Section 2.2: Fixed-Point Iteration

Def The number p is a fixed-point for a given function g if $g(p)=p$.

We can transition from root-finding problems to fixed-point problems easily

ex If we have a root-finding problem $f(p)=0$ we can define g with a fixed point p in many ways

$$g(x)=x-f(x)$$

ex we can transition if g has a fixed point at p, then the function

$$f(x)=x-g(x)$$

has a zero at p.

ex $\ln(x)=x$ ← the value that solves this is a fixed point

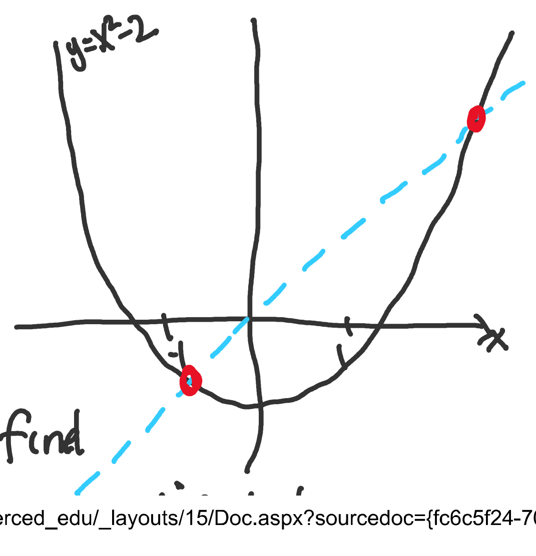
ex Find fixed point(s) of $g(x)=x^2-2$

set $g(p)=p$

$p^2-2=p$

$p^2-p-2=0$

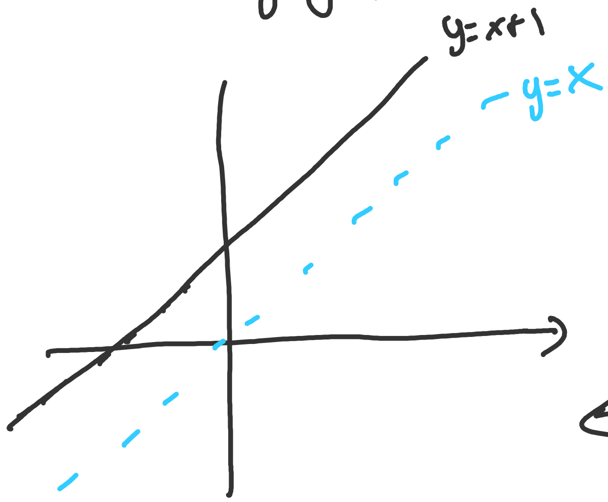
$(p-2)(p+1)$



Graphically we can find

fixed points as the intersection between

- $y = x$
- $y = g(x)$

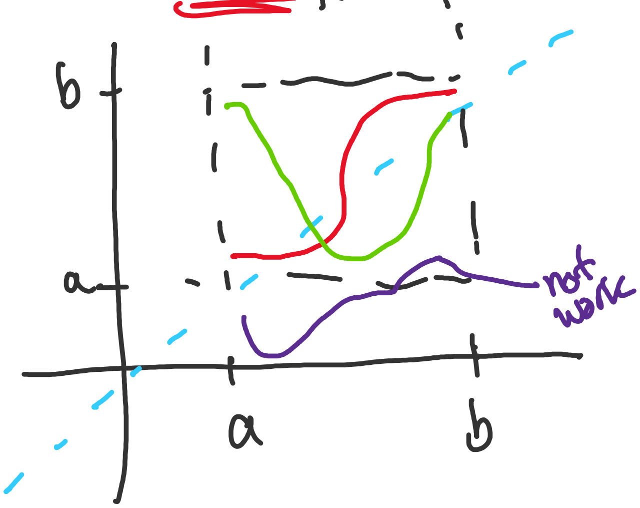


Note: Fixed points do not always exist for all functions

When do fixed points exist?

