SAMSUNG

Samsung Innovation Campus

Artificial Intelligence Course

Together for Tomorrow!
Enabling People
Education for Future Generations

Chapter 2.

Math for Data Science

Artificial Intelligence Course

Chapter Description

Chapter objectives

- ✓ Be able to install Anaconda to learn data science.
- ✓ Be able to know the types of numbers and math symbols that are required for data science.
- ✓ Be able to define and know various equations in basic mathematics and algebra required for data science.
- ✓ Learn about sequences, absolute values, functions, graphs, linear algebra (vector and matrix), and derivatives and prepare for mathematical concepts and practices needed for AI.

Chapter contents

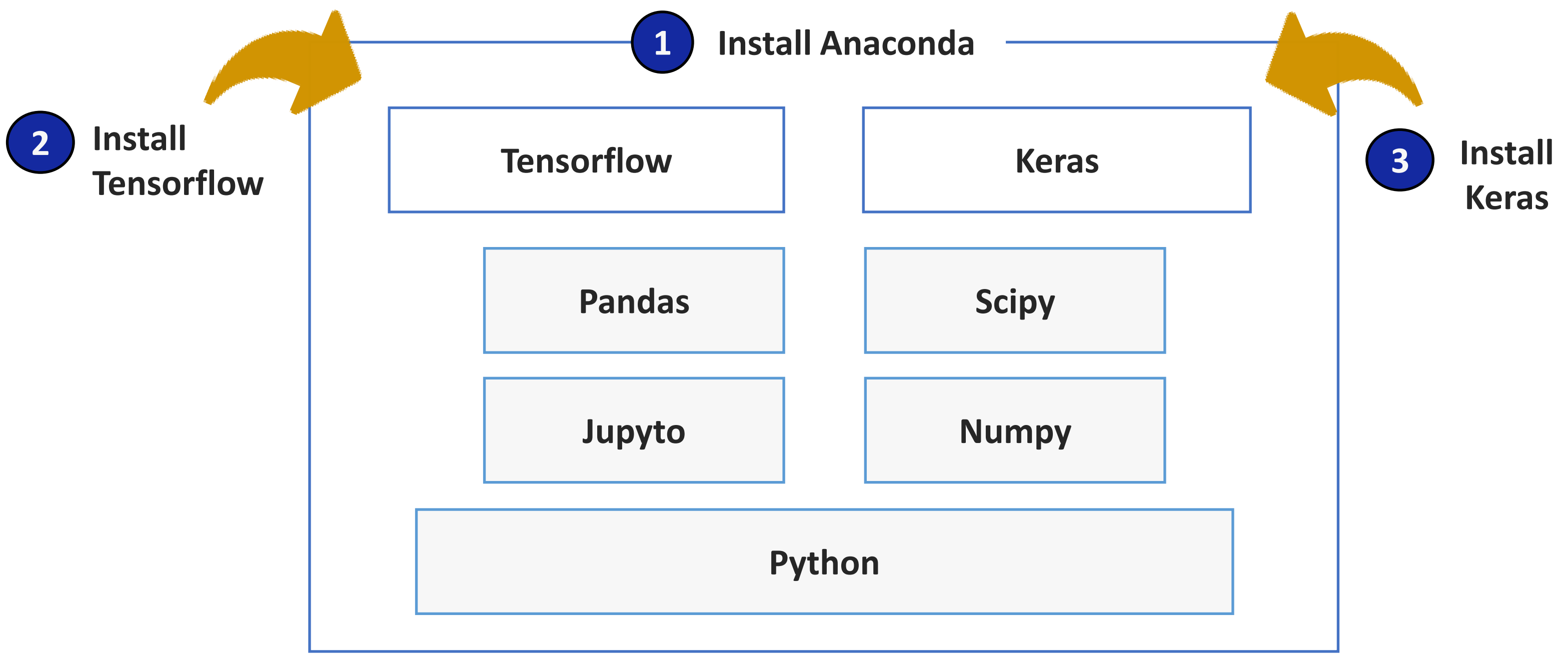
- ✓ Unit 1. Introduction
- ✓ Unit 2. Basic Math for Data Science
- ✓ Unit 3. Understanding Data Science: Vector
- ✓ Unit 4. Understanding Data Science: Matrix
- ✓ Unit 5. Understanding Deep Learning: Derivatives

Unit 1.

Introduction

Installing Anaconda Jupyter Notebook

Environment Setting for Data Analysis



Installing Anaconda

- www.anaconda.com

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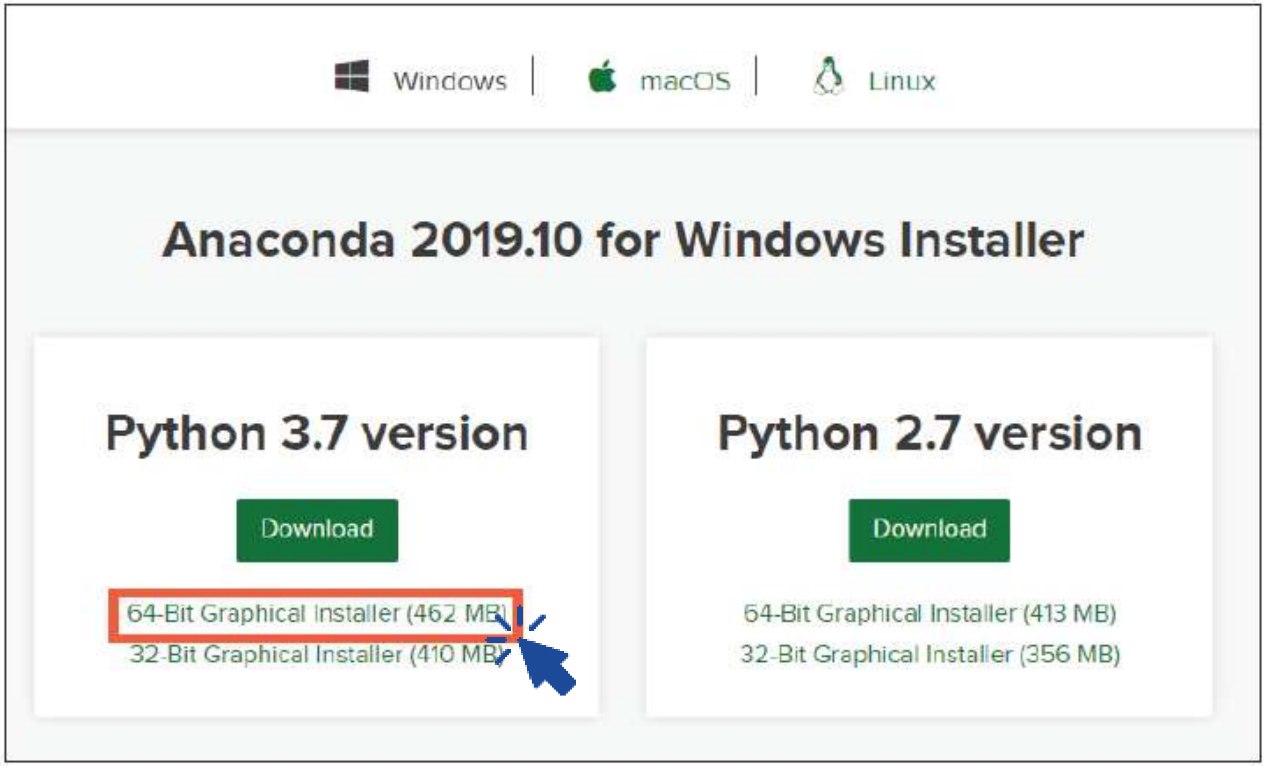

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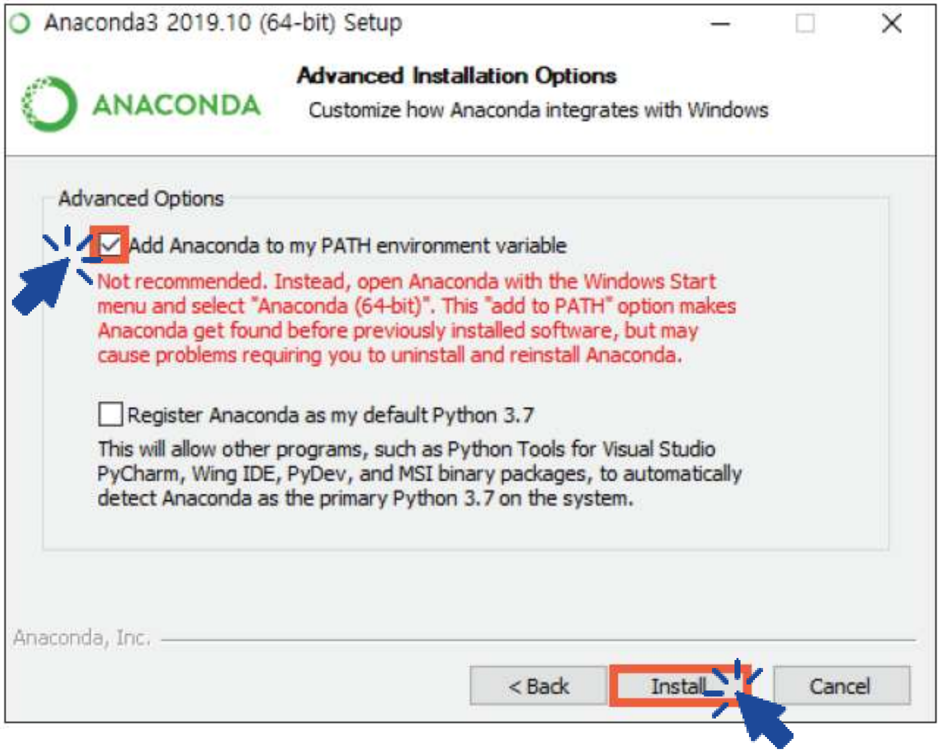
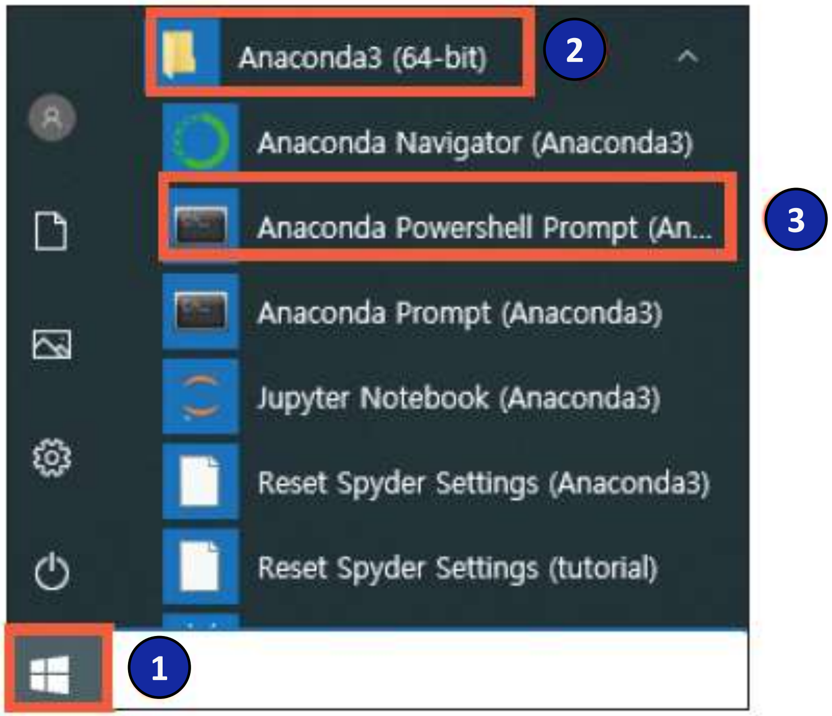


※ If you are not familiar with PATH environment, please click “Register Anaconda…”.

Installing Jupyter Notebook

- Once installation is complete, click the Windows Start menu, and then click Anaconda3 (64-bit).
- Select Anaconda Powershell Prompt.



