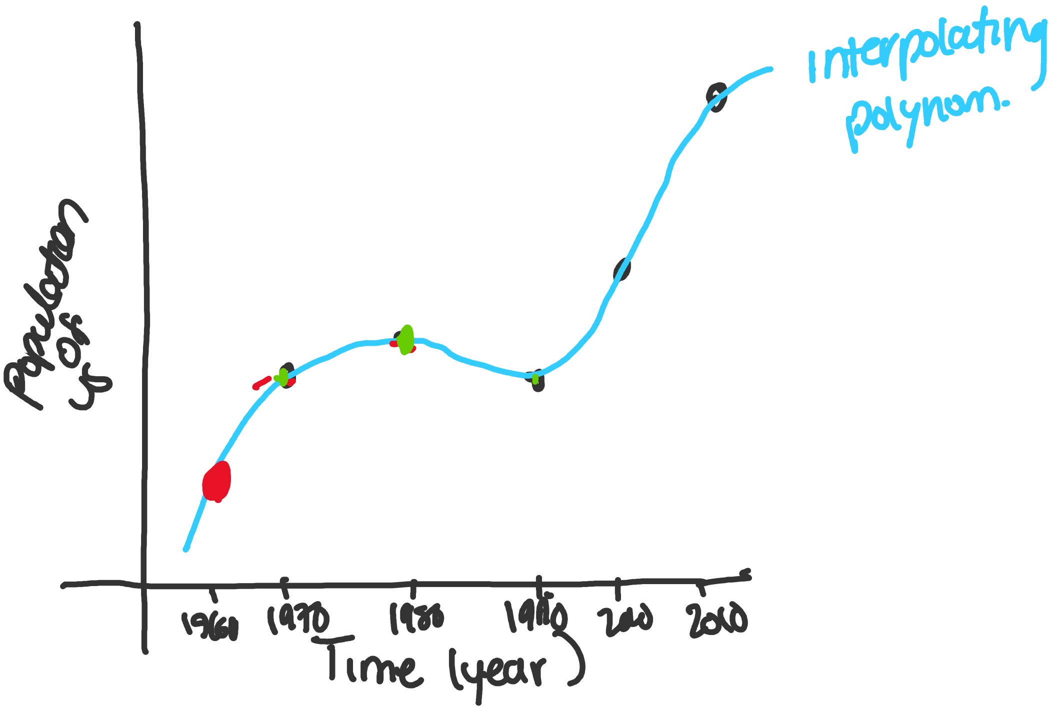
Section 4.1: Numerical differentiation

Tuesday, February 23, 2021 8:23 AM



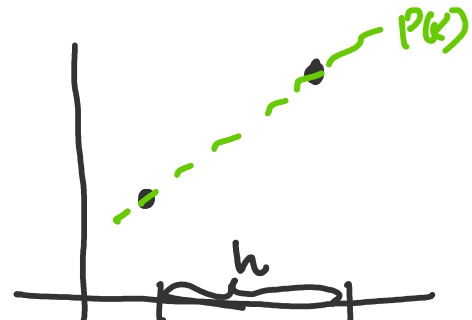
Q: what is the rate at which population is increasing in 1980?

A: what about using Lagrange interpolating polynomial & differentiating it?

Let's consider 2 points $(x_0, f(x_0))$, $(x_1, f(x_1))$

where $x_1 = x_0 + h$.

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$



$$P(x) = f(x_0)\left(\frac{x - x_1}{x_0 - x_1}\right) + f(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)$$

$+ \text{err}\left(\frac{(x-x_0)(x-x_1)}{2!} f''(\xi)\right)$

from §3.1

let $x_1 = x_0 + h$

$$P(x) = f(x_0)\frac{(x-x_0-h)}{-h} + f(x_0+h)\frac{(x-x_0)}{h} + \frac{(x-x_0)(x-x_0-h)}{2}f''(\xi)$$

differentiate

$$P'(x) = \frac{f(x_0)}{-h}[1] + \frac{f(x_0+h)}{h}[1] + \text{Error}$$

$$P'(x) = \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}_{\text{def of deriv.}} + \text{error}$$

If we evaluate at x_0

$$P'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \underbrace{\frac{h}{2}f''(\xi)}_{\text{error.}} \qquad \xi \in [x_0, x_0+h]$$

Note: This formula is called forward difference formula ($h>0$) or backward difference formula ($h<0$). Also called a 2 point formula (since it used 2 pts for interpolating polynom).

ex Use forward diff. formula to estimate derivative of $\ln(x)$ at $x_0=1.8$ using $h=0.1$, $h=0.05$, $h=0.01$.