Thm Existence / Uniqueness of fixed point for $g(x)$ on $x \in [a,b]$

If g is cont on $[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$ then g has at least 1 fixed point in $[a,b]$. If, in addition, $g'(x)$ exists on (a,b) and $|g'(x)| \leq k$ for all $x \in (a,b)$, $k<1$, then there exist one unique fixed point in $[a,b]$



pf let $h(x) = g(x) - x$

$h(a) > 0$

$h(b) < 0$

$\Rightarrow$ by IVT there is a $p \in (a,b)$ such that $h(p) = 0$

unique $\Rightarrow$ MVT

ex Show that $g(x) = \frac{(x^2-1)}{3}$ has a unique fixed point in the interval $[-1,1]$.

(1) is $g(x)$ cont. on $[-1,1]$? ✓

(2) does $g(x) \in (-1,1)$ when $x \in (-1,1]$? ✓

recall Calc I

If we find minima & maxima, evaluate endpoints

find critical points ($g'(x)=0$)

$$g'(x) = \frac{2}{3}x$$

$$g'(x) = 0 \text{ if } x = 0$$

evaluate g at critical points & endpoints

$$g(-1) = \frac{((-1)^2-1)}{3} = 0$$

$$g(0) = \frac{(0^2-1)}{3} = -\frac{1}{3}$$

$$g(1) = \frac{(1^2-1)}{0} = 0.$$

$$\Rightarrow g(x) \in [-\tfrac{1}{3}, 0] \text{ for } x \in [-1,1]$$

(3) Is $|g'(x)| < 1$? (uniqueness) ✓

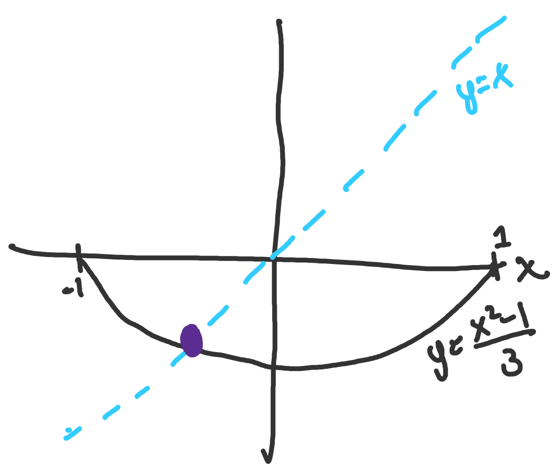
$$g'(x) = \frac{2x}{3}$$

$$|g'(x)| = \left|\frac{2x}{3}\right|$$

if $x \in (-1,1)$

$\leq \frac{2}{3} < 1$

$x=1$ or $x=-1$ "worst case scenario"



The fixed point iteration:

Start by choosing an initial guess p_0 & generate a sequence $\{p_k\}_{k=0}^{\infty}$ by defining $p_k = g(p_{k-1})$. If the sequence converges to p & g is cont.

$$p = \lim_{k\to\infty} p_k = \lim_{k\to\infty} g(p_{k-1})$$

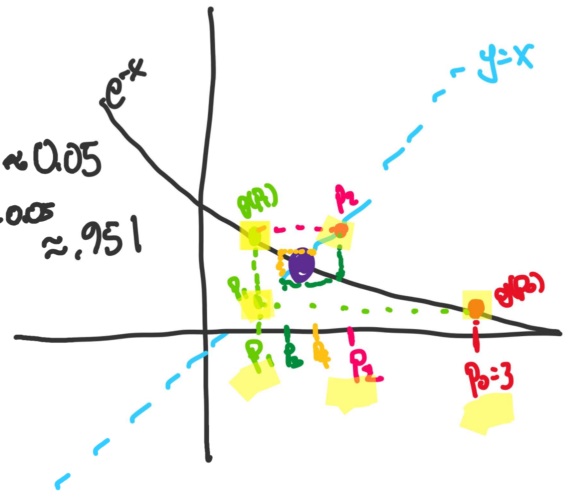
ex ★ USE FOR DEBUGGING CODE ★

Find the fixed point of e^{-x} using FPI with $p_0 = 3$. Find p_2.