Installing Jupyter Notebook

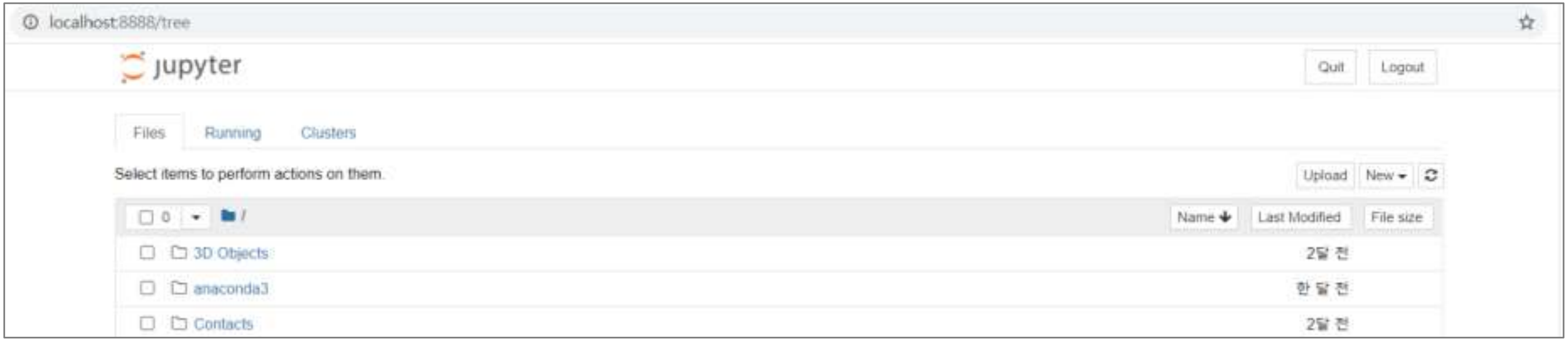
- Enter “jupyter notebook” as instructed in the Anaconda prompt.

Anaconda Prompt (Anaconda3)

```
(base) C:\Users\it>jupyter notebook
```

Installing Jupyter Notebook

- The Juyper Notebook screen is shown as follows.



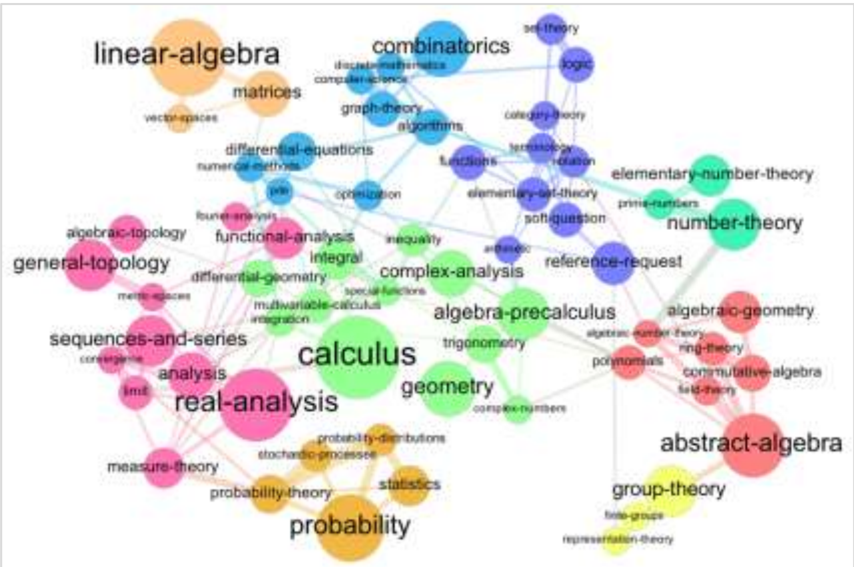
Unit 1.

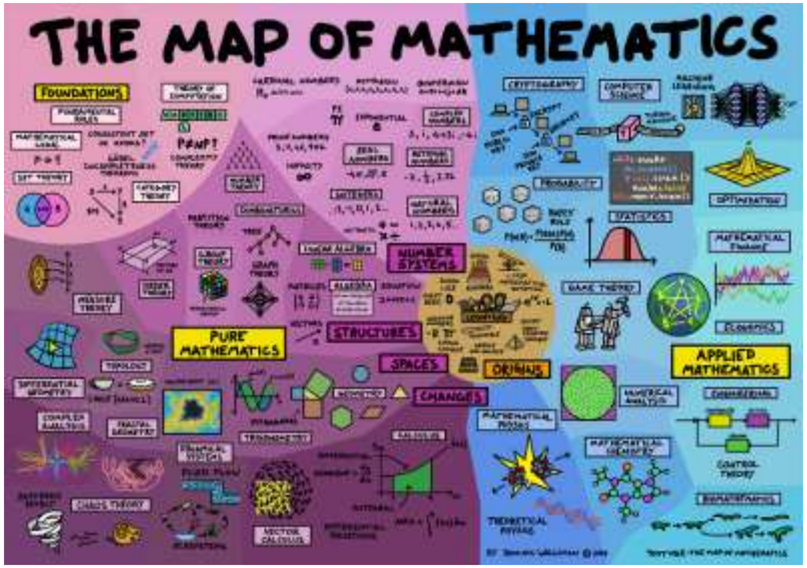
Introduction

Classification of Mathematics

Mathematics, in short, is a study of quantity. Traditionally, mathematics is primarily divided into two areas: arithmetic and geometry.

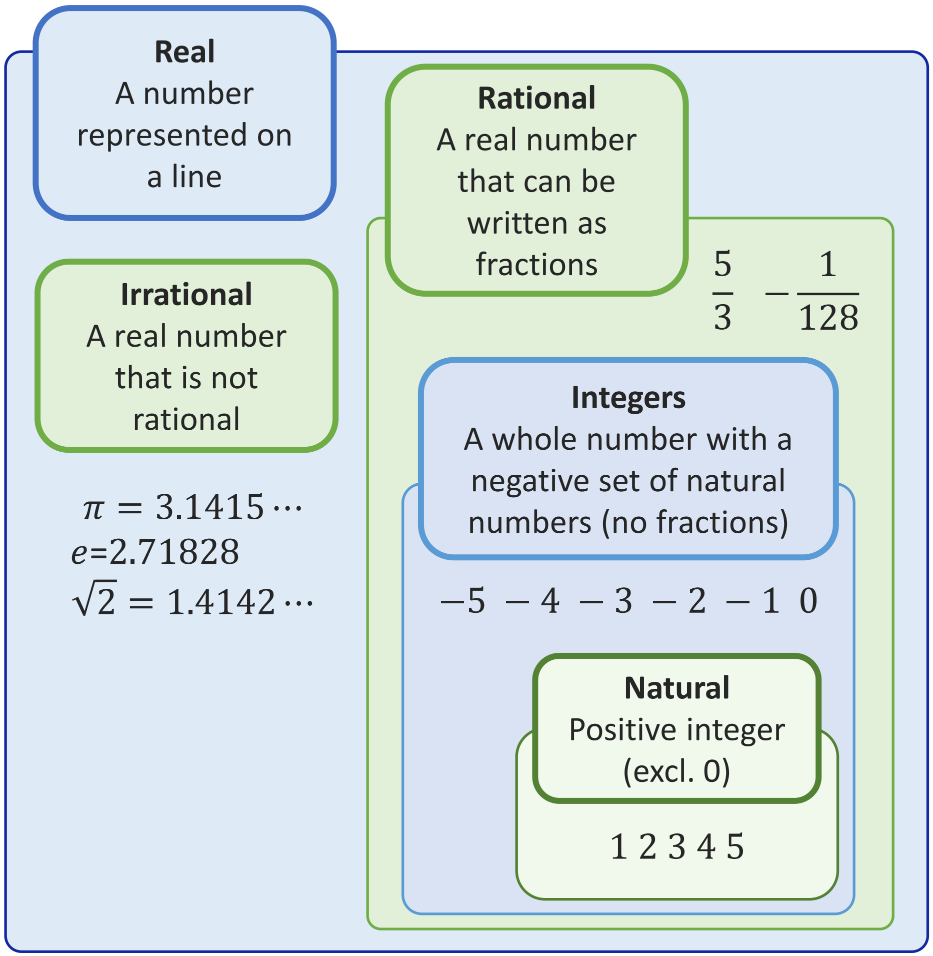
- Arithmetic is a branch of mathematics that deals with numerical calculations. It refers to a method of calculation using integers, rational numbers, real numbers, and complex numbers.
- Arithmetic covers all laws that combine two or more numbers.





https://www.quora.com/Is-there-any-diagram-or-tree-of-all-fields-of-mathematics-or-mathematics-evolution

The types of numbers are shown below.



Type	Description
Natural Numbers	Common counting numbers (0 is not a natural number per se, but it is often included.)
Integers	Whole numbers with a negative set of natural numbers (no fractions)
Rational Numbers	All numbers which can be written as fractions
Irrational Numbers	All numbers which cannot be written as fractions
Real Numbers	A set of rational and irrational numbers which can be represented on a line

Unit 1.

Introduction

1.1. Installing Anaconda for Python

1.2. Intro to Mathematics

1.3. Mathematical Symbols

Reading and Writing Math Symbols

Learning how to read and write symbols in mathematics is important because most of them are Greek letters.

Uppercase	Lowercase	Pronunciation	Uppercase	Lowercase	Pronunciation	Uppercase	Lowercase	Pronunciation
A	α	Alpha	I	ι	Iota	P	ρ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
Δ	δ	Delta	M	μ	Mu	Υ	υ	Upeilon
E	ε	Epsilon	N	ν	Nu	Φ	φ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	omikron	Ψ	ψ	Psi
Θ	θ	theta	Π	π	mi	Ω	ω	omega

Unit 2.

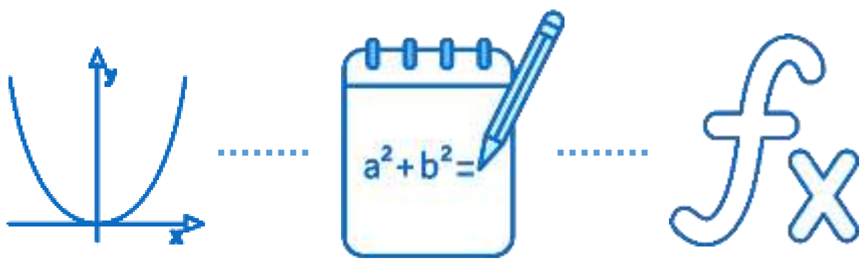
Basic Math for Data Science

- 2.1. Algebra
- 2.2. Sequence
- 2.3. Absolute Value and Euclidean Distance
- 2.4. Sets
- 2.5. Concept of Functions
- 2.6. Exponential and Logarithmic Functions
- 2.7. Natural Logarithms
- 2.8. Sigmoid Functions
- 2.9. Trigonometric Functions

Algebra

Algebra is a study of numerical relationships, properties, and laws of calculation using general characters, which represent numbers instead of individual numbers.

- Currently, it encompasses the study of algebra, a set where combinations between elements such as addition and multiplication are defined.
- In mathematics, an expression represents a rule, principle, and fact with mathematical symbols.
- Equations usually represent a clear and unchanging relationship between certain quantities expressed in letters with algebraic symbols.



Algebra

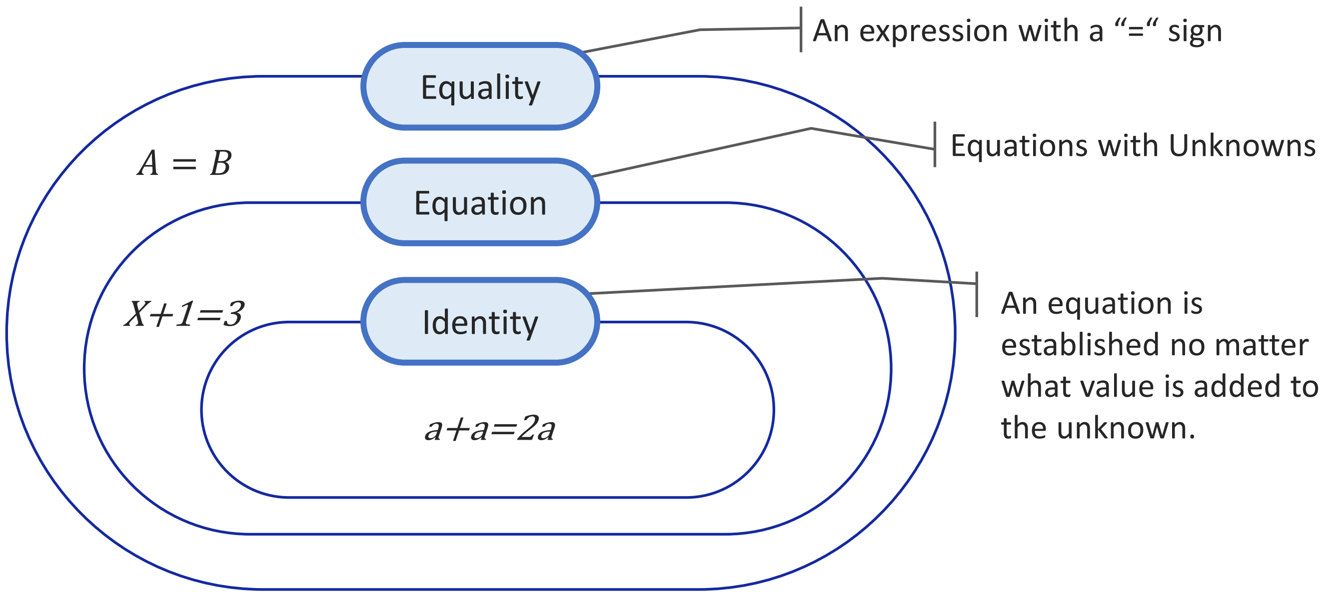
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What is a Mathematical Expression?

