CHAPTER 4

Section 4.1

1. (a) Using integration by parts twice, the integral can be written as

$$\begin{aligned}\int x^2 \sin x\, dx = -x^2 \cos x + \int 2x \cos x\, dx &= -x^2 \cos x + 2x \sin x - \int 2 \sin x\, dx \\ &= -x^2 \cos x + 2x \sin x + 2\cos x + C \\ &= 2x \sin x - \left(x^2 - 2\right) \cos x + C\end{aligned}$$

(b) Making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\begin{aligned}\int \frac{x}{1+x^4}\, dx = \frac{1}{2} \int \frac{2x}{1+\left(x^2\right)^2}\, dx = \frac{1}{2} \int \frac{1}{1+u^2}\, du &= \frac{1}{2} \tan^{-1} u + C \\ &= \frac{1}{2} \tan^{-1} x^2 + C\end{aligned}$$

(c) Using partial fraction expansion, we can write

$$\begin{aligned}\int \frac{1}{(x-1)(x-2)}\, dx = \int \left(-\frac{1}{x-1} + \frac{1}{x-2}\right) dx &= -\int \frac{dx}{x-1} + \int \frac{dx}{x-2} \\ &= -\ln(x-1) + \ln(x-2) + C \\ &= \ln \frac{x-2}{x-1} + C\end{aligned}$$

(d) Making the substitution $u = \sqrt{x-1}$ so that $2udu = dx$, the integral can be written as

$$\begin{aligned}\int \frac{1}{1+\sqrt{x-1}}\, dx = 2\int \frac{u}{1+u}\, du &= 2 \int \frac{-1+1+u}{1+u}\, du \\ &= 2 \int \left(-\frac{1}{1+u} + 1\right) du \\ &= -2 \int \frac{du}{1+u} + 2 \int du \\ &= -2 \ln(1+u) + 2u + C \\ &= 2\left[\sqrt{x-1} - \ln\left(1+\sqrt{x-1}\right)\right] + C\end{aligned}$$

2. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\, dx = \int_0^1 \cos^2\theta\, d\theta = \frac{1}{2}\int_0^1 (1+\cos 2\theta)\, d\theta &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\sin 2\theta}{4}\right|_0^{\pi/2} \\ &= \left.\frac{\theta}{2}\right|_0^{\pi/2} + \left.\frac{\cos\theta \sin\theta}{2}\right|_0^{\pi/2} = \frac{\pi}{4}\end{aligned}$$

(b) Using the identity $\sin mx \sin nx = (1/2)\cos[(m-n)x] - (1/2)\cos[(m+n)x]$, the integral can be written as

$$\begin{aligned}\int_0^{\pi} \sin 2x \sin 3x \, dx = \frac{1}{2}\int_0^{\pi} (\cos x - \cos 5x)\, dx &= \frac{1}{2}\int_0^{\pi} \cos x\, dx - \frac{1}{2}\int_0^{\pi} \cos 5x \, dx \\ &= \left.\frac{\sin x}{2}\right|_0^{\pi} - \left.\frac{\sin 5x}{10}\right|_0^{\pi} = 0\end{aligned}$$

(c) Using integration by parts twice, the integral can be written as

$$\begin{aligned}\int_0^1 \left(2x^2 - 3x + 1\right) e^x \, dx &= \left(2x^2 - 3x + 1\right) e^x\Big|_0^1 - \int_0^1 (4x-3)\, e^x \, dx \\ &= \left(2x^2 - 3x + 1\right) e^x\Big|_0^1 - (4x-3)\, e^x\Big|_0^1 + \int_0^1 4e^x \, dx \\ &= \left(2x^2 - 3x + 1\right) e^x\Big|_0^1 - (4x-3)\, e^x\Big|_0^1 + 4e^x\Big|_0^1 = 3e - 8\end{aligned}$$