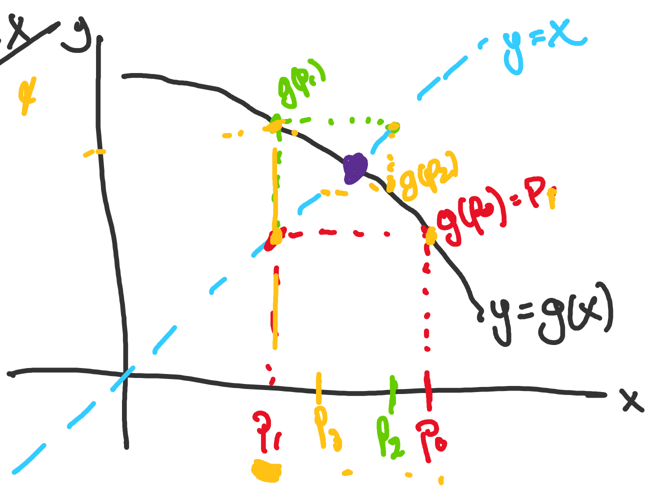
$p_0 = 3$

$p_1 = g(p_0) = e^{-3} \approx 0.05$

$p_2 = g(p_1) = e^{-.05} \approx .951$

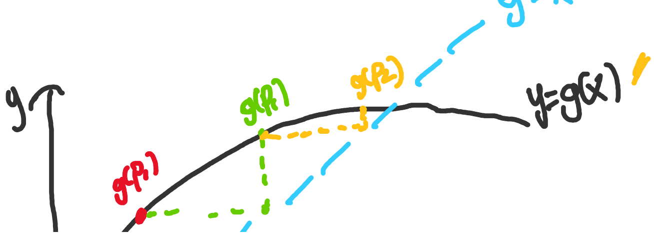


ex

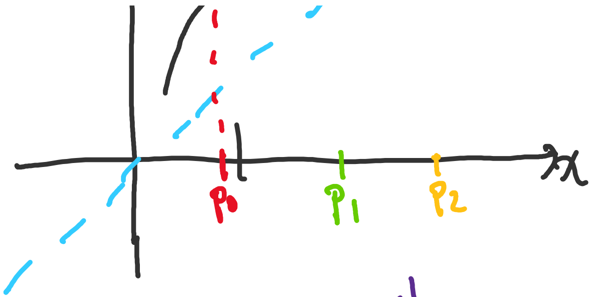


We expect to converge to the fixed point

ex

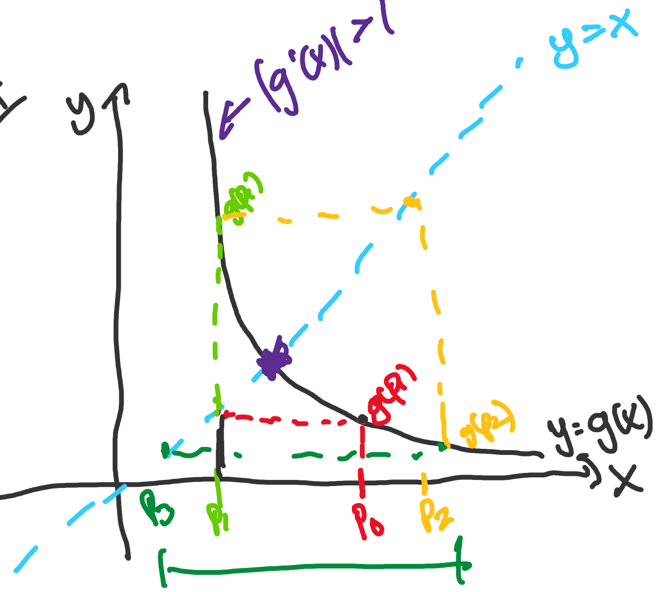


We expect to converge to



the fixed point.

ex



we do not expect to converge to the fixed point.

Thm Guaranteed to converge to fixed point.

Let our conditions for existence of a unique fixed point hold. Then the sequence $p_n = g(p_{n-1})$ converges to the unique fixed point.

PSEUDOCODE for FPI

Inputs: p_0, N, g, tol

initial guess (p_0), max # iterations (N), function to find fixed point in (g), error tolerance we want to achieve (tol)

Outputs: c, err, n

approx of fixed point (c), # of iterations we ran (n)

Note: we estimate error as

$|p_n - p_{n-1}|$

$|g(p_n) - p_n|$

initialize n, err

while err>tol and $\boxed{n \le N}$

  $p = g(p_0)$

  err $= |p - p_0|$

  $n = n+1$

set $p_0 = p$

end %while

I want to continue doing the method if error is larger than I'd like or if the maximum # iters has not been reached