- It is an expression using numbers, variables, or both.

$$y$$
$$7$$
$$9-5$$
$$5 \times x - 9$$
$$4 + 8 \times (5 - 4)$$
$$X + 5 \times (7 - x)$$

Equality, Equation, Identity



What is an Equation?

- It is an equation including unknowns.
- The equation is simply expressed by putting an equal sign between two equations.
- All of the following examples are equations.

$$7=7$$
$$x=9$$
$$y+9=14$$
$$x\text{-}4=15-x$$
$$5xy = 8xy^2 + 4$$

Algebraic Equation

- Variables or constants are combined with addition, subtraction, multiplication, and division (except when divided by 0).
- Variables are classified into independent and dependent variables. An independent variable is an amount that increases or decreases or an amount that has countless values in the same equation. The dependent variable also changes, but its amount is generated according to the change in the independent variable.

Ex In y=f(x), x is an independent variable, and y is a dependent variable.

- This type of algebraic equation is also called a "polynomial equation." In many algebraic equations, mathematical and scientific expressions have several variables that are conventionally used as follows.

  - n represents a natural number or integer.
  - x represents a real number.
  - z represents a complex number.

What is a Term?

It is an expression of a number or letter or a product of the two. For example:

$$3, a, 3a, -4ab, \frac{x}{4}, a^2$$

The number of times each term is multiplied by a variable is called a degree.

In each term, the part excluding the characters corresponding to the variables is called a coefficient.

- The order of terms multiplied by two or more variables can be added to the indices of each variable.

Ex If a term is x2y3, the order for the variables x and y is $2 + 3 = 5$ which is 5.

What is a Polynomial Equation?

- It is an equation that includes the sum of powers with one or more unknowns.
- In this equation, the unknowns and numbers on both sides of the equal sign are polynomials.
- The below equation is a polynomial equation.

$$(x-2)^3 = x^3 \ - 6x^2 + 12x \ - 8$$

- The coefficient and the degree of the equation

4 Terms

	$4a$	$+ (-3b)$	$+ \ 4a^2b$	$+ \ 6$
Coefficient	4	-3	4	6
Degree	1	1	3	0

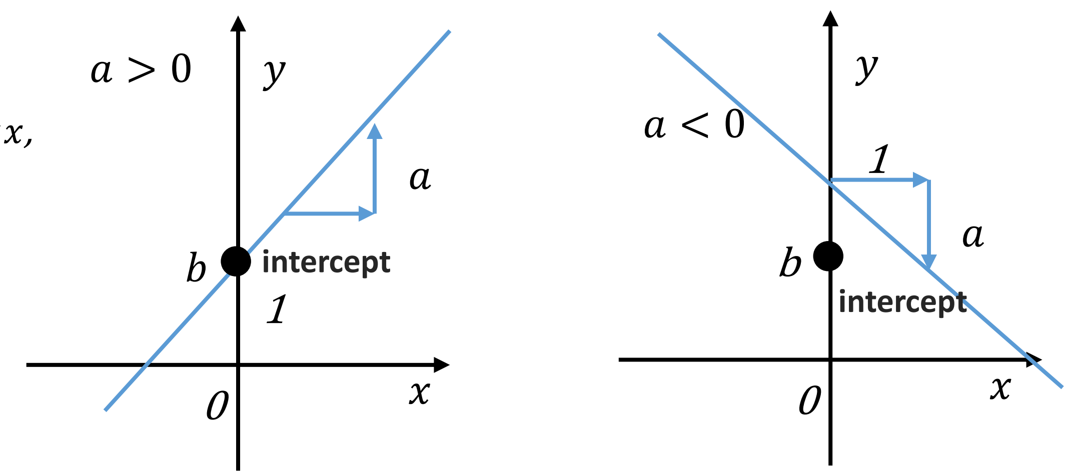
First-Degree Equation

- The first-degree equation is represented by a straight line, and the second is represented by a parabolic graph.
- The graph's shape varies depending on whether the coefficient in front of the largest n-degree is positive or negative.

Graph of a First-Degree Equation

$$First-Degree\ Equation\ for\ x,$$
$$ax + b\ (but, a \neq 0)$$

a *slope*
b *intercept*



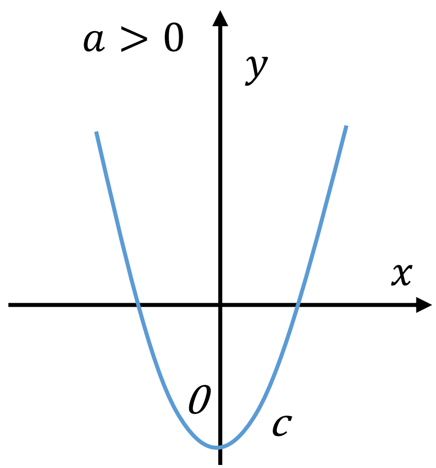
- a and b are treated as constants.
- The degree of term ax is 1, and the order of term b is zero because there is no letter x. Thus, this equation becomes a first-degree equation.

Second-Degree Equation

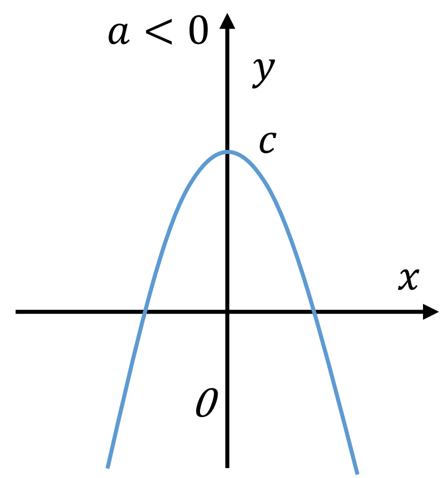
- If the coefficient a is positive, the parabola will convex downward. If a is negative, the parabola will convex upward.

Graph of a Second-Degree Equation

Second-Degree Equation for x,
$ax2 + bx + c$
$(but\ a \neq 0)$



Convex Downward



Convex Upward

- a and b are treated as constants. The degree of ax is 2, the degree of b is 1, and the degree of c is 0 because there is no letter x. Thus, this equation is a second-degree equation.

Unit 2.

Basic Math for Data Science

Sequence

What is a Sequence?

- It is a list of real numbers according to a rule in the order of natural numbers.
- Each number listed is referred to as the “term” of the sequence. Each term is distinguished by a comma.
- The sequence is usually placed in parentheses {} to represent the nth term.

Ex Example of a Sequence

$$1,2,3,4$$
$$x_1, x_2, x_3, x_4$$
$$x_1, x_2, \cdots\ x_n$$

The Sum and Product of Sequences

- Σ (sigma): It is a symbol that briefly represents the sum of sequences and is read as a sum.
- The Π symbol is an uppercase letter of the Greek letter pi. However, it is read as a product rather than pi.
- The starting index value is displayed below with the sum and multiplication symbols, and the final index value is displayed above.
- Multiplication sign can be confused with the letter x, so it is marked as a dot (ex. a·b), not a x b.

$$\sum_{i=1}^{N} x_i = x_1 + x_2 + \cdots + x_N$$

$$\prod_{i=1}^{N} x_i = x_1 \cdot x_2 \cdot \cdots \cdot x_N$$

Arithmetic Sequence

- An arithmetic sequence is created by adding a certain number d to the first term a. The arithmetic sequence $\{a_n\}$ satisfies the following for the natural number n.

$$a_{n+1} = a_n + d(n = 1,2,3,\cdots)$$

- In the definition of an arithmetic sequence, a constant number d is called a common difference of the sequence.

The Sum of Arithmetic Sequence

- Suppose the first term is a, and the common difference is d in the arithmetic sequence $\{a_n\}$. The sum, S_n, from its first term to the nth term is as follows.

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n(2a + (n-1)d)}{2}$$

Find S_n for the arithmetic sequence $\{a_n\}$, and then solve for S_{10}.

- The first term is 5, and the common difference is 3.

In [1]:
```
N = 10
a = 5
d = 3

sum1 = 0
for n in range(1,N+1):
    sum1 = sum1 + (a+(n-1)*d)
print("S10 = ", sum1, "by summation")
```

```
S10 =  185 by summation
```

Geometric Sequence

- The sequence produced by multiplying the first term a by a constant number r in turn is called a geometric sequence. The sequence $\{a_n\}$ satisfies the following for the natural number n.

$$a_{n+1} = a_n r \ (n = 1,2,3, \cdots)$$

- In the definition of a geometric sequence, the constant number r is called a common ratio.

The Sum of Geometric Sequences

- Suppose the first term is a, and the common ratio is r in the geometric sequence $\{a_n\}$. The sum, S_n, from its first term to the nth term is as follows.

$$S_n = \begin{cases} na & , r = 1 \\ \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1} & , r \neq 1 \end{cases}$$

Properties of Σ

- The symbol for the sum of sequences, Σ, has properties as below.

For sequences $\{a_n\}$, $\{b_n\}$, and constant c, the symbol Σ satisfies the following properties.

$$(1) \sum_{k=1}^{n}(a_k + bk) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$(2) \sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$(3) \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

$$(4) \sum_{k=1}^{n} c = cn$$

The Sum of Powers of Natural Numbers

- The sum of the powers of the natural number is often used to find the sum S_n for a general sequence $\{a_n\}$. The following represents such formulas.

(1) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(2) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Find the sum below.

$$\sum_{k=1}^{10}(k^2-k)$$

In [2]:
```python
N = 10

sum1 = 0
for k in range(1,N+1):
    sum1 = sum1 + (k**2-k)
print("S10 = ", sum1, "by summation")
```

```
S10 =  330 by summation
```

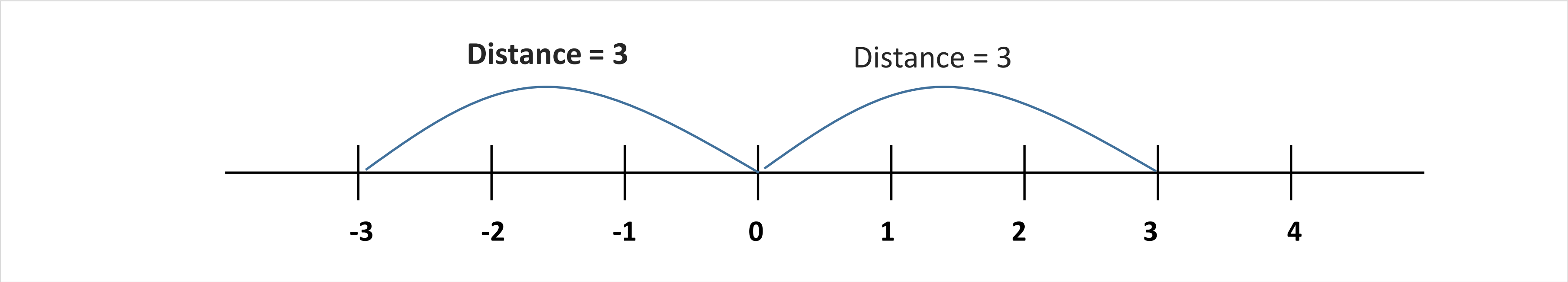
Unit 2.

Basic Math for Data Science

Definition of Absolute Value

Absolute Value

- The absolute value of a certain number means the distance between the number and zero on a vertical line.



Definition of the Euclidean Distance

Euclidean Distance

- It is the length of a line segment between two points as if measured with a ruler.
- The Euclidean distance in two dimensions is as follows.
- When there are points $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ represented by orthogonal coordinate systems, the distance between p and q are calculated is as follows using the two Euclidean norms.