(d) Using integration by parts, the fact that $(d/dx)\tan^{-1} x = 1/(1+x^2)$ and making the substitution $u = x^2$ so that $du = 2xdx$, the integral can be written as

$$\begin{aligned}\int_0^1 \tan^{-1} x \, dx = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx &= x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx \\ &= x\tan^{-1} x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{du}{1+u} \\ &= x\tan^{-1} x\Big|_0^1 - \left.\frac{\ln(1+u)}{2}\right|_0^1 \\ &= x\tan^{-1} x\Big|_0^1 - \left.\frac{\ln(1+x^2)}{2}\right|_0^1 = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}\end{aligned}$$

3. (a) Making the substitution $x = \sin\theta$ so that $dx = \cos\theta d\theta$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, the integral can be written as

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 0^+}\int_b^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 0^+}\int_b^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\, d\theta &= \lim_{b\to 0^+}\int_b^{\pi/2} d\theta \\ &= \lim_{b\to 0^+} \theta\Big|_b^{\pi/2} \\ &= \lim_{b\to 0^+}\left(\frac{\pi}{2} - b\right) = \frac{\pi}{2}\end{aligned}$$

(b) Making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\int_0^{\infty} e^{-x}\, dx = \lim_{b\to\infty}\int_0^b e^{-x}\, dx = \lim_{b\to -\infty} -\int_0^b e^u \, du = \lim_{b\to-\infty} -e^u\Big|_0^b = \lim_{b\to-\infty}\left(-e^b + 1\right) = 1$$

(c) Using integration by parts, the integral can be written as

$$\begin{aligned}\int_0^1 \ln x\,dx = \lim_{b\to 0^+}\int_b^1 \ln x\,dx = \lim_{b\to 0^+} x\ln x\Big|_b^1 - \lim_{b\to 0^+}\int_b^1 dx &= \lim_{b\to 0^+}(x\ln x - x)\Big|_b^1\\ &= \lim_{b\to 0^+}(-1 - b\ln b + b)\\ &= -1 - \lim_{b\to 0^+} b\ln b = -1\end{aligned}$$

where the last step follows from the fact that

$$\lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{1/b} \overset{LH}{=} \lim_{b\to 0^+}\frac{1/b}{-1/b^2} = \lim_{b\to 0^+} -b = 0$$

using L'Hopital's rule.

(d) Making the substitutions $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, $2u = \theta$ so that $2du = d\theta$, $v = \cos u$ so that $dv = -\sin u du$ and $w = \sin u$ so that $dw = \cos u du$ and the identities $1 + \tan^2\theta = \sec^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, the integral can be written as

$$\begin{aligned}\int_1^\infty \frac{dx}{x\sqrt{1+x^2}} &= \lim_{b\to\infty}\int_1^b \frac{dx}{x\sqrt{1+x^2}}\\ &= \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{\sec^2\theta}{\tan\theta\sqrt{1+\tan^2\theta}}\,d\theta\\ &= \lim_{b\to\pi/2}\int_{\pi/4}^b \frac{d\theta}{\sin\theta}\\ &= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{2du}{\sin 2u}\\ &= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{du}{\sin u\cos u}\\ &= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin^2 u + \cos^2 u}{\sin u\cos u}\,du\\ &= \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\sin u}{\cos u}\,du + \lim_{b\to\pi/4}\int_{\pi/8}^b \frac{\cos u}{\sin u}\,du\\ &= \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2+\sqrt{2}}/2}^b -\frac{dv}{v} + \lim_{b\to\sqrt{2}/2}\int_{\sqrt{2-\sqrt{2}}/2}^b \frac{dw}{w}\\ &= \lim_{b\to\sqrt{2}/2} -\ln v\Big|_{\sqrt{2+\sqrt{2}}/2}^b + \lim_{b\to\sqrt{2}/2}\ln w\Big|_{\sqrt{2-\sqrt{2}}/2}^b\\ &= \lim_{b\to\sqrt{2}/2} -\ln b + \ln\frac{\sqrt{2+\sqrt{2}}}{2} + \lim_{b\to\sqrt{2}/2}\ln b - \ln\frac{\sqrt{2-\sqrt{2}}}{2}\\ &= \ln\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \frac{1}{2}\ln\left(3+2\sqrt{2}\right) = \frac{1}{2}\ln\left(1+\sqrt{2}\right)^2 = \ln\left(1+\sqrt{2}\right)\end{aligned}$$

