Section 5.2: Euler's Method

Tuesday, March 9, 2021 10:27 AM

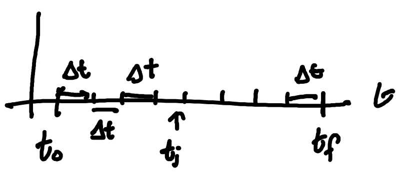
We want to solve $y(t)$ the IVP

$y' = f(t,y)$ ODE

$t_0 \le t \le t_f$ Time domain

$y(t_0) = \alpha$ initial condition.

(1) Discretize over time.



if we sample N points then

$\Delta t = \dfrac{t_f - t_0}{N}$

i.e., $t_j = t_0 + j\Delta t$ for $j = 0, \dots, N$.

$t_{j+1} = t_0 + (j+1)\Delta t = t_j + \Delta t$.

(2) Discretize the equation

$y'(t) = f(t,y)$

how can we approximate $y'(t)$?

numerical diff!

let's use forward difference! (§4.1)

$$y'(t) \approx \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

plug in to ODE

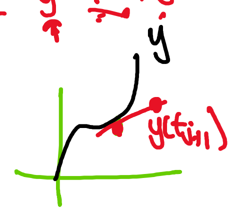
$y'(t_j) = f(t_j, y(t_j))$, $j = 0, \dots, N$

$$\frac{\boxed{y(t_j+\Delta t)} - y(t_j)}{\Delta t} = f(t_j, y(t_j))$$

$$y(t_j+\Delta t) = y(t_j) + \Delta t\, f(t_j, y(t_j))$$

$$y(t_{j+1}) = y(t_j) + \Delta t\, \underbrace{f(t_j, y(t_j))}_{\text{slope.}}$$

$y' = f(t,y)$



Euler's Method is given by

- $y_0 = \alpha$

  $\Delta t = \dfrac{t_f - t_0}{N}$, $t_j = t_0 + j\Delta t$

- $y_{j+1} = y_j + \Delta t\, f(t_j, y_j)$ (note $y_{j+1} = y(t_{j+1})$)

For loop for computing.

Ex Consider IVP $y' = y - e^t$, $[0 \le t \le 1]$, $y(0) = 2$.

★ (1) Well posed? $\quad \Delta t = \frac{t_f - t_0}{N}$

(2) Find exact sol. $\quad 0.25 = \frac{1-0}{N}$

★ (3) Estimate soln using Euler's Method w/ $\Delta t = 0.25$. $(N = 4)$

(1) Well posed? Yes.

- domain convex?

✓

- Lipschitz cond.

$\underline{|\partial f / \partial y| < L?}$

$f(t, y) = y - e^t$

$\frac{\partial f}{\partial y} = 1$

$|\partial f / \partial y| \le 1$ ✓

$\Rightarrow L = 1$

(2) Exact sol?

$y(t) = -e^t(t-2)$ (integrating factors) For student

★ Test case (3) Euler's Method $\Delta t = 0.25$

$f(t,y) = \text{RHS} = \underline{y} - e^t$

$y_0 = 2.$

$y_1 = y_0 + \Delta t (f(t_0, y_0))$

$= 2 + \frac{1}{4}[f(0, 2)]$

$= 2 + \frac{1}{4}[2 - e^0] = 2.25$

t	approx y	exact y
0	2	2
0.25	2.25	2.247
0.5	2.4915	2.473
0.75	2.7022	2.6463
1	2.8485	2.7183

$y_2 = y_1 + \Delta t f(t_1, y_1)$

$= 2.25 + \frac{1}{4}[f(0.25, 2.25)]$

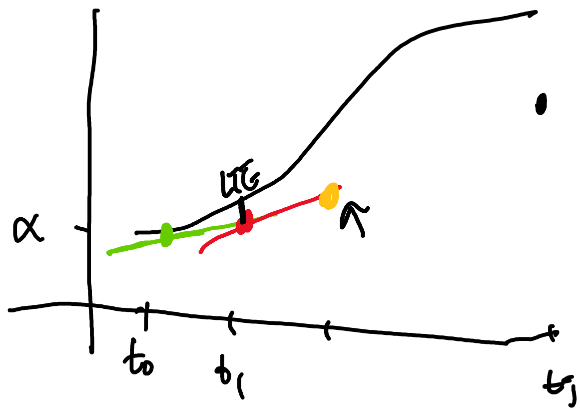
$= 2.25 + \frac{1}{4}[2.25 - e^{0.25}] \approx 2.4915$