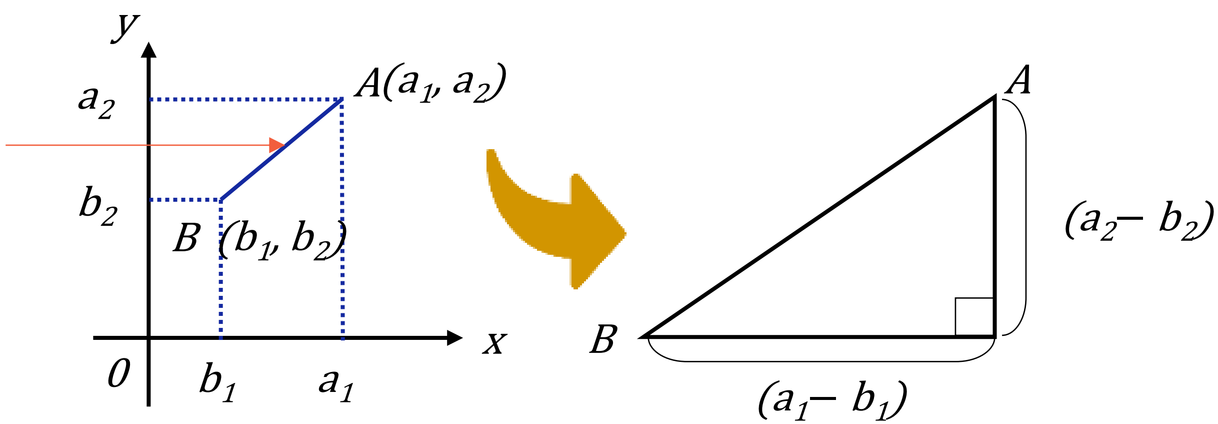
$$\| p - q \| = \sqrt{(p - q) \cdot (p - q)} = \sqrt{\| p \|^2 + \| q \|^2 - 2p \cdot q}$$

- The Euclidean distance between point A and the origin can be expressed as $\| A \|$ using the symbol '‖.' The distance between point A and point B can be expressed as $\| A - B \|$.

Euclidean Distance

- As shown in the below figure, the length of the hypotenuse (segment AB) is calculated using the Pythagorean Theorem.

The distance of segment AB is the distance between point A and point B.



Unit 2.

Basic Math for Data Science

The Concept of Sets

Sets and Elements

- Set: A group in which the target is clearly determined by a given condition
- Element: Each object that makes up the set
- If A is a set of factors of 5, then 1 and 5 are the elements for set A. In this case, the elements 1 and 5 belong to set A and are denoted by the symbols 1∈A and 5∈A. Numbers like 2, 3, and 4 are not elements because they do not belong to set A. In this case, they are denoted by the ∈ symbol with a slash.

How to Represent a Set

- Enumeration: A method of representing a set by listing all elements in the set in { } (When representing a set in an enumeration, each element is written only once regardless of order. If there are too many elements and they have a particular rule in order, some of them can be omitted using the "...."

Ex {1, 2, 3}

- Set-Builder Notation: A method of expressing common properties of a set's elements as a condition

Ex $\{x|x$ is a multiple of 3$\}$

- Venn Diagram: A method of representing a set as a figure, such as a circle, rectangle, etc.

The Inclusive Relation Between Sets

- Suppose there are two sets, A and B. If all elements in set B belong to set A, then set B is said to be a subset of set A. This case can b expressed as B⊂A, meaning that "Set B is included in Set A." or "Set A contains Set B."
- Every set is a subset of itself, and an empty set is a subset of all other sets.
- If set B is not a subset of set A, it is denoted by a single slash in the ⊂ symbol.
- When the elements in sets A and B are identical, sets A and B are said to be the same (i.e., A=B). In this case, B is a subset of A (A⊂ B), and A is a subset of B (B⊂A).
- When the elements in sets A and B are not identical, they are expressed with a single slash in the = symbol.
- Set A is called a true subset of set B if set A is a subset of but not the same as set B (i.e., A⊂B and A=/=B). Here, =/= means "is not equal to".

Calculation of Sets

Intersection and Union

- Suppose there are two sets, A and B. A set of elements belonging to both sets A and B is called the intersection of A and B. It can be represented as A∩B. If this is expressed in the set-builder notation, it would be $\{x|x \in A \text{ and } x \in B\}$.
- A set of elements that belong to set A or B is called the set of A and B. It can be represented as A∪B. If this is expressed in the set-builder notation, it would be $\{x|x \in A \text{ or } x \in B\}$.
- In general, when sets A and B are finite sets, n(A∪B)=n(A)+n(B)-n(A∩B).

Universal Set and Complement Set

- Considering a subset of a given set, the initial set is called a universal set, expressed as U.
- Suppose that set A is a subset of the whole set U. The set of elements belonging to set U but not to set A is called a complement set of A, denoted as A^C. If this is expressed in the set-builder notation, it would be $A^C = \{x | x \in U$ and $x \in/ A$(does not belong)$\}$; and $n(A^C)=n(U)-n(A)$.
- Suppose there are two sets, A and B. A set of all elements that belong to set A but not to set B is called the difference of set B for set A, expressed as A-B. If the difference of set B for set A is expressed with set-builder notation, it would be A-B= $\{x | x \in A$ and $x \in/ B$(does not belong)$\}$ and $n(A-B)=n(A)-n(A\cap B)$. The A-B and B-A, here, are different.

Unit 2.

Basic Math for Data Science

Concept of Functions

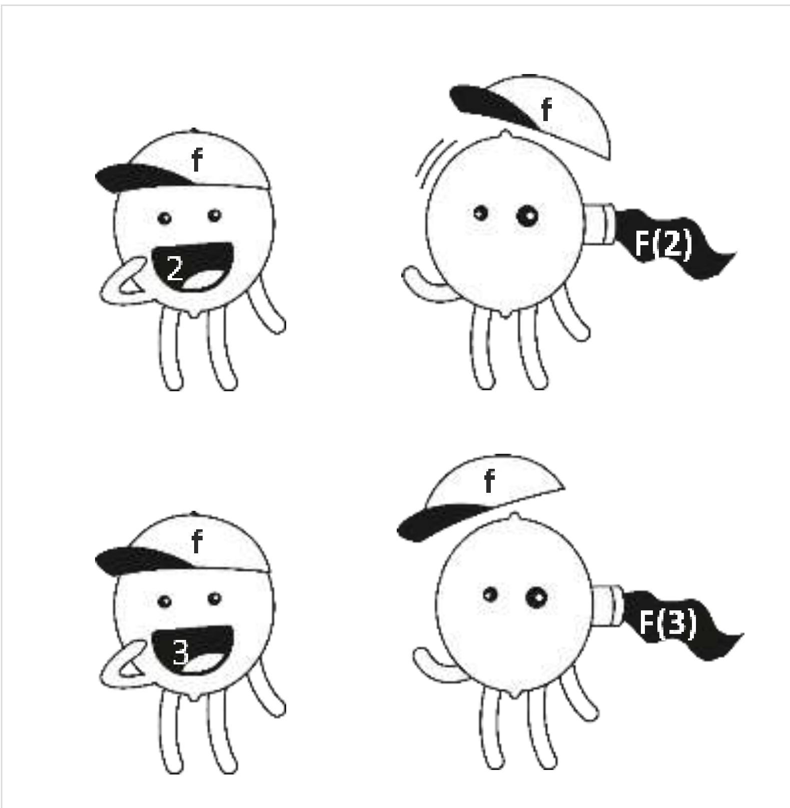
A function is a rule that corresponds each element x in set A to the only element y in set B.

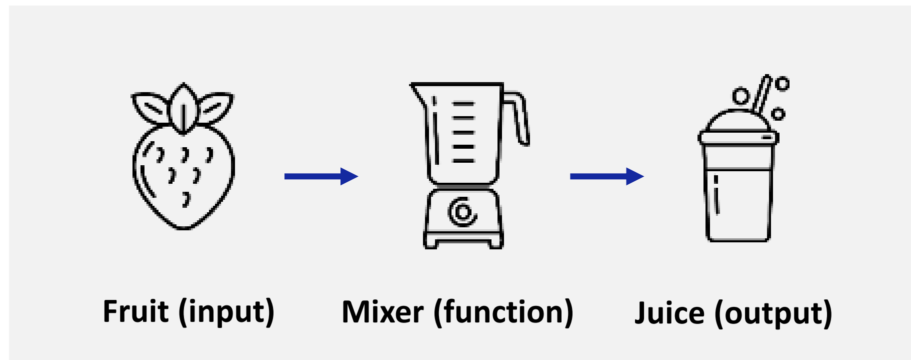
- Denoted by this symbol: f: A → B
- y corresponding to x is called y=f(x)
- The function is not always written as f(x), but also as g(x) depending on the equation.

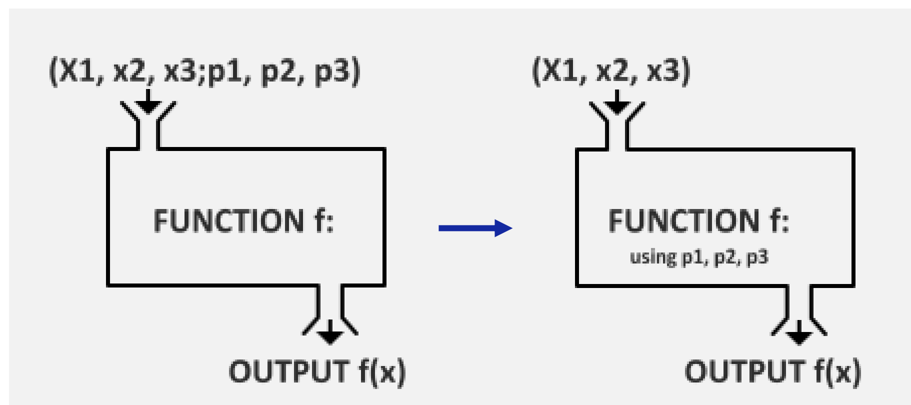
  However, the equation $x^2 + y^2 = 9$ is not a function because, in this case, both x and y are independent variables.

A Comparison Between a Function and a Computer

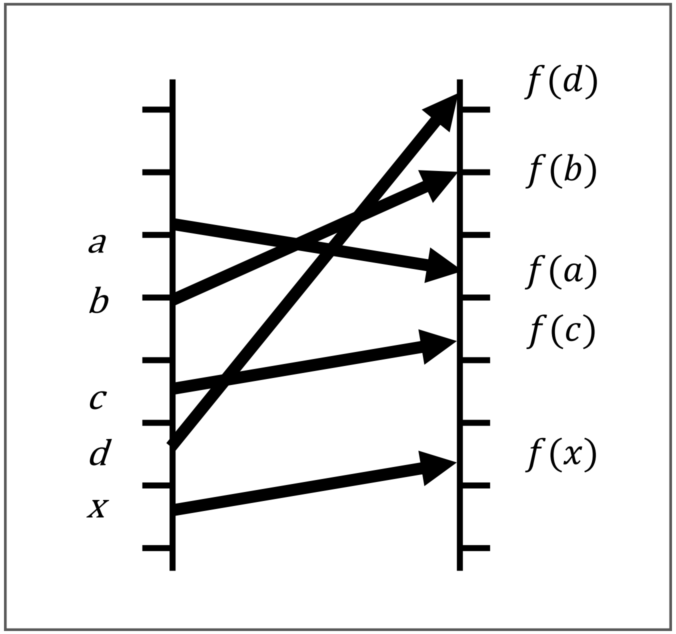
- When explaining functions and their correlation to computers, it is often compared to a mixer. It is also described as an input-output device or a kind of number processor.







Function: A Set of Arrows



- As shown in the figure, a function can be seen simply as a set of arrows pointing from one number to another.
- The arrow comes from each x in the domain of f and points to the value f(x).
- A function is an equation representing the relationship between variables and includes only algebraic operations.

Graph of a Function

Defining the Graph of a Function

- If f is a function with set A as a domain, we define the function f's graph as a set of ordered pairs as follows.

$$\{\, (x, f(x)) \mid x \in A \,\}$$

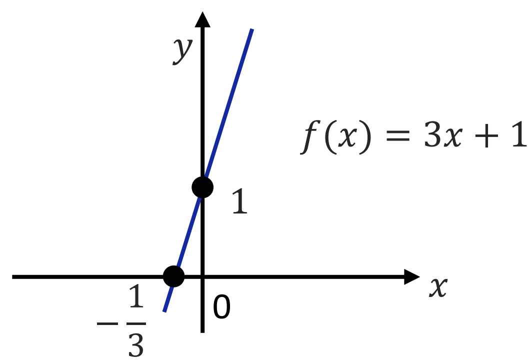
Find the domain and range of the following function and draw its graph.

$$f(x) = 3x + 1$$

- The given function f is defined for all real numbers x, and the phrase 3x+1 of x by f is also defined for all real numbers.
- Therefore, the domain and the range of function f is the real number set $\mathbb{R}$.

