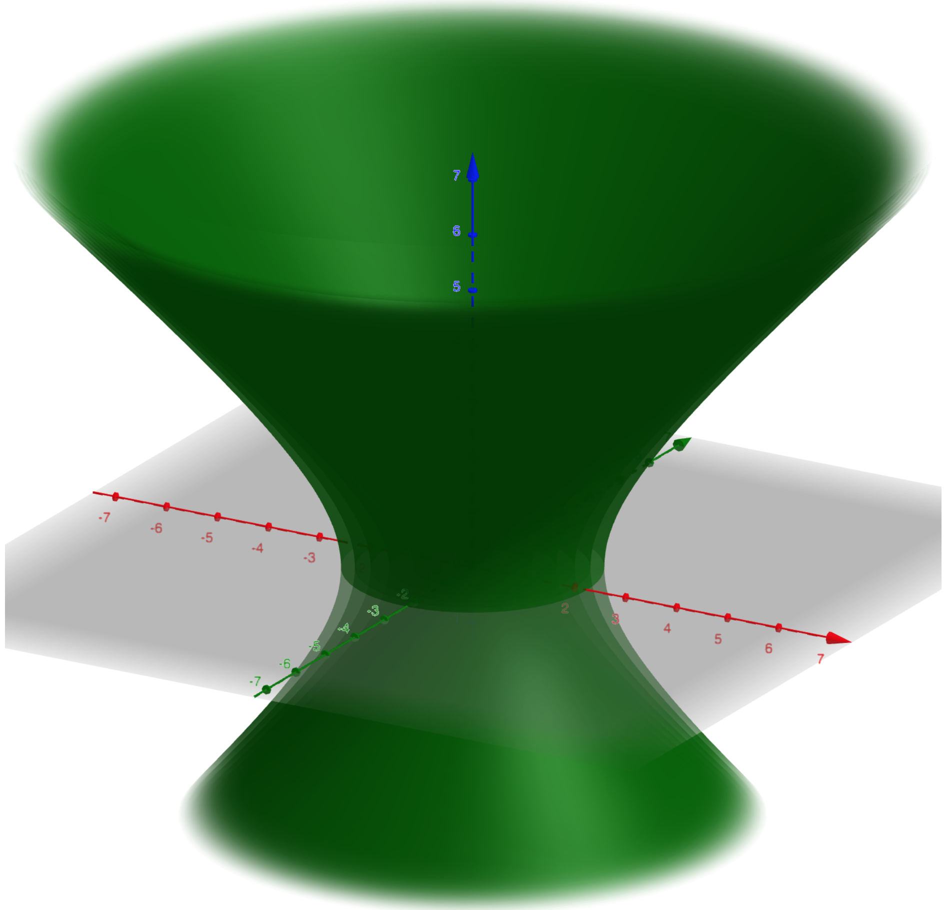


$\mathbf{v} = -x\mathbf{i} + -y\mathbf{j} - z\mathbf{k}$

2. (a) 

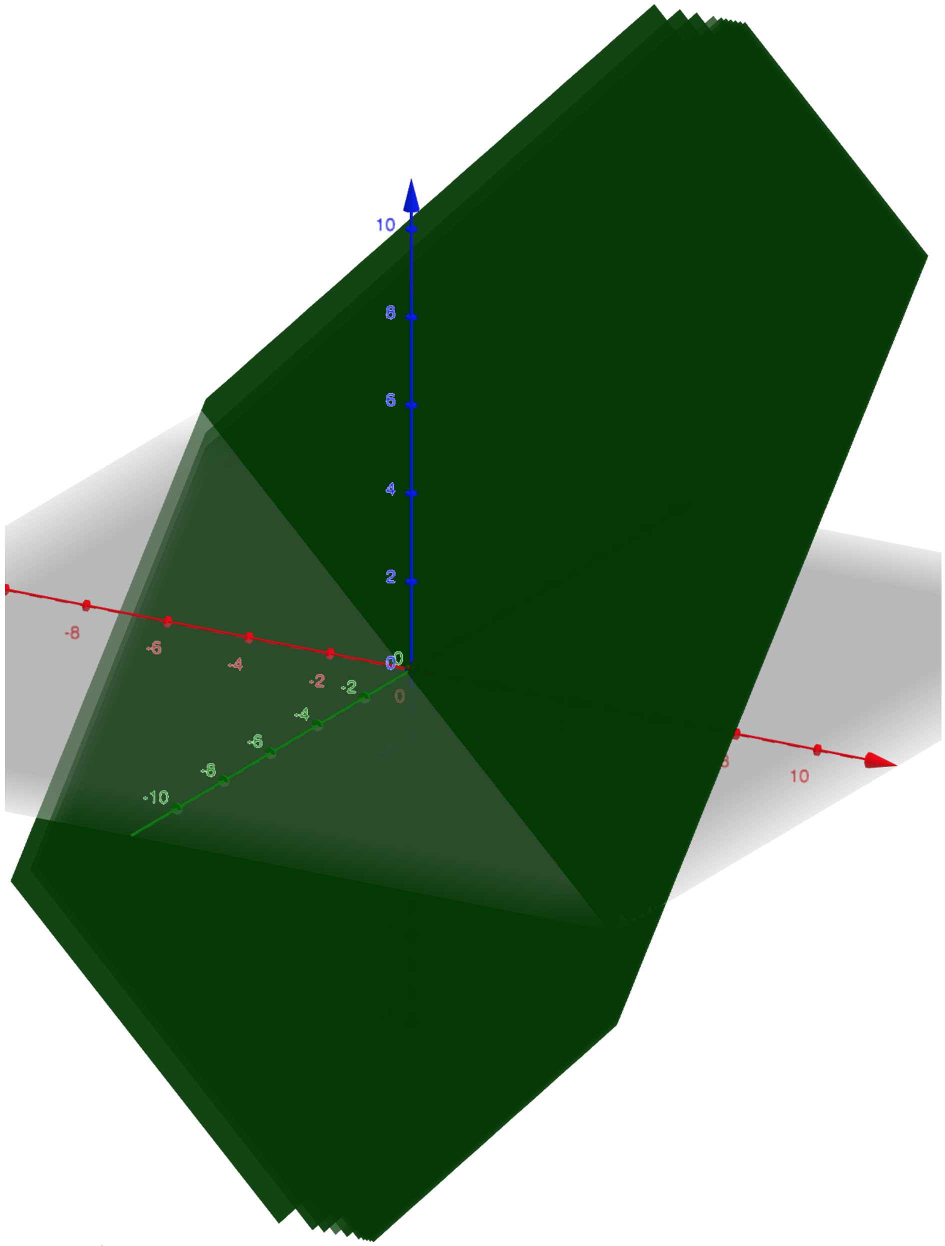
$f = xy$

(b)