$h=0.1$

$$f'(1.8) \approx \frac{f(1.9)-f(1.8)}{0.1} = 0.5407$$

$h=0.05$

$$f'(1.8) \approx \frac{f(1.85)-f(1.8)}{} = 0.5480$$

$.05$

$h = 0.01$

$$f'(1.8) \approx \frac{f(1.81) - f(1.8)}{0.01} = 0.5540$$

True answer $= \frac{1}{1.8}$

Q: How can we get more accurate?

A: Add more points to interpolating polynomial?

Recall Lagrange Polynoms

$$f(x) = \sum_{k=0}^{n} f(x_k) L_k(x) + \frac{(x-x_0)(x-x_1) \cdots (x-x_n)}{(n+1)!} f^{(n+1)}(\xi(x))$$

Now take deriv.

$$f'(x) = \sum_{k=0}^{n} f(x_k) L_k'(x) + D[\quad]$$

If we look at a specific data point x_j

$$f'(x_j) = \underbrace{\sum_{k=0}^{n} f(x_k) L_k'(x_j)}_{\substack{(n+1)\text{-point formula to} \\ \text{approximate } f'(x_j)}} + \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \prod_{\substack{k=0 \\ k \neq j}}^{n} (x_j - x_k)$$

ex Lets try with 3 points.

recall

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

took derivative with respect to

x.

$$L_0'(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)}$$

Similarly

$$L_1'(x) = \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)}, \quad L_2'(x) = \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)}$$

$$\Rightarrow f'(x_j) = f(x_0)\left(\frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)}\right) + f(x_1)\left(\frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)}\right)$$
$$+ f(x_2)\left(\frac{2x_j - x_0 - x_1}{(x_2 - x_1)(x_2 - x_0)}\right) + \frac{1}{6} f^{(3)}(\xi_j)(x_j - x_0)(x_j - x_1)$$

Yikes!

ex Assume equally spaced points: $x_1 = x_0 + h$, $x_2 = x_0 + 2h$

THEN we can simplify

3pt endpoint $$f'(x_0) = \frac{1}{2h}\left[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)\right] + \frac{h^2}{3} f^{(3)}(\xi_0)$$

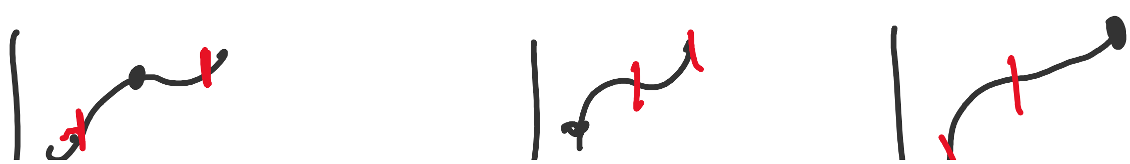
$\xi_0 \in [x_0, x_0 + 2h]$

3pt midpoint $$f'(x_0) = \frac{1}{2h}\left[f(x_0 + h) - f(x_0 - h)\right] - \frac{h^2}{6} f^{(3)}(\xi_1)$$

$\xi_1 \in [x_0 - h, x_0 + h]$

Q: Why is error halved in the midpoint formula

A: midpoint use information from both sides



midpoint endpoint endpoint.

There are also 5-pt formula

5pt endpoint

$$f'(x_0) = \frac{1}{12h}\Big[-25f(x_0) + 48f(x_0+h) - 36f(x_0+2h) + 16f(x_0+3h) - 3f(x_0+4h)\Big] + \underbrace{\frac{h^4}{5}f^{(5)}(\xi)}_{\text{error}}$$

5pt midpoint

$$f'(x_0) = \frac{1}{12h}\Big[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\Big] + \underbrace{\frac{h^4}{30}f^{(5)}(\xi)}_{\text{error}} \quad \xi \in [x_0-2h, x_0+2h]$$

What about higher order derivs?

consider 3rd Taylor polynomial about a point x_0

WANT: $f''(x_0)$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \boxed{\frac{f''(x_0)}{2}}(x-x_0)^2 + \frac{1}{6}f'''(\xi)(x-x_0)^3$$

lets eval @ x_0+h $\quad (x_0+h-x_0)$

$$\Rightarrow f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(\xi)h^3$$

$$\Rightarrow f(x_0-h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \text{error}$$

add these eqns together

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + f''(x_0)h^2 + \frac{h^4}{24}\left[f^{(4)}(\xi) + f^{(4)}(\xi)\right]$$

solve for $f''(x_0)$

$$\boxed{f''(x_0) = \frac{1}{h^2}\Big[f(x_0-h) - 2f(x_0) + f(x_0+h)\Big] - \frac{h^2}{12}f^{(4)}(\xi)}$$

(eq 1.2)

$O(h^3)$

$$2f(x_0+h) = 2f(x_0) + 2f'(x_0)h + f''(x_0)h^2 + h^3$$

$$\frac{1}{h^2}\left[f(0) - 2f(1) + f(2)\right]$$