Section 3.2: Newton's divided differences

Tuesday, February 16, 2021 8:37 AM

Lagrange polynomials are prone to error since we are subtracting possibly close terms.

We want to construct the same interpolating polynomial $P(x)$,

To do this, we'll write

$$P(x) = a_0 + a_1\underline{(x-x_0)} + a_2\underline{(x-x_0)(x-x_1)} + \dots$$
$$+ \dots a_n\underline{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}$$

We'll need to find a_i

What is a_0?

$$a_0 = P(x_0) = y_0 = f(x_0)$$

b/c must go thru (x_0, y_0) data point.

What is a_1?

$$P(x_1) = a_0 + a_1(x_1 - x_0) = y_1 = f(x_1)$$

$$a_1 = \underline{\frac{f(x_1)-f(x_0)}{x_1 - x_0}}$$

"slope-y. derivative-y"

What is a_2?

$$P(x_2) = a_0 + a_1(x_2-x_0) + a_2(x_2-x_0)(x_2-x_1)$$

$$\vdots$$

$$a_2 = \frac{f(x_2) - f(x_0) - \overbrace{\boxed{\left(\frac{f(x_1)-f(x_0)}{x_1-x_0}\right)}}^{a_1}(x_2-x_0)}{(x_2-x_0)(x_2-x_1)}$$

ugh complicated

Introduce notation.

$$f[x_0] = f(x_0)$$
$$f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1 - x_0} - \text{"}a_1\text{"}$$
$$f[x_j, x_{j+1}] = \frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j}$$

Using this notation, lets go back to a_2.

$$a_2 = \frac{f[x_1,x_2]-f[x_1,x_0]}{x_2-x_0} \;\; : \;\; \underline{f[x_0,x_1,x_2]}$$

In general this formula is

$$f[x_j, x_{j+1}, \dots, x_{j+k}] = \frac{f[x_{j+1}, \dots, x_{j+k}] - f[x_j, \dots, x_{j+k-1}]}{x_{j+k} - x_j}$$

Okay so let's put it all together

$$a_0 = f[x_0]$$
$$a_1 = f[x_0, x_1]$$
$$a_2 = f[x_0, x_1, x_2]$$
$$\vdots$$
$$a_k = f[x_0, x_1, x_2, \dots, x_k]$$

Then we can write interpolating polynomial

$$P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1 \dots x_k](x - x_0) \cdots (x - x_{k-1})$$

$$= f[x_0] + \sum_{k=1}^{n} f[x_0 \dots x_k] \prod_{i=0}^{k-1} (x - x_i)$$

$N - j + 1$

col 2. $4 - 2 + 1 = 3$

col 3 $4 - 3 + 1$ 2

$$F = \begin{pmatrix} 12 & 1 & -1/6 & \frac{1}{20} \\ 13 & \frac{1}{3} & 2/15 & NaN \\ 14 & 1 & NaN & NaN \\ 16 & NaN & NaN & NaN \end{pmatrix}$$

ex Given the table of values

x	5	6	9	11
y	12	13	14	16

find $P(x)$ using NDD & then estimate the data if $x = 7$.

Test case

F.

i	x_i	$f[x_i]$ (a_0)	$f[x_{i-1}, x_i]$ (a_1)	$f[x_{i-2}, x_{i-1}, x_i]$ (a_2)	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$ (a_3)
0	5	12.	$f[x_0, x_1] = \frac{13-12}{6-5} = 1$	$\frac{\frac{1}{3} - 1}{9 - 5} = -\frac{1}{6}$	$\frac{\frac{2}{15} - (-\frac{1}{6})}{11-5} = \frac{1}{20}$
1	6	13.	$f[x_1, x_2] = \frac{14-13}{9-6} = \frac{1}{3}$	$x_2 - x_0$ $\frac{1 - \frac{1}{3}}{11-6} = \frac{2}{15}$	
2	9	14.	$f[x_2, x_3] = \frac{16-14}{11-9} = 1$	$x_3 - x_1$	
3	11	16.			

Now can construct $P(x)$

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$P(x) = 12 + 1(x - 5) - \frac{1}{6}(x - 5)(x - 6)$$

$$+\frac{1}{20}(x-5)(x-6)(x-9)$$

Test case

$$P(7) = 12 + 1(7-5) - \frac{1}{6}(7-5)(7-6) + \frac{1}{20}(7-5)(7-6)(7-9)$$

$$= 13.4\overline{66}$$

Pseudocode

inputs: datx, daty, x

data points

column vector that are the points we want to interpolate over

outputs: F, y

P evaluated at the x points

NDD Table

initialize

N = length of datx.

F is Nulls of size (N×N)

1st col of F = daty.

for j = 2 to N "columns of F"

for i = 1 to N - j + 1 "rows of F"

$$F(i,j) = \frac{F(i+1, j-1) - F(i, j-1)}{datx(i+j-1) - datx(i)}$$

end %for

end %for

create divided diff table F

initialize G (size length(x), N)

for k = 2:N

for j = 1:k-1

G(k^th col) = G(k^th col) * (x - datx(j))

end %for

end %for

create polynomial

initialize y (colum vector of size n)

for k=1 to N

$$y = y + \underbrace{G(k+n\,col)}_{(x-x_0)\cdots(x-x_{k-1})} \times \underbrace{F(1,k)}_{a_k}$$

end % for

interpolating polynom