$f = x^2 + y^2 - z^2$

(c)



$f = e^{x+y-z}$

3. If $f = xy$ then ∇f is given by

$$\nabla f = y\mathbf{i} + x\mathbf{j}$$

If $f = x^2 + y^2 - z^2$ then ∇f is given by

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$$

If $f = e^{x+y-z}$ then ∇f is given by

$$\nabla f = e^{x+y-z}\mathbf{i} + e^{x+y-z}\mathbf{j} - e^{x+y-z}\mathbf{k}$$

4. Let $f = kMm/r$, where $r = \sqrt{x^2+y^2+z^2}$ be the equation for the gravitational potential. Then

$$
\begin{aligned}
\nabla f &= \nabla\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{kMm}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{x}{r}\mathbf{i} - \frac{kMm}{r^2}\frac{y}{r}\mathbf{j} - \frac{kMm}{r^2}\frac{z}{r}\mathbf{k} \\
&= -\frac{kMm}{r^2}\frac{\mathbf{r}}{r}
\end{aligned}
$$

is a vector equation for the gravitational field.

5. Let f be given by

$$
f = \ln\frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}}
$$

Then

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}\frac{\partial}{\partial x}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{x-1}{(x-1)^2+y^2} - \frac{x+1}{(x+1)^2+y^2} \\
&= \frac{2\left(x^2-y^2-1\right)}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]} \\
\frac{\partial f}{\partial y} &= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}\frac{\partial}{\partial y}\left[\left((x-1)^2+y^2\right)^{1/2}\left((x+1)^2+y^2\right)^{-1/2}\right] \\
&= \frac{y}{(x-1)^2+y^2} - \frac{y}{(x+1)^2+y^2} \\
&= \frac{4xy}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}
\end{aligned}
$$

Hence,

$$
\nabla f = \frac{1}{\left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]}\left[2\left(x^2-y^2-1\right)\mathbf{i} + 4xy\mathbf{j}\right]
$$

6.

$$\begin{aligned}
\nabla(f+g) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(f+g) \\
&= \frac{\partial}{\partial x}(f+g)\,\mathbf{i}+\frac{\partial}{\partial y}(f+g)\,\mathbf{j}+\frac{\partial}{\partial z}(f+g)\,\mathbf{k} \\
&= \left(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial x}\right)\mathbf{i}+\left(\frac{\partial f}{\partial y}+\frac{\partial g}{\partial y}\right)\mathbf{j}+\left(\frac{\partial f}{\partial z}+\frac{\partial g}{\partial z}\right)\mathbf{k} \\
&= \left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)+\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}+\frac{\partial g}{\partial z}\mathbf{k}\right) \\
&= \nabla f+\nabla g
\end{aligned}$$

$$\begin{aligned}
\nabla(fg) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)(fg) \\
&= \frac{\partial}{\partial x}(fg)\,\mathbf{i}+\frac{\partial}{\partial y}(fg)\,\mathbf{j}+\frac{\partial}{\partial z}(fg)\,\mathbf{k} \\
&= \left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)\mathbf{i}+\left(f\frac{\partial g}{\partial y}+g\frac{\partial f}{\partial y}\right)\mathbf{j}+\left(f\frac{\partial g}{\partial z}+g\frac{\partial f}{\partial z}\right)\mathbf{k} \\
&= \left(f\frac{\partial g}{\partial x}\mathbf{i}+f\frac{\partial g}{\partial y}\mathbf{j}+f\frac{\partial g}{\partial z}\mathbf{k}\right)+\left(g\frac{\partial f}{\partial x}\mathbf{i}+g\frac{\partial f}{\partial y}\mathbf{j}+g\frac{\partial f}{\partial z}\mathbf{k}\right) \\
&= f\nabla g+g\nabla f
\end{aligned}$$

7. Let $f(x,y,z)$ be a composite function $F(u)$, where $u=g(x,y,z)$. Then

$$\begin{aligned}
\nabla f=\nabla F=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)F(u) &= \frac{\partial}{\partial x}F(u)\,\mathbf{i}+\frac{\partial}{\partial y}F(u)\,\mathbf{j}+\frac{\partial}{\partial z}F(u)\,\mathbf{k} \\
&= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial z}\mathbf{k} \\
&= \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial u}{\partial z}\mathbf{k}\right) \\
&= F'(u)\,\nabla g
\end{aligned}$$

8.

$$\begin{aligned}
\nabla\frac{f}{g} &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\frac{f}{g}\\
&= \frac{\partial}{\partial x}\frac{f}{g}\mathbf{i}+\frac{\partial}{\partial y}\frac{f}{g}\mathbf{j}+\frac{\partial}{\partial z}\frac{f}{g}\mathbf{k}\\
&= \frac{gf_x-fg_x}{g^2}\mathbf{i}+\frac{gf_y-fg_y}{g^2}\mathbf{j}+\frac{gf_z-fg_z}{g^2}\mathbf{k}\\
&= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)-f\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}+\frac{\partial g}{\partial z}\mathbf{k}\right)\right]\\
&= \frac{1}{g^2}\left(g\nabla f-f\nabla g\right)
\end{aligned}$$

9. (a) If $f(x,y,z)=w=x^3y-y^3z$ then H is given by

$$H=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)=\begin{bmatrix} w_{xx} & w_{xy} & w_{xz}\\ w_{yx} & w_{yy} & w_{yz}\\ w_{zx} & w_{zy} & w_{zz}\end{bmatrix}=\begin{bmatrix} 6xy & 3x^2 & 0\\ 3x^2 & -6yz & -3y^2\\ 0 & -3y^2 & 0\end{bmatrix}$$

If $f(x,y,z)=w=x_1^2+2x_1x_2+5x_1x_3+2x_2x_1+4x_2^2+x_2x_3+5x_3x_1+x_3x_2+2x_3^2$ then H is given by

$$H=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)=\begin{bmatrix} w_{x_1x_1} & w_{x_1x_2} & w_{x_1x_3}\\ w_{x_2x_1} & w_{x_2x_2} & w_{x_2x_3}\\ w_{x_3x_1} & w_{x_3x_2} & w_{x_3x_3}\end{bmatrix}=\begin{bmatrix} 2 & 4 & 10\\ 4 & 8 & 2\\ 10 & 2 & 4\end{bmatrix}$$

(b) As long as the function $f(x_1,\ldots,x_n)$ has continuous second partial derivatives then $\partial^2 f/(\partial x_i\partial x_j)=\partial^2 f/(\partial x_j\partial x_i)$, which implies that H will be symmetric.

