Section 5.3: Taylor Methods

Tuesday, March 16, 2021 8:57 AM

Recall Taylor Polynomials

lets do a TP about t_j for y and evaluate at $t_j+\Delta t$.

ex Recall TP of f at x_0 evaluated at x is given by

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\dots$$

To do TP of y at t_j evaluated at $t_j+\Delta t$

replace f by y.
x_0 by t_j
x by $t_j+\Delta t$

$$y(t_j+\Delta t)=y(t_j)+y'(t_j)(t_j+\Delta t-t_j)+\frac{y''(t_j)}{2}(t_j+\Delta t-t_j)^2+\dots$$

$$=y(t_j)+y'(t_j)\Delta t+\frac{y''(t_j)}{2}\Delta t^2+\frac{y'''(t_j)}{3!}\Delta t^3+\dots\frac{y^{(n)}(t_j)}{n!}\Delta t^n+\text{err}$$

where $\text{err}=\frac{y^{(n+1)}(\xi)}{(n+1)!}\Delta t^{n+1}$ $\quad \xi\in[t_j,t_j+\Delta t]$

The Taylor Method of order 1 is Euler's Method

$$y(t_j+\Delta t)=y(t_j)+\underbrace{y'(t_j)}_{f(t_j,y_j)}\Delta t+\text{err}$$

$\longrightarrow$ since IVP $y'=f(t,y)$

How can we compare methods?

Local Truncation Error (LTE)

if model/method

$$w_{j+1}=w_j+\Delta t(\phi(t_j,w_j))$$

the actual soln is

$y'(t_j)=f(t_j,y_j)$ so LTE

$$\tau_{j+1}(\Delta t)=\frac{y(t_{j+1})-y(t_j)}{\dots}-\phi(t_j,w_j)$$

Δt

"local truncation error"

ex For Euler's Method (TM of order 1), what is LTE?

$$y(t_{j+1}) = y(t_j) + \Delta t f(t_j, y_j) + \underbrace{\frac{y''(\xi)}{2}}\Delta t^2$$

$$y(t_{j+1}) - y(t_j) - \Delta t f(t_j, y_j) = \frac{y''(\xi)}{2}\Delta t^2 \text{ error.}$$

$$\frac{y(t_{j+1}) - y(t_j)}{\Delta t} - f(t_j, y_j) = \frac{y''(\xi)}{2}\Delta t$$

$\phi(t_j, y_j)$

we can also get ϕ from Method:

$$y(t_{j+1}) = y(t_j) + \Delta t\,\boxed{f(t_j, y_j)}$$

LTE is $O(\Delta t)$

ex Find Taylor Method of order 2 & its LTE.

(1) Write TP of order 2.

$$y(t_j + \Delta t) = \underbrace{y(t_j) + \Delta t\, y'(t_j) + \frac{\Delta t^2}{2} y''(t_j)}_{\text{Method}} + \underbrace{\boxed{\frac{y'''(\xi)}{3!}\Delta t^3}}_{\text{err.}} + \frac{y^{(4)}(\cdot)}{4!}$$

$$\boxed{w_{j+1} = w_j + \Delta t\left[\underbrace{f(t_j, y_j) + \frac{\Delta t}{2} f'(t_j, y_j)}_{\phi}\right]}$$

$t_{j+1} = t_j + \Delta t$

Also

$$w_{j+1} = w_j + \Delta t f(t_j, y_j) + \frac{\Delta t^2}{2} f'(t_j, y_j)$$

(2) Rewrite to find LTE.

$$y(t_j + \Delta t) - y(t_j) - \Delta t\, y'(t_j) - \frac{\Delta t^2}{2} y''(t_j) = \frac{y'''(\xi)}{6}\Delta t^3.$$

$$\underbrace{\frac{y(t_j+\Delta t)-y(t_j)}{\Delta t}}_{\text{approx of } y'(t_j)} - y'(t_j) - \frac{\Delta t}{2}y''(t_j) = \frac{y'''(\xi)}{6}\Delta t^2$$

$$\boxed{LTE = \frac{y'''(\xi)}{6}\Delta t^2}$$

$$O(\Delta t^2)$$

$$y' = f(t,y)$$

ex Apply TM2 for $y' = y - t^2 + 1$, $0 \le t \le 2$, $y(0) = 0.5$ & $N = 10$.

Test case for code

$$\Delta t = \frac{t_f - t_0}{N} = \frac{2-0}{10} = 0.2$$

$$f(t,y) = y - t^2 + 1$$

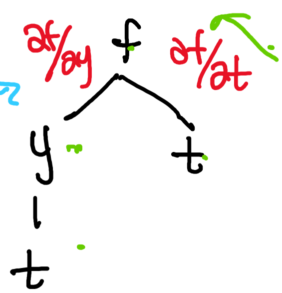
$$\begin{cases} w_0 = 0.5 \\ w_{j+1} = w_j + \Delta t\, f(t_j, y_j) + \frac{\Delta t^2}{2} f'(t_j, y_j) \end{cases}$$

$f'(t,y) = ?$ chain rule

$$= \frac{\partial f}{\partial y}\cdot\underbrace{\frac{dy}{dt}}_{y'=f(t,y)} + \frac{\partial f}{\partial t}$$

$$= 1\cdot(y - t^2 + 1) + (-2t)$$

$$= y - t^2 + 1 - 2t.$$



$$w_{j+1} = w_j + \Delta t\left[w_j - t_j^2 + 1\right] + \frac{\Delta t^2}{2}\left[w_j - t_j^2 + 1 - 2t_j\right]$$

$$w_0 = 0.5$$

$$w_1 = w_0 + 0.2\left[0.5 - 0^2 + 1\right] + \frac{0.2^2}{2}\left[0.5 - 0 + 1 - 0\right]$$

"Think Pseudocode"

TM2

Inputs: f, t_0, t_f, N, f' ← you have to calculate :(

B.O. Find TM3 & its associated LIE. check out Assignment 5.

(1) Include names of all present

(2) doc/image of what your group found.