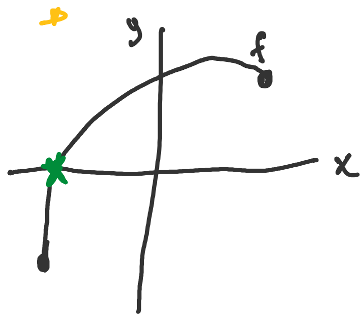
Section 2.3: Newton's Method

Draw an example of a function for which bisection method would be inefficient & $f(-1)=-1$, $f(1)=1$



what properties does this function have?

Steep near root

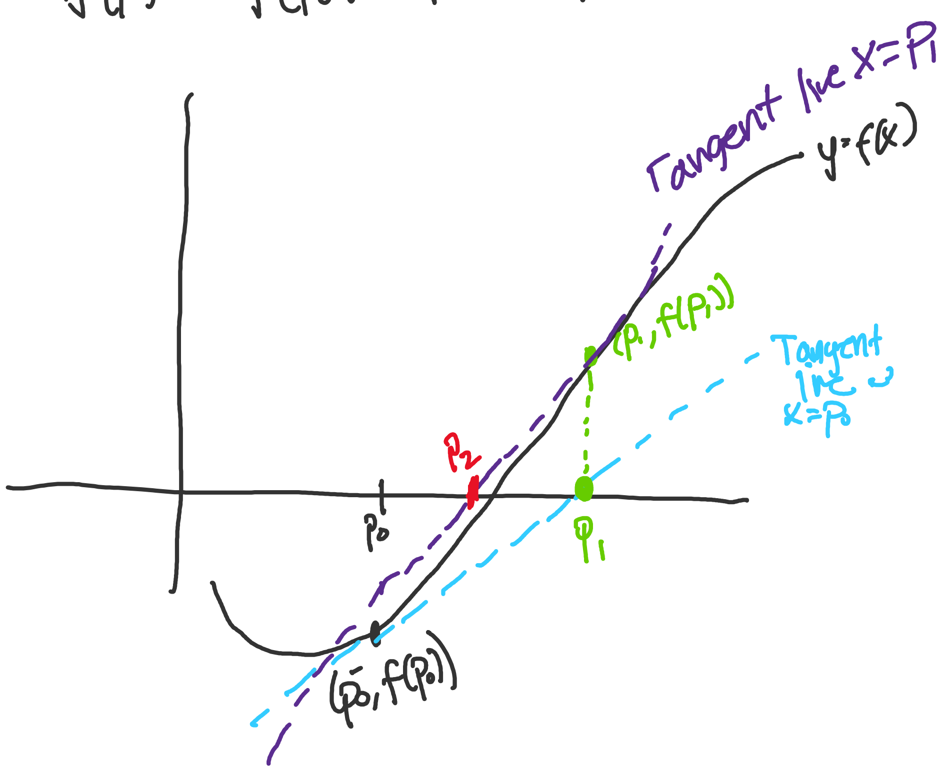
↑ maybe derivatives can give information.

Newton's Method

based on 1^{st} Taylor Polynomial (i.e, Tangent line)

Recall 1^{st} TP of f about p_0

$$f(p) \approx f(p_0) + f'(p_0)(p - p_0)$$



Recall

$$f(p) = f(p_0) + f'(p_0)(p - p_0)$$

Since $f(p) = 0$

$$0 = f(p_0) + f'(p_0)(p - p_0)$$

$$0 \approx f(p_0) + p f'(p_0) - p_0 f'(p_0)$$

solve for p

$$p \approx \underbrace{p_0 - \frac{f(p_0)}{f'(p_0)}}\ \underbrace{:=}_{\text{define calling}}\ p_1$$

Newton's Method

Start w/an initial guess p_0 & generate a sequence $\{p_n\}_{n=0}^{\infty}$ by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad n \geq 1$$

ex/ Apply Newton's method to approximate the root of $f(x) = x^2 - 3 = 0$ w/Newton's Method and an initial guess of $p_0 = 2$.

$$f'(x) = 2x.$$

$$p_0 = 2$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 2 - \frac{(2^2-3)}{2 \cdot 2} = 2 - \frac{1}{4} = 1.75$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = \frac{7}{4} - \frac{(7/4^2 - 3)}{2 \cdot 7/4} \approx 1.732143$$

$p_3 = \ldots 1.73205081$ (8 digits accurate!)

Newton's Method can fail if $f'(p_n) = 0$
(b/c tangent line never crosses axis, divide by 0)

Thm
Let f & its derivative be cont on $[a,b]$ such that $f(p)=0$, $f'(p) \neq 0$. Then $\exists$ a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial guess $p_0 \in [p-\delta, p+\delta]$

"we will converge if sufficiently close to real root"

Pseudocode

inputs: p_0, tol, N, f, f'

output: p, err, n

initialize err, n

while $\underline{n < N}$ and $\underline{\text{err} > \text{tol}}$

* check if $f'(p_0) = 0$.

1. generate next app. $p = p_0 - \frac{f(p_0)}{f'(p_0)}$

2. calculate new err: err $= |p - p_0|$

3. increment n : $n = n+1$

4. reset p_0 : $p_0 = p$.

end % while.

One thing that is difficult about Newton's Method is that we need to calculate derivative by hand

How can we get around this?

<u>Approximate</u> the derivative.

Recall limit def of derivative.

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}$$

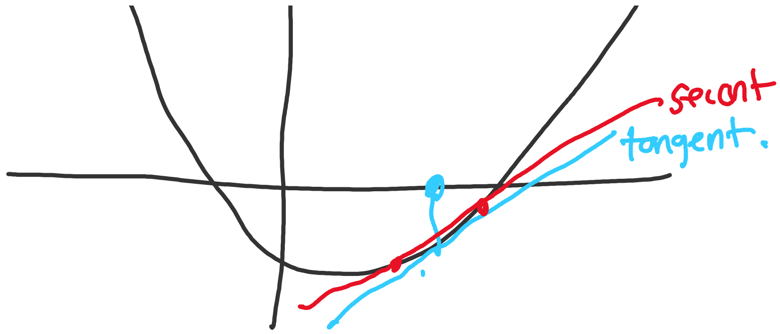
Say p_{n-2} is close to p_{n-1}, then

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{\underbrace{p_{n-2} - p_{n-1}}}$$ "Secant Line"

Plug into Newton's Method

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$ "Secant Method"

We no longer need a derivative, <u>BUT</u> we need 2 initial guesses


secant
tangent.