(c) As discussed in Section 2.14, the directional derivative of a function $f(x,y)$ in a given direction can be written as

$$\nabla_\alpha f=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\sin\alpha=\nabla f\cdot\mathbf{u}$$

where $\mathbf{u}=\cos\alpha\mathbf{i}+\sin\alpha\mathbf{j}$ is a unit vector that makes an angle α with the positive x-axis. Hence,

$$\begin{aligned}
\nabla_\alpha\nabla_\beta f &= \nabla_\alpha\left(\frac{\partial f}{\partial x}\cos\beta+\frac{\partial f}{\partial y}\sin\beta\right)\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cos\beta+\frac{\partial f}{\partial y}\sin\beta\right)\cos\alpha+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\cos\beta+\frac{\partial f}{\partial y}\sin\beta\right)\sin\alpha\\
&= \cos\beta\left(\frac{\partial^2 f}{\partial x^2}\cos\alpha+\frac{\partial^2 f}{\partial x\partial y}\sin\alpha\right)+\sin\beta\left(\frac{\partial^2 f}{\partial y\partial x}\cos\alpha+\frac{\partial^2 f}{\partial y^2}\sin\alpha\right)\\
&= \begin{bmatrix}\cos\beta & \sin\beta\end{bmatrix}\begin{bmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{bmatrix}\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix}\\
&= \begin{bmatrix}\cos\beta & \sin\beta\end{bmatrix}H\begin{bmatrix}\cos\alpha & \sin\alpha\end{bmatrix}^\top
\end{aligned}$$

Section 3.6

1.

$$
\begin{aligned}
\nabla \cdot (\mathbf{u}+\mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left[(u_x+v_x)\,\mathbf{i}+(u_y+v_y)\,\mathbf{j}+(u_z+v_z)\,\mathbf{k}\right] \\
&= \frac{\partial}{\partial x}(u_x+v_x)+\frac{\partial}{\partial y}(u_y+v_y)+\frac{\partial}{\partial z}(u_z+v_z) \\
&= \left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right) \\
&= \nabla\cdot\mathbf{u}+\nabla\cdot\mathbf{v}
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot(f\mathbf{u}) &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot(fu_x\mathbf{i}+fu_y\mathbf{j}+fu_z\mathbf{k}) \\
&= \frac{\partial}{\partial x}(fu_x)+\frac{\partial}{\partial y}(fu_y)+\frac{\partial}{\partial z}(fu_z) \\
&= f\frac{\partial u_x}{\partial x}+u_x\frac{\partial f}{\partial x}+f\frac{\partial u_y}{\partial y}+u_y\frac{\partial f}{\partial y}+f\frac{\partial u_z}{\partial z}+u_z\frac{\partial f}{\partial z} \\
&= f\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\left(u_x\frac{\partial f}{\partial x}+u_y\frac{\partial f}{\partial y}+u_z\frac{\partial f}{\partial z}\right) \\
&= f\,(\nabla\cdot\mathbf{u})+(\nabla f\cdot\mathbf{u})
\end{aligned}
$$

2. Recognizing that $\mathbf{v}=\rho\mathbf{u}$ and using (3.22), then (3.17) can be written as

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{v}=\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=\frac{\partial\rho}{\partial t}+(\nabla\rho\cdot\mathbf{u})+\rho\,(\nabla\cdot\mathbf{u})=0$$

According to Problem 12 of Section 2.8, the first two terms can be written as

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{u}=\frac{\partial\rho}{\partial t}+u_x\frac{\partial\rho}{\partial x}+u_y\frac{\partial\rho}{\partial y}+u_z\frac{\partial\rho}{\partial z}=\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}\frac{dx}{dt}+\frac{\partial\rho}{\partial y}\frac{dy}{dt}+\frac{\partial\rho}{\partial z}\frac{dz}{dt}=\frac{d\rho}{dt}=\frac{D\rho}{Dt}$$

Hence, (3.17) can be written as

$$\frac{\partial\rho}{\partial t}+(\nabla\rho\cdot\mathbf{u})+\rho\,(\nabla\cdot\mathbf{u})=\frac{D\rho}{Dt}+\rho\,(\nabla\cdot\mathbf{u})=0$$

When $\rho\equiv a$, where a is some arbitrary constant, then $D\rho/dt\equiv Da/dt=0$ and the equation above reduces to

$$\rho\,(\nabla\cdot\mathbf{u})\equiv a\,(\nabla\cdot\mathbf{u})=0$$

Since $\rho\equiv a\neq 0$, the only way for this equation to make sense is if $\nabla\cdot\mathbf{u}=0$.

3.

$$
\begin{aligned}
\nabla \times (\mathbf{u} + \mathbf{v}) &= \left[\frac{\partial}{\partial y} (u_z + v_z) - \frac{\partial}{\partial z} (u_y + v_y) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} (u_x + v_x) - \frac{\partial}{\partial x} (u_z + v_z) \right] \mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x} (u_y + v_y) - \frac{\partial}{\partial y} (u_x + v_x) \right] \mathbf{k} \\
&= \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \mathbf{k} \right] \\
&\quad + \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k} \right] \\
&= (\nabla \times \mathbf{u}) + (\nabla \times \mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times (f\mathbf{u}) &= \left[\frac{\partial}{\partial y} (f u_z) - \frac{\partial}{\partial z} (f u_y) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} (f u_x) - \frac{\partial}{\partial x} (f u_z) \right] \mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x} (f u_y) - \frac{\partial}{\partial y} (f u_x) \right] \mathbf{k} \\
&= \left[f \frac{\partial u_z}{\partial y} + u_z \frac{\partial f}{\partial y} - \left(f \frac{\partial u_y}{\partial z} + u_y \frac{\partial f}{\partial z} \right) \right] \mathbf{i} + \left[f \frac{\partial u_x}{\partial z} + u_x \frac{\partial f}{\partial z} - \left(f \frac{\partial u_z}{\partial x} + u_z \frac{\partial f}{\partial x} \right) \right] \mathbf{j} \\
&\quad + \left[f \frac{\partial u_y}{\partial x} + u_y \frac{\partial f}{\partial x} - \left(f \frac{\partial u_x}{\partial y} + u_x \frac{\partial f}{\partial y} \right) \right] \mathbf{k} \\
&= f \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \mathbf{k} \right] \\
&\quad + \left[\left(u_z \frac{\partial f}{\partial y} - u_y \frac{\partial f}{\partial z} \right) \mathbf{i} + \left(u_x \frac{\partial f}{\partial z} - u_z \frac{\partial f}{\partial x} \right) \mathbf{j} + \left(u_y \frac{\partial f}{\partial x} - u_x \frac{\partial f}{\partial y} \right) \mathbf{k} \right] \\
&= (f \nabla \times \mathbf{u}) + (\nabla f \times \mathbf{u})
\end{aligned}
$$

4.

$$
\begin{aligned}
\nabla \times (\nabla f) &= \left[\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) - \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial y} \right) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} \left(\frac{\partial f}{\partial x} \right) - \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z} \right) \right] \mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \right] \mathbf{k} \\
&= \left(\frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} \right) \mathbf{k} \\
&= \mathbf{0}
\end{aligned}
$$

5. (a) If $\mathbf{v} = 2xyz\mathbf{i} + x^2 z\mathbf{j} + x^2 y\mathbf{k}$, then

$$
\begin{aligned}
\nabla \times \mathbf{v} &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k} \\
&= \left(x^2 - x^2 \right) \mathbf{i} + (2xy - 2xy) \mathbf{j} + (2xz - 2xz) \mathbf{k} \\
&= \mathbf{0}
\end{aligned}
$$

Let $f = x^2yz + a$, where a is an arbitrary constant. Then $\nabla f = \mathbf{v}$.

