A technical appendix to accompany 'On the political economy of immigration and income redistribution'

Jim Dolmas and Gregory W. Huffman

1 Details of the underlying two-party competition

There are two parties, or candidates, participating in the two rounds of elections. In the first, candidates compete over the issue of immigration (M); in the second, candidates compete over the tax rate (θ). The elections are decided by majority rule, with voters voting solely on the basis of their indirect utilities over M and θ. The candidates only care about winning, have identical preferences, and their utility functions are time-separable across the two elections. In particular, each candidate's instantaneous utilities from winning, tying (if both candidates adopt the same platform) or losing an election are $+1$, 0 and -1, and the candidate's lifetime utility is a weighted sum of these instantaneous utilities from the two elections.

At the start of period one, the two candidates simultaneously announce values of M, their platforms for the first election. An election takes place, and a winner and loser are decided based on the preferences of natives in the economy. The winning immigration policy—call it M^*—is then known, and implemented at the start of period two. Immigrants arrive, production takes places, and agents receive their second-period incomes. The candidates then simultaneously announce values of θ, a second election is held, a winner and a loser decided, and the winning tax policy—call it θ^*—is implemented in period three.

The (pure) strategies of the two candidates take the form of a value for M and a policy function $\theta(\cdot)$ which prescribes a tax platform to announce contingent on the outcome of the first election, including the winning immigration policy M^* and (potentially) any resulting characteristics of the economy—*e.g.*, the distribution of income in period two when M^* is adopted in period one (call this distribution $\mu_2(\cdot; M^*)$).

We can find an equilibrium by using backwards induction to reduce the game, starting from any outcome of the first-period simultaneous moves by the two players; in fact, the economic environment and the candidates' available strategies and pay-offs are the same at any two nodes at this point in the game's extensive form game which share the same winning first-round policy M^*. Since voters' preferences over θ are single-peaked, and candidates' utilities time-separable, announcing the tax rate preferred by the median second-round voter—the median of the distribution of period-two income $\mu_2(\cdot; M^*)$—is a dominant strategy for both candidates.[1] The expected (incremental) utility to both candidates from the second round of elections is then zero, and we are, in effect, then left with a single election over the single issue of immigration. That is, in any Nash equilibrium of the full game, we can essentially ignore the second round of elections—the candidates will both play the tax rate preferred by the median of $\mu_2(\cdot; M^*)$ in the second round, whatever M^* wins the first round, earning an incremental payoff of zero.

We assume that natives in the economy understand this aspect of the game, so that when voting in the first election—in which M is the issue—agents use the tax rate preferred by the median of $\mu_2(\cdot; M)$ as their forecast of the third period tax rate if policy M is adopted in period one. The indirect utilities over M that result from natives' rationally solving out the subsequent equilibrium of the economy for each M, given this method of forecasting θ, are precisely the utility functions we calculate and present in our numerical experiments.

In the first-round of elections, then, candidates announce platforms with respect to M and receive (lifetime) pay-offs of either $+1$, 0 or -1 as the fraction of natives who prefer their platform to that of their opponent is either greater than, equal to or less than 50%. For the parametrizations of our model, natives lifetime utilities as a function of M are generally either increasing or decreasing in M over an interval $\left[0, \bar{M}\right]$. When that is the case, and the issue space is restricted to $\left[0, \bar{M}\right]$, candidates' strategy sets can, in effect, be reduced to the binary set $\left\{0, \bar{M}\right\}$, and the value preferred by the median native arises as the equilibrium platform of each candidate.

2 Details of the model

To correctly and meticulously analyze the equilibrium solution to this recursive problem, we must investigate how expectations of future variables influence contemporaneous decisions. Additionally, it must be insured that there is consistency between the decisions made in periods 2, and the expectations of these decisions that are formulated in period 1. Therefore, because of the recursive nature of this problem, we must begin by analyzing the decision problem of an agent in period 2. The optimization problem faced by an agent at the start of the second period, whether an existing resident or immigrant, is to maximize

$$\log(c_2) + \beta \log(c_3) \tag{1}$$

subject to the budget constraints given by

[1] In fact, given the assumption that candidates' lifetime utilities are weighted sums of their instantaneous utilities, the remainder of the game from any node after the first election has the form of the usual symmetric two-party zero-sum election game over the single issue of θ.

$$c_2 + s_3 = y_2, \tag{2}$$

$$c_3 = (1-\theta)(r_3 s_3 + w_3) + \tau. \tag{3}$$

The solution to the optimization problem of an agent with income y_2 in period two is the following savings rule

$$s_3(y_2; \Phi) = \frac{\beta}{1+\beta} y_2 - \frac{1}{1+\beta}\Phi, \tag{4}$$

where

$$\Phi \equiv \frac{w_3}{r_3} + \frac{\tau}{(1-\theta) r_3}.$$

The quantity of domestic capital is then determined by adding up the savings of agents in the economy, using the period-two distribution of income μ_2:

$$K_3^D = \int s_3(y_2; \Phi)\mu_2(dy_2). \tag{5}$$

Substituting equation (4) into equation (5) then yields

$$K_3 = \left(\frac{\beta}{1+\beta}\right) L\bar{y}_2 - \left(\frac{1}{1+\beta}\right)\Phi \tag{6}$$

where average income in period 2 is given by $\bar{y}_2 \equiv \int y_2 \mu_2 (dy_2)$.

