Section 3.1: Lagrange Polynomials

Tuesday, February 9, 2021 8:50 AM

We use polynomials b/c they are "nice" (easy to differentiate or integrate)

Thm Weierstrauss Approx. Thm

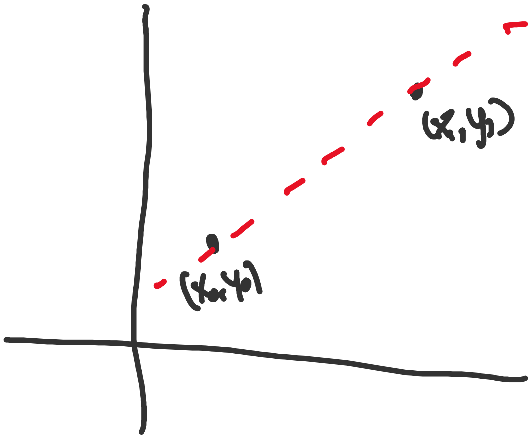
Suppose f is defined & cont on $[a,b]$. For each $\varepsilon > 0$ $\exists$ a polynomial $P(x)$ such that

$$\underbrace{|f(x)-P(x)|}_{\text{abs error.}} < \varepsilon \quad \forall x \in [a,b].$$

"there exists"

"for all"

ex Find equation of line passing thru (x_0, y_0), (x_1, y_1)



find slope:

$$m = \frac{(y_1 - y_0)}{(x_1 - x_0)}$$

find intercept

$$b = y_0 - \frac{(y_1 - y_0)}{(x_1 - x_0)} x_0$$

To put all together

$$y = \frac{(y_1 - y_0)}{(x_1 - x_0)} x + y_0 - \frac{(y_1 - y_0)}{(x_1 - x_0)} x_0$$

We can do this in a slightly different way.

$x=x_1$ at $x=x_0$

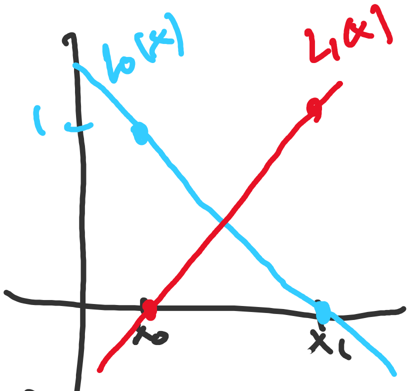
Define

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \begin{cases} 0 \text{ at } x \\ 1 \text{ at } x = x_0 \end{cases} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0} \quad \begin{cases} 0 \\ 1 \text{ at } x = x_1 \end{cases}$$

$L_0(x)$, $L_1(x)$ are linear,

$L_0(x) = 0$ if $x = x_1$

$L_1(x) = 0$ if $x = x_0$



Then the linear Lagrange interpolating polynomial

$$P(x) = L_0(x) \cdot \underbrace{f(x_0)}_{y_0} + L_1(x) \cdot \underbrace{f(x_1)}_{y_1}$$

Sanity check!

Does $P(x_0) = y_0$?

Does $P(x_1) = y_1$?

eg Find Lagrange Interpolating polynomial thru $\underset{\sim}{(2,4)}$, $(5,1)$

$$\left.\begin{aligned} L_0(x) &= \frac{x - x_1}{x_0 - x_1} = \frac{x-5}{2-5} \\ L_1(x) &= \frac{x - x_0}{x_1 - x_0} = \frac{x-2}{5-2} \end{aligned}\right\} \text{"Lagrange Polynomials"}$$

$$P(x) = L_0(x) \cdot y_0 + L_1(x)(y_1)$$

$$= \left(\frac{x-5}{-3}\right) 4 + \left(\frac{x-2}{3}\right) \cdot 1$$

$= 6-x.$

What if we have more then 2 points?

Generalize!

To construct P(x) such that it passes thru

$\{(x_0,y_0),(x_1,y_1) \ldots\ldots, (x_n,y_n)\}$

we will create a <u>basis</u> of polynomials of degree n such that $L_k(x)=1$ when $x=x_k$ and $L_k(x)=0$ when $x=x_j,\ j\neq k$.