(b) If $\mathbf{v} = e^{xy}[(2y^2 + yz^2)\mathbf{i} + (2xy + xz^2 + 2)\mathbf{j} + 2z\mathbf{k}]$, then

$$\begin{aligned}\nabla \times \mathbf{v} &= (2xze^{xy} - 2xze^{xy})\,\mathbf{i} + (2yze^{xy} - 2yze^{xy})\,\mathbf{j} \\ &\quad + \left[ye^{xy}\left(2xy + xz^2 + 2\right) + e^{xy}\left(2y + z^2\right) - xe^{xy}\left(2y^2 + yz^2\right) - e^{xy}\left(4y + z^2\right)\right]\mathbf{k} \\ &= \mathbf{0}\end{aligned}$$

Let $f = e^{xy}(2y + z^2) + a$, where a is an arbitrary constant. Then $\nabla f = \mathbf{v}$.

6.

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\ &\quad \cdot \left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\ &= \frac{\partial^2 v_z}{\partial y \partial x} - \frac{\partial^2 v_y}{\partial z \partial x} + \frac{\partial^2 v_x}{\partial z \partial y} - \frac{\partial^2 v_z}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_x}{\partial y \partial z} \\ &= 0\end{aligned}$$

7. (a) If $\mathbf{v} = 2x\mathbf{i} + y\mathbf{j} - 3z\mathbf{k}$, then

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 2 + 1 - 3 = 0$$

Since $\nabla \cdot \mathbf{v} = 0$, the vector $\mathbf{v} = \nabla \times \mathbf{u}$ for some vector $\mathbf{u}$. Furthermore, since by (3.32) $\nabla \times (\nabla f) = \mathbf{0}$, we can safely assume that $\mathbf{u}$ is of the form $\mathbf{u} = \mathbf{u}_0 + \nabla f$, where f is an arbitrary scalar function and $\mathbf{u}_0$ is any one vector whose curl is $\mathbf{v}$, as then $\nabla \times \mathbf{u} = \nabla \times (\mathbf{u}_0 + \nabla f) = (\nabla \times \mathbf{u}_0) + [\nabla \times (\nabla f)] = \nabla \times \mathbf{u}_0$.
Next, assume $\mathbf{u}_0 \cdot \mathbf{k} = 0$, which implies $u_{0z} = 0$. Equating the components of $\nabla \times \mathbf{u}_0$ to those of $\mathbf{v}$ then gives

$$\frac{\partial u_{0z}}{\partial y} - \frac{\partial u_{0y}}{\partial z} = -\frac{\partial u_{0y}}{\partial z} = 2x, \quad \frac{\partial u_{0x}}{\partial z} - \frac{\partial u_{0z}}{\partial x} = \frac{\partial u_{0x}}{\partial z} = y, \quad \frac{\partial u_{0y}}{\partial x} - \frac{\partial u_{0x}}{\partial y} = -3z$$

from which we may deduce that $\mathbf{u}_0 = yz\mathbf{i} - 2xz\mathbf{j}$.

(b) If $\mathbf{v} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, then going through the exact same steps as for part (a) gives $\mathbf{u}_0 = (z^2/2)\mathbf{i} + [(x^2 - 2yz)/2]\mathbf{j}$.

8.

$$
\begin{aligned}
\operatorname{div}\operatorname{grad} f = \nabla\cdot(\nabla f) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z} \\
&= \nabla^2 f \\
&= \Delta f
\end{aligned}
$$

Let $f = 1/\sqrt{x^2+y^2+z^2}$. Then

$$
\nabla^2 f = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}} - \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} = 0
$$

9.

$$
\begin{aligned}
\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left[\left(u_y v_z - u_z v_y\right)\mathbf{i} + \left(u_z v_x - u_x v_z\right)\mathbf{j} + \left(u_x v_y - u_y v_x\right)\mathbf{k}\right] \\
&= \frac{\partial}{\partial x}\left(u_y v_z - u_z v_y\right) + \frac{\partial}{\partial y}\left(u_z v_x - u_x v_z\right) + \frac{\partial}{\partial z}\left(u_x v_y - u_y v_x\right) \\
&= u_y\frac{\partial v_z}{\partial x} + v_z\frac{\partial u_y}{\partial x} - u_z\frac{\partial v_y}{\partial x} - v_y\frac{\partial u_z}{\partial x} + u_z\frac{\partial v_x}{\partial y} + v_x\frac{\partial u_z}{\partial y} - u_x\frac{\partial v_z}{\partial y} - v_z\frac{\partial u_x}{\partial y} \\
&\quad + u_x\frac{\partial v_y}{\partial z} + v_y\frac{\partial u_x}{\partial z} - u_y\frac{\partial v_x}{\partial z} - v_x\frac{\partial u_y}{\partial z} \\
&= \left(v_x\frac{\partial u_z}{\partial y} - v_x\frac{\partial u_y}{\partial z} + v_y\frac{\partial u_x}{\partial z} - v_y\frac{\partial u_z}{\partial x} + v_z\frac{\partial u_y}{\partial x} - v_z\frac{\partial u_x}{\partial y}\right) \\
&\quad + \left(u_x\frac{\partial v_y}{\partial z} - u_x\frac{\partial v_z}{\partial y} + u_y\frac{\partial v_z}{\partial x} - u_y\frac{\partial v_x}{\partial z} + u_z\frac{\partial v_x}{\partial y} - u_z\frac{\partial v_y}{\partial x}\right) \\
&= \left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right)\cdot\left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&\quad - \left(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}\right)\cdot\left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\
&= \mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v})
\end{aligned}
$$

10.