Now, foreign capital located in the domestic economy in period three is determined by the following equation

$$K_3^F = K_2^F + \lambda^{-1}\left[\beta(1-\theta) r_3 - 1\right], \tag{7}$$

given K_2^F.

Aggregate capital in place for period-three production is then

$$K_3 = K_3^D + K_3^F. \tag{8}$$

Substituting equations (6) and (7) into equation (8) yields

$$K_3 = \left(\frac{\beta}{1+\beta}\right) L\bar{y}_2 - \left(\frac{1}{1+\beta}\right)\Phi + K_2^F + \lambda^{-1}\left[\beta(1-\theta) r_3 - 1\right]. \tag{9}$$

The conditions describing the market returns for capital and labor in either of the two production periods are written as follows:

$$r_i = F_1(K_i, L_i) = \alpha A (K_i/L_i)^{\alpha-1}, \tag{10}$$

and

$$w_i = F_2(K_i, L_i) = (1-\alpha) A (K_i/L_i)^{\alpha}, \tag{11}$$

implying that the equilibrium wage-rental ratios obey $w_i/r_i = [(1-\alpha)/\alpha](K_i/L_i)$.

As government tax revenue is $\theta(r_3 K_3 + w_3 L)$, we have that the lump-sum transfer can be written as $\tau = \theta(r_3(K_3/L) + w_3)$, implying that

$$\begin{aligned}\frac{\tau}{(1-\theta) r_3} &= \frac{\theta}{1-\theta}\left(\frac{K_3}{L} + \frac{w_3}{r_3}\right) \\ &= \frac{\theta}{\alpha(1-\theta)}\left(\frac{K_3}{L}\right).\end{aligned}$$

Thus, Φ can be written as

$$\begin{aligned}\Phi &= \frac{w_3}{r_3} + \frac{\tau}{(1-\theta) r_3} \\ &= \frac{1-\alpha}{\alpha}\left(\frac{K_3}{L}\right) + \frac{\theta}{\alpha(1-\theta)}\left(\frac{K_3}{L}\right) \\ &= \left[\frac{1-\alpha(1-\theta)}{\alpha(1-\theta)}\right]\left(\frac{K_3}{L}\right).\end{aligned} \tag{12}$$

Substituting this expression for Φ, along with equation (10) into equation (9), and rearranging slightly, produces the following equation:

$$\left(\frac{\beta}{1+\beta}\bar{y}_2 + \frac{K_2^F - \lambda^{-1}}{L}\right) - \left[1 + \frac{1}{1+\beta}\left(\frac{1-\alpha(1-\theta)}{\alpha(1-\theta) L}\right)\right]\left(\frac{K_3}{L}\right) + \left[\frac{\alpha\beta A}{\lambda L}(1-\theta)\right]\left(\frac{K_3}{L}\right)^{\alpha-1} = 0. \tag{13}$$

As $\bar{y}_2$ and L are taken here as given, and K_2^F—as we show below—can be expressed implicitly in terms of $\bar{y}_2$ and L, we can write this equation more succinctly as $G(\theta, K_3/L; \bar{y}_2, L) = 0$. This equilibrium condition implies a unique value of the capital-labor for each value of $\theta \in [0, 1)$, and will serve as a constraint on the choice of preferred tax rate by the median voter in the second period.

Substitution of equations (2), (3) and (4) into the utility function (1) then yields the following representation for the expected utility of a resident with initial period-two income equal to y_2:

$$(1+\beta)\log(y_2 + \Phi) + \beta \log((1-\theta) r_3). \tag{14}$$

Using the factor price relations, and the expression (12) for Φ, the utility from period-two onward, for an agent with income y_2, can be expressed as a function of the tax rate θ and the third-period capital-labor ratio K_3/L:

$$v(\theta, K_3/L; y_2) \equiv (1+\beta)\log\left[y_2 + \left[\frac{1-\alpha(1-\theta)}{\alpha(1-\theta)}\right]\left(\frac{K_3}{L}\right)\right] + \beta\log(1-\theta) - \beta(1-\alpha)\log(K_3/L). \tag{15}$$

