CHAPTER 5

Section 5.3

1. (a) From the given end points $(0,0)$, $(2,2)$ it follows that we can represent the curve C in the form $y = x$, $0 \leq x \leq 2$. Hence, by (5.6) we find

$$\int_{(0,0)}^{(2,2)} y^2\,dx = \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

(b) Given the end points $(2,1)$, $(1,2)$ we will parameterise the curve C according to: $x = 2 - t$, $y = 1 + t$, $0 \leq t \leq 1$. Then by (5.4) we find

$$\int_{(2,1)}^{(1,2)} y\,dx = -\int_0^1 (1+t)\;dt = -\left[t + \frac{t^2}{2}\right]_0^1 = -\frac{3}{2}$$

(c) Given the end points $(1,1)$, $(2,1)$ we will parameterise the curve C according to $x = 1 + t$, $y = 1$, $0 \leq t \leq 1$. Then by (5.5) we find

$$\int_{(1,1)}^{(2,1)} x\,dy = \int_0^1 (1+t)\,(0)\;dt = 0$$

2. (a) Let us represent the curve $C : x = \sqrt{1 - y^2}$ in the form $x = \cos t$, $y = \sin t$, $-\pi/2 \leq t \leq \pi/2$. Then by (5.4) and (5.5)

$$\begin{aligned}\int_{(0,-1)}^{(0,1)} y^2\,dx + x^2\,dy &= \int_{-\pi/2}^{\pi/2} -\sin^3 t\,dt + \cos^3 t\,dt \\ &= \int_{-\pi/2}^{\pi/2} -\left(1 - \cos^2 t\right)\sin t + \left(1 - \sin^2 t\right)\cos t\,dt \\ &= \left[\cos t - \frac{\cos^3 t}{3} + \sin t - \frac{\sin^3 t}{3}\right]_{-\pi/2}^{\pi/2} = \frac{4}{3}\end{aligned}$$

(b) Let C be the parabola $y = x^2$. Then by (5.6) and (5.7) we find

$$\int_{(0,0)}^{(2,4)} y\,dx + x\,dy = \int_0^2 \left(x^2 + 2x^2\right)\;dx = \left[\frac{x^3}{3} + \frac{2}{3}x^3\right]_0^2 = 8$$

(c) Let C be the curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq \pi/2$ and let us use the substitution $u = \tan^3 t$. Then by (5.4) and (5.5) we can rewrite the integral as

$$\begin{aligned}\int_{(1,0)}^{(0,1)} \frac{y\,dx - x\,dy}{x^2 + y^2} &= -3\int_0^{\pi/2} \frac{\sin^4 t\cos^2 t + \sin^2 t\cos^4 t}{\cos^6 t + \sin^6 t}\,dt = \int_0^{\pi/2} \frac{-3\sin^2 t\cos^2 t}{\cos^6 t + \sin^6 t}\,dt \\ &= -\int_0^{\infty} \frac{\cos^6 t}{\cos^6 t + \sin^6 t}\,du = -\int_0^{\infty} \frac{du}{1+u^2} = \lim_{b\to\infty} -\int_0^b \frac{du}{1+u^2} \\ &= \lim_{b\to\infty} -\tan^{-1} u\Big|_0^b = \lim_{b\to\infty} -\tan^{-1} b = -\frac{\pi}{2}\end{aligned}$$

3. (a) Let C be the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$. Then the integral

$$\oint_C y^2\,dx + xy\,dy$$

can be evaluated by computing the sum of the four integrals

$$\underbrace{\int_{(1,1)}^{(-1,1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(-1,1)}^{(-1,-1)} xy\,dy}_{dx=0} \qquad \underbrace{\int_{(-1,-1)}^{(1,-1)} y^2\,dx}_{dy=0} \qquad \underbrace{\int_{(1,-1)}^{(1,1)} xy\,dy}_{dx=0}$$

Hence,

$$\begin{aligned}\oint_C y^2\,dx + xy\,dy &= \int_1^{-1} dx - \int_1^{-1} y\,dy + \int_{-1}^{1} dx + \int_{-1}^{1} y\,dy \\ &= x\big|_1^{-1} - \left.\frac{y^2}{2}\right|_1^{-1} + x\big|_{-1}^{1} + \left.\frac{y^2}{2}\right|_{-1}^{1} = 0\end{aligned}$$