$$
\begin{aligned}
\nabla \times (\nabla \times \mathbf{u}) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \\
&\quad \times \left[\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}\right] \\
&= \left[\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\right]\mathbf{i} \\
&\quad + \left[\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\right]\mathbf{k} \\
&= \left(\frac{\partial^2 u_y}{\partial x \partial y} - \frac{\partial^2 u_x}{\partial y^2} - \frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 u_z}{\partial y \partial z} - \frac{\partial^2 u_y}{\partial z^2} - \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_x}{\partial y \partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial z \partial x} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_y}{\partial z \partial y}\right)\mathbf{k} \\
&= \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} - \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial^2 u_x}{\partial y^2} - \frac{\partial^2 u_x}{\partial z^2}\right)\mathbf{i} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial y \partial x} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z} - \frac{\partial^2 u_y}{\partial x^2} - \frac{\partial^2 u_y}{\partial y^2} - \frac{\partial^2 u_y}{\partial z^2}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial^2 u_x}{\partial z \partial x} + \frac{\partial^2 u_y}{\partial z \partial y} + \frac{\partial^2 u_z}{\partial z^2} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2} - \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k} \\
&= \left[\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 u_x}{\partial y \partial x} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z}\right)\mathbf{j}\right. \\
&\quad \left. + \left(\frac{\partial^2 u_x}{\partial z \partial x} + \frac{\partial^2 u_y}{\partial z \partial y} + \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k}\right] \\
&\quad - \left[\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right)\mathbf{j}\right. \\
&\quad \left. + \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right)\mathbf{k}\right] \\
&= \nabla\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \left(\nabla^2 u_x \mathbf{i} + \nabla^2 u_y \mathbf{j} + \nabla^2 u_z \mathbf{k}\right) \\
&= \nabla\left(\nabla \cdot \mathbf{u}\right) - \nabla^2 \mathbf{u}
\end{aligned}
$$

11. (a)

$$\begin{aligned}\nabla\cdot[\mathbf{u}\times(\mathbf{v}\times\mathbf{w})] &= \nabla\cdot\underbrace{[(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}-(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}]}_{(1.19)}\\ &= \underbrace{\nabla\cdot[(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}]-\nabla\cdot[(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}]}_{(3.21)}\\ &= \underbrace{(\mathbf{u}\cdot\mathbf{w})\,(\nabla\cdot\mathbf{v})+[\nabla\,(\mathbf{u}\cdot\mathbf{w})]\cdot\mathbf{v}}_{(3.22)}-(\mathbf{u}\cdot\mathbf{v})\,(\nabla\cdot\mathbf{w})-[\nabla\,(\mathbf{u}\cdot\mathbf{v})]\cdot\mathbf{w}\\ &= (\mathbf{u}\cdot\mathbf{w})\,(\nabla\cdot\mathbf{v})-(\mathbf{u}\cdot\mathbf{v})\,(\nabla\cdot\mathbf{w})+[\nabla\,(\mathbf{u}\cdot\mathbf{w})]\cdot\mathbf{v}-[\nabla\,(\mathbf{u}\cdot\mathbf{v})]\cdot\mathbf{w}\end{aligned}$$

(b)

$$\begin{aligned}\nabla\cdot[(\nabla f)\times(f\nabla g)] &= \underbrace{(f\nabla g)\cdot[\nabla\times(\nabla f)]-(\nabla f)\cdot[\nabla\times(f\nabla g)]}_{(3.35)}\\ &= (f\nabla g)\cdot\underbrace{\mathbf{0}}_{(3.31)}-(\nabla f)\cdot[\nabla\times(f\nabla g)]\\ &= -\,(\nabla f)\cdot[\nabla\times(f\nabla g)]\\ &= -\,(\nabla f)\cdot\underbrace{[f\,(\nabla\times(\nabla g))+(\nabla f)\times(\nabla g)]}_{(3.28)}\\ &= -\,(\nabla f)\cdot\left[f\,\underbrace{\mathbf{0}}_{(3.31)}+(\nabla f)\times(\nabla g)\right]\\ &= -\,(\nabla f)\cdot(\nabla f)\times(\nabla g)\\ &= (\nabla f)\cdot(\nabla g)\times(\nabla f)\\ &= \underbrace{(\nabla g)\cdot(\nabla f)\times(\nabla f)}_{(1.34)}\\ &= (\nabla g)\cdot\underbrace{\mathbf{0}}_{(1.19)}\\ &= 0\end{aligned}$$

(c)

$$\begin{aligned}\nabla\times[(\nabla\times\mathbf{v})+\nabla f] = \underbrace{\nabla\times(\nabla\times\mathbf{v})+\nabla\times(\nabla f)}_{(3.27)} &= \nabla\times(\nabla\times\mathbf{v})+\underbrace{\mathbf{0}}_{(3.31)}\\ &= \nabla\times(\nabla\times\mathbf{v})\end{aligned}$$

(d)

$$\nabla^2 f=\mathbf{0}+\nabla^2 f=\nabla\times\nabla\cdot\mathbf{v}+\nabla\cdot\nabla f=\underbrace{\nabla\cdot\nabla\times\mathbf{v}}_{(1.34)}+\nabla\cdot\nabla f=\underbrace{\nabla\cdot[(\nabla\times\mathbf{v})+\nabla f]}_{(3.21)}$$

12. (a) Let $\mathbf{u}$ be a unit vector, such that

$$\mathbf{u} = \frac{u_x}{|\mathbf{u}|}\mathbf{i} + \frac{u_y}{|\mathbf{u}|}\mathbf{j} + \frac{u_z}{|\mathbf{u}|}\mathbf{k} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

That is, $u_x/|\mathbf{u}|$, $u_y|\mathbf{u}|$, $u_z/|\mathbf{u}|$ are, by Section 1.2, simply the direction cosines of $\mathbf{u}$. Hence by (2.114),

$$(\mathbf{u}\cdot\nabla) f = \frac{u_x}{|\mathbf{u}|}\frac{\partial f}{\partial x} + \frac{u_y}{|\mathbf{u}|}\frac{\partial f}{\partial y} + \frac{u_z}{|\mathbf{u}|}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma = \nabla_u f$$

(b)

$$[(\mathbf{i}-\mathbf{j})\cdot\nabla] f = (\mathbf{i}-\mathbf{j})\cdot(\nabla f) = (\mathbf{i}-\mathbf{j})\cdot\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$$

(c) Let $\mathbf{v} = x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}$. Then

$$[(x\mathbf{i} - y\mathbf{j})\cdot\nabla]\left(x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}\right) = x\frac{\partial \mathbf{v}}{\partial x} - y\frac{\partial \mathbf{v}}{\partial y} = 2\left(x^2\mathbf{i} + y^2\mathbf{j}\right)$$

13.

$$\begin{aligned}
\nabla(\mathbf{u}\cdot\mathbf{v}) &= \nabla(u_x v_x + u_y v_y + u_z v_z) \\
&= \nabla(u_x v_x) + \nabla(u_y v_y) + \nabla(u_z v_z) \\
&= u_x\nabla v_x + v_x\nabla u_x + u_y\nabla v_y + v_y\nabla u_y + u_z\nabla v_z + v_z\nabla u_z \\
&= (u_x\nabla v_x + u_y\nabla v_y + u_z\nabla v_z) + (v_x\nabla u_x + v_y\nabla u_y + v_z\nabla u_z)
\end{aligned}$$

Let us for a moment focus on the first three terms $u_x\nabla v_x + u_y\nabla v_y + u_z\nabla v_z = \mathbf{a}$.

