Section 5.1: Initial Value Problems

Tuesday, March 9, 2021 8:58 AM

Def An initial value problem (IVP) is of the form

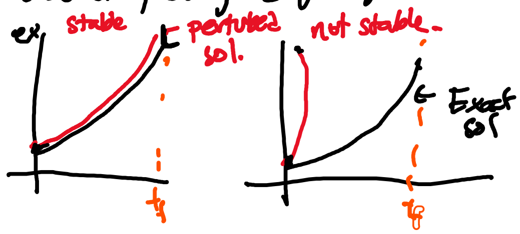
$$\frac{dy}{dt} = f(t, y) \quad \leftarrow \text{ODE}$$
$$y(t_0) = \alpha \quad \leftarrow \text{I.C.}$$
$$t_0 \le t \le t_f \quad \leftarrow \text{time domain}$$

Some questions we might ask about IVP

1) Does the IVP have a solution? (Existence)
2) Is the solution unique? (Uniqueness)
3) Is the solution stable? (stability)

↑ if we perturb the initial condition slightly, does the solution also only change slightly.

ex,



An ODE/IVP is "well-posed" if it exists, is unique, and is stable (1–3)

Def A function $f(t, y)$ satisfies Lipschitz condition in variable y on a domain $D \in \mathbb{R}^2$ if $\exists$ a constant $L > 0$ s.t.

"such that" "there exists"

$$|f(t, y_1) - f(t, y_2)| \le L\,|y_1 - y_2|$$

if $(t, y_1), (t, y_2) \in D$.

we can kind of rewrite

$$\frac{|f(t, y_1) - f(t, y_2)|}{|y_1 - y_2|} \le L$$

Math 23! Calc 3

$\approx \left|\partial f/\partial y\right| \leftarrow$ the partial deriv of f with respect to y.

with respect to y

ex/ Show that $f(t,y) = t|y|$ satisfies Lipschitz condition on $D = \{(t,y),\ 1 \le t \le 2,\ -3 \le y \le 4\}$

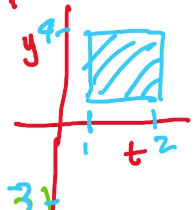
plug in t, y_1, y_2 into my f function

$$|f(t,y_1) - f(t,y_2)| = \big|\, t|y_1| - t|y_2| \,\big|$$
$$= |t|\,\big|\, |y_1| - |y_2| \,\big|$$
$$\le 2\, |y_1 - y_2|$$

since $t \in (1,3)$

"worst case scenario" $\le$

Domain



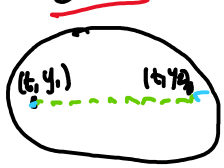
$\Rightarrow$ $t|y|$ satisfies Lipschitz condition on D with a Lipschitz constant of 2 (L)

ex if in prev. prob $-4 \le t \le 2$, then L constant $= 4$

Thm If f is defined on a convex set & if $\exists\, L \ge 0$ s.t. $|\partial f/\partial y| \le L$, then f satisfies the Lipschitz condition & the IVP is "well-posed".

$D = \{(1$

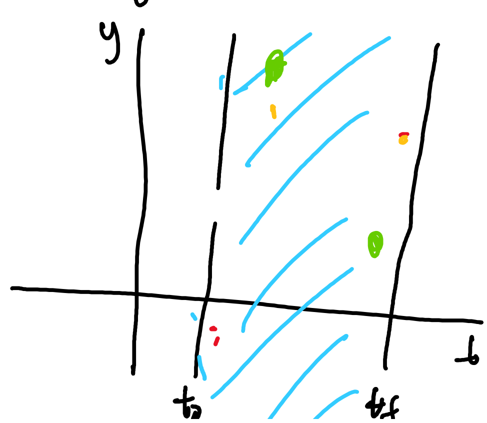
Convex



Not Convex

Note: Usually we consider Domain $D = \{(t,y),\ t_0 \le t, t_f,\ -\infty < y < \infty\}$

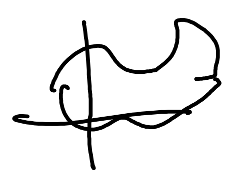
This domain is convex!



calc3

$-2 \le x \le 2$

$f(x) \le y \le g(x)$



ex Is the IVP $y' = f(t,y)$

$$y' = y\cos(t), \quad 0 \le t \le \pi, \quad y(0) = 1$$

well posed?

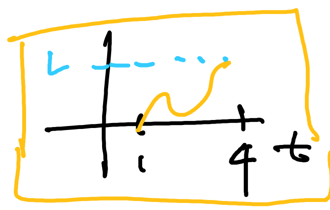
(1) Is domain convex? ✓

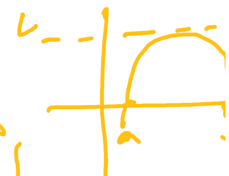
(2) Lipschitz cond $\left(\left|\frac{\partial f}{\partial y}\right| \le L ? \right)$ or $\left(|f(t,y_1) - f(t,y_2)| \le \right.$

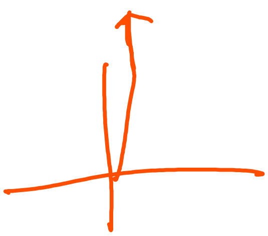
Is partial $\frac{\partial f}{\partial y}$ bounded?

$$\frac{\partial f}{\partial y} = \cos t$$

$$\left|\frac{\partial f}{\partial y}\right| = |\cos(t)|$$

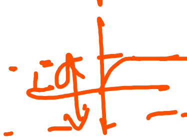






$$\le 1$$

↑ since $\cos(t) \in (-1,1)$ on interval $[0,\pi]$



$L = 1$.

Yes well-posed.

$\frac{\pi}{4} \le t \le \frac{\pi}{2}$

max of $\cos(t)$ on $\frac{\pi}{4} \le t \le \frac{\pi}{2}$

$L = \frac{\sqrt{2}}{2}$

prev. def

Try to find L s.t

$$|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$$

→ plug in f.

$$|f(t,y_1) - f(t,y_2)| = |y_1\cos t - y_2\cos t|$$

$$= |\cos(t)(y_1 - y_2)|$$

$$= |\cos(t)||y_1 - y_2|$$

↑ highest value $\cos(t)$ can take on interval $[0,\pi]$

$$\le 1 \cdot |y_1 - y_2|$$

if $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$

$$= |\cos(t)||y_1 - y_2|$$

$$\leq \frac{\sqrt{2}}{2}|y_1 - y_2|$$

↑ since highest val of $\cos(t)$ on interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.