CHAPTER 2

Section 2.4

1. Some examples are:

$$\begin{aligned} \text{volume of a cylinder}: &\quad \pi r^2 h \\ \text{surface area of a cone}: &\quad \pi r\,(l + r) \\ \text{volume of a cuboid}: &\quad lwh \end{aligned}$$

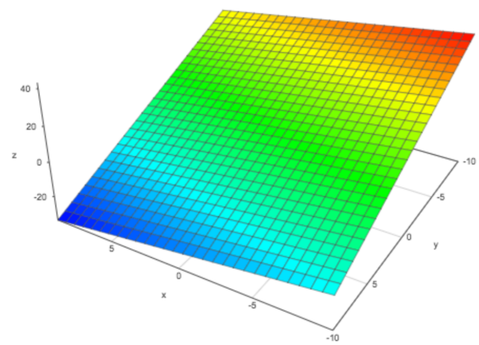

Figure 1: $z = 3 - x - 3y$

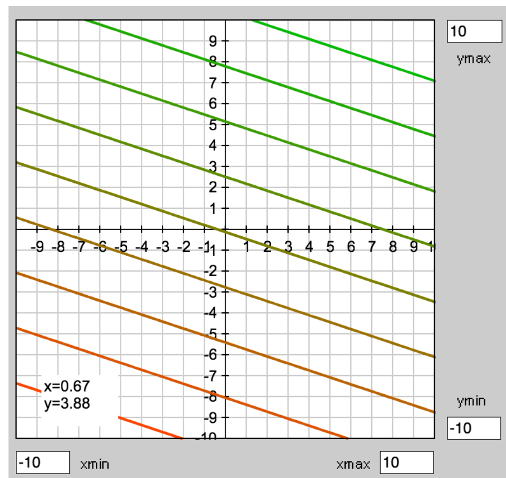

Figure 2: $z = 3 - x - 3y$

2. (a)

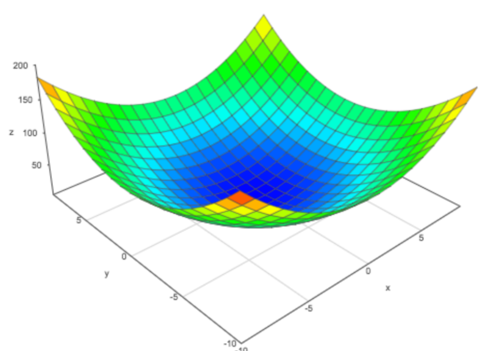

Figure 3: $z = x^2 + y^2 + 1$

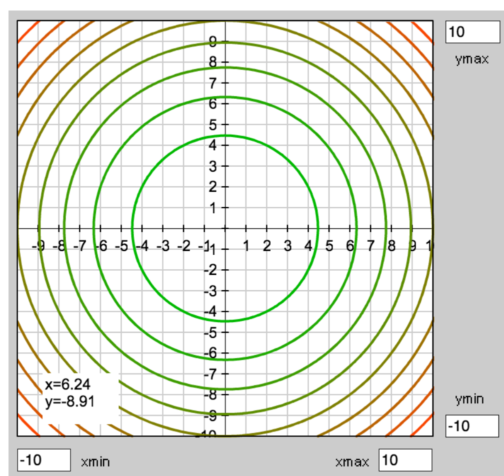

Figure 4: $z = x^2 + y^2 + 1$

(b)

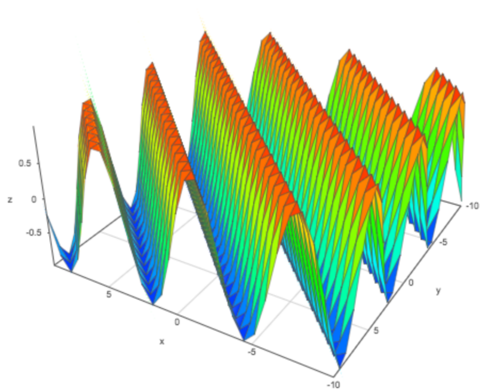

Figure 5: $z = \sin(x + y)$

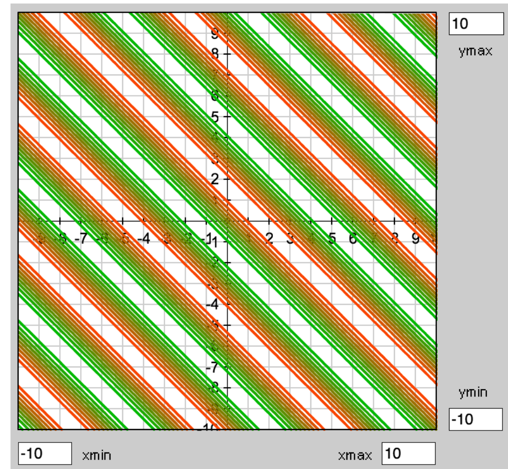

Figure 6: $z = \sin(x + y)$

(c)

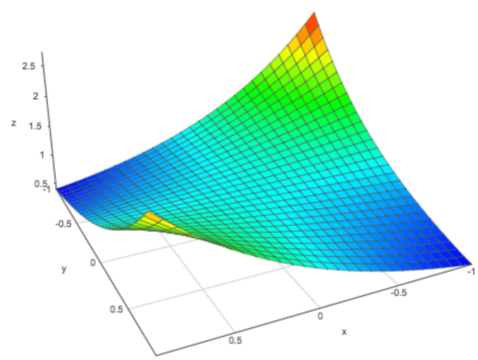

Figure 7: $z = e^{xy}$

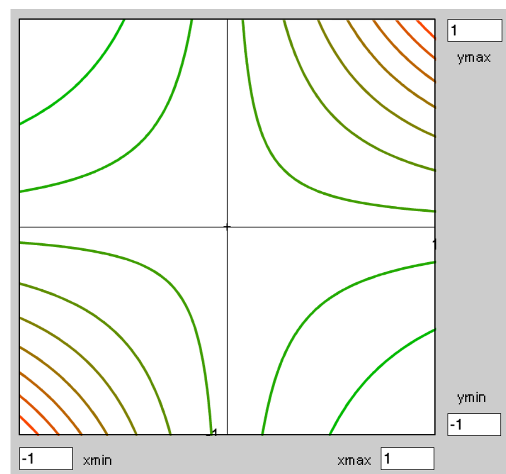

Figure 8: $z = e^{xy}$

(d)

3. (a) The level surfaces of $u = x^2 + y^2 + z^2$ are spheres centered at the point $(x, y, z) = (0, 0, 0)$ and with radius $r = \sqrt{u}$.

   (b) The level surfaces of $u = x + y + z$ are planes, where a particular value for u denotes the point of intersection of the plane with the x, y and z axes.

   (c) The level surfaces of $w = x^2 + y^2 - z$ are hyperbolic paraboloids with the saddle point located at point $(x, y, z) = (0, 0, -w)$.

   (d) The level surface of $w = x^2 + y^2$ is a hyperbolic paraboloid with its saddle point located at at point $(x, y, z) = (0, 0, 0)$.