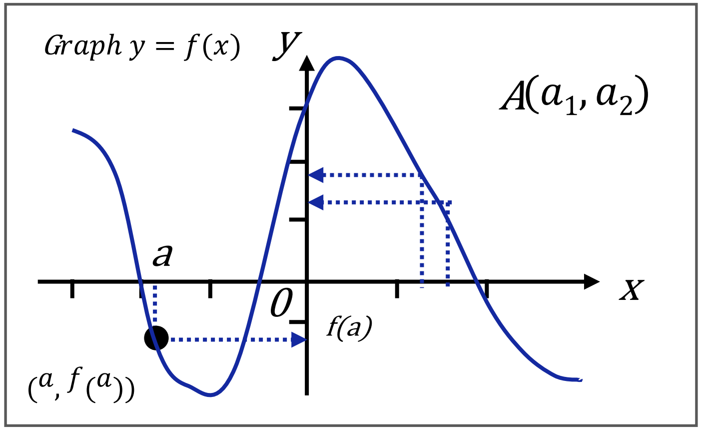
In [3]:

```
import numpy as np
import matplotlib.pyplot as plt
#     f(x)=3x+1
x= np.linspace(-5,5,1001)
fx = 3*x+1
plt.plot(x,fx)
plt.show()
```



Graphic Representation of Functions

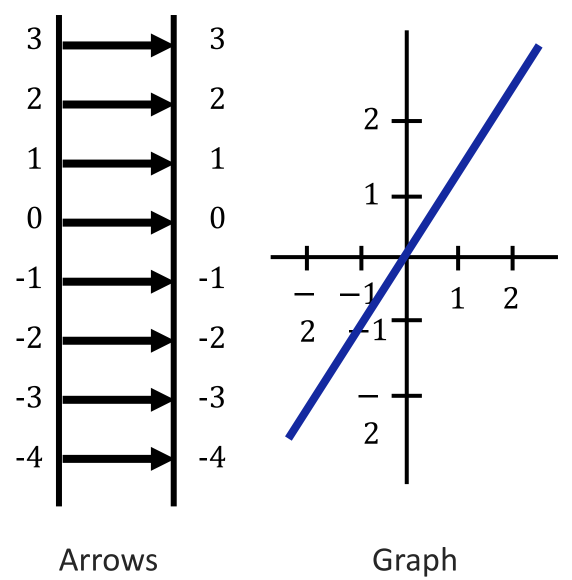
- When the first vertical line (or axis) is laid down, the function can be viewed as a graph.
- The input value x is placed on the horizontal axis, and the output value y is placed on the vertical axis.
- And then the position of the point$(a, f(a))$ above (or below) the point a is determined by the y coordinate that corresponds to the function f's value at point a and point a itself.



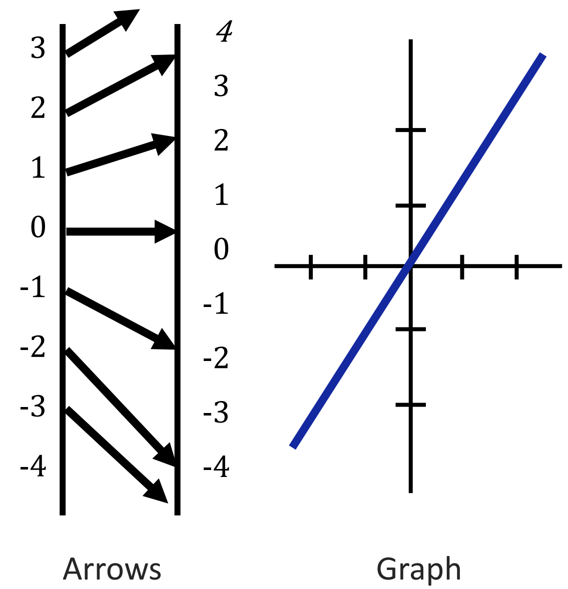
- The curve consists of all points (x, y) determined by $y = f(x)$ which is called the graph $y = f(x)$.

Examples of Graphic Representation of Functions

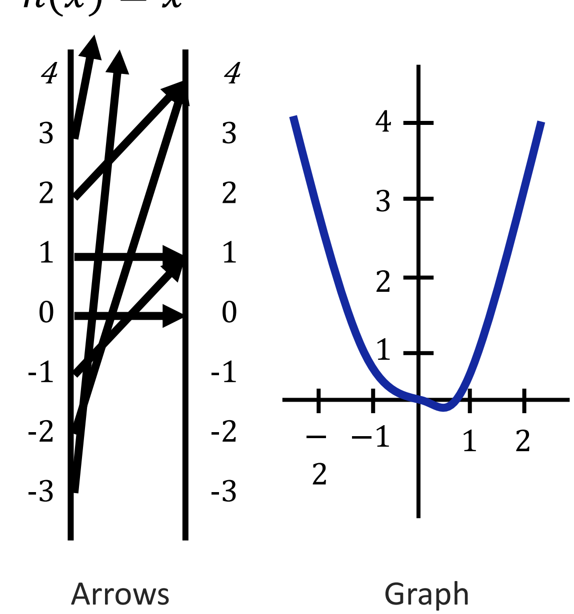
$F(x) = x$



$g(x) = 2x$



$h(x) = x^2$

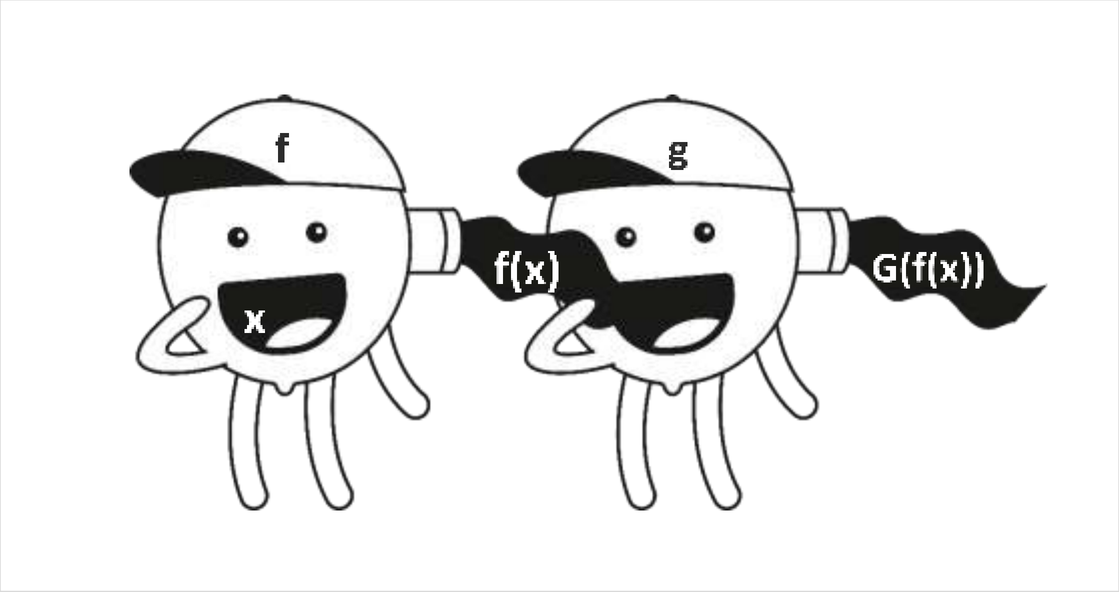


What is a Composite Function?

Composite Function

- It is also possible to “insert” one function into another.

Ex $\mathrm{f(x)} = 10 + x^2$ $g(x) = 3 + x$

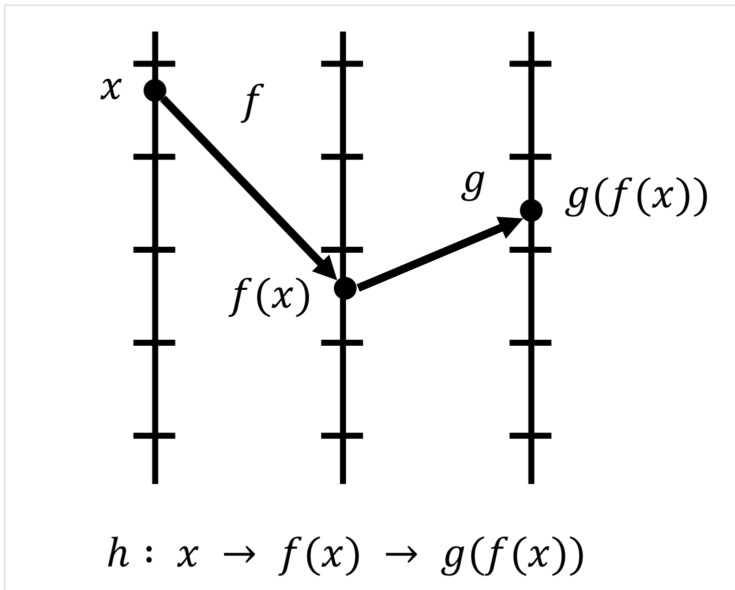


Composite Function

- It is possible to substitute a value for x, or a function for x. In other words, the combination of several functions as follows is called a composite function.

$$\mathrm{f(g(x))} = 10 + g(x)^2 = 10 + (3 + x)^2$$

$$\mathrm{g(f(x))} = 3 + \mathrm{f(x)} = 3 + (10 + x^2)$$



Find $(g \circ f)(2), (f \circ g)(2), (g \circ g)(2)$ for the functions f and g in $f(x) = x^3 + 1$ and $g(x) = \sqrt{x+2}$, respectively.

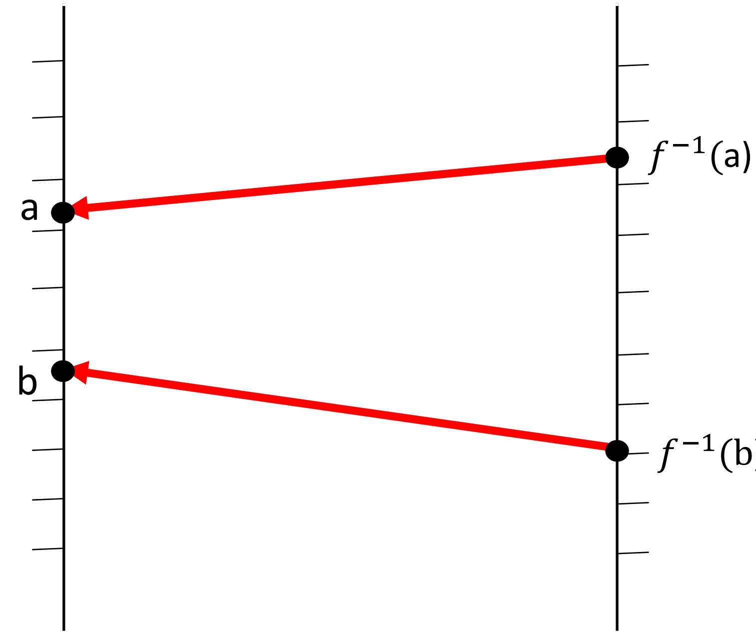
```
In [4]: # f(x)=x^3+1
        def f(x):
            return x**3 + 1

        # g(x)=sqrt(x+2)
        def g(x):
            return np.sqrt(x+2)

        # (g o f)(2), (f o g)(2), (g o g)(2)
        print("(g o f)(2) =", g(f(2)))
        print("(f o g)(2) =", f(g(2)))
        print("(g o g)(2) =", g(g(2)))

        (g o f)(2) = 3.3166247903554
        (f o g)(2) = 9.0
        (g o g)(2) = 2.0
```

What is an Inverse Function?

Inverse Function

- When combining two functions, sometimes, nothing happens.
- If $f(x) = x^{\frac{1}{3}}$ and $g(y) = y^3$, then $h(x) = g(f(x)) = (x^{\frac{1}{3}})^3 = x$.
- If x is inserted into g ° f, then x is returned. This is because h cubes the cube root. Eventually, g returns the result of f to its original state.
- If f is a one-to-one function, we can create a new function, f^{-1}, in other words, an inverse function of f.
- For every x in the domain, f^{-1} is defined as follows.

$$f^{-1}(f(x)) = x$$



Unit 2.

Basic Math for Data Science

Exponents

An exponent is the number multiplied by a specific number called a base.

- The number expressed by a base and exponent is called a power. It is usually represented in a power of 10.

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

- As shown above, 2 is multiplied 3 times by the number of exponents. In this case, 3 is an exponent, and 2 is a base.

Logarithms: The Opposite of Exponentials

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

- In the above case, the index 3 signifies how many times the base 2 must be multiplied. The value meaning 'how many times does 2 need to be multiplied to become 8?' is called a log. It can be represented with the symbol $log_2 8 = 3$. At this time, 2 is called the base, and 8 is called the antilogarithm.