Then the interpolating polynomial as

$$P(x)= y_0 L_0(x)+y_1L_1(x)+\cdots+y_nL_n(x).$$

ex/ Let's start with 3 points.

Want to find $L_0(x)$, such that

$\underline{L_0(x_0)=1}$, $\underline{L_0(x_1)=0}$, $\underline{L_0(x_2)=0}$.

gives 0s at $x=x_1, x=x_2$

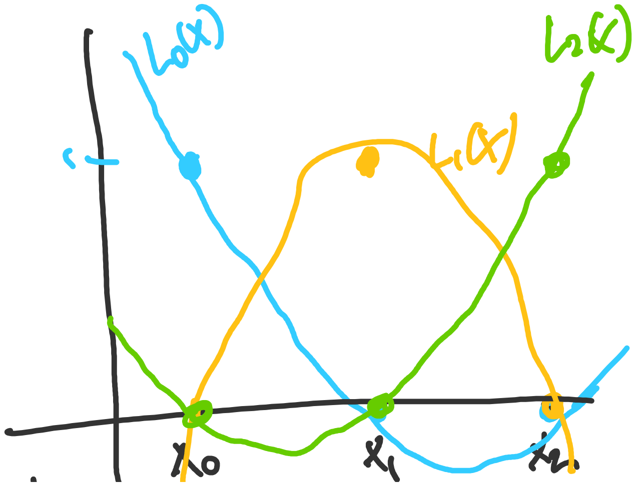
$(x-x_1)(x-x_2)$

if I plug in x_0 above

$(x_0-x_1)(x_0-x_2)$

ensures $L_0(x_0)=1$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$



Now find $L_1(x)$ such that

$$L_1(x_0)=0 \quad L_1(x_1)=1, \; L_1(x_2)=0.$$

$$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$ *check*

Now find $L_2(x)$: $L_2(x)=\dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

To find $P(x)$

$$P(x)=y_0L_0(x)+y_1L_1(x)+y_2L_2(x)$$

Thm

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose vales are given at those number $(y_0, y_1, \ldots, y_n)$ then a unique polynomial $P(x)$ of degree at most n exists w/

$$f(x_k)=y_k=P(x_k) \quad \text{for } k=0,1,\ldots,n$$

& the polynomial is given by "Lagrange Interpolating Polynomial"

$$P(x_0)=f(x_0)\underbrace{\boxed{L_0(x_0)}}_{1}+\underbrace{f(x_1)L_1(x_0)}_{=0}\cdots$$

$$*\; P(x)=f(x_0)L_0(x)+\cdots+f(x_n)L_n(x)=\sum_{k=0}^{n}f(x_k)L_k(x),$$

where

$$*\; L_k(x)=\prod_{\substack{i=0\\ i\neq k}}^{\boxed{n}}\frac{(x-x_i)}{(x_k-x_i)}$$

ensures that $L_k(x_i)=0$

"Lagrange Polynomials"

ensures $L_k(x_k)=1$

$$L_0(x)=\prod_{\substack{i=0\\ i\neq 0}} \rightarrow \prod_{i=1}$$

x= [0:01:2]

ex/ Given $(0, 1.3)$, $(1, -1.2)$, $(2, \sqrt{3})$ find the Lagrange

xdat ydat xdat ydat xdat ydat

Intepolating polynomial going thru all points.