4. (a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{1+x^2+y^2} = \frac{0+0}{1+0+0} = 0$$

(b) Let $x=y$. Then the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x}{2x^2} = \lim_{(x,y)\to(0,0)} \frac{1}{2x} = \infty$$

Next, let $x=0$. Then the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{0+y^2} = \frac{0}{0+0} = 0$$

Hence, the limit does not exist.

(c)

$$\lim_{(x,y)\to(0,0)} \frac{(1+y^2)\sin x}{x} = \left(\lim_{y\to 0} 1+y^2\right) \lim_{x\to 0} \frac{\sin x}{x} = (1)\,(1) = 1$$

To show that $\lim_{x\to 0} \sin x/x = 1$ we use the sandwich theorem:

$$\sin x \le x \le \tan x \quad \to \quad 1 \le \frac{x}{\sin x} \le \frac{\tan x}{\sin x} \quad \to \quad 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

Next, note that

$$\lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1$$

Hence,

$$\lim_{x\to 0} \frac{x}{\sin x} = 1 \qquad \Longrightarrow \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{1+x-y}{x^2+y^2} = \frac{1+0-0}{0+0} = \infty$$

5. (a) Let us consider the limit

$$\lim_{(x,y)\to(0,0)} z = \lim_{(x,y)\to(0,0)} \frac{x}{x-y} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x}{x-y}\right) = \lim_{x\to 0} \frac{x}{x-0} = \lim_{x\to 0} 1 = 1$$

However

$$\lim_{(x,y)\to(0,0)} z = \lim_{(x,y)\to(0,0)} \frac{x}{x-y} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x}{x-y}\right) = \lim_{y\to 0} \frac{0}{0-y} = 0$$

Hence, the limit at $(x,y)=(0,0)$ does not exist and so the function z is discontinuous at that point.

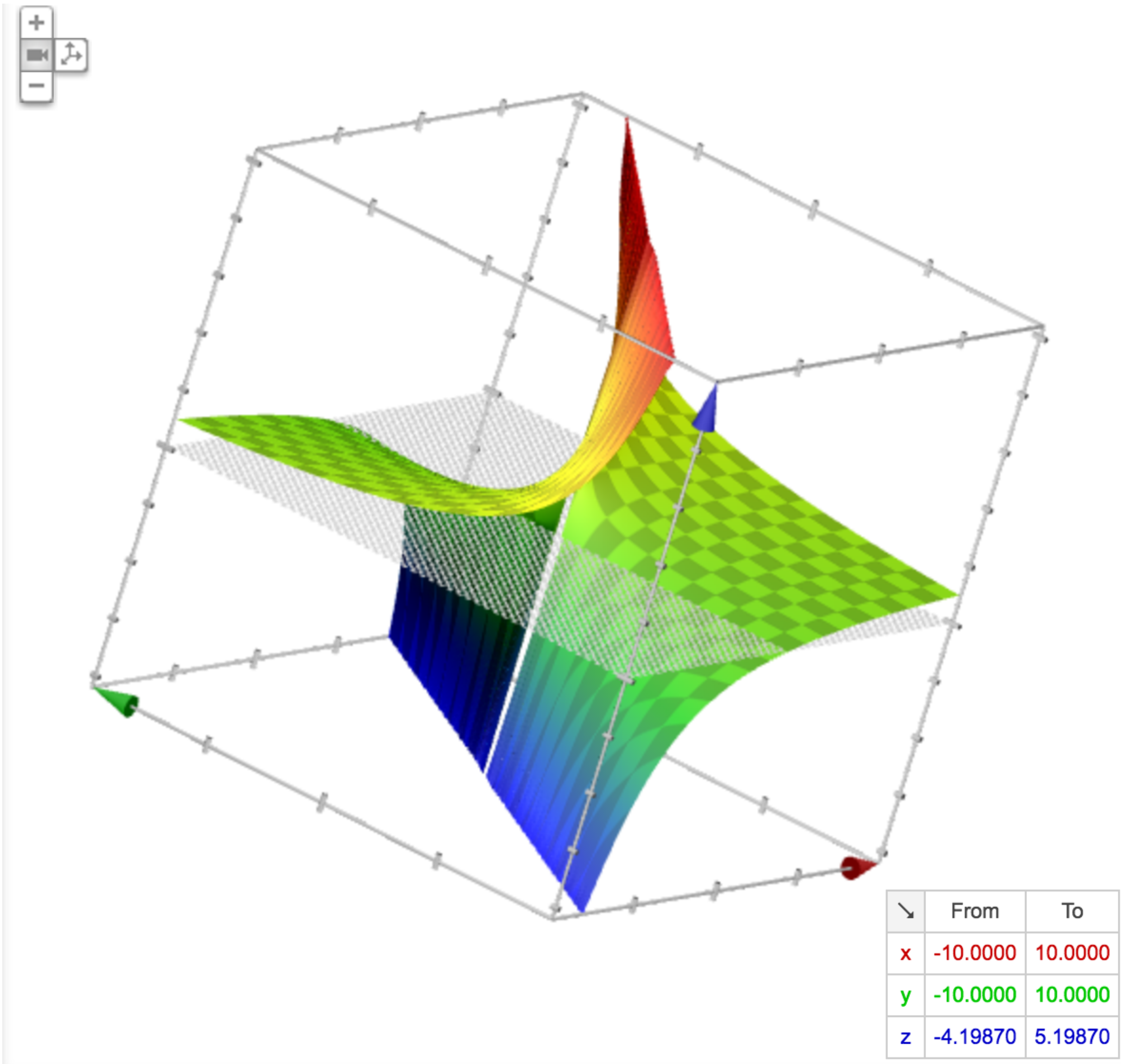

Figure 9: $z = x/(x-y)$

(b) Let us consider the limit

$$\lim_{(x,y)\to(0,0)} z = \lim_{(x,y)\to(0,0)} \ln\left(x^2+y^2\right) = \lim_{x\to 0}\left(\lim_{y\to 0} \ln\left(x^2+y^2\right)\right) = \lim_{x\to 0} \ln x^2 = -\infty$$

However, the point $(x, y) = (0, 0)$ is not in the domain of z, as the function is not defined there. Hence, strictly speaking z is continuous over its domain of definition $x, y \in (0, \infty)$ and has an infinite discontinuity at the point $(x, y) = (0, 0)$ as it is not defined there.