$\vdots$

$y_3 = 2.7022$

$y_4 = 2.8485$

As j increases, error increases (i.e., errors accumulate over time)



$$y_{j+1} = y_j + \Delta t \underbrace{f(t_j, y_j)}$$

$$\underbrace{y_{j+1} = y_j + \Delta t\, y'(t_j)}$$

Tangent line

"error" is linear in Δt.

Thm Error bound for Euler's Method

Suppose $f \in C(D)$ satisfies Lipschitz cond. on D. Moreover, suppose $\exists\ M > 0$ such that $|\partial^2 f/\partial t^2| \leq M$ for $M \in [t_0, t_f]$ Then

$y''(\xi)$

$$\underbrace{|y(t_j) - y_E(t_j)|} \leq \frac{\Delta t M}{2L}\left(e^{L(t_j - t_0)} - 1\right)$$

Q: Why does error increase as time increases?

$O(\Delta t)$

ex Find error bound for previous problem & compare to actual error.

$$|y(0.25) - y_E(0.25)| \leq \frac{\Delta t M}{2L}\left[e^{L(t_j - t_0)} - 1\right]$$

$L = 1$ b/c. our previous prob.

find M

recall $f(t, y) = y - e^t$

$\frac{\partial f}{\partial t} = -e^t$

$$\frac{\partial f}{\partial t} = -e^{t}$$

$$\frac{\partial^2 f}{\partial t^2} = -e^{t}$$

$$\left|\frac{\partial^2 f}{\partial t^2}\right| = |-e^{t}| = |e^{t}|$$

when is this largest? what t value gives largest?

t=1, is worst case

$$|\partial^2 f/\partial t^2| \leq e \Rightarrow M = e$$

$$|y(0.25) - y_E(0.25)| \leq \frac{1}{4} \cdot \frac{e}{2 \cdot 1}\left[e^{1\cdot(0.25-0)} - 1\right]$$

$$\leq \sim 0.0965$$

$$|y(0.5) - y_E(0.5)| \leq \frac{1}{4} \cdot \frac{e}{2}\left[e^{(0.5)} - 1\right] \sim 0.2204$$

Q: How could we improve?

A: let's rederive Euler's method from Taylor Series.

Ex Find 1st Taylor Polynomial of y at t_j evaluated at $t_j + \Delta t$

$$\boxed{y(t_j + \Delta t) = y(t_j) + \Delta t\, y'(t_j) + \underbrace{\frac{\Delta t^2}{2} y''(\xi)}_{\text{error}}}$$

isolate

$$\underbrace{\frac{y(t_j + \Delta t) - y(t_j)}{\Delta t}}_{\text{estimate}} - \underbrace{y'(t_j)}_{\text{exact}} = \underbrace{\frac{y''(\xi)}{2}\Delta t}_{\text{error}}$$

$O(\Delta t)$

Q: How can we improve our method?

A: Use Taylor Polynomial w/ more terms

Recall EM.

$$y_{t_{j+1}} = y_{t_j} + \Delta t\, f(t_j, y_j)$$

$$\underbrace{y_{t_{j+1}}}_{y(t_j+\Delta t)} \qquad \underbrace{f(t_j, y_j)}_{y'(t_j)}$$

$t_j = t_0 + j\cdot$

$t_1 = t_0 + \Delta t$

$t_2 = t_1 + \Delta t$

$t_{j+1} = t_j + \Delta$

$$y(t_j+\Delta t) = y(t_j) + \Delta t\, y'(t_j)$$

Recall TP of f about x_0 evaluated at

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)$$

replacing f by y. x_0 by t , x at $t+\Delta t$

$$y(t+\Delta t) = y(t) + y'(t)(t+\Delta t - t) + \frac{y''}{2}$$

$$= y(t) + y'(t)\Delta t + \frac{y''(t)}{2}(\Delta t)^2$$