(b) Let C be the circle $x^2 + y^2 = 1$. Using the parameterization $x = \cos t$, $y = \sin t$ where $0 \le t \le 2\pi$, then by (5.4) and (5.5) the integral

$$\oint_C y\,dx - x\,dy$$

may be written as

$$\begin{aligned}\oint_C y\,dx - x\,dy = \int_0^{2\pi} -\sin^2 t\,dt - \cos^2 t\,dt = -\int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right)\,dt = -\int_0^{2\pi} dt \\ = -2\pi\end{aligned}$$

(c) Let C be the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. Then the integral

$$\oint_C x^2y^2\,dx - xy^3\,dy$$

can be evaluated by computing the sum of the three integrals

$$\underbrace{\int_{(0,0)}^{(1,0)} x^2y^2\,dx = 0}_{dy=0} \qquad \underbrace{-\int_{(1,0)}^{(1,1)} xy^3\,dy}_{dx=0} \qquad \int_{(1,1)}^{(0,0)} x^2y^2\,dx - xy^3\,dy$$

Hence,

$$\begin{aligned}\oint_C x^2y^2\,dx - xy^3\,dy &= -\int_0^1 y^3\,dy + \int_0^1 x^4\,dx - \int_0^1 y^4\,dy \\ &= -\left.\frac{y^4}{4}\right|_0^1 + \left.\frac{x^5}{5}\right|_0^1 - \left.\frac{y^5}{5}\right|_0^1 = -\frac{1}{4}\end{aligned}$$

4. (a) Let C be the circle $x^2 + y^2 = 4$. Then using the parametrisation $x = 4\cos t$, $y = 4\sin t$, where $0 \le t \le 2\pi$ and (5.12) the integral

$$\oint_C \left(x^2 - y^2\right)\, ds$$

may be written as

$$\oint_C \left(x^2 - y^2\right)\, ds = 64 \int_0^{2\pi} \left(\cos^2 t - \sin^2 t\right)\, dt = 64 \int_0^{2\pi} \cos 2t\, dt = 32 \sin 2t\Big|_0^{2\pi} = 0$$

(b) Let C be the line $y = x$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) the integral

$$\int_{(0,0)}^{(1,1)} x\, ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} x\, ds = \sqrt{2} \int_0^1 x\, dx = \frac{\sqrt{2}}{2} x^2 \Big|_0^1 = \frac{1}{\sqrt{2}}$$

(c) Let C be the parabola $y = x^2$ with endpoints $(0,0)$, $(1,1)$. Then by (5.14) and using the substitution $x = (1/2)\tan u$, such that $dx = (1/2)\sec^2 u\, du$ the integral

$$\int_{(0,0)}^{(1,1)} ds$$

may be written as

$$\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1 + 4x^2}\, dx = \frac{1}{2} \int_0^{\tan^{-1} 2} \sec^3 u\, du$$

In order to solve the integral on the right hand side, let us solve the indefinite integral

$$\begin{aligned} \int \sec^3 x\, dx = \int_0 \sec^2 x \sec x\, dx &= \sec x \tan x - \int \sec x \tan^2 x\, dx + C \\ &= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right)\, dx + C \\ &= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx + C \end{aligned}$$

Adding the term $\int \sec^3 x\, dx$ to both sides and dividing by two then gives

$$\begin{aligned} \int \sec^3 x\, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x\, dx + C \\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C \end{aligned}$$

Substituting in the original equation then gives

$$\begin{aligned}
\int_{(0,0)}^{(1,1)} ds = \int_0^1 \sqrt{1+4x^2}\, dx &= \frac{1}{2}\int_0^{\tan^{-1} 2} \sec^3 u\, du \\
&= \frac{1}{4}\sec u \tan u \Big|_0^{\tan^{-1} 2} + \frac{1}{4}\ln|\sec u + \tan u| \Big|_0^{\tan^{-1} 2} \\
&= \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5}+2\right)}{4}
\end{aligned}$$