Exponentials

$$8 = 2^3$$
$$100 = 10^2$$
$$81 = 3^4$$
$$0.01 = 10^{-2}$$

Logarithms

$$3 = \log_2 8$$
$$2 = \log_{10} 100$$
$$4 = \log_3 81$$
$$-2 = \log_{10} 0.01$$

Exponents **Log**

Base

Antilog

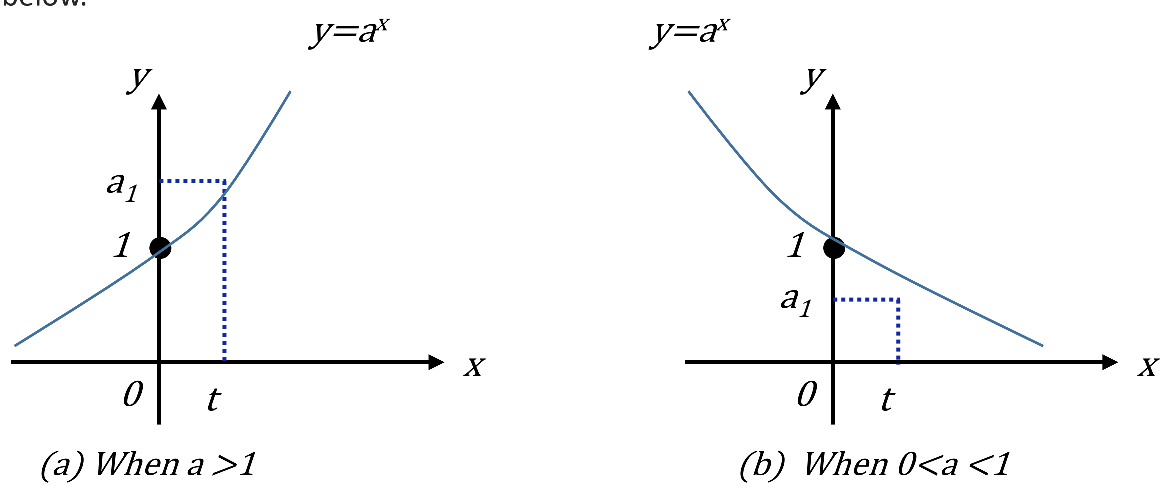
Exponential Functions

- A function in the form as shown below

$$f(x) = a^x$$

- Here, the "base" a is a constant, and the exponent x is a variable.

For the real number a, satisfying $a > 0, a \neq 1$, the graph of the exponential function $y = ax$ with the range a can be drawn as shown below.



The graph of the exponential function $y = a^2$ with the range a

- When the base is 1 in the exponent, it means nothing, so it is taken out altogether.

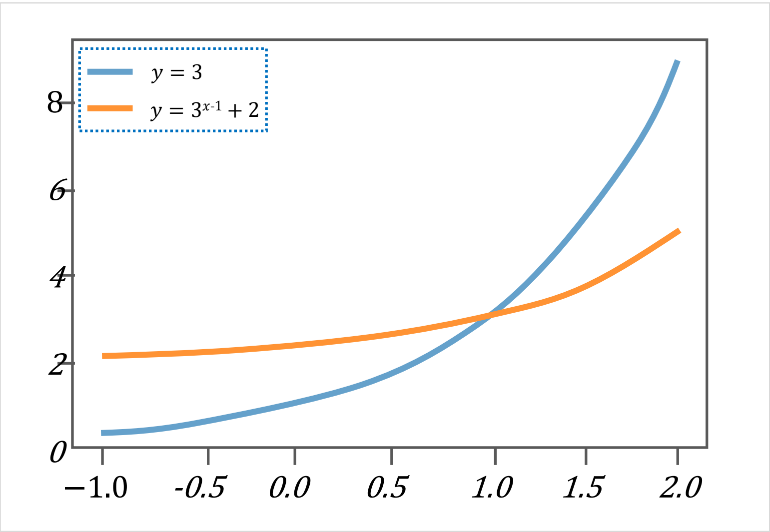
Draw Graphs of Exponential Functions

(a) $y = 3^{x-1} + 2$ b) $y = 2^{-x-2}$

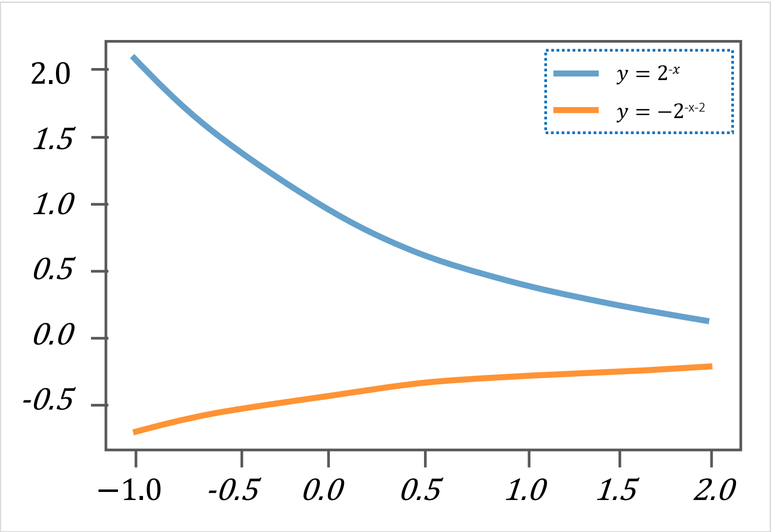
In [5]:
```
import numpy as np
import matplotlib.pyplot as plt

# (a) y1=3^x, y2=3^(x-1)+2
x = np.linspace(-1,2,301)
y1 = 3**x
y2 = 3**(x-1)+2

# For visualization
graph1, = plt.plot(x,y1)
graph2, = plt.plot(x,y2)
plt.legend(handles=(graph1,graph2), labels=(r'$y=3^x$',r'$y=3^{x-1}+2$'))
plt.show()

# (b) y2=2**(-x-2)
x = np.linspace(-1,2,301)
y1 = 2**(-x)
y2 = -2**(-x-2)

# For visualization
graph1, = plt.plot(x,y1)
graph2, = plt.plot(x,y2)
plt.legend(handles=(graph1,graph2), labels=(r'$y=2^{-x}$',r'$y= 2^{-X-2}$ '))
plt.show()
```

Draw Graphs of Exponential Functions

(a) $y = 3^{x-1} + 2$



b) $y = 2^{-x-2}$



Logarithms (log)

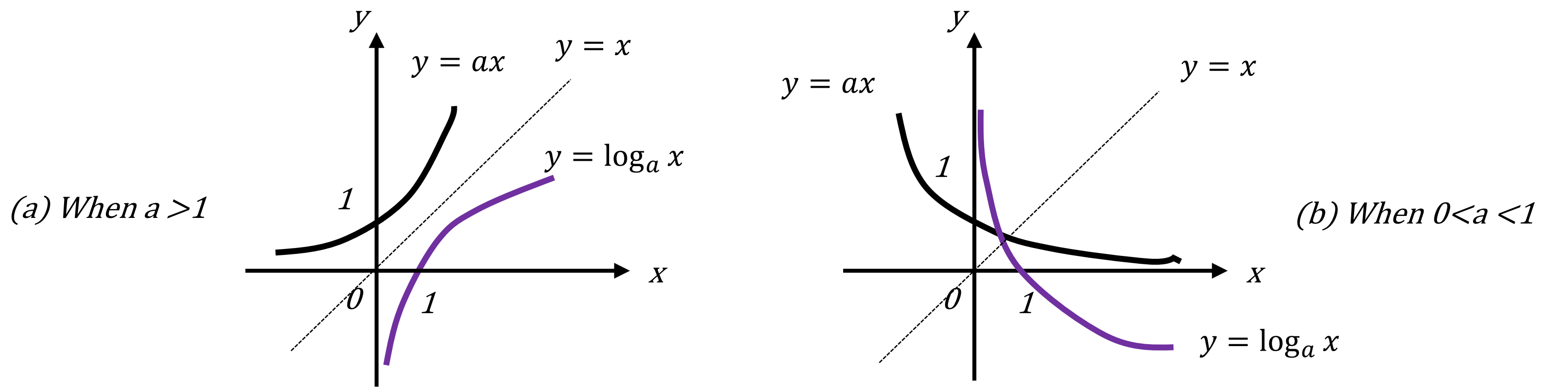
Logarithmic Function (log function)

- The log function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ for the real number a (where $a > 0, a \neq 1$) and the set of real numbers $\mathbb{R}$ and set $\mathbb{R}^+$ is as follows.

$$f(x) = log_a x$$

- According to the definition of logarithms, the inverse function of $y = a^x$ is the inverse function of the exponential function $x = log_a y$ and the inverse function of $y = a^x$ (logarithmic function).

For the real number a, satisfying $a > 0, a \neq 1$, the graph of the logarithmic function $\mathrm{y} = \log_{\mathrm{a}} \mathrm{x}$ with the range a can be drawn as shown below.



The Graph of the Logarithmic Function $y = logax$ with the range a

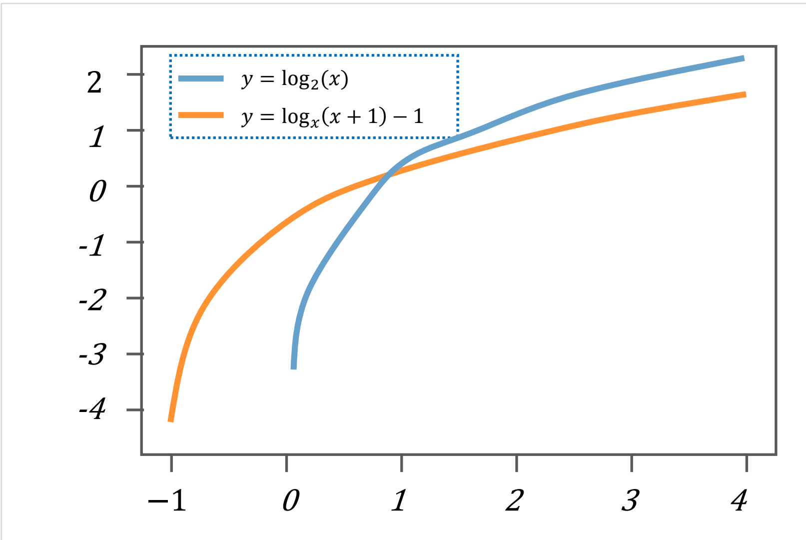
Draw the Graph of the Function Below.

- Draw the graph of the function y = $log_2(x+1)-1$.

In [6]:
```
import numpy as np
import matplotlib.pyplot as plt

# y1=log_2(x), y2=log_2(x+1)-1
x1 = np.linspace(0.1,4,401)
y1 = np.log2(x1)
x2 = np.linspace(-0.9,4,501)
y2 = np.log2(x2+1)-1

# For visualization
graph1, = plt.plot(x1,y1)
graph2, = plt.plot(x2,y2)
plt.legend(handles=(graph1,graph2), labels=(r'$y=log_2(x)$',r'$y=log_2(x+1)-1$'))
plt.show()
```

Draw the Graph of the Function Below.

- Draw the graph of the function $y = log_2(x+1) - 1$.



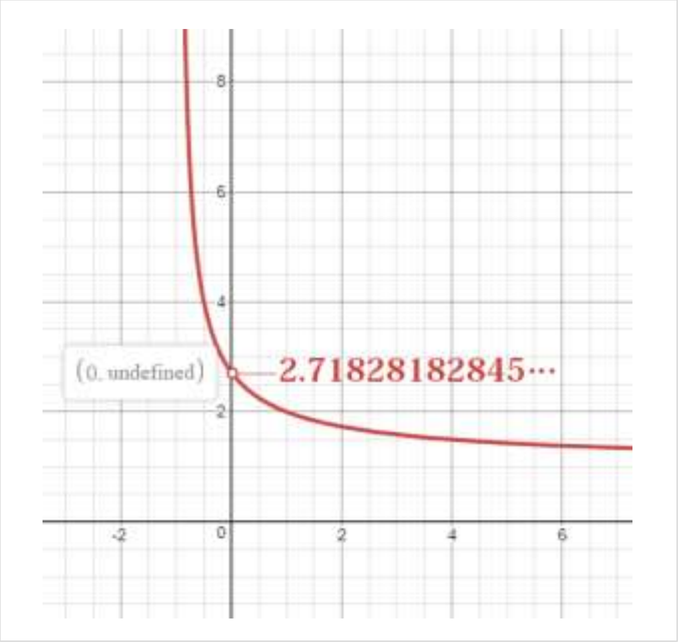
Unit 2.

Basic Math for Data Science

What is Euler's Constant?

Euler's Constant

- e is also called the base of the natural logarithm and is an irrational constant expressed as 2.7182818284.... For our sake, let us consider that it is an irrational number with an approximate value of 2.7182.



x	(1+x)^(1/x)
0.01	2.70481383
0.0099	2.7049473
0.0098	2.70508079
0.0097	2.70521431
0.0096	2.70534785
0.0095	2.70548142
0.0094	2.70561501
0.0093	2.70574863
0.0092	2.70588227
0.0091	2.70601593
0.009	2.70614962
0.0089	2.70628333
0.0088	2.70641707
0.0087	2.70655083
0.0086	2.70668461
0.0085	2.70681842
0.0084	2.70695225
0.0083	2.70708611
0.0082	2.70722
0.0081	2.70735390
0.008	2.70748783
0.0079	2.70762179
0.0078	2.70775577
0.0077	2.70788977
0.0076	2.70802380
0.0075	2.70815785
0.0074	2.70829193
0.0073	2.70842603
0.0072	2.70856016

x	(1+x)^(1/x)
0.0001	2.71814593
0.00009	2.7181595
0.00008	2.71817311
0.00007	2.71818669
0.00006	2.71820028
0.00005	2.71821387
0.00004	2.71822746
0.00003	2.71824106
0.00002	2.71825465
0.00001	2.71826824
0	#DIV/0!
-1.E-05	2.71829542
-2.E-05	2.71830901
-3.E-05	2.71832260
-4.E-05	2.71833620
-5.E-05	2.71834979
-6.E-05	2.71836338
-7.E-05	2.71837697
-8.E-05	2.71839057
-9.E-05	2.71840416
-0.0001	2.71841776
-0.0001	2.71843135
-0.0001	2.71844494
-0.0001	2.71845854
-0.0001	2.71847213
-0.0002	2.71848573
-0.0002	2.71849932
-0.0002	2.71851292
-0.0002	2.71852651

What are Natural Logarithms?