We assume that the agent with the median level of period-two income (y_2^m) will ultimately determine the third-period tax rate. Let $z(\theta;\bar{y}_2,L)$ denote the third-period capital-labor ratio implied by a given value of θ, when average period-two income is $\bar{y}_2$ and the period-two population is L. Precisely, $z(\theta;\bar{y}_2,L)$ is defined implicitly by (13), *i.e.*, $G(\theta,z;\bar{y}_2,L)=0$. We can then write the equilibrium level of taxation in the following form

$$\theta^*(y_2^m,\bar{y}_2,L)=\underset{0\le\theta<1}{\arg\max}\left\{v\left(\theta,z\left(\theta;\bar{y}_2,L\right);y_2^m\right)\right\}. \tag{16}$$

Note that, given y_2^m, $\bar{y}_2$ and L, the median voter's preferred tax rate $\theta^*(y_2^m,\bar{y}_2,L)$ and the constraint $G(\theta,K_3/L;\bar{y}_2,L)=0$ imply an associated third-period capital-labor ratio, and hence third-period wage and rental rates. For given values of y_2^m, $\bar{y}_2$ and L, then, any agent's utility from period two onward is given by (14), with Φ, θ and r_3 evaluated at the values implied by setting $\theta=\theta^*(y_2^m,\bar{y}_2,L)$ and K_3/L consistent with the constraint $G(\theta,K_3/L;\bar{y}_2,L)=0$ for that value of θ.

A native agent in period one, then, taking as given L and a conjecture about the period-two income statistics $(y_2^m,\bar{y}_2)$—which imply values of θ, K_3/L and Φ in the manner just described—chooses savings for period two so as to solve

$$\max\left\{\log\left(k_1-s_2\right)+\beta\left(1+\beta\right)\log\left(r_2s_2+w_2+\Phi\right)\right\},$$

which utilizes the indirect utility function given in (14). This optimization implies a savings decision rule of the form:

$$s_2\left(k_1;y_2^m,\bar{y}_2,L\right)=\frac{\beta\left(1+\beta\right)}{1+\beta\left(1+\beta\right)}k_1-\frac{1}{1+\beta\left(1+\beta\right)}\left(\frac{w_2+\Phi}{r_2}\right).$$

Let $\delta\equiv\beta(1+\beta)$. Adding these savings choices up using the distribution $\mu_1$of initial capital gives:

$$K_2^D=\int s_2(k_1;y_2^m,\bar{y}_2,L)\mu_1(dk_1) \tag{17}$$

$$=\frac{\delta}{1+\delta}\bar{k}_1-\frac{1}{1+\delta}\left(\frac{w_2+\Phi}{r_2}\right) \tag{18}$$

and so,

$$\begin{aligned}K_2&=K_2^D+K_2^F\\&=\frac{\delta}{1+\delta}\bar{k}_1-\frac{1}{1+\delta}\left(\frac{w_2+\Phi}{r_2}\right)+K_1^F+\lambda^{-1}\left(\beta r_2-1\right).\end{aligned} \tag{19}$$

By substituting the factor price relations into equation (19), and rearranging terms, we obtain the following equation determining the equilibrium period-two capital-labor ratio, given L and Φ:

$$K_2/L=\frac{\alpha\delta\bar{k}_1+\alpha\left(1+\delta\right)\left(K_1^F-\lambda^{-1}+\beta\alpha A\lambda^{-1}\left(K_2/L\right)^{\alpha-1}\right)-\left(\Phi/A\right)\left(K_2/L\right)^{1-\alpha}}{1-\alpha+\alpha\left(1+\delta\right)L} \tag{20}$$

The period-two level of average income can be written as follows:

$$\begin{aligned}\bar{y}_2 &= r_2\left(K_2^D/L\right)+w_2\\ &= r_2\left(K_2/L\right)+w_2-r_2\left(K_2^F/L\right)\\ &= r_2(K_2/L)+w_2-r_2\left(\frac{K_1^F+\lambda^{-1}\left(\beta r_2-1\right)}{L}\right)\end{aligned}$$

Using the factor price relations, this can be re-written as follows

$$\bar{y}_2 = A\left(K_2/L\right)^{\alpha}-\alpha A(K_2/L)^{\alpha-1}\left(\frac{K_1^F+\lambda^{-1}\left(\beta\alpha A\left(K_2/L\right)^{\alpha-1}-1\right)}{L}\right) \tag{21}$$

Note that, from (21), for given values of $\bar{y}_2$ and L, there is unique value of K_2/L which is consistent with these values for average income and population, hence a unique value of r_2, and so of $K_2^F = K_1^F+\lambda^{-1}\left(\beta r_2-1\right)$. This verifies the claim made above regarding the relationship between K_2^F and $\bar{y}_2$ and L, which justifies writing the function $G\left(\theta, K_3/L;\bar{y}_2, L\right)$ as dependent on $\bar{y}_2$ and L alone, rather than $\bar{y}_2$, L and K_2^F.