5. Let a path $x = \phi(t)$, $y = \psi(t)$, $h \le t \le k$, where x and y are continuous and have continuous derivatives for $h \le t \le k$ like (5.1) be given. Next, let us make a change of parameter by the equation $t = g(\tau)$, $\alpha \le \tau \le \beta$, where $g'(\tau)$ is continuous and positive in the interval and $g(\alpha) = h$, $g(\beta) = k$. Then by (5.4) the line integral $\int_C f(x,y)\, dx$ on the path $x = \phi(g(\tau))$, $y = \psi(g(\tau))$, such that $dx = (d/d\tau)\phi(g(\tau))\, d\tau$, is given by

$$\begin{aligned}
\int\limits_C f(x,y)\, dx &= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right), \psi\left(g\left(\tau\right)\right)\right] \frac{d}{d\tau}\phi\left(g\left(\tau\right)\right)\, d\tau \\
&= \int_\alpha^\beta f\left[\phi\left(g\left(\tau\right)\right), \psi\left(g\left(\tau\right)\right)\right] \frac{d\phi}{dt}\frac{d}{d\tau} g\left(\tau\right)\, d\tau \\
&= \int_h^k f\left[\phi\left(t\right), \psi\left(t\right)\right] \frac{d\phi}{dt}\frac{dt}{d\tau}\, d\tau = \int_h^k f\left[\phi\left(t\right), \psi\left(t\right)\right] \phi'\left(t\right)\, dt
\end{aligned}$$

6. (a) Using (a), the integral $\int P\, dx + Q\, dy$ along the path $C \to ABFG$ may be approximated as

$$\begin{aligned}
\int\limits_C P\, dx + Q\, dy \sim &\left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right] \\
&+ \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right] = 7
\end{aligned}$$

(b) Using (a), the integral $\int P\, dx + Q\, dy$ along the path $C \to AFGKH$ may be approximated as

$$\begin{aligned}
\int\limits_C P\, dx + Q\, dy \sim &\left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(1+4)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 1 + \frac{1}{2}(4+6)\cdot 0\right] \\
&+ \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+9)\cdot 1\right] + \left[\frac{1}{2}(0+2)\cdot 1 + \frac{1}{2}(9+8)\cdot -1\right] \\
= 5 &
\end{aligned}$$

(c) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABCDHLSONMIEA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+8)\cdot 1 + \frac{1}{2}(2+3)\cdot 0\right] \\
& + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+4)\cdot 0\right] + \left[\frac{1}{2}(5+2)\cdot 0 + \frac{1}{2}(4+8)\cdot 1\right] \\
& + \left[\frac{1}{2}(2+1)\cdot 0 + \frac{1}{2}(8+2)\cdot 1\right] + \left[\frac{1}{2}(1+4)\cdot 0 + \frac{1}{2}(2+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(4+3)\cdot -1 + \frac{1}{2}(6+2)\cdot 0\right] + \left[\frac{1}{2}(3+7)\cdot -1 + \frac{1}{2}(2+8)\cdot 0\right] \\
& + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] \\
& + \left[\frac{1}{2}(8+3)\cdot 0 + \frac{1}{2}(3+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right] \\
= & \; 8
\end{aligned}$$

(d) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to AFJNMIJFA$ may be approximated as

$$\begin{aligned}
\int\limits_C P\,dx + Q\,dy \sim & \left[\frac{1}{2}(0+0)\cdot 1 + \frac{1}{2}(4+1)\cdot 1\right] + \left[\frac{1}{2}(0+5)\cdot 0 + \frac{1}{2}(4+6)\cdot 1\right] \\
& + \left[\frac{1}{2}(5+7)\cdot 0 + \frac{1}{2}(6+8)\cdot 1\right] + \left[\frac{1}{2}(7+2)\cdot -1 + \frac{1}{2}(8+4)\cdot 0\right] \\
& + \left[\frac{1}{2}(2+8)\cdot 0 + \frac{1}{2}(4+3)\cdot -1\right] + \left[\frac{1}{2}(8+5)\cdot 1 + \frac{1}{2}(3+6)\cdot 0\right] \\
& + \left[\frac{1}{2}(5+0)\cdot 0 + \frac{1}{2}(6+4)\cdot -1\right] + \left[\frac{1}{2}(0+0)\cdot -1 + \frac{1}{2}(4+1)\cdot -1\right] \\
= & \; \frac{11}{2}
\end{aligned}$$