Natural Logarithms

- The log of the number represented by the symbol e is called the “natural logarithm.”
- The natural logarithm of a certain number x is represented as $\ln x$.

The Euler's Constant is the Base of Natural Log

Natural log is a log that has a base of e.

- Natural logarithms, logs with e as its base, are used in many fields.
- By its definition, a log function can have multiple bases. But in general, $\log x$ that does not have a separate base signifies a natural logarithm. However, due to the confusion with common logarithms, it is now marked as $\ln x$.

What are Common Logarithms?

Common Logarithms

- The logarithm is the number of powers that are multiplied by the base to obtain a certain positive number. For example, the logarithm of 100, whose base is 10, is 2 or $log_{10}100 = 2$. This is because $10^2 = 100$. The common logarithm is a positive value using 10 as its base and is expressed as log x.

Unit 2.

Basic Math for Data Science

Sigmoid Functions

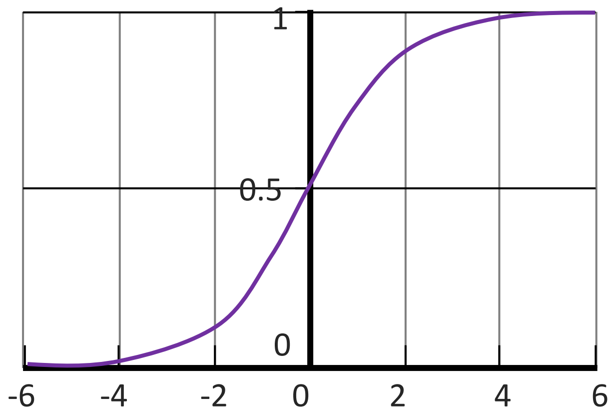
A sigmoid function has the entire real number as a domain, and the return value generally increases monotonically but can also decrease monotonically.

- The return value (y-axis) of the sigmoid function often ranges from 0 to 1.
  For example, the logistic function below is defined by the following formula.

$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$

- Alternatively, it may range from -1 to 1.

Logistic Curve

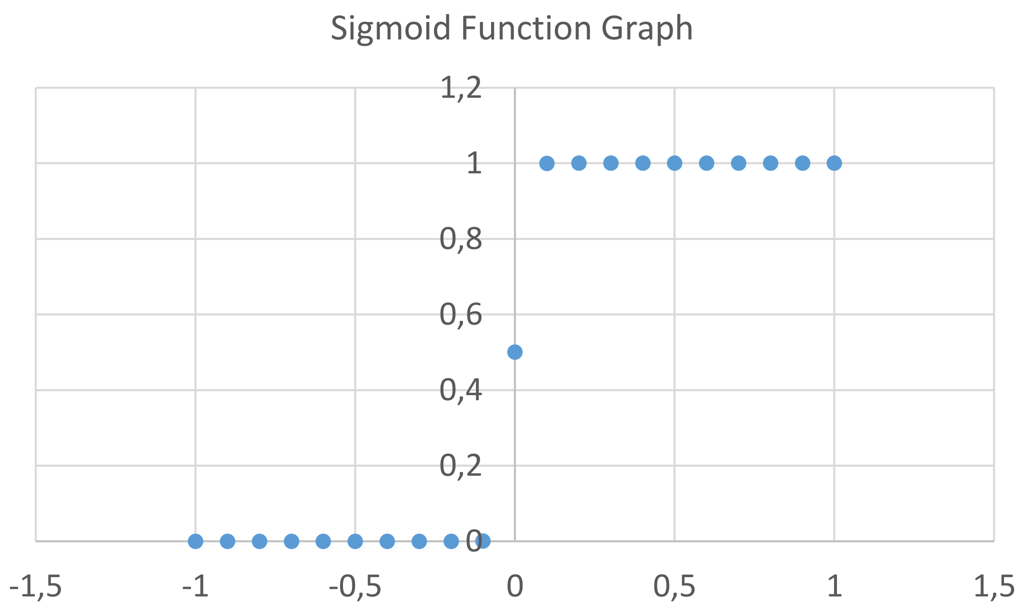


It can also be created through Excel.

- A sigmoid function is a function that is commonly seen in the field of artificial intelligence. When a is 1, it is called a standard sigmoid function.

$$\frac{1}{1+\exp(-ax)}$$

x	y
-1	3.72008E-44
-0.9	8.19401E-40
-0.8	1.80485E-35
-0.7	3.97545E-31
-0.6	8.75651E-27
-0.5	1.92875E-22
-0.4	4.24835E-18
-0.3	9.35762E-14
-0.2	2.06115E-09
-0.1	4.53979E-05
0	0.5
0.1	0.999954602
0.2	0.999999998
0.3	1
0.4	1
0.5	1
0.6	1
0.7	1
0.8	1
0.9	1
1	1



Unit 2.

Basic Math for Data Science

What are Trigonometric Functions?

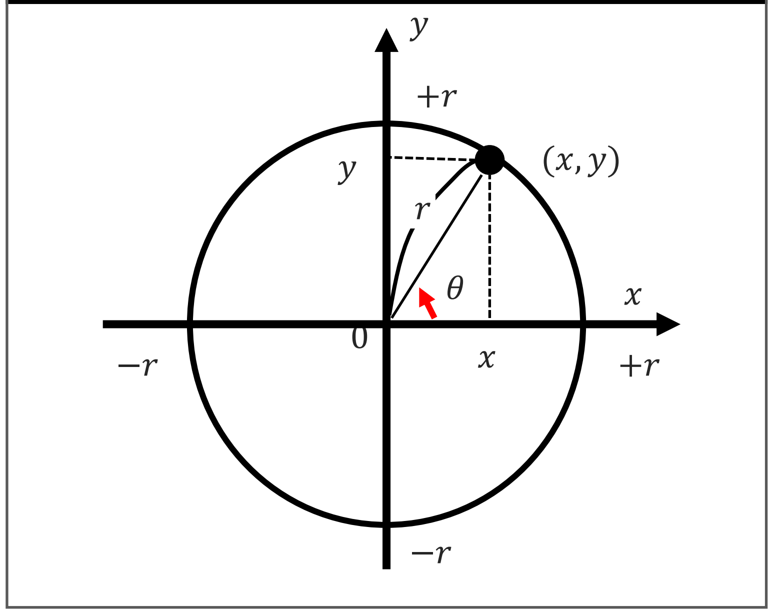
A trigonometric function refers to a function whose value varies depending on the size of the angle, or in other words, a function whose size is a variable.

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}$$

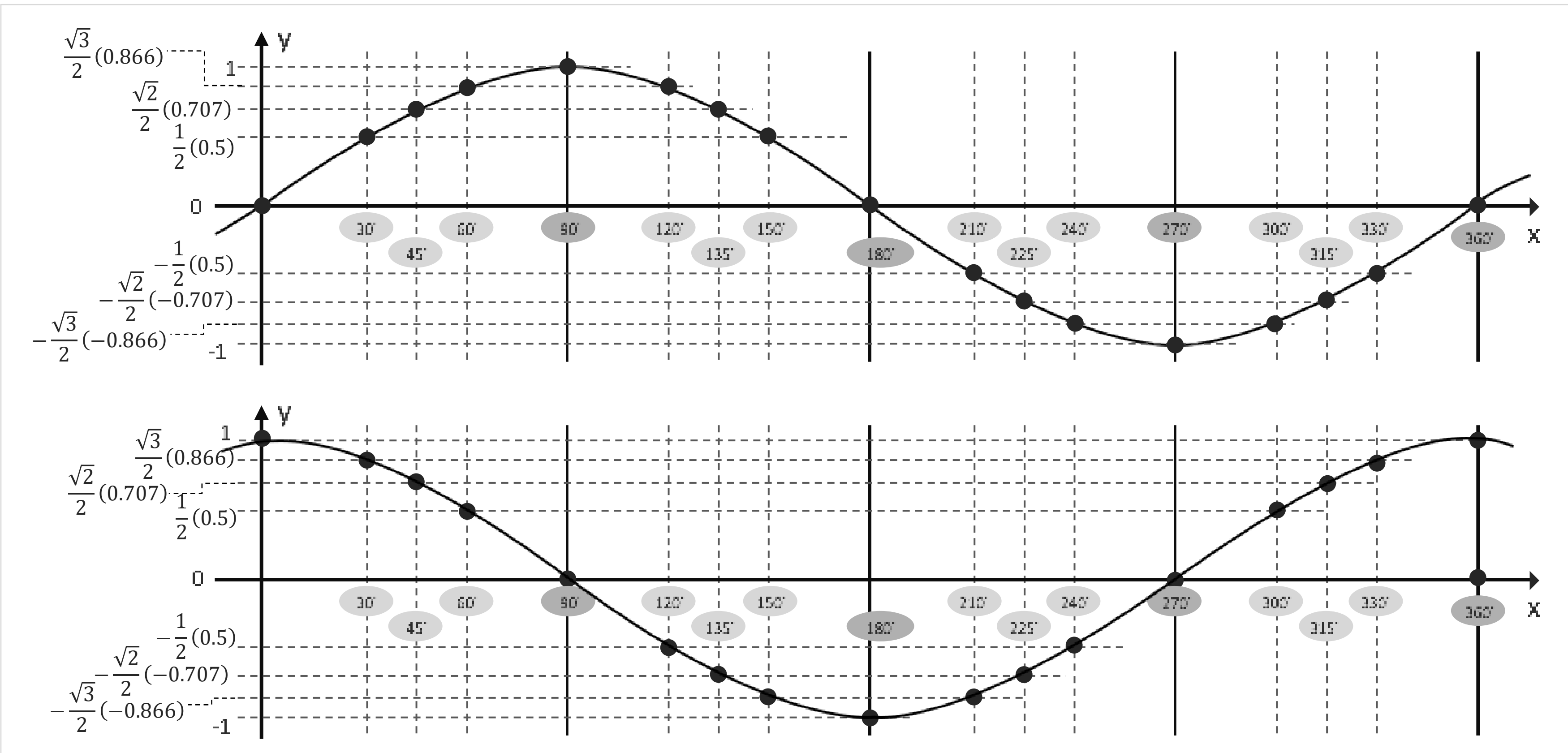
- The trigonometric function is defined using a point (x, y) on a circle with the radius r.

Definition of a Trigonometric Function



Graph of Trigonometric Functions

Graphs of $y = \sin x$ and $y = \cos x$.



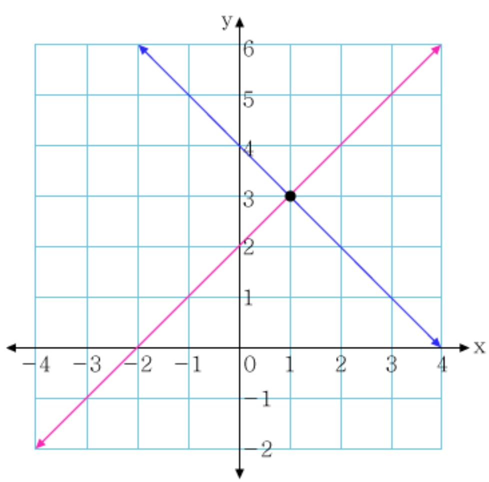
Unit 3.