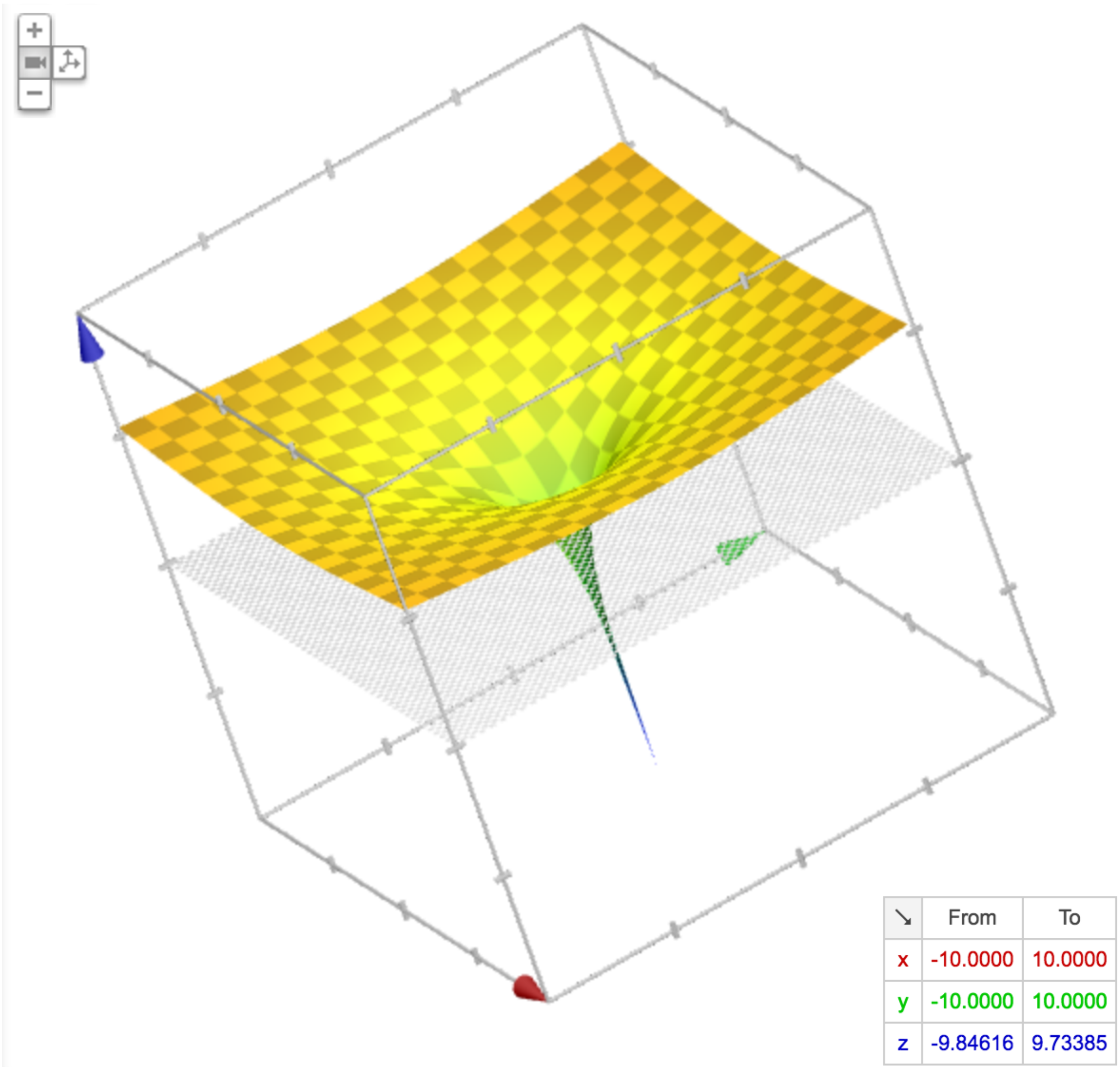

Figure 10: $z = x/(x - y)$

6. (a) The function e^a where a is an arbitrary real valued scalar is defined for any a, positive or negative. Hence, the domain may be formally defined as $\{x, y \in \mathbb{R} \mid \infty < x, y < \infty\}$.

   (b) The domain for the function $z = \ln(x^2+y^2-1)$ is given by $\{x, y \in \mathbb{R} \mid x^2+y^2 > 1\}$.

   (c) The set in which the function $z = \sqrt{1 - x^2 - y^2}$ is defined is the closed region $\{x, y \in \mathbb{R} \mid x^2 + y^2 \leq 1\}$.

   (d) The set in which the function $u = xy/z$ is defined is an open set, excluding the points lying in the xy plane (i.e. $z = 0$). It is not a domain, since not all points in the open set can be joined by a broken line.

7. Let $f(x, y)$ be defined in domain D and continuous at the point (x_1, y_1) of D, so that $\lim_{(x,y)\to(x_1,y_1)} f(x, y) = c = f(x_1, y_1)$. Substituting $\epsilon = (1/2)f(x_1, y_1)$ in (2.3) then gives

$$|f(x,y) - f(x_1,y_1)| < \frac{1}{2}f(x_1,y_1)$$

which is equivalent to stating that there is a neighbourhood of (x_1, y_1) in which $f(x, y) > (1/2)f(x_1, y_1) > 0$. Rewriting the absolute inequality gives

$$\begin{aligned} -\frac{1}{2}f(x_1,y_1) < f(x,y) - f(x_1,y_1) &< \frac{1}{2}f(x_1,y_1) \\ 0 < f(x,y) - \frac{1}{2}f(x_1,y_1) &< f(x_1,y_1) \\ f(x_1,y_1) > f(x,y) - \frac{1}{2}f(x_1,y_1) &> 0 \end{aligned}$$

Focusing on the second inequality we find

$$f(x,y) > \frac{1}{2}f(x_1,y_1)$$

And since the starting assumption was that $f(x_1, y_1) > 0$ clearly $(1/2)f(x_1, y_1) > 0$ as well.

8. Suppose that the domain D could consist of two open sets E_1 and E_2 with no point in common. Next let us choose point P in E_1 and Q in E_2 and join them by a broken line in D. We will regard this line as a path from point P to Q and let s be the distance from P along the path so that the path is given by continuous functions $x = x(s)$ and $y = y(s)$, where $0 \le s \le L$, with $s = 0$ at point P and $s = L$ at point Q. Now consider a function $f(s)$ and let $f(s) = -1$ if $(x(s), y(s))$ is in E_1 and let $f(s) = 1$ if $(x(s), y(s))$ is in E_2. Furthermore, let this function $f(s)$ be some linear combination of $x(s)$ and $y(s)$, i.e. $f(s) = ax(s)+by(s)$, where a and b are arbitrary scalars. Now since both $x(s)$ and $y(s)$ are continuous for $0 \le s \le L$ then according to (2.7) so will be $f(s)$. Next, we apply the *intermediate value theorem*: If $f(x)$ is continuous for $a \le x \le b$ and $f(a) < 0$, $f(b) > 0$, then $f(x) = 0$ for some x between a and b. Hence, since $f(s)$ is continuous for $0 \le s \le L$ and $f(0) = -1 < 0$ and $f(L) = 1 > 0$, then $f(s) = 0$ for some $s = s_0$ between 0 and L. But $f(s_0) = 0$ does not correspond to a point $(x(s_0), y(s_0))$ lying in either E_1 or E_2. In other words, a section of the path representing the broken line connecting points P and Q and given by continuous functions $x(s)$ and $y(s)$ doesn't belong to either E_1 or E_2. But this contradicts the definition of a domain D, which states that two points P and Q belonging to two different non-overlapping open sets E_1 and E_2 cannot be joined by a broken line.