The median level of period-two income is determined as follows. In our benchmark case, immigrants enter at the bottom of the second-period distribution of income, so that the agent with the median income level in period-two must have initial capital $\hat{k}_1\left(M\right)$, given by the solution to

$$M+\int_{\underline{k}}^{\hat{k}_1(M)}\mu_1\left(dk_1\right)=\frac{1}{2}\left(1+M\right),$$

where $\underline{k}$ denotes the minimum initial capital holding among natives. The period-two income of this agent is then

$$\begin{aligned}y_2^m &= r_2 s_2\left(\hat{k}_1\left(M\right);y_2^m,\bar{y}_2,L\right)+w_2\\ &= \frac{\delta}{1+\delta}r_2\hat{k}_1\left(M\right)-\frac{1}{1+\delta}\left(w_2+\Phi\right)+w_2.\end{aligned} \tag{22}$$

Note that the realized median value of period-two income depends, through Φ, on natives' conjectured median (and mean) level of period-two income.

3 Description of solution algorithm

The algorithm which we describe in this section is to be conducted for each value of the variable L or M. Hence, L should be thought of as a constant in what follows. The initial level of foreign capital, K_1^F, and the initial capital distribution, $\mu_1\left(\cdot\right)$, are also given. The method for numerically solving for an equilibrium is then as follows:

1. Pick initial values for the following variables: K_2, θ and K_3. Also pick small positive values ε_1, ε_2 and ε_3, which will be measures of convergence for various parts of the algorithm.

2. Given θ and K_3, set $\Phi = \left[\left(1-\alpha\left(1-\theta\right)\right)/\alpha\left(1-\theta\right)\right]\left(K_3/L\right)$.

3. Find the value of K_2 consistent with Φ:

   (a) Calculate the values of w_2 and r_2 implied by the current value of K_2, using (10) and (11).

   (b) Using these and Φ, evaluate the right-hand side of (19) to obtain a new value K_2'.

   (c) If $|K_2 - K_2'| \leq \varepsilon_1$, stop iterating; else, set $K_2 = K_2'$ and go back to step 3(a).

4. Given the value of K_2 which is consistent with Φ, calculate the implied factor prices w_2 and r_2 using (10) and (11), and evaluate the implied average and median period-two incomes using (21) and (22). These are the relevant statistics describing the period-two income distribution when natives make their first-period choices conditional on Φ.

5. Update θ and K_3:

   (a) Update θ to θ', where θ' is a value preferred to θ by the native with period-two income y_2^m. In our programs, we use a simple gradient method which selects a direction of improvement using the derivatives of $v\left(\theta, z; y_2^m\right)$ and the constraint function $G\left(\theta, z; \bar{y}_2, L\right)$; in particular, θ' is a tax rate lying in the direction $v_\theta - \left(G_\theta/G_z\right)v_z$ from θ.[2]

   (b) Find the value of K_3, call it K_3', which is consistent with θ'—*i.e.*, obeying (13) when θ is set to the value θ'. The procedure is essentially the same iterative process as that used on K_2 in step 3—*i.e.*, taking the current K_3 into a new K_3'. In this case, however, it is based on iterating on equation (9) while using (12) to substitute for Φ, and evaluating θ at the updated value θ'. Iterations stop when the tolerance criterion ε_2 is met.

6. Check if $\left|\theta - \theta'\right| + |K_3 - K_3'| \leq \varepsilon_3$. If so, stop; else, return to step 2 with $\theta = \theta'$ and $K_3 = K_3'$.

To calculate the economy's behavior over a grid of immigration levels, we simply repeat this procedure for other values of the variable L.[3] Once the equilibrium values of K_2, K_3, and θ are determined, the resulting factor prices for both periods can be calculated from equations (10) and (11). Additionally, K_3^F is determined by equation(7) and finally we have $K_2^F = K_1^F + \lambda^{-1}\left[\beta r_2 - 1\right]$.

[2] Let $d = v_\theta - \left(G_\theta/G_z\right)v_z$. The actual step $\theta' - \theta$ is of the form $md/\left(1+|d|\right)$, where m is a maximum-step-size parameter.

[3] Except for step (1)—in evaluating equilibrium at points $L_1 < L_2 < \cdots < L_N$, the conditions with which we initialize the routine for L_{i+1} $(i \geq 1)$ are simply the solutions for L_i.

