Section 4.2: Numerical Integration

Thursday, February 25, 2021 9:46 AM

Numerical Quadrature uses sums $\sum_{i=0}^{n} a_i f(x_i)$ to estimate $\int_a^b f(x)dx$.

Similar to §4.1, we start w/ Lagrange interpolating polynomial between 2pt $(x_0, f(x_0))$, $(x_1, f(x_1))$

(a) Create L.I.P.

$x^2 \to \frac{x^3}{3} \to \frac{h^3}{3}$

$$P(x) = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1) + \frac{(x-x_0)(x-x_1)}{2} f''(\xi)$$

integrate

(b) Integrate.

$$\int_{x_0}^{x_1} P(x)dx = \int_{x_0}^{x_1} \left[\frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)\right] dx + \int err$$

$$= \left[\frac{(x-x_1)^2}{2(x_0-x_1)} f(x_0) + \frac{(x-x_0)^2}{2(x_1-x_0)} f(x_1)\right]\Bigg|_{x_0}^{x_1} + \int err$$

plug in endpoints

$$= \frac{(x_1-x_0)}{2}[f(x_0)+f(x_1)] - \frac{h^3}{12} f''(\xi)$$

d.o.p: 1

x^3

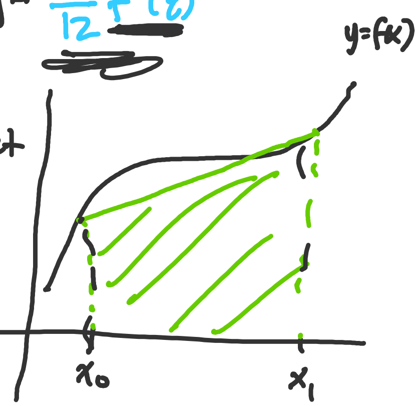
1st der: $3x^2$

2nd der: $6x \neq 0$

So, letting $h = x_1 - x_0$ we get

Trapezoid Rule

$$\int_a^b f(x)dx = \boxed{\frac{h}{2}[f(a)+f(b)]} + \frac{h^3}{12} f''(\xi)$$



x

1st der: 1

2nd der: 0

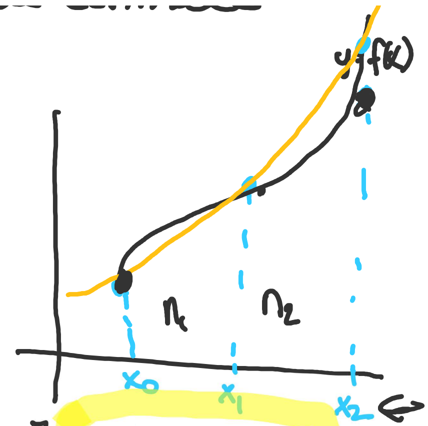
Q: when would error be 0?

A: if $f''(x) = 0$ (i.e., linear functions).

If we use more points in our L.I.P, we expect a "better" / more accurate estimate. w/

a better / more accurate estimate

ex 3 points means our L.I.P. is quadratic



If we create LIP & do simplifying we get

Simpson's Rule

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)] - \boxed{\frac{h^5}{90} f^{(4)}(\xi)}$$

Q: when would error for S.R. be 0?

when $f^{(4)}=0$ so if f is polynom of deg. 3 or less.

x^3

1st deriv: $3x^2$

2nd der: $6x$

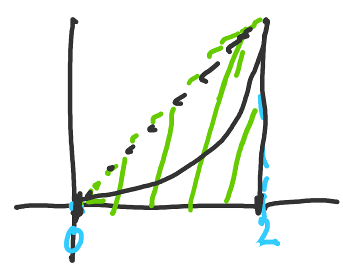
3rd der: 6

4th der: 0

ex

Compare T.R & S.R. to approximate

$\int_0^2 x^2 dx.$



exact $\frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}.$

T.R. - Trapezoid Rule

$\frac{h}{2}[f(x_0)+f(x_1)]$

$h=2,\ x_0=0,\ x_1=2$

$\frac{2}{2}[f(0)+f(2)]$

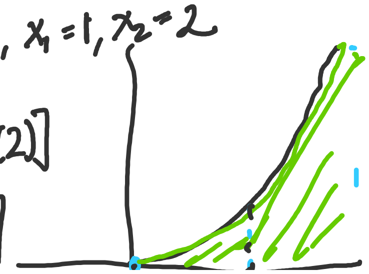
$1[0+4] = 4.$

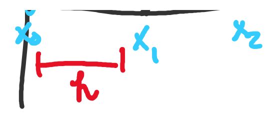
S.R. $\int_0^2 x^2 dx = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]$

$h=1 \quad x_0=0,\ x_1=1,\ x_2=2$

$=\frac{1}{3}[f(0)+4f(1)+f(2)]$

$=\frac{1}{3}[0^2+4(1)^2+(2)^2]$





$= 8/3$

Wow! It's exact!
(b/c 4^{th} der. of $f = 0$).
(or degree of f is 3 or less)

Def
The degree of accuracy, or precision of a quadrature formula is the largest positive integer n such that the formula is exact for x^k, $k = 0, .., n$. $\Rightarrow$ err $= 0$

If formula is exact for x, x^2, x^3
d.o.p $= 3$.
.. for x, x^2, d.p $= 2$.

Trap. Rule

$$\int_a^b f(x)dx = \frac{h}{2}[f(a) + f(b)] + \underbrace{\frac{h^3}{12} f''(\xi)}_{\text{err.}}$$

we have 0 err if $f''(x) = 0$.
$\Rightarrow f$ is linear d.o.p $= 1$.

Simpson's Rule

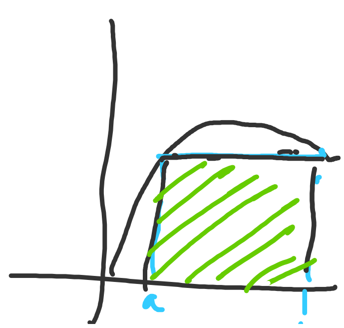
$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{h^5}{90} f^{(4)}(\xi)$$

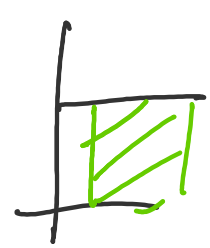
$\Rightarrow$ dop $= 3$. "degree of precision"

$\Rightarrow h$ is distance between points (equispaced)

Riemann Sum?

degree 0.





Similarly to derivatives we have an $(n+1)$-point formula

Thm Suppose that $\sum_{i=0}^{n} a_i f(x_i)$ denotes the $(n+1)$-point formula w/ $x_0 = a$, $x_n = b$, $h = \frac{(b-a)}{n}$. Then there exist $\xi \in (a,b)$ s.t.

x^{n+1}

If n is even

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \underbrace{\frac{h^{n+3}\boxed{f^{(n+2)}}(\xi)}{(n+2)!}\int_0^n t(t-1)(t-2)\dots(t-n)dt}_{\text{error}}$$

If n is odd

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2}\boxed{f^{(n+1)}}(\xi)}{(n+1)!}\int_0^n t(t-1)\dots(t-n)dt.$$

If n is even, deg. of. precision is $\underline{n+1}$

If n is odd, deg. of precision is $\underline{n}$

If n is even, the error will be 0 if the $(n+2)^{nd}$ derivative of f is 0. $\Rightarrow$ x^{n+1} is highest deg. polynomial $\Rightarrow$ d.o.p $n+1$.

"degree of precision"