9. Let the set A consist of all points (x, y) for which the continuous function $f(x, y) > 0$ in domain D. Let (x_1, y_1) be such a point. We can choose $f(x_1, y_1)$ arbitrarily small

as long as $f(x_1, y_1) > 0$. Then with the help of the answer to Problem 7 we can verify that there is a neighborhood of $f(x_1, y_1)$ in which $f(x, y) > (1/2)f(x_1, y_1) > 0$. In other words, no matter how small $f(x_1, y_1)$ is, as long as $f(x_1, y_1) > 0$ and $f(x, y)$ is continuous, there will always exist a neighborhood of (x_1, y_1) of radius δ where $f(x, y) > (1/2)f(x_1, y_1) > 0$. Hence, the set A is an open set. A similar reasoning can be applied to conclude that the set B is an open set. Together, A and B form two non-overlapping open sets. Next, imagine choosing a point $(x, y) = P$ in A and a point $(x, y) = Q$ in B and join them by a continuous (broken) line in D. Let s, $0 \leq s \leq L$ denote the distance along the path in the same way as for Problem 8, i.e. the path is from P to Q and is given by the continuous functions $x = x(s)$ and y = y(s). Now we apply the *intermediate value theorem*; since $f(s)$ is continuous for $0 \leq s \leq L$ and $f(0) > 0$ and $f(L) < 0$, then $f(s) = 0$ for some $s = s_0$ between $s = 0$ and $s = L$. Let us suppose the opposite; that $f(s) \neq 0$ for any s. This would imply that D consists of the two non-overlapping open sets A and B only, which as we concluded in Problem 8 contradicts the definition of a domain D; stating that two points P and Q belonging to two different non-overlapping open sets A and B cannot be joined by a (broken) line.

10. (a) Let $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$ in V^n. If $|\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_n^2} < \epsilon$ then

$$|\mathbf{x}|^2 = x_1^2 + \cdots + x_n^2 < \epsilon^2 \qquad \Longrightarrow \qquad |x_1| < \epsilon, \ldots, |x_n| < \epsilon$$

For $n = 2$, the result may be geometrically interpreted by stating that if the length of a vector $\mathbf{x}$ with origin at point $(x_1, x_2) = (0, 0)$, i.e. $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$ is smaller than some $\epsilon > 0$, then there exists a neighborhood of $(0, 0)$ where $|x_1| < \epsilon$ and $|x_2| < \epsilon$.

(b) Suppose that $|x_1| < \delta, \ldots, |x_n| < \delta$ then $x_1^2 < \delta^2, \ldots, x_n^2 < \delta^2$ and so

$$x_1^2 + \cdots + x_n^2 < \delta^2 + \cdots + \delta^2 = n\delta^2 \qquad n \geq 0$$

Taking square roots next gives

$$\sqrt{x_1^2 + \cdots + x_n^2} = |\mathbf{x}| < \sqrt{n}\delta < n\delta$$

where the right most inequality clearly holds, since $\sqrt{n} < n$.

(c) To show continuity of the mapping $\mathbf{y} = \mathbf{f}(\mathbf{x})$ at the point $\mathbf{x}^0$, where $\mathbf{x} \in V^n$ and $\mathbf{y} \in V^m$, we choose a $\delta > 0$ for a given $\epsilon > 0$ small enough such that

$$|f_1(x_1, \ldots, x_n) - f_1\left(x_1^0, \ldots, x_n^0\right)| < \frac{\epsilon}{m}, \ldots, |f_m(x_1, \ldots, x_n) - f_m\left(x_1^0, \ldots, x_n^0\right)| < \frac{\epsilon}{m}$$

for $|\mathbf{x} - \mathbf{x}^0| = \sqrt{(x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0)^2} < \delta$. Squaring and summing gives

$$\left[f_1(x_1, \ldots, x_n) - f_1\left(x_1^0, \ldots, x_n^0\right)\right]^2 + \cdots + \left[f_m(x_1, \ldots, x_n) - f_m\left(x_1^0, \ldots, x_n^0\right)\right]^2 < \frac{\epsilon^2}{m}$$

Finally, taking square roots of both sides of the inequality gives

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \frac{\epsilon}{\sqrt{m}} < \epsilon$$

In conclusion, since we haven chosen δ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \epsilon/\sqrt{m}$, it will certainly satisfy $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}^0)| < \epsilon$, since $\epsilon > \epsilon/\sqrt{m}$.

(d) Squaring the inequality that signifies continuity for the mapping $\mathbf{y} = \mathbf{f}(\mathbf{x})$ gives

$$\left[f_1(x_1, \ldots, x_n) - f_1\left(x_1^0, \ldots, x_n^0\right)\right]^2 + \cdots + \left[f_m(x_1, \ldots, x_n) - f_m\left(x_1^0, \ldots, x_n^0\right)\right]^2 < \epsilon^2$$

which implies that

$$\left[f_1(x_1, \ldots, x_n) - f_1\left(x_1^0, \ldots, x_n^0\right)\right]^2 < \epsilon^2, \ldots, \left[f_m(x_1, \ldots, x_n) - f_m\left(x_1^0, \ldots, x_n^0\right)\right]^2 < \epsilon^2$$

Taking square roots of both sides then finally results in

$$|f_1(\mathbf{x}) - f_1\left(\mathbf{x}^0\right)| < \epsilon, \ldots, |f_m(\mathbf{x}) - f_m\left(\mathbf{x}^0\right)| < \epsilon$$

from which we may conclude that each of the functions $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ is continuous at $(x_1^0, \ldots, x_n^0)$.

11. Let us for the moment assume that the limit $P_n \to P_0$ is not unique so that there exists a $P_n \to P_0'$, $P_0' \neq P_0$ and lets take $\epsilon = (1/3)d(P_0, P_0') = (1/3)|P_0' - P_0|$. We thus have $|P_n - P_0| < \epsilon$ and $|P_n - P_0'| < \epsilon$. Then

$$|P_0' - P_0| = \underbrace{|P_n - P_0 + P_0' - P_n| \leq |P_n - P_0| + |P_0' - P_n|}_{\text{triangle inequality}} < \frac{2}{3}|P_0' - P_0|$$

which is clearly a contradiction and so we must conclude that in fact $P_0 = P_0'$, i.e. the limit P_0 is unique.

12. To show that a set E in the plane is closed if and only if for every convergent sequence of points P_n in E the limit of the sequence is in E we will try to prove the opposite and see that it produces a contradiction. First, suppose E is closed and $P_0 \to P_0$, with P_n in E for all n, but the limit P_0 not in E (i.e. $P_0 \in \mathbb{R} \setminus E$). According to section (2.2), since E is closed, $\mathbb{R} \setminus E$ is open. Now since $P_0 \in \mathbb{R} \setminus E$ and the set is open there will exist a neighborhood of P_0 of radius ϵ such that $\mathrm{d}(P, P_0) < \epsilon$ which is completely contained in $\mathbb{R} \setminus E$ and so implies $P \notin E$. But this would mean there exists an N such that for all $n \geq N$, $P_n \in \mathbb{R} \setminus E$, which contradicts the assumption that the sequence P_n is entirely contained in E.