Remark 1 *This routine is surprisingly quick. For each level of the immigration variable M, the resulting equilibrium can be calculated in about a second, using the Mathworks's* MATLAB *software on a Pentium II machine.*

Remark 2 *It is possible to see from equation (4) that if r_3 or y_2 are relatively low, it is conceivable for some agents to wish to borrow rather than lend or save. Since this would permit them to be subsidized through the tax system because interest income is taxable at the rate θ, we experimented with versions of the model in which individual saving was restricted to be non-negative (*i.e., *s_2,$s_3 \geq 0$). This is actually a non-trivial complication since there turn out to be various types of agents whose decision rules must be characterized (*i.e., *those for whom $s_2 > 0$ and $s_3 = 0$, or $s_2 = 0$ and $s_3 > 0$, or $s_2 = 0$ and $s_3 = 0$, or $s_2 > 0$ and $s_3 > 0$). This greatly complicates the expressions for the aggregate capital stocks, and significantly slows the computations, as the fractions of agents in each group need to be calculated at each pass.*[4] *It turned out that imposing this restriction had very little impact on the results because very few agents were constrained, and those that were constrained obviously had low levels of income.*

Remark 3 *For the case in which it is not the period-two median agent who is critical in determining the voting outcome, the procedure is slightly different, and in general simpler. For example, in the case in which immigrants are not permitted to vote then the median voter over the tax policy is just the median native*—i.e., *the native whose period-one capital-holding is $\hat{k}_1(0) \equiv \hat{k}_1$, defined by*

$$\int_{\underline{k}}^{\hat{k}_1} \mu_1(dk_1) = \frac{1}{2}.$$

In this case, there is no need—except for purposes of comparison—to calculate the actual median level of period-two income at all.

Remark 4 *In the case in which the immigrants arrive with some amount of capital, then the distribution of period-two income, $\mu_2(\cdot)$, and period-two average income, $\bar{y}_2$, must be calculated so as to take this into account. The median period-two income recipient at $M = 0$ is of course simply the median native, and, if immigrants arrive with capital* greater *than the initial median holding among natives, then the initial capital-holding that identifies the median period-two income recipient* rises *as M is increased from $M = 0$. From a computational standpoint, the modification this engenders is straightforward when M is restricted to values such that the median period-two income recipient is a native who is poorer than an immigrant. For example, if immigrants arrive with capital which would place them at the cut-off for the top p fraction of the initial capital distribution (μ_1), where $p < 1/2$, then $M + p \leq \frac{1}{2}(1 + M)$, or $M \leq 1 - 2p \equiv \bar{M}(p)$, is a sufficient condition for the median period-two income recipient to be a native who is poorer than an immigrant.*[5]

[4] With a log-normal distribution, this means performing numerical integrations at each step.

[5] There is no problem in theory with allowing M to exceed this value. In terms of computation, for $M < \bar{M}(p)$, it is

Remark 5 *In the case in which immigrants do not receive the transfer payment τ (but still pay taxes), we have*

$$\begin{aligned}\tau &= \theta \left(r_3 K_3 + w_3 L\right) \\ &= \left(1+M\right)\theta\left(r_3 \frac{K_3}{L} + w_3\right),\end{aligned}$$

so that the expression (12) for Φ must be modified to

$$\begin{aligned}\Phi &= \frac{\tau}{\left(1-\theta\right) r_3} + \frac{w_3}{r_3} \\ &= \left(1+M\right)\frac{\theta}{1-\theta}\left(\frac{K_3}{L} + \frac{w_3}{r_3}\right) + \frac{w_3}{r_3} \\ &= \left[\frac{1-\alpha\left(1-\theta\right)+\theta M}{\alpha\left(1-\theta\right)}\right]\frac{K_3}{L}.\end{aligned}$$

4 Determination of the tax rate in a special case

In the special case where there is no foreign capital, and agents have only capital income in the final period, it is possible to derive a simple closed-form solution for agents' preferred tax rates. In this section of the appendix, we present that solution, which is useful for garnering some intuition regarding the relationship between the equilibrium tax rate and median and average second-period income.