(e) Using (a), the integral $\int P\,dx + Q\,dy$ along the path $C \to ABFEAEFBA$ may be approximated as

$$\int_C P\,dx + Q\,dy \sim \left[\frac{1}{2}(0+3)\cdot 1 + \frac{1}{2}(1+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+4)\cdot 1\right]$$

$$+ \left[\frac{1}{2}(0+3)\cdot -1 + \frac{1}{2}(4+2)\cdot 0\right] + \left[\frac{1}{2}(3+0)\cdot 0 + \frac{1}{2}(2+1)\cdot -1\right]$$

$$+ \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(1+2)\cdot 1\right] + \left[\frac{1}{2}(3+0)\cdot 1 + \frac{1}{2}(2+4)\cdot 0\right]$$

$$+ \left[\frac{1}{2}(0+3)\cdot 0 + \frac{1}{2}(4+2)\cdot -1\right] + \left[\frac{1}{2}(3+0)\cdot -1 + \frac{1}{2}(2+1)\cdot 0\right]$$

$$= 0$$

7. Let C be a smooth curve in the xy-plane and let $f(x,y) > 0$ be a continuous function defined over a region of the xy-plane containing the curve C. The equation $z = f(x,y)$ then is the equation of a surface that lies above the region of the xy-plane containing the curve C. Next, we imagine moving a straight line along C perpendicular to the xy-plane, effectively tracing out a "wall" standing on C, orthogonal to the xy-plane. This "wall" cuts the surface $z = f(x,y)$, forming a curve on it that lies above the curve C. In fact, the curve C may be interpreted as the projection of the surface curve onto the xy-plane. Using (5.11), the line integral

$$\int_C f(x,y)\;ds = \lim_{\substack{n\to\infty \\ \max \Delta_i s \to 0}} \sum_{i=1}^{n} f\left(x_i^*, y_i^*\right)\Delta_i s$$

then may be interpreted as an infinite sum of the length of each straight line directed from C to the surface curve lying above it in the limit where the distance Δs between each subsequent line becomes infinitely small, effectively tracing out a "wall" with height at each point (x,y) given by $f(x,y)$. This may be interpreted the as the area of the cylindrical surface $0 \le z \le f(x,y)$, (x,y) on C.

Section 5.5

1. Let the vector $v = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ be given. Then by (5.25) and (5.29)

   (a) The integral $\int_C v_T\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int_C v_T\,ds = \int_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6)\,(5.9)}{=} \int_0^1 2x^2\,dx + \int_0^1 2y^2\,dy = \frac{4}{3}$$

   (b) The integral $\int_C v_T\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

   $$\int_C v_T\,ds = \int_C \left(x^2+y^2\right)\,dx + 2xy\,dy \overset{(5.6)\,(5.7)}{=} \int_0^1 \left(x^2+5x^4\right)\,dx = \frac{4}{3}$$

(c) The integral $\int_C v_T\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\int\limits_C v_T\,ds = \int\limits_C \left(x^2+y^2\right)\,dx + 2xy\,dy$$
$$= \int_{(0,0)}^{(1,0)} \left(x^2+y^2\right)\,dx + 2xy\,dy + \int_{(1,0)}^{(1,1)} \left(x^2+y^2\right)\,dx + 2xy\,dy$$
$$= \int_0^1 x^2\,dx + \int_0^1 2y\,dy = \frac{4}{3}$$

2. Let $\mathbf{v} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be the same vector as given in Problem 1, and let $\mathbf{n}$ be the unit normal vector 90° behind the tangent vector $\mathbf{T}$ as given by (5.37). Then the normal component of $\mathbf{v}$ is given by $v_n = \mathbf{v}\cdot\mathbf{n} = (P\mathbf{i}+Q\mathbf{j})\cdot(y_s\mathbf{i} - x_s\mathbf{j}) = -Qx_s + Py_s$. Then by (5.25) and (5.29)

(a) The integral $\int_C v_n\,ds$ along the path $C \to y = x$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.9)}{=} \int_0^1 -2x^2\,dx + \int_0^1 2y^2\,dy = 0$$

(b) The integral $\int_C v_n\,ds$ along the path $C \to y = x^2$ from $(0,0)$ to $(1,1)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy \overset{(5.6)\,(5.7)}{=} \int_0^1 2x^5\,dx = \frac{1}{3}$$

(c) The integral $\int_C v_n\,ds$ along the broken line from $(0,0)$ to $(1,1)$ with corner at $(1,0)$ may be evaluated as

$$\int\limits_C v_n\,ds = \int\limits_C -2xy\,dx + \left(x^2+y^2\right)\,dy$$
$$= \int_{(0,0)}^{(1,0)} -2xy\,dx + \left(x^2+y^2\right)\,dy + \int_{(1,0)}^{(1,1)} -2xy\,dx + \left(x^2+y^2\right)\,dy$$
$$= \int_0^1 \left(1+y^2\right)\,dy = \frac{4}{3}$$

3.