Next, suppose E is such that whenever $P_n \in E$ and $P_n \to P_0$, then $P_0 \in E$. To show that E is closed, we need to prove that $\mathbb{R} \setminus E$ is open, meaning that a neighborhood of a point $P \in \mathbb{R} \setminus E$ of radius $\epsilon > 0$ is contained entirely in $\mathbb{R} \setminus E$. Let us suppose the opposite however, that P_0 is a point not in E ($P_0 \in \mathbb{R} \setminus E$), but has at least one point of its neighborhood in E. In other words, suppose a neighborhood of P_0 of arbitrary radius $\epsilon > 0$ will contain at least one point that lies in E, in particular consider $\epsilon = 1, \epsilon = 1/2, \ldots, \epsilon = 1/n$. Let $P_n \in E$ be such a point and let its distance to $P_0 \in \mathbb{R} \setminus E$ satisfy the condition $\mathrm{d}(P_n, P_0) < 1/n$. Then for $\epsilon = 1/n$ we arrive at $P_n \to P_0$ (see Problem 11 for the definition of the limit of a convergent series), which implies $P_n \in \mathbb{R} \setminus E$. But this is contradictory to the original assumption that $P_n \in E$. Hence, this proves that E is closed and in conclusion we have proven that a set E is closed if and only if for every convergent sequence of point P_n in E, the limit of the sequence P_0 is in E.

13. (a) A set is called open if we can form a neighborhood of a point in the set of radius ϵ that is contained entirely in the set. In other words, this neighborhood does not contain any elements that are not part of the set. Since by definition the empty set does not contain any elements, the above statement can be applied to it without any problems and so it can be considered to be open.

    (b) To show that a set E in the plane and its boundary is closed is equivalent to showing that the complement to this is an open set $\mathbb{R} \setminus \bar{E}$ where the set $\bar{E}$ denotes the union of E and its boundary. To show that $\mathbb{R} \setminus \bar{E}$ is open is equivalent to showing that a neighborhood of a point $P \in \mathbb{R} \setminus \bar{E}$ of radius $\epsilon > 0$ is contained entirely in $\mathbb{R} \setminus \bar{E}$. We have already proven this as part of the second part of the solution to Problem 12 and so we won't repeat it again. Hence, we may conclude that a set E in the plane and its boundary are indeed closed.

Section 2.6

1. (a)

$$\frac{\partial z}{\partial x} = -\frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial z}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

   (b)

$$\frac{\partial z}{\partial x} = y^2 \cos xy \qquad \frac{\partial z}{\partial y} = \sin xy + xy \cos xy$$

   (c)

$$\frac{\partial z}{\partial x} = \frac{3x^2+2xy-2xz}{x^2-3z^2} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{x^2-3z^2}$$

   (d)

$$\frac{\partial z}{\partial x} = \frac{e^{x+2y}}{2\sqrt{e^{x+2y}-y^2}} \qquad \frac{\partial z}{\partial y} = \frac{e^{x+2y}-y}{\sqrt{e^{x+2y}-y^2}}$$

(e)

$$\frac{\partial z}{\partial x} = 3x\sqrt{x^2+y^2} \qquad \frac{\partial z}{\partial y} = 3y\sqrt{x^2+y^2}$$

(f)

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x+2y)^2}} \qquad \frac{\partial z}{\partial y} = \frac{2}{\sqrt{1-(x+2y)^2}}$$

(g)

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^z+1} \qquad \frac{\partial z}{\partial y} = \frac{2e^y}{e^z+1}$$

(h)

$$\frac{\partial z}{\partial x} = -\frac{y+z}{x+2z} \qquad \frac{\partial z}{\partial y} = -\frac{2xy+z^2+xz}{2yz+xy}$$

2. Using a forward difference:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} \approxeq \frac{f(2,1)-f(1,1)}{1} = \frac{2-(-1)}{1} = 3$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} \approxeq \frac{f(1,2)-f(1,1)}{1} = \frac{-3-(-1)}{1} = -2$$

Using a backward difference:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} \approxeq \frac{f(1,1)-f(0,1)}{1} = \frac{-1-(-2)}{1} = 1$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} \approxeq \frac{f(1,1)-f(1,0)}{1} = \frac{-1-1}{1} = -2$$

Using a centered difference:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} \approxeq \frac{f(2,1)-f(0,1)}{2} = \frac{2-(-2)}{2} = 2$$
$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} \approxeq \frac{f(1,2)-f(1,0)}{2} = \frac{-3-(-1)}{2} = -2$$

3. (a)

$$\left(\frac{\partial u}{\partial x}\right)_y = 2x \qquad \left(\frac{\partial v}{\partial y}\right)_x = -2$$

(b)

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\partial x}{\partial u} = e^u \cos v \qquad\qquad \left(\frac{\partial y}{\partial v}\right)_u = e^u \cos v$$

(c)

$$\left(\frac{\partial x}{\partial u}\right)_y = \left[\frac{\partial}{\partial u}(u+2y)\right]_y = 1 \qquad\qquad \left(\frac{\partial y}{\partial v}\right)_u = \left[\frac{1}{2}\frac{\partial}{\partial v}(u-v)\right]_u = -\frac{1}{2}$$

(d)

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2+y^2}} \qquad\qquad \left(\frac{\partial r}{\partial \theta}\right)_x = \frac{x\sin\theta}{\cos^2\theta} = x\sec\theta\tan\theta$$

4. (a)

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{ydx - xdy}{y^2}$$

(b)

$$dz = \frac{xdx + ydy}{x^2+y^2}$$

(c)

$$dz = \frac{(y-y^2)\,dx + (x-x^2)\,dy}{(1-x-y)^2}$$

(d)

$$dz = (x-2y)^4 e^{xy}\left[\left(5+xy-2y^2\right)dx + \left(-10-2xy+x^2\right)dy\right]$$

(e)

$$dz = \frac{-ydx + xdy}{x^2+y^2}$$

(f)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = -\frac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{3/2}}$$

5. (a)

$$\begin{aligned}
\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) = (x+\Delta x)^2 + 2(x+\Delta x)(y+\Delta y) - x^2 - 2xy\big|_{(1,1)} \\
&= 2(x+y)\Delta x + 2x\Delta y + 2\Delta x\Delta y + \overline{\Delta x}^2\big|_{(1,1)} \\
&= 2(1+1)\Delta x + 2\Delta y + 2\Delta x\Delta y + \overline{\Delta x}^2 \\
&= 4\Delta x + 2\Delta y + 2\Delta x\Delta y + \overline{\Delta x}^2 \\
dz &= 2(x+y)\Delta x + 2x\Delta y \\
&= 2(1+1)\Delta x + 2\Delta y \\
&= 4\Delta x + 2\Delta y
\end{aligned}$$

(b)