Suppose that, then, there is no foreign capital—equivalently, that $\lambda = +\infty$ and $K_1^F = 0$—and that third-period production possibilities are given by $F\left(K\right) = AK$, so that the equilibrium after-tax return to saving is $\left(1-\theta\right)A$. Consider an agent—either a native or immigrant—who begins period two with income equal to y_2. Given values for the tax rate θ and transfer payment τ, the problem faced by such an agent is

$$\max_{s_3}\left\{\log\left(y_2 - s_3\right) + \beta\log\left[\left(1-\theta\right)As_3 + \tau\right]\right\}. \tag{23}$$

It is straightforward to verify that the optimal choice of savings is given by

$$s_3 = \frac{1}{1+\beta}\left(\beta y_2 - \frac{\tau}{\left(1-\theta\right)A}\right).$$

By aggregating this expression across all agents, and using the fact that the lump-sum transfer (τ) equals the tax rate times the amount of per-capita capital income, it can be shown that

$$\tau = \frac{\theta A\beta\left(1-\theta\right)}{\left(1-\theta\right)\left(1+\beta\right)+\theta}\bar{y}_2.$$

By substituting this expression, together with the equations determining optimal consumption and saving decisions, back into the utility function (23), we then get an indirect utility function that describes preferences

straightforward to describe how the capital-holding identifying the median income-recipient changes as M changes; when M exceeds $\bar{M}\left(p\right)$, however, one needs to take account of the point-mass of immigrants in the second-period income distribution.

over these two periods, and this can be written as follows:

$$w\left(\theta; y_2/\bar{y}_2\right) = \eta + (1+\beta)\log\left(\bar{y}_2\right) + (1+\beta)\log\left(\frac{(1+\beta)\frac{y_2}{\bar{y}_2} + \theta\beta\left(1-\frac{y_2}{\bar{y}_2}\right)}{1+\beta-\theta\beta}\right) + \beta\log\left(1-\theta\right),$$

where $\eta = -(1+\beta)\log(1+\beta) + \beta\log(\beta A)$.Written in this manner, it becomes clear that an individual's indirect utility depends not only on the parameters θ and A, but also on his income *relative to the average.* The reason for this is clear: the agent's resulting transfer payment depends on the average level of income.

It is straightforward to verify that this expression is differentiable in θ, and is decreasing in θ when $y_2/\bar{y}_2 \geq 1$—*i.e.*, when the agent is richer than average. The preferred tax rate of any agent with $y_2 \geq \bar{y}_2$ is always $\theta = 0$.

For agents with $y_2/\bar{y}_2 < 1$, one can verify that w is strictly concave in θ on $(0, 1)$. Concavity is easier to see if the two last terms involving θ are written out as

$$(1+\beta)\log\left[(1+\beta)\frac{y_2}{\bar{y}_2} + \theta\beta\left(1-\frac{y_2}{\bar{y}_2}\right)\right] - (1+\beta)\log\left(1+\beta-\theta\beta\right) + \beta\log\left(1-\theta\right),$$

which is the sum of three strictly concave functions.

Hence a poorer-than-average agent who wishes to calculate his or her most preferred value of θ subject to the constraint $\theta \in [0, 1]$, could perform this calculation by merely taking the derivative of $w\left(\theta; y_2/\bar{y}_2\right)$, and setting it equal to zero. This yields a quadratic expression and after some tedious algebra it can be shown that the most preferred tax rate of an agent with $y_2/\bar{y}_2 \equiv z < 1$ is given by

$$\theta^*(z) = \frac{1+\beta}{\beta}\frac{1+2\beta(1-z) - \sqrt{1+4\beta(1-z)}}{2\beta(1-z)}.$$

Some properties of $\theta^*(z)$ are worth noting. First, $\theta^*(z)$ is decreasing in z: the relatively poorer is an agent, the higher is his or her preferred tax rate (and transfer). Also, $\lim_{z\to 1}\theta^*(z) = 0$, so there is no discontinuity in the preferred tax rates as we move from the relatively wealthy—those agents with $z \geq 1$, whose preferred tax rate is zero—to the relatively poor. While $\theta^*(z)$ varies monotonically with z, it's dependence on the time-preference parameter β is more complex. Finally, as is shown in Dolmas and Huffman [1], for sufficiently small values of z, $\theta^*(z)$ may be on the 'wrong' side of the economy's Laffer curve—*i.e.*, for small z, $\theta^*(z)$ may be higher than the value of θ which maximizes total tax revenue.

5 Details of the simultaneous-voting case

The key to verifying that the median's preferred point $(\theta^m, 0)$ is a local majority-rule equilibrium in the sense of Plott [2], is the inequality ordering of the marginal rates of substitution between θ and M for wealthy and poor natives. In this section, we provide the steps which lead to the assertion that

$$-\frac{V_\theta(x')}{V_M(x')} > -\frac{V_\theta(x'')}{V_M(x'')}$$

when $x' > \hat{x} > x^m > x''$. The x's here, as in section 7.3 of the paper, denote ratios of initial wealth to average initial wealth, with x^m being that of the median native, and $\hat{x}$ the critical value at which natives' marginal utilities from immigration, at $(\theta^m, 0)$, switch from being negative (if $x < \hat{x}$) to positive (if $x > \hat{x}$). The existence of such an $\hat{x}$ can be confirmed by inspection of the marginal utility of immigration

$$V_M(\theta^m, 0; x) = -\frac{\Pi_M}{\Pi} \times \left\{ \beta(1-\alpha)(1+\beta+\alpha\beta) - (1+\delta) \frac{(1-\alpha+\psi\xi)\Pi}{\alpha x + (1-\alpha+\psi\xi)\Pi} \right\},$$

where δ, ψ, ξ and Π are as defined in section 7.3.