Expanding these gives

$$
\begin{aligned}
\mathbf{a} &= u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i}+\frac{\partial v_x}{\partial y}\mathbf{j}+\frac{\partial v_x}{\partial z}\mathbf{k}\right)+u_y\left(\frac{\partial v_y}{\partial x}\mathbf{i}+\frac{\partial v_y}{\partial y}\mathbf{j}+\frac{\partial v_y}{\partial z}\mathbf{k}\right)+u_z\left(\frac{\partial v_z}{\partial x}\mathbf{i}+\frac{\partial v_z}{\partial y}\mathbf{j}+\frac{\partial v_z}{\partial z}\mathbf{k}\right)\\
&= u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i}+\frac{\partial v_x}{\partial y}\mathbf{j}+\frac{\partial v_x}{\partial z}\mathbf{k}\right)+u_y\left(\frac{\partial v_y}{\partial x}\mathbf{i}+\frac{\partial v_y}{\partial y}\mathbf{j}+\frac{\partial v_y}{\partial z}\mathbf{k}\right)+u_z\left(\frac{\partial v_z}{\partial x}\mathbf{i}+\frac{\partial v_z}{\partial y}\mathbf{j}+\frac{\partial v_z}{\partial z}\mathbf{k}\right)\\
&\quad+u_x\left(\frac{\partial v_y}{\partial x}\mathbf{j}-\frac{\partial v_y}{\partial x}\mathbf{j}+\frac{\partial v_z}{\partial x}\mathbf{k}-\frac{\partial v_z}{\partial x}\mathbf{k}\right)+u_y\left(\frac{\partial v_x}{\partial y}\mathbf{i}-\frac{\partial v_x}{\partial y}\mathbf{i}+\frac{\partial v_z}{\partial y}\mathbf{k}-\frac{\partial v_z}{\partial y}\mathbf{k}\right)\\
&\quad+u_z\left(\frac{\partial v_x}{\partial z}\mathbf{i}-\frac{\partial v_x}{\partial z}\mathbf{i}+\frac{\partial v_y}{\partial z}\mathbf{j}-\frac{\partial v_y}{\partial z}\mathbf{j}\right)\\
&= u_x\left(\frac{\partial v_x}{\partial x}\mathbf{i}+\frac{\partial v_y}{\partial x}\mathbf{j}+\frac{\partial v_z}{\partial x}\mathbf{k}\right)+u_y\left(\frac{\partial v_x}{\partial y}\mathbf{i}+\frac{\partial v_y}{\partial y}\mathbf{j}+\frac{\partial v_z}{\partial y}\mathbf{k}\right)+u_z\left(\frac{\partial v_x}{\partial z}\mathbf{i}+\frac{\partial v_y}{\partial z}\mathbf{j}+\frac{\partial v_z}{\partial z}\mathbf{k}\right)\\
&\quad+\left[u_y\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)-u_z\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right]\mathbf{i}+\left[u_z\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-u_x\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right]\mathbf{j}\\
&\quad+\left[u_x\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)-u_y\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\right]\mathbf{k}\\
&= u_x\frac{\partial \mathbf{v}}{\partial x}+u_y\frac{\partial \mathbf{v}}{\partial y}+u_z\frac{\partial \mathbf{v}}{\partial z}+\left\{\mathbf{u}\times\left[\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\mathbf{k}\right]\right\}\\
&= (\mathbf{u}\cdot\nabla)\,\mathbf{v}+[\mathbf{u}\times(\nabla\times\mathbf{v})]
\end{aligned}
$$

Then, clearly

$$
v_x\nabla u_x+v_y\nabla u_y+v_z\nabla u_z=(\mathbf{v}\cdot\nabla)\,\mathbf{u}+[\mathbf{v}\times(\nabla\times\mathbf{u})]
$$

And so we may conclude that

$$
\nabla(\mathbf{u}\cdot\mathbf{v})=(\mathbf{u}\cdot\nabla)\,\mathbf{v}+[\mathbf{u}\times(\nabla\times\mathbf{v})]+(\mathbf{v}\cdot\nabla)\,\mathbf{u}+[\mathbf{v}\times(\nabla\times\mathbf{u})]
$$

14.