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = \left.\frac{x+\Delta x}{x+\Delta x + y\Delta y} - \frac{x}{x+y}\right|_{(1,1)}$$
$$= \frac{1+\Delta x}{2+\Delta x+\Delta y} - \frac{1}{2}$$
$$= \frac{\Delta x - \Delta y}{2(2+\Delta x+\Delta y)}$$
$$= \frac{(\Delta x-\Delta y)(2+\Delta x+\Delta y) - (\Delta x - \Delta y)(\Delta x+\Delta y)}{4(2+\Delta x+\Delta y)}$$
$$= \frac{\Delta x - \Delta y}{4} - \frac{(\Delta x-\Delta y)(\Delta x+\Delta y)}{4(2+\Delta x+\Delta y)}$$
$$dz = \frac{\Delta x - \Delta y}{4}$$

6. Given the data for point $(x,y) = (1,2)$ we get $\Delta x = 0.1$ and $\Delta y = -0.2$ for the point $(x,y) = (1.1, 1.8)$, and so

$$dz = f_x(1,2)\Delta x + f_y(1,2)\Delta y = 2(0.1) + 5(-0.2) = -0.8$$

which gives the estimate

$$f(1.1, 1.8) = f(1,2) + dz = 3 - 0.8 = 2.2$$

Next, for the point $(x,y) = (1.2, 1.8)$ we have $\Delta x = 0.2$ and $\Delta y = -0.2$ and so

$$dz = 2(0.2) + 5(-0.2) = -0.6$$

which gives the estimate

$$f(1.2, 1.8) = 3 - 0.6 = 2.4$$

And lastly, for the point $(x,y) = (1.3, 1.8)$ we have $\Delta x = 0.3$ and $\Delta y = -0.2$ and so

$$dz = 2(0.3) + 5(-0.2) = -0.4$$

which gives the estimate

$$f(1.3, 1.8) = 3 - 0.4 = 2.6$$

7. First off, we will show that the limit at the point $(x,y) = (0,0)$ does not exist for $z = f(x,y)$. Let us consider approaching the point $(x,y) = (0,0)$ along the line $x = y$, such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Similarly, approaching the point $(x,y) = (0,0)$ along the line $x = -y$ result in

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{-x^2}{2x^2} = \lim_{x\to 0} -\frac{1}{2} = -\frac{1}{2}$$

Combined with the fact that $f(0,0) = 0$, we may conclude that there does not exists a unique limit at the point $(x, y) = (0, 0)$ and so the function $z = f(x, y) = xy/(x^2 + y^2)$ is discontinuous at this point. Taking partial derivatives gives

$$\frac{\partial f}{\partial x} = -\frac{y\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2} \qquad\qquad \frac{\partial f}{\partial y} = \frac{x\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$$

Since $f(x, y)$ is discontinuous at the point $(x, y) = (0, 0)$, so will be $\partial f/\partial x$ and $\partial f/\partial y$ (i.e. the partial derivatives do not exist at this point). However, we can show this explicitly as well by once again taking limits. First, we will take the one-sided limit, approaching zero for positive y along the line $x = 0$ of $\partial f/\partial x$, giving

$$\lim_{y\to 0^+} \frac{\partial f}{\partial x} = -\frac{y\left(0 - y^2\right)}{\left(0 + y^2\right)^2} = \lim_{y\to 0^+} \frac{y^3}{y^4} = \lim_{y\to 0^+} \frac{1}{y} = \infty$$

However, approaching zero for negative y along the line $x = 0$ gives

$$\lim_{y\to 0^-} \frac{\partial f}{\partial x} = -\frac{y\left(0 - y^2\right)}{\left(0 + y^2\right)^2} = \lim_{y\to 0^-} \frac{y^3}{y^4} = \lim_{y\to 0^-} \frac{1}{y} = -\infty$$

Hence, the limit does not exist. A similar analysis for $\partial f/\partial y$ reveals that the limit for $\partial f/\partial y$ at the point $(x, y) = (0, 0)$ does not exist either and so we may conclude that $\partial f/\partial x$ and $\partial f/\partial y$ exist for all (x, y) and are continuous except at the point $(x, y) = (0, 0)$.

The fundamental lemma states that if a function $z = f(x, y)$ has continuous partial derivatives in D, then z has a differential

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

at every point (x, y) of D. Since we have just verified that the function $z = f(x, y) = xy/(x^2 + y^2)$ has continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ except at $(x, y) = (0, 0)$ (as $f(x, y)$ is discontinuous there), we may conclude that $z = f(x, y)$ has a differential for $(x, y) \neq (0, 0)$, which is of the form

$$dz = -\frac{y\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}\Delta x + \frac{x\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}\Delta y = \frac{x^2 - y^2}{\left(x^2 + y^2\right)^2}\left(-y\Delta x + x\Delta y\right)$$

Section 2.7

1. (a)

$$\begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 & 2x_2 \\ 3x_2 & 3x_1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ 2x_1x_3 & 0 & x_1^2 \end{bmatrix}$$

(d)

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \\ 2x & 0 \end{bmatrix}$$

(e)

$$\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 2xyz & x^2z & x^2y \end{bmatrix}$$

(f)

$$\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & -2z \end{bmatrix}$$

(g)

$$\begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} 2t \\ 3t^2 \\ 4t^3 \end{bmatrix}$$

2. (a)

$$\begin{aligned} d\mathbf{y} = \mathbf{f}_{\mathbf{x}}|_{\mathbf{x}=(2,1)}d\mathbf{x} &= \begin{bmatrix} \left.\frac{\partial y_1}{\partial x_1}\right|_{x_1=2,x_2=1} & \left.\frac{\partial y_1}{\partial x_2}\right|_{x_1=2,x_2=1} \\ \left.\frac{\partial y_2}{\partial x_1}\right|_{x_1=2,x_2=1} & \left.\frac{\partial y_2}{\partial x_2}\right|_{x_1=2,x_2=1} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 2x_1|_{x_1=2} & 2x_2|_{x_2=1} \\ x_2|_{x_2=1} & x_1|_{x_1=2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.04 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.06 \end{bmatrix} \\ \mathbf{f}(2.04, 1.01) = \mathbf{f}(2,1) + d\mathbf{y} &= \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.18 \\ 0.06 \end{bmatrix} = \begin{bmatrix} 5.18 \\ 2.06 \end{bmatrix} \end{aligned}$$