In order to calculate marginal rates of substitution between θ and M, we need $V_\theta(\theta^m, 0; x)$, the marginal utility of θ. Differentiating a native's lifetime utility—

$$\begin{aligned} V(\theta, M; x) &= (1+\delta)\log(\alpha x + (1-\alpha+\psi(\theta)\xi(\theta))\Pi(\theta, M)) + \beta^2 \log(1-\theta) \\ &\quad -\beta(1-\alpha)(1+\beta+\alpha\beta)\log(\Pi(\theta, M)) - \beta^2(1-\alpha)\log(\xi(\theta)), \end{aligned}$$

—with respect to θ gives

$$V_\theta = \frac{\Pi_\theta}{\Pi_M} V_M + (1+\delta)\frac{\psi\xi\Pi}{\alpha x + (1-\alpha+\psi\xi)\Pi}\left(\frac{\psi'}{\psi} + \frac{\xi'}{\xi}\right) - \frac{\beta^2}{1-\theta} - \beta^2(1-\alpha)\frac{\xi'}{\xi},$$

where we have suppressed the arguments of ψ, ξ, Π and V.

As

$$\psi(\theta) = \frac{1-\alpha(1-\theta)}{\alpha(1-\theta)},$$

we have

$$\frac{\psi'(\theta)}{\psi(\theta)} = \frac{\alpha}{1-\alpha(1-\theta)} + \frac{1}{1-\theta}.$$

With ξ defined by

$$\xi(\theta) = \frac{\alpha\beta(1-\theta)}{1+\alpha\beta(1-\theta)},$$

we have

$$\frac{\xi'(\theta)}{\xi(\theta)} = -\frac{1}{1-\theta} + \frac{\alpha\beta}{1+\alpha\beta(1-\theta)}.$$

Thus,

$$\begin{aligned} \frac{\psi'(\theta)}{\psi(\theta)} + \frac{\xi'(\theta)}{\xi(\theta)} &= \frac{\alpha}{1-\alpha(1-\theta)} + \frac{\alpha\beta}{1+\alpha\beta(1-\theta)} \\ &= \frac{\alpha(1+\beta)}{(1-\alpha(1-\theta))(1+\alpha\beta(1-\theta))}. \end{aligned}$$

Further,

$$\begin{aligned} \frac{\beta^2}{1-\theta} + \beta^2(1-\alpha)\frac{\xi'(\theta)}{\xi(\theta)} &= \beta^2\left[\frac{1}{1-\theta} - \frac{1-\alpha}{1-\theta} + \frac{\alpha\beta(1-\alpha)}{1+\alpha\beta(1-\theta)}\right] \\ &= \alpha\beta^2\left[\frac{1+\beta(1-\theta)}{(1-\theta)(1+\alpha\beta(1-\theta))}\right]. \end{aligned}$$

Hence,

$$V_\theta = \frac{\Pi_\theta}{\Pi_M} V_M + (1+\delta) \frac{\psi\xi\Pi}{\alpha x + (1-\alpha+\psi\xi)\Pi} \left(\frac{\alpha(1+\beta)}{(1-\alpha(1-\theta))(1+\alpha\beta(1-\theta))} \right.$$
$$\left. -\alpha\beta^2 \left[\frac{1+\beta(1-\theta)}{(1-\theta)(1+\alpha\beta(1-\theta))} \right] \right).$$

As one can tell, these expressions are quickly becoming quite large. To conserve on space, let z stand for $1-\theta$, and let $\Phi \equiv (1+\delta)(1-\alpha+\psi\xi)\Pi / [\alpha x + (1-\alpha+\psi\xi)\Pi]$. With this notation—the utility of which will become clear momentarily—the second term in the expression for V_θ may be written

$$\frac{\psi\xi}{1-\alpha+\psi\xi} \left(\frac{\alpha(1+\beta)}{(1-\alpha z)(1+\alpha\beta z)} \right) \Phi.$$

Since

$$\frac{\psi\xi}{1-\alpha+\psi\xi} = \frac{\beta(1-\alpha z)}{(1-\alpha)(1+\alpha\beta z)+\beta(1-\alpha z)}$$

this second term is actually:

$$\left(\frac{\alpha\beta(1+\beta)}{[(1-\alpha)(1+\alpha\beta z)+\beta(1-\alpha z)](1+\alpha\beta z)} \right) \Phi.$$

