Section 1.2: Taylor Series

Recall Taylor Series of $f(x)$ about the point x_0

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Taylor Series approximate functions

Def

Suppose $f \in \underbrace{C[a,b]}_{\text{continuous on } [a,b]}$, $f^{(n+1)}$ exists on $[a,b]$.

For every $x \in [a,b]$ $\exists$ "there exists" a number $\xi \in [a,b]$ such that

$$f(x) = \underbrace{T_n(x)}_{\text{Taylor Polynomial of degree } n} + \underbrace{R_n(x)}_{\text{Remainder term.}}$$

where

$$T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$R_n(x)$ is our 'remainder' (error)

"Truncation error"

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \quad \xi \in [a,b]$$

ex

Let $f(x) = \cos(x)$. Determine the $\underline{2^{nd}}$ Taylor Polynomial of f about $\underline{x_0 = 0}$.

$$f(x) \approx T_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{\quad}{2}(x-x_0)^2$$

$$= \cos(0) + (-\sin(0))(x-0) - \frac{\cos(0)}{2}(x-0)^2$$

$$\boxed{T_2(x) = 1 - \frac{1}{2}x^2}$$

Look @ Remainder term

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-0)^3$$

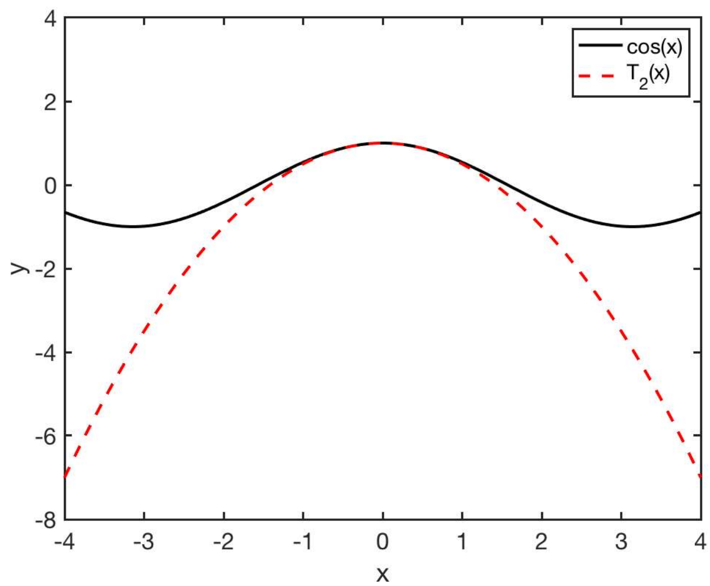
$$= \frac{\sin(\xi)}{6}x^3$$

$$R_2(x) \le \frac{|\sin(\xi)|}{6}x^3$$

$$R_2(x) \le \frac{x^3}{6} \qquad \xi \in [a,b]$$

$$[-\pi, \pi]$$

Comparison between $f(x) = \cos(x)$ and $T_2(x) = 1 - \frac{x^2}{2}$



ex) Find 4^{th} Taylor Polynomial of $\cos(x)$ about $x_0 = 0$.

a) $T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 +$$

$$\frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f''''(x_0)}{4!}(x-x_0)^4$$

$$= \cos(0) - \cancel{\sin(0)(x-0)} - \frac{\cos(0)}{2}(x-0)^2 + \cancel{\frac{\sin(0)}{6}(x-0)^3} + \frac{\cos(0)}{24}(x-0)^4$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

b) $R_4(x) = \dfrac{f^{(5)}(\xi)}{5!}(x)^5$

$$= \frac{-\sin(\xi)}{120}x^5$$

$$R_4(x) \leq \frac{|-\sin(\xi)|}{120}x^5$$

when is $|-\sin(x)|$ largest over $[-\pi, \pi]$?

$$R_4(x) \leq \frac{x^5}{120}$$

What is error bound of $R_4(x)$ if $x \in [0, \frac{\pi}{4}]$?

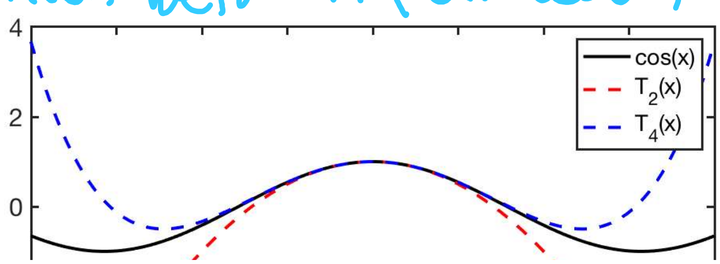
$$R_4(x) \leq \frac{|-\sin(\xi)|}{120}x^5$$

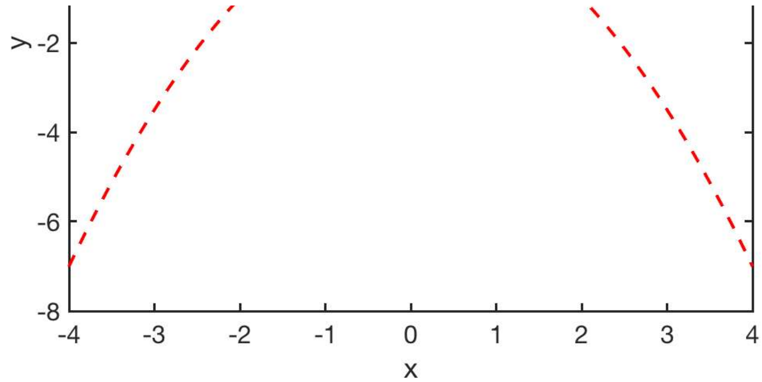
when is $|-\sin(x)|$ largest over $[0, \frac{\pi}{4}]$?

$$|-\sin(\tfrac{\pi}{4})| = \frac{\sqrt{2}}{2}$$

$$R_4(x) \leq \frac{\sqrt{2}}{240}x^5$$

Comparison between $f(x) = \cos(x)$, $T_2(x)$, $T_4(x)$




y
-2
-4
-6
-8
-4 -3 -2 -1 0 1 2 3 4
x