$$\underset{k=0}{L_0(x)} = \prod_{\substack{i=0\\ i\neq 0}}^{2} \frac{(x-x_i)}{(x_0-x_i)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{(x-1)(x-2)}{2}$$

$$L_1(x) = \frac{x^2-2x}{-1}$$

$$L_2(x) = \frac{x^2-x}{2}$$

left as exercise for students ☺

step 1 Find Ls

$$P(x) = \underbrace{1.3}_{y_0} \underbrace{\frac{(x-1)(x-2)}{2}}_{L_0(x)} - \underbrace{1.2}_{y_1} \underbrace{\frac{x(x-2)}{-1}}_{L_1(x)} + \underbrace{\sqrt{3}}_{y_2} \underbrace{\frac{x(x-1)}{2}}_{L_2(x)}$$

step 2 Find P

$$L_1(x) = \prod_{\substack{i=0\\ i\neq 1}}^{2} \frac{(x-x_i)}{(x_1-x_i)}$$

$$= \frac{(x-x_0)}{(x_1-x_0)} \cdot \frac{(x-x_2)}{(x_1-x_2)}$$

Pseudocode

```
inputs: xdat, ydat, x
        data points.
        x: column vector of the x locations we want to interpolate over.
outputs: L, y
         L: matrix
         y: column vector of P evaluated of x

   initialize L, y.
   L: (length(x), length(datx))
   for k going from 1 to N
       N: length(datx)
       col. each Lk
```

Step 1: creating $L_k(x)$

```
    for i going from 1 to N          -- Multiplies $L_k$ by $\left(\frac{x-x_i}{x_k-x_i}\right)$
      for j going from 1 to length(x)
        if i = k
          L = L   (skip)
        else
          $L(j,k) = L(j,k) \cdot \frac{x(j) - datx(i)}{datx(k) - datx(i)}$
kth column -> $L_k = L_k \left(\frac{x - datx(i)}{datx(k) - datx(i)}\right)$    $\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)$
        end % if
      end % for
    end % for
  end % for
```

L =

Step 2: creating interpolant.

```
for k going from 1 to N
  y = y + daty(k) • $L_k$    <- kth col of L
end % for
```

$$\begin{array}{c} 0.25 \\ 1 \\ 1.2 \\ 0.5 \end{array} \rightarrow L = \begin{array}{c|cccc} & (0.5) & (0.7) & (1.3) & (1.5) \\ & L_0 & L_1 & L_2 \cdots & L_n \\ \hline x(1) & L_0(x(1)) & L_1(x(1)) & \cdots & L_n(x(1)) \\ x(2) & L_0(x(2)) & L_1(x(2)) & \cdots & L_n(x(2)) \\ \vdots & \vdots & & & \vdots \\ x(m) & L_0(x(m)) & \cdots & \cdots & L_n(x(m)) \end{array}$$

$$y = \begin{pmatrix} P(x(1)) \\ P(x(2)) \\ \vdots \\ P(x(m)) \end{pmatrix}$$

(daty(1) • L(

$P(x(1)) : daty(1) \cdot L_0(x(1))$

$+ daty(2) \cdot L_1(x(1))$

ex/

4 data points

$(.5, .7)(.7, 1), (1.3, 2)(1.5, 2.2)$

Let's create L.I.P

(1) Make Lagrange polynomial, $L_k(x)$.

want $\boxed{\begin{array}{l} L_k(x_k) = 1 \\ L_k(x_i) = 0 \end{array}}$, $i \neq k$

$L_0(x_0) = 1$ $L_1(x_0) = 0$

$L_0(x_1) = 0$ $L_1(x_1) = 1$

$L_0(x_2) = 0$ $L_1(x_2) = 0$

$L_0(x_3) = 0$ $L_1(x_3) = 0$

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

for loop

$$L_0(x_0) = \frac{x_0-x_1)\cancel{(x_0-x_2)}(x_0-x_3)}{(x_0-x_1)\cancel{(x_0-x_2}(x_0-x_3} = 1$$

$$L_0(x) = \frac{(x-0.7)(x-1.3)(x-1.5)}{(.5-0.7)(0.5-1.3)(0.5-1.5)},$$

(2) . $L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$

$$P(x) = {}_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$P(x_0) = y_0 \underbrace{L_0(x_0)}_{1} + \underbrace{y_1 L_1(x_0)}_{0} + \underbrace{y_2 L_2(x_0)}_{0} + \underbrace{y_3 L_3(x_0)}_{0}$$

$$\downarrow$$

$$P(x_0) = y_0$$