Section 4.3: Composite Numerical Integration

Tuesday, March 2, 2021 9:55 AM

ex

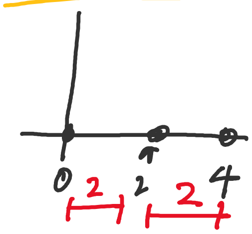
a) Use Simpson's Rule to approximate $\int_0^4 e^x dx$

exact $\int_0^4 e^x dx = e^x\Big|_0^4 = e^4 - 1$

$h = 2$, $x_0 = 0$, $x_1 = 2$, $x_2 = 4$.

$\frac{2}{3}(e^0 + 4e^2 + e^4) \approx 56.76958$

err ~ 3.17143



n=2.

b) Estimate $\int_0^2 e^x dx + \int_2^4 e^x dx$

$h = 1$. $h = \frac{b-a}{2}$ (for S.R)

$\int_0^2 e^x dx + \int_2^4 e^x dx = \frac{1}{3}(e^0 + 4e + e^2) + \frac{1}{3}(e^2 + 4e^3 + e^4)$

$= \frac{1}{3}(e^0 + 4e + 2e^2 + 4e^3 + e^4)$

≈ 53.86385 (err ~ 0.2657).

n=4

c) Estimate $\int_0^1 e^x dx + \int_1^2 e^x dx + \int_2^3 e^x dx + \int_3^4 e^x dx$

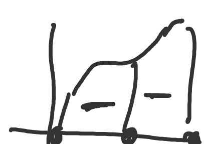
$h = 1/2$ $\left(\frac{b-a}{2}\right)$

$\frac{1}{6}(e^0 + 4\ldots) + \frac{1}{6}(e^1 + \ldots) + \frac{1}{6}(e^2 \ldots) + \frac{1}{6}(e^3 + \ldots)$

≈ 53.61622 (err ~ -0.01807)

n=8

Let's generalize to any number of subintervals n (where n is even) for $\int_a^b f(x)dx$.



$h = \frac{(b-a)}{n}$

$x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$ CSR

$x_0 = a, \; x_1 = a + h, \ldots$

$x_j = a + jh.$

CTR

$\int_a^b f(x)dx =$

$$\underbrace{\sum_{j=1}^{n/2} \underbrace{\frac{h}{3}\left[f(x_{2j-1}) + 4f(x_{2j}) + f(x_{2j+1})\right]}_{\text{SR (not composite)}}}_{\text{composite}}$$

$\frac{h}{3}[f$

$$= \underbrace{\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}_{j=1} + \underbrace{\frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)]}_{j=2}$$

$$+ \ldots + \underbrace{\frac{h}{3}[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]}_{j = N/2}$$

Simpson's Rule ext. int

we can simplify since x_{2j} term shows up twice.

$$= \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)\right]$$

Formula.

even terms except beginning & end

odd terms

this sum goes to $n/2 - 1$ b/c if it went to $n/2$, we would get $2f(x_n)$

Composite Simpson's Rule

f and its derivs (up to 4th deriv.) are cont. on $[a,b]$ $(f \in C[a,b])$

Let $f \in C^4[a,b]$, n even, $h = \frac{(b-a)}{n}$ and $\boxed{x_j = a + jh}$ for $j = 0, \ldots, n$. Then the CSR for n subintervals can be written as

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{N/2-1} f(x_{2j}) + 4\sum_{j=1}^{N/2} f(x_{2j-1}) + f(b)\right]$$

$$\underbrace{- \frac{(b-a)}{180} h^4 f^{4}(\xi)}_{\text{err.}} \quad \xi \in [a,b]$$

ex/ Estimate $\int_0^4 e^x dx$ w/ $N=4$.