$$
\begin{aligned}
\nabla \times (\mathbf{u} \times \mathbf{v}) &= \nabla \times \left[(u_y v_z - u_z v_y)\,\mathbf{i} + (u_z v_x - u_x v_z)\,\mathbf{j} + (u_x v_y - u_y v_x)\,\mathbf{k} \right] \\
&= \left[\nabla \times (u_y v_z \mathbf{i} + u_z v_x \mathbf{j} + u_x v_y \mathbf{k}) \right] - \left[\nabla \times (u_z v_y \mathbf{i} + u_x v_z \mathbf{j} + u_y v_x \mathbf{k}) \right] \\
&= \left[\frac{\partial}{\partial y}(u_x v_y) - \frac{\partial}{\partial z}(u_z v_x) \right] \mathbf{i} + \left[\frac{\partial}{\partial z}(u_y v_z) - \frac{\partial}{\partial x}(u_x v_y) \right] \mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x}(u_z v_x) - \frac{\partial}{\partial y}(u_y v_z) \right] \mathbf{k} - \left[\frac{\partial}{\partial y}(u_y v_x) - \frac{\partial}{\partial z}(u_x v_z) \right] \mathbf{i} \\
&\quad - \left[\frac{\partial}{\partial z}(u_z v_y) - \frac{\partial}{\partial x}(u_y v_x) \right] \mathbf{j} - \left[\frac{\partial}{\partial x}(u_x v_z) - \frac{\partial}{\partial y}(u_z v_y) \right] \mathbf{k} \\
&= \left(u_x \frac{\partial v_y}{\partial y} + v_y \frac{\partial u_x}{\partial y} - u_z \frac{\partial v_x}{\partial z} - v_x \frac{\partial u_z}{\partial z} - u_y \frac{\partial v_x}{\partial y} - v_x \frac{\partial u_y}{\partial y} + u_x \frac{\partial v_z}{\partial z} + v_z \frac{\partial u_x}{\partial z} \right) \mathbf{i} \\
&\quad + \left(u_y \frac{\partial v_z}{\partial z} + v_z \frac{\partial u_y}{\partial z} - u_x \frac{\partial v_y}{\partial x} - v_y \frac{\partial u_x}{\partial x} - u_z \frac{\partial v_y}{\partial z} - v_y \frac{\partial u_z}{\partial z} + u_y \frac{\partial v_x}{\partial x} + v_x \frac{\partial u_y}{\partial x} \right) \mathbf{j} \\
&\quad + \left(u_z \frac{\partial v_x}{\partial x} + v_x \frac{\partial u_z}{\partial x} - u_y \frac{\partial v_z}{\partial y} - v_z \frac{\partial u_y}{\partial y} - u_x \frac{\partial v_z}{\partial x} - v_z \frac{\partial u_x}{\partial x} + u_z \frac{\partial v_y}{\partial y} + v_y \frac{\partial u_z}{\partial y} \right) \mathbf{k} \\
&= \left(u_x \frac{\partial v_y}{\partial y} - u_z \frac{\partial v_x}{\partial z} - u_y \frac{\partial v_x}{\partial y} + u_x \frac{\partial v_z}{\partial z} \right) \mathbf{i} + \left(u_y \frac{\partial v_z}{\partial z} - u_x \frac{\partial v_y}{\partial x} - u_z \frac{\partial v_y}{\partial z} + u_y \frac{\partial v_x}{\partial x} \right) \mathbf{j} \\
&\quad + \left(u_z \frac{\partial v_x}{\partial x} - u_y \frac{\partial v_z}{\partial y} - u_x \frac{\partial v_z}{\partial x} + u_z \frac{\partial v_y}{\partial y} \right) \mathbf{k} \\
&\quad + \left(v_y \frac{\partial u_x}{\partial y} - v_x \frac{\partial u_z}{\partial z} - v_x \frac{\partial u_y}{\partial y} + v_z \frac{\partial u_x}{\partial z} \right) \mathbf{i} + \left(v_z \frac{\partial u_y}{\partial z} - v_y \frac{\partial u_x}{\partial x} - v_y \frac{\partial u_z}{\partial z} + v_x \frac{\partial u_y}{\partial x} \right) \mathbf{j} \\
&\quad + \left(v_x \frac{\partial u_z}{\partial x} - v_z \frac{\partial u_y}{\partial y} - v_z \frac{\partial u_x}{\partial x} + v_y \frac{\partial u_z}{\partial y} \right) \mathbf{k}
\end{aligned}
$$

Let us for a moment focus on the first three terms

$$
\begin{aligned}
\mathbf{a} = &\left(u_x \frac{\partial v_y}{\partial y} - u_z \frac{\partial v_x}{\partial z} - u_y \frac{\partial v_x}{\partial y} + u_x \frac{\partial v_z}{\partial z} \right) \mathbf{i} + \left(u_y \frac{\partial v_z}{\partial z} - u_x \frac{\partial v_y}{\partial x} - u_z \frac{\partial v_y}{\partial z} + u_y \frac{\partial v_x}{\partial x} \right) \mathbf{j} \\
&+ \left(u_z \frac{\partial v_x}{\partial x} - u_y \frac{\partial v_z}{\partial y} - u_x \frac{\partial v_z}{\partial x} + u_z \frac{\partial v_y}{\partial y} \right) \mathbf{k}
\end{aligned}
$$

These may be further manipulated to get

$$
\begin{aligned}
\mathbf{a} &= \left(u_x \frac{\partial v_y}{\partial y} - u_z \frac{\partial v_x}{\partial z} - u_y \frac{\partial v_x}{\partial y} + u_x \frac{\partial v_z}{\partial z}\right)\mathbf{i} + \left(u_y \frac{\partial v_z}{\partial z} - u_x \frac{\partial v_y}{\partial x} - u_z \frac{\partial v_y}{\partial z} + u_y \frac{\partial v_x}{\partial x}\right)\mathbf{j} \\
&\quad + \left(u_z \frac{\partial v_x}{\partial x} - u_y \frac{\partial v_z}{\partial y} - u_x \frac{\partial v_z}{\partial x} + u_z \frac{\partial v_y}{\partial y}\right)\mathbf{k} + \left(u_x \frac{\partial v_x}{\partial x} - u_x \frac{\partial v_x}{\partial x}\right)\mathbf{i} + \left(u_y \frac{\partial v_y}{\partial y} - u_y \frac{\partial v_y}{\partial y}\right)\mathbf{j} \\
&\quad + \left(u_z \frac{\partial v_z}{\partial z} - u_z \frac{\partial v_z}{\partial z}\right)\mathbf{k} \\
&= u_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{i} + u_y \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{j} + u_z \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\mathbf{k} \\
&\quad - u_x \left(\frac{\partial v_x}{\partial x}\mathbf{i} + \frac{\partial v_y}{\partial x}\mathbf{j} + \frac{\partial v_z}{\partial x}\mathbf{k}\right) - u_y \left(\frac{\partial v_x}{\partial y}\mathbf{i} + \frac{\partial v_y}{\partial y}\mathbf{j} + \frac{\partial v_z}{\partial y}\mathbf{k}\right) - u_z \left(\frac{\partial v_x}{\partial z}\mathbf{i} + \frac{\partial v_y}{\partial z}\mathbf{j} + \frac{\partial v_z}{\partial z}\mathbf{k}\right) \\
&= (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) - u_z \frac{\partial \mathbf{v}}{\partial x} - u_y \frac{\partial \mathbf{v}}{\partial y} - u_z \frac{\partial \mathbf{v}}{\partial z} \\
&= \mathbf{u}\left(\nabla \cdot \mathbf{v}\right) - \left[\left(\mathbf{u} \cdot \nabla\right)\mathbf{v}\right]
\end{aligned}
$$

In a similar way it may be shown that the remaining three terms can be written as $-\mathbf{v}(\nabla \cdot \mathbf{u}) + [(\mathbf{v} \cdot \nabla)\mathbf{u}]$, and hence, we may conclude that

$$
\nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}\left(\nabla \cdot \mathbf{v}\right) - \mathbf{v}\left(\nabla \cdot \mathbf{u}\right) + \left[\left(\mathbf{v} \cdot \nabla\right)\mathbf{u}\right] - \left[\left(\mathbf{u} \cdot \nabla\right)\mathbf{v}\right]
$$

15. Let the sphere be given by $F(x, y, z) = x^2 + y^2 + z^2 = 9$. The unit outer normal vector to the sphere is then given by

$$
\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right)
$$

Next, let $\mathbf{u} = (x^2 - z^2)(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$. Then, with the help of (2.117)

$$
\begin{aligned}
\frac{\partial}{\partial n}\left(\nabla \cdot \mathbf{u}\right) = \nabla\left(\nabla \cdot \mathbf{u}\right) \cdot \mathbf{n} &= \nabla\left[\frac{\partial}{\partial x}\left(x^2 - z^2\right) - \frac{\partial}{\partial y}\left(x^2 - z^2\right) + 3\frac{\partial}{\partial z}\left(x^2 - z^2\right)\right] \cdot \mathbf{n} \\
&= \nabla\left(2x - 6z\right) \cdot \mathbf{n} \\
&= (2\mathbf{i} - 6\mathbf{k}) \cdot \frac{1}{\sqrt{x^2 + y^2 + z^2}}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right) \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}}\left(2x - 6z\right)
\end{aligned}
$$

Evaluating the result at the point $(2, 2, 1)$ then finally gives $-2/3$.