(b)

$$
\begin{aligned}
d\mathbf{y} = \mathbf{f}_{\mathbf{x}}|_{\mathbf{x}=(3,2,1)} d\mathbf{x} &= \begin{bmatrix} x_2|_{x_2=2} & x_1|_{x_1=3} & -2x_3|_{x_3=1} \\ [x_2+x_3]_{x_2=2,x_3=1} & x_1|_{x_1=3} & x_1|_{x_1=3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 3 & -2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0.01 \\ -0.01 \\ 0.03 \end{bmatrix} = \begin{bmatrix} -0.07 \\ 0.09 \end{bmatrix} \\
\mathbf{f}(3.01, 1.99, 1.03) = \mathbf{f}(3,2,1) + d\mathbf{y} &= \begin{bmatrix} 5 \\ 9 \end{bmatrix} + \begin{bmatrix} -0.07 \\ 0.09 \end{bmatrix} = \begin{bmatrix} 4.93 \\ 9.09 \end{bmatrix}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} &= \begin{bmatrix} \left.\dfrac{\partial u}{\partial x}\right|_{x=0,y=\pi/2} & \left.\dfrac{\partial u}{\partial y}\right|_{x=0,y=\pi/2} \\ \left.\dfrac{\partial v}{\partial x}\right|_{x=0,y=\pi/2} & \left.\dfrac{\partial v}{\partial y}\right|_{x=0,y=\pi/2} \\ \left.\dfrac{\partial w}{\partial x}\right|_{x=0,y=\pi/2} & \left.\dfrac{\partial w}{\partial y}\right|_{x=0,y=\pi/2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\
&= \begin{bmatrix} e^x \cos y|_{x=0,y=\pi/2} & -e^x \sin y|_{x=0,y=\pi/2} \\ e^x \sin y|_{x=0,y=\pi/2} & e^x \cos y|_{x=0,y=\pi/2} \\ 2e^x|_{x=0} & 0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 1.6 - (\pi/2) \end{bmatrix} \approxeq \begin{bmatrix} -0.03 \\ 0.1 \\ 0.2 \end{bmatrix} \\
\begin{bmatrix} u(0.1, 1.6) \\ v(0.1, 1.6) \\ w(0.1, 1.6) \end{bmatrix} &= \begin{bmatrix} u(0, \pi/2) \\ v(0, \pi/2) \\ w(0, \pi/2) \end{bmatrix} + \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \approxeq \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -0.03 \\ 0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -0.03 \\ 1.1 \\ 2.2 \end{bmatrix}
\end{aligned}
$$

(d)

$$d\mathbf{y} = \mathbf{f}_{\mathbf{x}}|_{\mathbf{x}=(1,0,\ldots,0)} d\mathbf{x} = \begin{bmatrix} \left.\dfrac{\partial y_1}{\partial x_1}\right|_{x_1=1,x_2=0,\ldots,x_n=0} & \cdots & \left.\dfrac{\partial y_1}{\partial x_n}\right|_{x_1=1,x_2=0,\ldots,x_n=0} \\ \vdots & & \vdots \\ \left.\dfrac{\partial y_n}{\partial x_1}\right|_{x_1=1,x_2=0,\ldots,x_n=0} & \cdots & \left.\dfrac{\partial y_n}{\partial x_n}\right|_{x_1=1,x_2=0,\ldots,x_n=0} \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2x_2|_{x_2=0} & \ldots & \ldots & 2x_n|_{x_2=0} \\ 2x_1|_{x_1=1} & 0 & 2x_3|_{x_3=0} & \ldots & 2x_n|_{x_n=0} \\ \vdots & & \ddots & & \vdots \\ 2x_1|_{x_1=1} & \ldots & 2x_{n-2}|_{x_{n-2}=0} & 0 & 2x_n|_{x_n=0} \\ 2x_1|_{x_1=1} & \ldots & \ldots & 2x_{n-1}|_{x_{n-1}=0} & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 2 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 2 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ \vdots \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbf{f}(1, 0.1, \ldots, 0.1) = \mathbf{f}(1, 0, \ldots, 0) + d\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

3. (a)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3(x^2 - y^2) & -6xy \\ 6xy & 3(x^2 - y^2) \end{vmatrix} = 9\left(x^2 - y^2\right)^2 + 36x^2y^2$$

$$= 9\left(x^2 + y^2\right)^2$$

(b)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} e^y \cos z & xe^y \cos z & -xe^y \sin z \\ e^y \sin z & xe^y \sin z & xe^y \cos z \\ e^y & xe^y & 0 \end{vmatrix} = 0$$

(c)

$$\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{vmatrix} = \begin{vmatrix} 2uvw & u^2w \\ 2uv^2 & 2u^2v \end{vmatrix} = 2u^3v^2w$$

(d)

$$\frac{\partial(f,g,h)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & 2 & 2z \\ yz & xz & xy \\ 0 & 0 & 2z \end{vmatrix} = 4z^2\left(x^2 - y\right)$$

4. (a) The Jacobian determinant is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} = e^{2x}\left(\cos^2 y + \sin^2 y\right) = e^{2x}$$

Evaluating the Jacobian determinant at the point $(x,y) = (1,0)$ then gives $\partial(u,v)/\partial(x,y) = e^2 \approxeq 7.39$.

(b) Squaring and adding the equations $u = e^x \cos y$ and $v = e^x \sin y$ gives

$$u^2 + v^2 = e^{2x}\left(\cos^2 y + \sin^2 y\right) = e^{2x} \qquad 0.9 \leq x \leq 1.1$$

Dividing the second equation by the first gives

$$\frac{v}{u} = \frac{\sin y}{\cos y} = \tan y \quad \Longrightarrow \quad v = (\tan y)\, u \qquad -0.1 \leq y \leq 0.1$$

These two equations describe a region R_{uv} which is bounded by arcs of the circles $u^2+v^2 = e^{1.8}$, $u^2+v^2 = e^{2.2}$ and the rays $v = (\tan -0.1)u$, $v = (\tan 0.1)u \to v = \pm(\tan 0.1)u$ (see the right (or left) half of Figure 11).

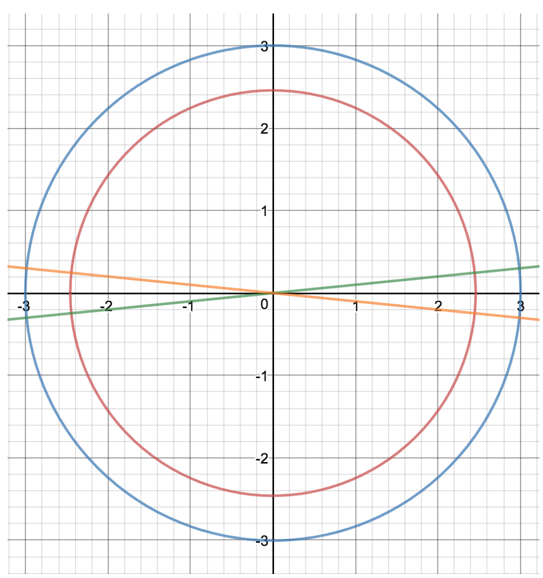

Figure 11: $u^2+v^2 = e^{1.8}$, $u^2+v^2 = e^{2.2}$, $v = \pm(\tan 0.1)\, u$

To find the area A_{uv} of this region we make use of the formula $A = (\theta/2)r^2$ and so

$$A_{uv} = 2\frac{0.1}{2}\left(e^{2.2} - e^{1.8}\right)$$

which gives for the ratio of the are of R_{uv} to that of R_{xy}

$$\frac{A_{uv}}{A_{xy}} = \frac{0.1}{0.04}\left(e^{2.2} - e^{1.8}\right) \approxeq 7.44$$

This answer is slightly higher than the value of the Jacobian determinant from part (a).