(e) Using integration by parts twice and making the substitution $u = -x$ so that $du = -dx$, the integral can be written as

$$\begin{aligned}
\int_0^\infty x^2 e^{-x}\,dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx &= \lim_{b\to-\infty}\int_0^b -u^2 e^u\,du \\
&= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty}\int_0^b 2ue^u\,du \\
&= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty}\int_0^b 2e^u\,du \\
&= \lim_{b\to-\infty} -u^2e^u\Big|_0^b + \lim_{b\to-\infty} 2ue^u\Big|_0^b - \lim_{b\to-\infty} 2e^u\Big|_0^b \\
&= \lim_{b\to-\infty}\left(-b^2e^b + 2be^b - 2e^b + 2\right) = 2
\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to-\infty} -b^2e^b = \lim_{b\to-\infty} -\frac{b^2}{e^{-b}} \stackrel{LH}{=} \lim_{b\to-\infty}\frac{2b}{e^{-b}} \stackrel{LH}{=} \lim_{b\to-\infty} -\frac{2}{e^{-b}} = 0$$

(f) Using integration by parts, the integral can be written as

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty} -\frac{\ln x}{x}\Big|_1^b + \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} \\
&= \lim_{b\to\infty} -\frac{\ln x}{x}\Big|_1^b - \lim_{b\to\infty}\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} + 1\right) = 1
\end{aligned}$$

where the last step follows from employing L'Hopital's rule:

$$\lim_{b\to\infty} -\frac{\ln b}{b} \stackrel{LH}{=} \lim_{b\to\infty} -\frac{1/b}{1} = \lim_{b\to\infty} -\frac{1}{b} = 0$$

4. (a)

$$\begin{aligned}
\int_{-1}^1 \frac{dx}{x^{1/3}} = \int_{-1}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} &= \lim_{b\to0^-}\int_{-1}^b \frac{dx}{x^{1/3}} + \lim_{b\to0^+}\int_b^1 \frac{dx}{x^{1/3}} \\
&= \lim_{b\to0^-}\frac{3}{2}x^{2/3}\Big|_{-1}^b + \lim_{b\to0^+}\frac{3}{2}x^{2/3}\Big|_b^1 \\
&= \lim_{b\to0^-}\frac{3}{2}\left(b^{2/3}-1\right) + \lim_{b\to0^+}\frac{3}{2}\left(1-b^{2/3}\right) = 0
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-1}^1 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3} &= \lim_{b\to0^-}\int_{-1}^b \frac{dx}{x^3} + \lim_{b\to0^+}\int_b^1 \frac{dx}{x^3} \\
&= \lim_{b\to0^-} -\frac{1}{4x^4}\Big|_{-1}^b + \lim_{b\to0^+}\frac{1}{4x^4}\Big|_b^1 \\
&= \lim_{b\to0^-}\frac{1}{4}\left(-b^{-4}+1\right) + \lim_{b\to0^+}\frac{1}{4}\left(1-b^{-4}\right) = -\infty
\end{aligned}$$

Hence, the integral is divergent.

(c) Making the substitution $x = \tan\theta$ so that $dx = \sec^2\theta d\theta$, the integral can be written as

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\pi/2}\int_0^b \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta = \lim_{b\to\pi/2}\int_0^b d\theta = \lim_{b\to\pi/2}\theta\Big|_0^b$$
$$= \lim_{b\to\pi/2} b = \frac{\pi}{2}$$

(d) Using a partial fraction expansion, the integral can be written as

$$\begin{aligned}\int_0^\infty \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx &= \lim_{b\to0^+}\int_b^1 \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx + \lim_{b\to\infty}\int_1^b \frac{x^2-x-1}{x\left(x^3+1\right)}\,dx \\ &= \lim_{b\to0^+}\int_b^1\left[\frac{4x-2}{3\left(x^2-x+1\right)} - \frac{1}{3\left(x+1\right)} - \frac{1}{x}\right]dx + \ldots \\ &= \lim_{b\to0^+}\frac{1}{3}\int_b^1 \frac{4x-2}{x^2-x+1}\,dx - \lim_{b\to0^+}\frac{1}{3}\int_b^1\frac{dx}{x+1} - \lim_{b\to0^+}\int_b^1\frac{dx}{x} + \ldots\end{aligned}$$