(prob 1b from BO room)

$h = \frac{b-a}{} \quad \frac{4-0}{} = 1$

$h = \frac{b-a}{n} = \frac{4-0}{4} = 1.$

$$= \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{1} f(x_{2j}) + 4\sum_{j=1}^{2} f(x_{2j-1}) + f(b)\right]$$

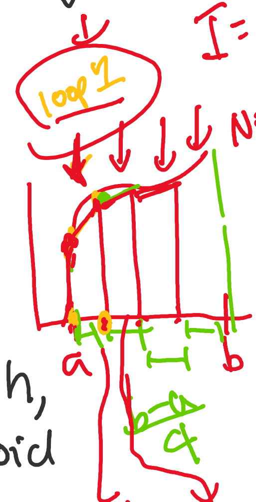
$x_0 = a,\ x_1 = a+h,\ x_2 = a+2h,\ x_3 = a+3h,\ x_4 = a+4h$

$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$

$$= \frac{1}{3}\left[f(0) + 2\underbrace{f(x_2)}_{f(2)} + 4[\underbrace{f(x_1)}_{j=1} + \underbrace{f(x_3)}_{j=2}] + f(x_4)\right]$$

$$= \frac{1}{3}\left[e^0 + 2e^2 + 4e + 4e^3 + e^4\right]$$

$I = I + \frac{h}{2}[f($



$x_1 = a+h,\ x_2 = a+2h$

$x_1 = a+h$

$\frac{b-a}{4}$

$\frac{h}{2}[f(x_j) + f(x_{j+1})]$

We can do same for TR.

Composite Trapezoid Rule

Let $f \in C^2[a,b]$, $h = \frac{(b-a)}{N}$ and $x_j = a + jh$, $j = 0, \dots, n$. Then the composite trapezoid rule for n subintervals is written as

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + \boxed{2\sum_{j=1}^{n-1} f(x_j)} + f(b)\right] - \frac{(b-a)}{12} h^2 f''(\xi) \quad \xi \in [a,b]$$

$\sum_{j=1}^{n-1} f(x_{j-1}) + 2\sum_{j=1}^{n-1} f(x_j$

$\frac{h}{2}\sum_{j=1}^{n-1}[f(x_{j-1}) + 2f(x_j) + f($

$f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f$

ex. Determine # subintervals (N) to ensure an error < 0.00002 when approximating $\int_0^{\pi} \sin(x)dx$ for

(a) Composite trapezoid Rule

(b) Composite Simpson's Rule.

$\frac{h}{2}\sum_{j=1}^{n}[f(x_j) + f(x$

$\int_a^b f(x)dx$

$\int_0^{\pi} \sin(x)dx$

$= f(x_1) + f(x_2) +$

$+ f(x_3)$

(a). Error term for CTR

$$\left|\frac{(b-a)}{12} \cdot h^2 \cdot f''(\xi)\right| \quad \xi \in [0,\pi]$$

$\left|\frac{(\pi-0)}{12} \cdot \frac{(\pi-0)^2}{} \cdot \sin(\xi)\right|$

$$\left|\frac{(\pi-0)}{12}\cdot\left(\frac{\pi-0}{N}\right)\cdot(-\sin(\xi))\right|$$