16. If a rigid body is rotating about the z-axis with angular velocity ω, then it is moving in a circular motion in the xy-plane. Hence, a particle of the body essentially follows a path equal to that of a point restricted to lie on a cylinder. Let r be the fixed radius

of the circle the path is constrained to move on in the xy-plane and let α be the initial angle of the particle in the xy-plane relative to the positive x-axis. Then, if ω is the angular velocity, at time t the particle will have moved through angle $\omega t + \alpha$. Since the particle is constrained to lie on the circle of radius r, its x-coordinate given by $r\cos(\omega t + \alpha)$ and its $y-$coordinate by $r\sin(\omega t + \alpha)$. As the particle is free to move in the z-plane, its z-coordinate is simply given by z. As such, a vector equation for the particle is given by

$$\overrightarrow{OP} = r\cos\left(\omega t + \alpha\right)\mathbf{i} + r\sin\left(\omega t + \alpha\right)\mathbf{j} + z\mathbf{k}$$

Next, let $\boldsymbol{\omega} = \omega\mathbf{k}$ be the angular velocity vector. Now the regular velocity of the particle is given by the vector $\mathbf{v}$, which is both perpendicular to the angular velocity vector (since by definition the angular velocity vector is perpendicular to the plane of rotation and hence, $\mathbf{v}$) and the position vector $\overrightarrow{OP}$. As such, it is given by

$$\begin{aligned}
\mathbf{v} &= \frac{d}{dt}\overrightarrow{OP} \\
&= -\omega r\sin\left(\omega t + \alpha\right)\mathbf{i} + \omega r\cos\left(\omega t + \alpha\right)\mathbf{j} \\
&= \left(\omega\mathbf{k}\right) \times \left[r\cos\left(\omega t + \alpha\right)\mathbf{i} + r\sin\left(\omega t + \alpha\right)\mathbf{j} + z\mathbf{k}\right] \\
&= \boldsymbol{\omega} \times \overrightarrow{OP}
\end{aligned}$$

Knowing this, the divergence and curl of $\mathbf{v}$ are given by

$$\begin{aligned}
\nabla \cdot \mathbf{v} = \nabla \cdot \left(\boldsymbol{\omega} \times \overrightarrow{OP}\right) &= \nabla \cdot \left[-\omega r\sin\left(\omega t + \alpha\right)\mathbf{i} + \omega r\cos\left(\omega t + \alpha\right)\mathbf{j}\right] \\
&= \nabla \cdot \left(-\omega y\mathbf{i} + \omega x\mathbf{j}\right) \\
&= \frac{\partial}{\partial x}\left(-\omega y\right) + \frac{\partial}{\partial y}\left(\omega x\right) \\
&= 0 \\
\nabla \times \mathbf{v} = \nabla \times \left(\boldsymbol{\omega} \times \overrightarrow{OP}\right) &= \nabla \times \left[-\omega r\sin\left(\omega t + \alpha\right)\mathbf{i} + \omega r\cos\left(\omega t + \alpha\right)\mathbf{j}\right] \\
&= -\omega\frac{\partial}{\partial z}r\cos\left(\omega t + \alpha\right)\mathbf{i} - \omega\frac{\partial}{\partial z}r\sin\left(\omega t + \alpha\right)\mathbf{j} \\
&\quad + \left[\omega\frac{\partial}{\partial x}r\cos\left(\omega t + \alpha\right) + \omega\frac{\partial}{\partial y}r\sin\left(\omega t + \alpha\right)\right]\mathbf{k} \\
&= -\omega\frac{\partial}{\partial z}x\mathbf{i} - \omega\frac{\partial}{\partial z}y\mathbf{j} + \left(\omega\frac{\partial}{\partial x}x + \omega\frac{\partial}{\partial y}y\right)\mathbf{k} \\
&= 2\omega\mathbf{k} \\
&= 2\boldsymbol{\omega}
\end{aligned}$$

17. Let a steady fluid in motion have the velocity vector $\mathbf{u} = y\mathbf{i} = d\mathbf{r}/dt$. Since $\mathbf{u}$ has no y or z components, the position of a point in the y and z directions does not

change with time (i.e. is constant). Hence, the position of a point at time t is given by $\mathbf{r}(t) = (c_2 t + c_1)\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, where c_1, c_2 and c_3 are some arbitrary constants. As such, the path of motion for each point of the vector field is a straight line when $c_2 \neq 0$. Furthermore, $\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \nabla \cdot (y\mathbf{i}) = (\partial/\partial x)y = 0$, and hence, the flow is incompressible. The relative rate of growth of a volume occupied by the fluid is roughly proportional to $\operatorname{div}\mathbf{u}$. To be exact; $\operatorname{div}\mathbf{u} = \lim_{\Delta t \to 0} \Delta V/(V\Delta t)$. Now since $\operatorname{div}\mathbf{u} = 0$ implies that $\Delta V = 0$, the volume occupied at time $t = 1$ will be the same as that at time $t = 0$, which is simply $V(t_0) = V(t_1) = 1$ for $t_0 = 0$ and $t_1 = 1$.

18. Let a steady fluid in motion have the velocity vector $\mathbf{u} = x\mathbf{i} = d\mathbf{r}/dt$. Since $\mathbf{u}$ has no y or z components (i.e. $dy/dt = 0$, $dz/dt = 0$), the position of a point in the y and z directions does not change with time. In other words, the position of a point has coordinates $y(t) = c_2$, $z(t) = c_3$, where c_2 and c_3 are arbitrary constants. For the x-coordinate however, we find that $dx/dt = x$, so that $x(t) = c_1 e^t$, where c_1 is the initial value of x at time $t = 0$. Hence, we find $\mathbf{r}(t) = c_1 e^t\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Furthermore, $\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \nabla \cdot (x\mathbf{i}) = (\partial/\partial x)x = 1$, and as such, the flow is *not* incompressible. Since

$$\operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = 1 = \frac{1}{V}\frac{dV}{dt} \qquad \Longrightarrow \qquad V(t) = V_0 e^t$$

The volume at $t = 0$ is $V(0) = V_0 = 1$. Hence, at time $t = 1$ the volume will be $V(1) = e$.

Section 3.8

1.