It is clear to see that the first two integrals (obtained by a partial fraction expansion) belonging to the first partial integral converge. However, the third diverges:

$$\lim_{b\to0^+}\int_b^1\frac{dx}{x} = \lim_{b\to0^+}\ln x\Big|_b^1 = -\lim_{b\to0^+}\ln b = \infty$$

Hence, since the first partial integral diverges, we conclude that the original integral is divergent.

(e)

$$\int_0^\infty \sin x\,dx = \lim_{b\to\infty}\int_0^b \sin x\,dx = \lim_{b\to\infty} -\cos x\Big|_0^b = \lim_{b\to\infty}\left(-\cos b + 1\right)$$

Since $\lim_{b\to\infty}\cos b$ does not exist the integral is divergent.

(f) Making the substitution $u = \cosh x$ so that $du = \sinh x dx$, the integral can be written as

$$\begin{aligned}\int_0^\infty \left(1-\tanh x\right)dx = \lim_{b\to\infty}\int_0^b\left(1-\tanh x\right)dx &= \lim_{b\to\infty}\int_0^b dx - \lim_{b\to\infty}\int_0^b \tanh x\,dx \\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_0^b\frac{\sinh x}{\cosh x}\,dx \\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\int_1^{\cosh b}\frac{du}{u} \\ &= \lim_{b\to\infty} x\Big|_0^b - \lim_{b\to\infty}\ln u\Big|_1^{\cosh b} \\ &= \lim_{b\to\infty}\left(b - \ln\cosh b\right) = \ln 2\end{aligned}$$

where the last step follows from the fact that

$$\lim_{b\to\infty}(b-\ln\cosh b)=\lim_{b\to\infty}\left[\ln e^b-\ln\left(e^b+e^{-b}\right)+\ln 2\right]=\lim_{b\to\infty}\left[\ln 2-\ln\left(1+e^{-2b}\right)\right]$$
$$=\ln 2$$

5. (a) The curves $y=0$, $y=1-x^2$ intersect at the point $(-1,0)$, $(1,0)$. Hence, the area between the curves is given by

$$A=\int_{-1}^{1}\left(1-x^2\right)\,dx=\left[x-\frac{x^3}{3}\right]_{-1}^{1}=\frac{4}{3}$$

(b) The curves $y=x^3$, $y=x^{1/3}$ intersect at the points $(-1,-1)$, $(0,0)$, $(1,1)$. Hence the area between the curves is given by

$$A=2\int_{0}^{1}\left(x^{1/3}-x^3\right)\,dx=\left[\frac{3x^{4/3}}{2}-\frac{x^4}{2}\right]_{0}^{1}=1$$

Note that we have used the fact that the intersection of the two curves is anti-symmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x=0$ to $x=1$ and multiply the result by two.

(c) The curves $y=6\sin^{-1}x$, $y=\pi\sin\pi x$ intersect at the points $(-1/2,-\pi)$, $(0,0)$, $(1/2,\pi)$. Hence, the area between the curves is given by

$$\begin{aligned}A&=2\int_{0}^{1/2}\left(\pi\sin\pi x-6\sin^{-1}x\right)\,dx\\&=2\pi\int_{0}^{1/2}\sin\pi x\,dx-12\int_{0}^{1/2}\sin^{-1}x\,dx\\&=-2\cos\pi x\Big|_{0}^{1/2}-12x\sin^{-1}x\Big|_{0}^{1/2}+\int_{0}^{1/2}\frac{12x}{\sqrt{1-x^2}}\,dx\\&=2-\pi-6\int_{1}^{3/4}\frac{du}{\sqrt{u}}\\&=2-\pi-6\int_{1}^{3/4}u^{-1/2}\,du\\&=2-\pi-12\sqrt{u}\Big|_{1}^{3/4}=14-\pi-6\sqrt{3}\end{aligned}$$

where we have used integration by parts and the substitution $u=1-x^2$ so that $du=-2xdx$ in order to solve the second integral. Note that we have used the fact that the intersection of the two curves is anti-symmetric with respect to the y-axis, and so in order to calculate the total area we can simply integrate from $x=0$ to $x=1/2$ and multiply the result by two.