$$\frac{\pi}{12}\cdot\frac{\pi^2}{N^2}\left|-\sin(\xi)\right|$$

since $\sin(x)\in[-1,1]$

$$\text{err.}\leq\frac{\pi^3}{12N^2}$$

so we want

$$\frac{\pi^3}{12N^2}<0.00002$$ solve for N

$$N>\sqrt{\frac{\pi^3}{12\cdot(0.00002)}}\approx 359.44$$

So we need ≥ 360 subintervals,

(b) Error term for CSR

$$\left|\frac{(b-a)}{180}h^4 f^4(\xi)\right|$$

$$\left|\frac{(b-a)}{180}\left(\frac{b-a}{N}\right)^4\sin(\xi)\right|$$

$$\left|\frac{(\pi-0)}{180}\left(\frac{\pi-0}{N}\right)^4\sin(\xi)\right|$$

$$\frac{\pi^5}{180N^4}\left|\sin(\xi)\right|$$

$$\text{err}\leq\frac{\pi^5}{180N^4}$$

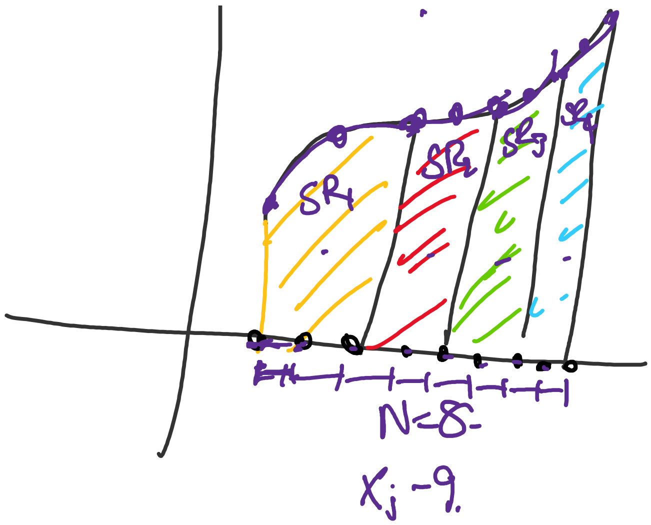
So now we <u>want</u> err <0.00002

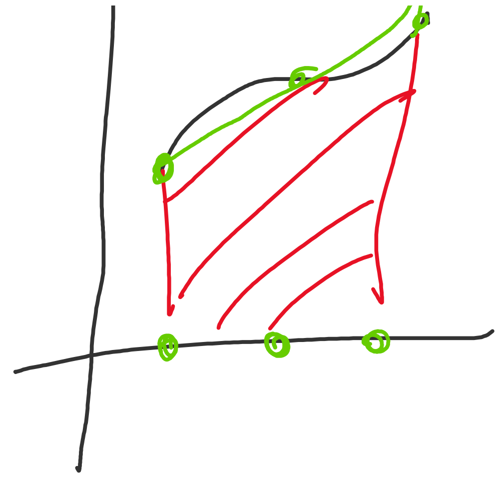
$$\frac{\pi^5}{180N^4}<0.00002$$ ← solve for N

$$N>\left(\frac{\pi^5}{180\,(0.00002)}\right)^{1/4}\approx 17.07$$

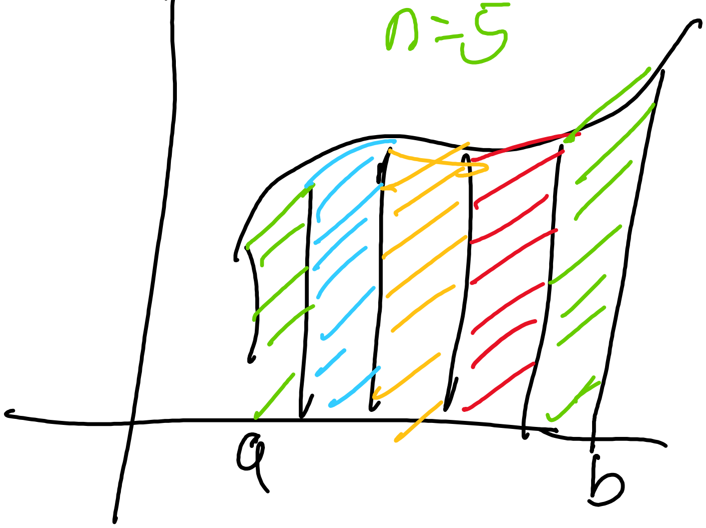
$\Rightarrow$ we need at least 18 subintervals.

CSR_1 SR

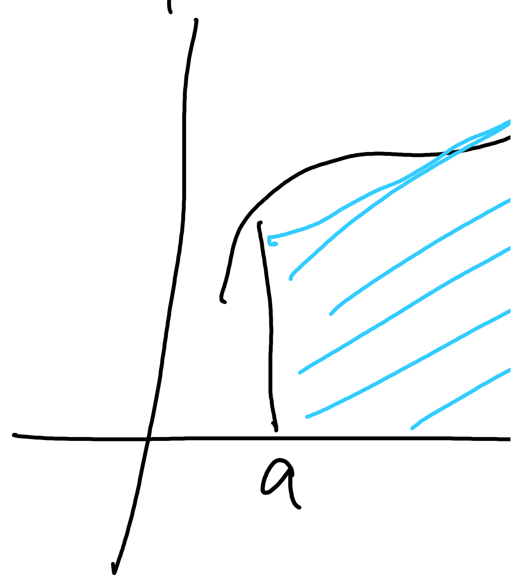




CTR



TR



$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$\int_0^4 e^x dx$ $\qquad$ $\int_4^0 e^x dx$