(c) The approximating linear mapping is given by

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

The region R'_{uv} corresponding to the square R_{xy} of part (b) under this linear mapping is a tilted square in the uv plane. For $dy = 0$ we have $du = e^x \cos y dx$ and $dv = e^x \sin$, so that (du, dv) follows a line of slope $\tan y$. For $dx = 0$ we have $du = -e^x \sin y dy$ and $dv = e^x \cos y dy$, so that (du, dv) follows a line of slope $-\cot y$. At the point $(x, y) = (1, 0)$ we have $du = edx$ and $dv = edy$ and so the area of the square region R'_{uv} is given by $A'_{uv} = dudv = e^2 dxdy$. The ratio of the area of R'_{uv} to that of R_{xy} then is

$$\frac{A'_{uv}}{A_{xy}} = e^2 \approxeq 7.39$$

This is the same answer as was found for part (a) and slightly smaller than the answer to part (b).

5. (a) Any two vectors $\mathbf{u}$ and $\mathbf{v}$ in V^2 that are not parallel (i.e. such that $\mathbf{u} \neq a\mathbf{v}$ for some arbitrary scalar a) are linearly independent. As Figure 12 shows, the sum of two vectors $\mathbf{u}$ and $\mathbf{v}$ forms the edges of a parallelogram, since $\mathbf{a} = \mathbf{v}$ and $\mathbf{u} = \mathbf{b}$. Now consider keeping the vector $\mathbf{v}$ fixed while scaling the vector $\mathbf{u}$ by some scalar $0 \leq a \leq 1$, such that $\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + \mathbf{v}$. As should be clear from looking at the figure, the point P will then lie somewhere on the line segment formed by the vector $\mathbf{b} = \mathbf{u}$, which is the rightmost edge of the parallelogram. Similarly, keeping the vector $\mathbf{u}$ fixed while scaling the vector $\mathbf{v}$ by some scalar $0 \leq b \leq 1$, such that $\mathbf{x} = \overrightarrow{OP} = \mathbf{u} + b\mathbf{v}$ will result in the point P lying somewhere on the line segment formed by the vector $\mathbf{v}$, which is the top edge of the parallelogram. Hence, it should not be hard to see that any combination $\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v}$, $0 \leq a \leq 1$, $0 \leq b \leq 1$ will result in a point P that is located somewhere inside or

on an edge of the parallelogram formed by the two linearly independent vectors $\mathbf{u} = \mathbf{b}$ and $\mathbf{v} = \mathbf{a}$.

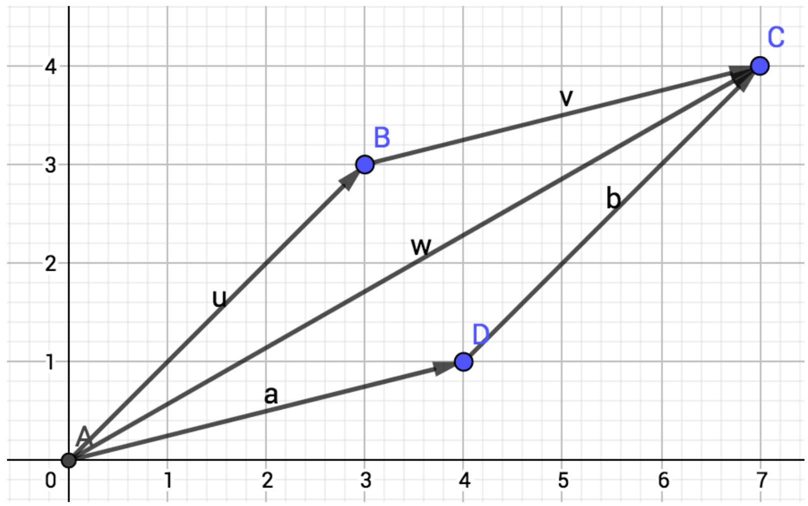

Figure 12: $\mathbf{w} = \mathbf{u} + \mathbf{v}$

(b) Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

be the matrix associated with the parallelogram of part (a) and the vector $\mathbf{x}$, such that

$$\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \qquad 0 \le a \le 1, 0 \le b \le 1$$

Then according to Section 1.4 the determinant of $\mathbf{B}$ may be interpreted as the area of the parallelogram: $A_{\mathbf{x}} = \det \mathbf{B}$. Similarly, let

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}\mathbf{u} & \mathbf{A}\mathbf{v} \end{bmatrix} = \mathbf{A}\mathbf{B}$$

be the matrix associated with the parallelogram obtained by the linear mapping $\mathbf{y} = \mathbf{A}\mathbf{x}$, such that

$$\mathbf{y} = \overrightarrow{OQ} = \mathbf{A}\left(a\mathbf{u} + b\mathbf{v}\right) = a\mathbf{A}\mathbf{u} + b\mathbf{A}\mathbf{v} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \qquad 0 \le a \le 1, 0 \le b \le 1$$

The area of this parallelogram then is given by

$$A_{\mathbf{y}} = \det \mathbf{C} = \det\left(\mathbf{A}\mathbf{B}\right) = \det \mathbf{A}\left(\det \mathbf{B}\right)$$

where the last equality holds because the determinant of a product of matrices is equal to the product of the determinant of each individual matrix. Hence, we observe that indeed as claimed $A_{\mathbf{y}} = \det \mathbf{A}\left(A_{\mathbf{x}}\right)$.

6. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors in V^3. A point P for which

$$\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \qquad 0 \leq a \leq 1, 0 \leq b \leq 1, 0 \leq c \leq 1$$

will then fill a parallelepiped in 3-dimensional space whose edges, properly directed, represent $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

be the matrix associated with the parallelepiped and the vector $\mathbf{x}$, such that

$$\mathbf{x} = \overrightarrow{OP} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad 0 \leq a \leq 1, 0 \leq b \leq 1, 0 \leq c \leq 1$$

Then according to Section 1.4 the determinant of $\mathbf{B}$ may be interpreted as the volume of the parallelepiped: $V_{\mathbf{x}} = \det \mathbf{B}$. Similarly, let

$$\mathbf{C} = \begin{bmatrix} \mathbf{Au} & \mathbf{Av} \end{bmatrix} = \mathbf{AB}$$

be the matrix associated with the parallelepiped obtained by the linear mapping $\mathbf{y} = \mathbf{Ax}$, such that

$$\begin{aligned} \mathbf{y} = \overrightarrow{OQ} = \mathbf{A}\left(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}\right) &= a\mathbf{Au} + b\mathbf{Av} + c\mathbf{Aw} \qquad 0 \leq a \leq 1, 0 \leq b \leq 1, 0 \leq c \leq 1 \\ &= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{aligned}$$

The volume of this parallelepiped then is given by

$$V_{\mathbf{y}} = \det \mathbf{C} = \det\left(\mathbf{AB}\right) = \det \mathbf{A}\left(\mathbf{B}\right)$$

where the last equality holds because the determinant of a product of matrices is equal to the product of the determinant of each individual matrix. Hence, we observe that indeed as claimed $V_{\mathbf{y}} = \det \mathbf{A}\left(V_{\mathbf{x}}\right)$.

Section 2.8

1.