6. (a)

$$\frac{1}{b-a}\int_a^b f(x)\ dx = \frac{2}{\pi}\int_0^{\pi/2}\sin x\, dx = -\frac{2\cos x}{\pi}\bigg|_0^{\pi/2} = \frac{2}{\pi}$$

(b)

$$\frac{1}{b-a}\int_a^b f(x)\ dx = \frac{2}{\pi}\int_{-\pi/2}^{0}\sin x\, dx = -\frac{2\cos x}{\pi}\bigg|_{-\pi/2}^{0} = \frac{2\cos x}{\pi}\bigg|_0^{-\pi/2} = -\frac{2}{\pi}$$

(c) Using the identities $\sin^2 x + \cos^2 x = 1$, $\cos 2x = 2\cos^2 x - 1$, we find

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\ dx = \frac{2}{\pi}\int_0^{\pi/2}\sin^2 x\, dx &= \frac{2}{\pi}\int_0^{\pi/2}\left(1-\cos^2 x\right)\ dx \\ &= \frac{1}{\pi}\int_0^{\pi/2}(1-\cos 2x)\ dx \\ &= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{1}{2}\end{aligned}$$

(d)

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\ dx = \frac{1}{x_2-x_1}\int_{x_1}^{x_2}(ax+b)\ dx &= \frac{1}{x_2-x_1}\left[\frac{ax^2}{2}+bx\right]_{x_1}^{x_2} \\ &= b + \frac{a}{2}(x_1+x_2)\end{aligned}$$

7. Let $f(x)$ and $g(x)$ be continuous for $a \le x \le b$ and $|g(x) - f(x)| \le \epsilon$ for $a \le x \le b$. Defining $h(x) = g(x) - f(x)$ so that $|h(x)| \le \epsilon$ and using (4.6) we then find

$$\left|\int_a^b h(x)\, dx\right| = \left|\int_a^b [g(x)-f(x)]\ dx\right| = \left|\int_a^b g(x)\, dx - \int_a^b f(x)\, dx\right| \le \epsilon\,(b-a)$$

8. (a)

$$\int_0^1 \sin x^2\, dx \approxeq \int_0^1\left(x^2 - \frac{x^6}{6}\right)\ dx = \left[\frac{x^3}{3} - \frac{x^7}{42}\right]_0^1 = \frac{13}{42} \approxeq 0.3095$$

The worst error is approximately 0.0081 at the point $x = 1$.

(b)

$$\int_0^1 e^{-x^2}\, dx \approxeq \int_0^1\left(1 - x^2 + \frac{x^4}{2}\right)\ dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1 = \frac{23}{30} \approxeq 0.7667$$

The worst error is approximately 0.1321 at the point $x = 1$.

9. Let $f(x)$ be continuous for $0 \leq x \leq 1$. Then (4.20) may be used to approximate the integral of $f(x)$ numerically:

$$\int_0^1 f(x)\, dx \sim \frac{1}{2n}\left[f(0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(1)\right]$$

where $0 < x_1 < x_2 \cdots < x_{n-1} < 1$. If we then let $n \to \infty$ and choose $x_1 = 1/n$, $x_2 = 2/n, \ldots, x_{n-1} = (n-1)/n$, $x_n = n/n$ such that the endpoints converge to $x = 0$ and $x = 1$ respectively, while at the same time choosing an infinite number of equally spaced, but infinitely close interior points $x_1, x_2, \ldots, x_{n-1}$ the finite sum converges to:

$$\lim_{n\to\infty} \frac{1}{2n}\left[f(0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(1)\right] =$$

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\, dx$$

Note that the end points of the first and second limits differ by a factor of $1/2$. However, since $2\infty = \infty$ this difference is of no importance.

10. (a)