$n=4$ $\qquad$ $n=4$

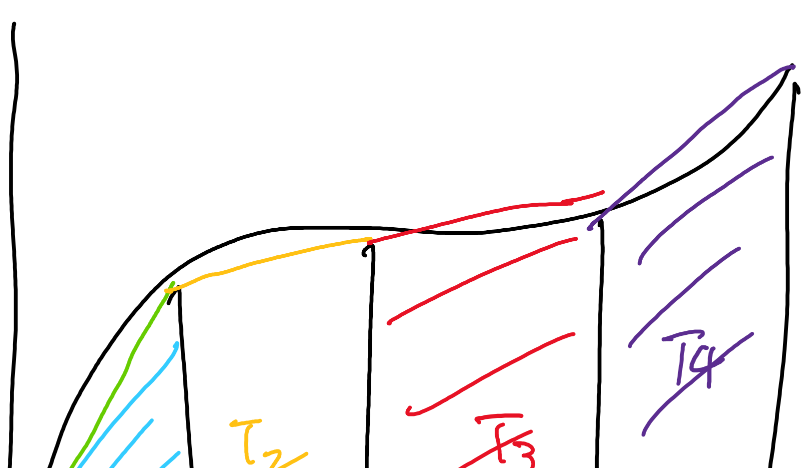
I $\qquad$ $-I$

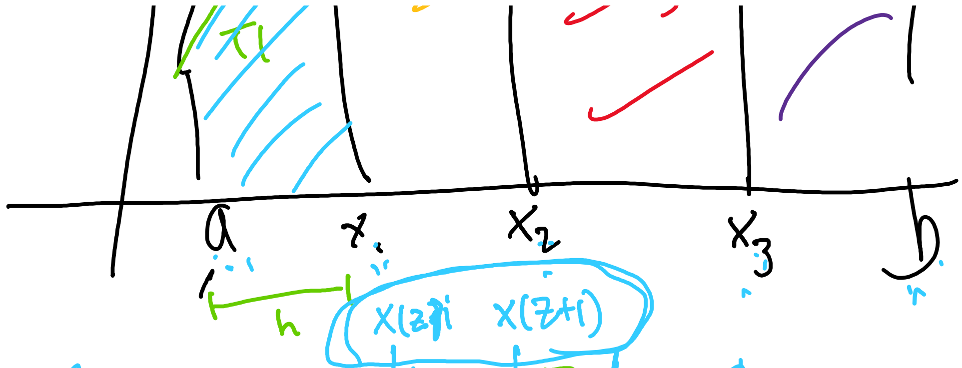
f=@(x) exp(x) $\qquad$ f=@(x) exp(x)

a=0 $\qquad$ a=4

b=4 $\qquad$ b=0

N=2 $\qquad$ N=2





Loop 1 $T_1 = \frac{h}{2}[f(a)+f(x_1)]$

Loop 2 $T_2 = \frac{h}{2}[f(x_1)+f(x_2)]$

Loop 3 $T_3 = \frac{h}{2}[f(x_2)+f(x_3)]$

Loop 4 $T_4 = \frac{h}{2}[f(x_3)+f(b)]$

1) make x vecto a to b in incremts of h.

2) For loop. add te nex T term.

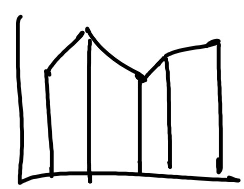
I=0 outside loop

Loop1 $I = I + \frac{h}{2}[f(a) + \cancel{2f(a)} + \cancel{f(b)}]$ $f(x_1)$

$+$ $f(x_1)$ $f(x_2)$

Loop2 $\frac{h}{2}[\cancel{f(a)} + \cancel{2f(x_1)} + \cancel{f(b)}]$

$+$



$e^x, 0, 4, n=2.$

$e^x, 0, 4, \underline{n=4}$

$x_j = a + j\cdot h$

in loop 1,

$j=0.$ $x_j = a.$ $x_i = (x_j + h)$