Thus,

$$V_\theta = \frac{\Pi_\theta}{\Pi_M} V_M + \left(\frac{\alpha\beta(1+\beta)}{[(1-\alpha)(1+\alpha\beta z)+\beta(1-\alpha z)](1+\alpha\beta z)} \right) \Phi - \alpha\beta^2 \left[\frac{1+\beta z}{z(1+\alpha\beta z)} \right].$$

We may write this as

$$V_\theta = \frac{\Pi_\theta}{\Pi_M} V_M + D \times [\Phi - A]$$

where

$$D \equiv \frac{\alpha\beta(1+\beta)}{[(1-\alpha)(1+\alpha\beta z)+\beta(1-\alpha z)](1+\alpha\beta z)}$$

and

$$A \equiv \alpha\beta^2 \left[\frac{1+\beta z}{z(1+\alpha\beta z)} \right] \frac{[(1-\alpha)(1+\alpha\beta z)+\beta(1-\alpha z)](1+\alpha\beta z)}{\alpha\beta(1+\beta)}$$
$$= \frac{\beta(1+\beta z)[(1-\alpha)(1+\alpha\beta z)+\beta(1-\alpha z)]}{(1+\beta)z}.$$

Next, note that, in the same way, V_M has the form

$$V_M = C \times [B - \Phi],$$

where

$$C \equiv -\frac{\Pi_M}{\Pi}$$

and

$$B \equiv \beta(1-\alpha)(1+\beta+\alpha\beta).$$

Now, consider $-\left(V_\theta/V_M\right)$:

$$
\begin{aligned}
-\frac{V_\theta}{V_M} &= -\frac{\left(\Pi_\theta/\Pi_M\right)V_M + D\times[\Phi - A]}{C\times[B-\Phi]} \\
&= -\frac{\Pi_\theta}{\Pi_M} + \frac{D}{C}\left[\frac{A-\Phi}{B-\Phi}\right].
\end{aligned}
$$

As both C and D are positive, and neither C, D nor Π depend on x, $-\left(V_\theta/V_M\right)$ for some native x' will be greater than the equivalent expression for some native x'' if and only if

$$
\frac{A-\Phi}{B-\Phi}
$$

for x' is greater than the corresponding expression for x''. Note that only the Φ in the last expression depends on x, and write $\Phi(x')$ and $\Phi(x'')$ for the values of Φ for natives x' and x''. Then,

$$
-\frac{V_\theta(x')}{V_M(x')} > -\frac{V_\theta(x'')}{V_M(x'')}
$$

if and only if

$$
\frac{A-\Phi(x')}{B-\Phi(x')} > \frac{A-\Phi(x'')}{B-\Phi(x'')}.
$$

Now, if $x' > \hat{x}$, both numerator and denominator on the left-hand side are positive, and if $x'' < x^m$,both numerator and denominator on the right-hand side are negative. We thus want to show that

$$
\frac{A-\Phi(x')}{B-\Phi(x')} > \frac{\Phi(x'')-A}{\Phi(x'')-B}.
$$

This inequality hold if and only if

$$
\left(A-\Phi(x')\right)\left(\Phi(x'')-B\right) > \left(\Phi(x'')-A\right)\left(B-\Phi(x')\right),
$$

which in turn will hold if and only if

$$
(A-B)\left(\Phi(x'')-\Phi(x')\right) > 0.
$$

Since $x' > x''$, and given the definition of Φ, $\Phi(x'')-\Phi(x') > 0$. Thus, the key inequality shown in section 7.3 holds if and only if $A > B$.

Is this the case? Recalling the definitions of A and B, we wish to verify that

$$
\frac{\beta(1+\beta z)\left[(1-\alpha)(1+\alpha\beta z)+\beta(1-\alpha z)\right]}{(1+\beta)z} > \beta(1-\alpha)(1+\beta+\alpha\beta).
$$

Brute-force algebra reveals that the last inequality holds if and only if

$$
0 > (\alpha\beta)^2 z^2 + \left[1+\beta-\alpha-(\alpha\beta)^2\right]z - (1-\alpha+\beta).
$$

The quadratic on the right-hand side is convex, with its minimum at a $z < 0$, so it is increasing for all $z > 0$. At $z = 0$ (or $\theta = 1$), it takes on the value $-(1-\alpha+\beta) < 0$, while at $z = 1$ (or $\theta = 0$), it is precisely zero. Thus, for any $z < 1$ ($\theta > 0$), it is in fact negative.

References

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[2] Plott, Charles R., A notion of equilibrium under majority rule, *American Economic Review* **57** (1967), 787–806.