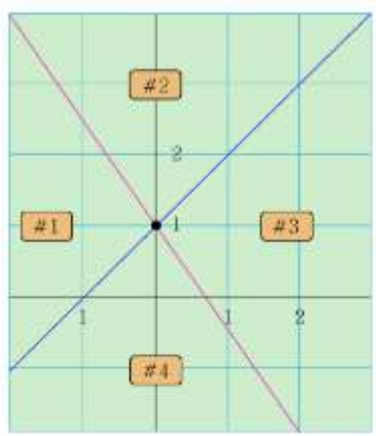
Understanding Data Science: Vector

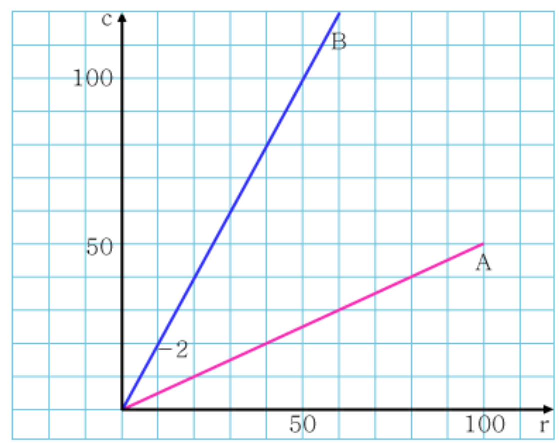
Concept of Vectors

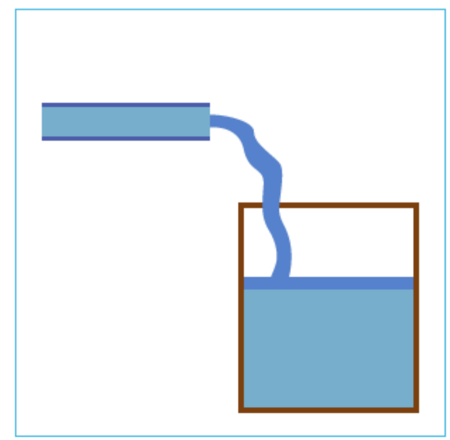
What is a vector?

- A vector refers to what can be expressed as a linear combination regarding the elements of set A.
- It refers to the case where the set A's element $x_1, x_2, x_3, \ldots, x_n$ is multiplied by the constant $a_1, a_2, a_3, \ldots, a_n$ and added into $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n$ within set A. This type of equation is called the linear combination of $x_1, x_2, x_3, \ldots, x_n$.









Linear Relationship between Time and Height of Water

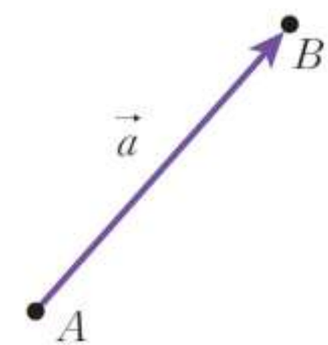
Various Definitions of Vector

Definition of Vector (1)

- A one-dimensional array of numbers or symbols

$$a = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}, \qquad b = [2 \quad 6 \quad 3], \qquad c = (2, 6, 3)$$

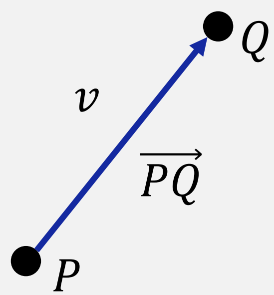
$$x = \begin{bmatrix} a \\ b \end{bmatrix}, \qquad y = [c \quad d], \qquad \vec{v} = (a, b, c)$$



- Vector is a quantity that has both size and direction.
- Starting Point of Arrow A: Tail, Ending Point of Arrow B: Head, Length of Arrow: Magnitude
- Its symbol is $\overrightarrow{AB}$ or $\vec{a}$.

Definition of Vector (2)

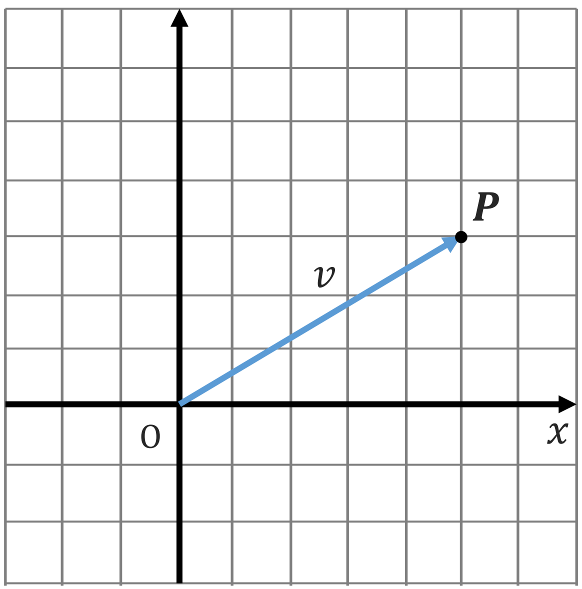
- The element of the vector space is called a vector, which intuitively represents an object whose ratio of direction and length is defined. As a point in space, the end of the arrow corresponds to the coordinates of the vector.
- A vector that represents a line segment with a starting point and an ending point



$$\overrightarrow{PQ} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \; or \; v = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Vector Space

- The vector space in linear algebra, mentioned here, refers to the space in which elements can be added to each other or increased or decreased to a given multiple. In other words, a space created by linear combinations is called a vector space.
- A vector representing the position in the coordinate space



$$v = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Unit 3.

Understanding Data Science: Vector

Vector Norm

Vector Norm

- A function for calculating the magnitude of a vector is called the vector norm. The norm of the vector u is expressed as $||u||$ and satisfies the following properties. Here, u and v are vectors, and α is a scalar.

(1) $||u|| \geq 0$

(2) $||\alpha u|| = |\alpha|||u||$

(3) $||u + v|| \leq ||u|| + ||v||$

(4) $||u|| = 0$ only when $u = 0$

Unit 3.

Understanding Data Science: Vector

Inner Product of Vectors

The vector is defined as "a quantity with size and direction." Thus, the multiplication of the vector should take both "size and direction" into account.

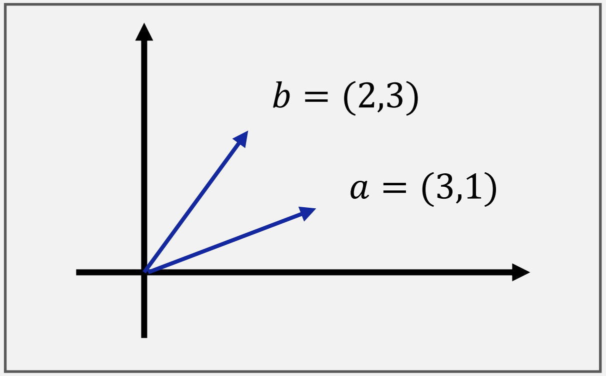
- The multiplication of vectors only concerned with the size (scalar) is called the inner product.
- The inner product of the two-dimensional vector can be calculated by the following equation.

$$\mathrm{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \mathrm{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

If you transpose the vector and multiply it, it is the same as calculating the inner product.

$$\mathbf{a} \cdot \boldsymbol{b} = 3 \cdot 2 + 1 \cdot 3 = 9$$



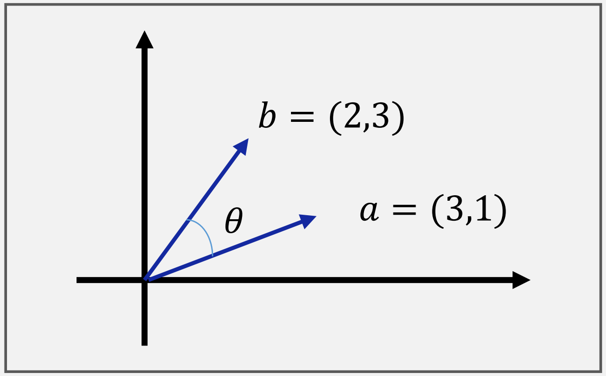
- The result of the vector's inner product is in a form of a real number (size) rather than in a form of a vector.
- This real number is sometimes referred to as scalar, and so this process is often called the scalar multiplication.
- The operator symbol of the inner product is not x, but ·, and it is called a "dot."

If the angle between the vectors a and b is θ, then the inner product can be expressed as follows.

$$\mathbf{a} \cdot \boldsymbol{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta$$

In the above equation, the notation |a| is the length of the vector. For example, the length of the vector $a = (a_1, a_2)$ is defined as follows.

$$|a| = \sqrt{a_1^2 + a_2^2}$$



Unit 3.

Understanding Data Science: Vector

Orthogonal Condition

Vectors with the inner product 0 are orthogonal to each other.

- If the angle between the vectors a and b is θ, the inner product can be expressed as follows.

$$<\mathbf{a}, \boldsymbol{b}> = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta$$

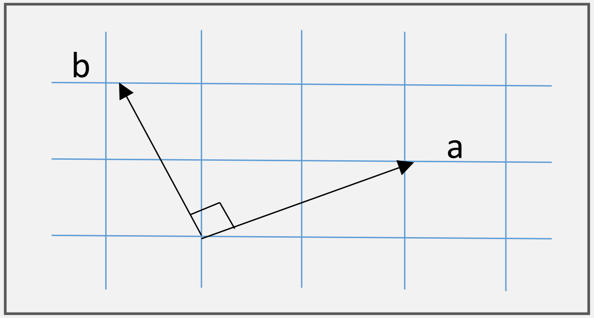
The fact that two vectors a and b are orthogonal (meeting vertically) means that the angle of the two vectors is at 90°.

In other words, <a,b> is 0.

The combination of $\cos 90° = 0$ is $<\mathbf{a}, \boldsymbol{b}> = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta = 0$.

For example, if we find the inner product of a=(2,1) and b=(-1,2) as shown in the figure,

$<a,b> = 2 \times (-1) + 1 \times 2 = 0$.

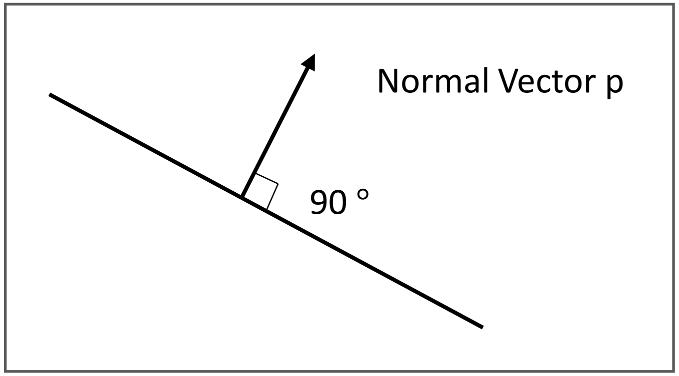


Unit 3.

Understanding Data Science: Vector

Normal Vector

A normal vector refers to a vector that is perpendicular to a certain straight line.



- As shown above, if the equation of the straight line is ax+by+c=0, the normal vector p becomes p=(a,b).

Unit 3.

Understanding Data Science: Vector

Cosine Similarity

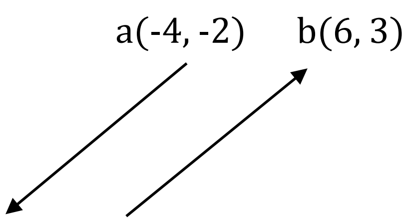
A high cosine similarity means that the vectors are more similar.

$$\langle a, \boldsymbol{b} \rangle = \|a\| \cdot \|b\| \cdot \cos\theta$$

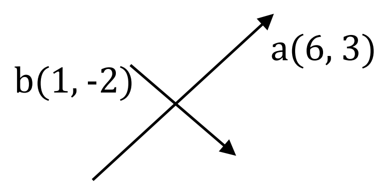
- The equation above solved around $\cos\theta$ looks like the below.

$$\cos(a, b) = \frac{\langle a, b \rangle}{\|a\| \cdot \|b\|}$$

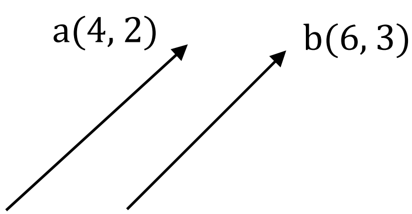
- The value of cosine similarity is within the interval of $-1 \le \cos \le 1$. When the similarity is -1, the two vectors are parallel to each other in opposite directions. When it is 0, the two vectors are orthogonal. When it is 1, the two vectors are parallel in the same direction.



Cosine Similarity is -1.



Cosine Similarity is 0.



Cosine Similarity is 1.

Unit 4.

Understanding Data Science: Matrix

Matrices

What is a matrix?

- It contains several vectors.
- A matrix containing a training set is called a design matrix.

Ex The 150 samples in the Iris data are represented by the design matrix X.

$$X = \begin{pmatrix} 5.1 & 3.5 & 1.4 & 0.2 \\ 4.9 & 3.0 & 1.4 & 0.2 \\ 4.7 & 3.2 & 1.3 & 0.2 \\ 4.6 & 3.1 & 1.5 & 0.2 \\ \vdots & \vdots & \vdots & \vdots \\ 6.2 & 3.4 & 5.4 & 2.3 \\ 5.9 & 3.0 & 5.1 & 1.8 \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\ x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \\ \vdots & \vdots & \vdots & \vdots \\ x_{149,1} & x_{149,2} & x_{149,3} & x_{149,4} \\ x_{150,1} & x_{150,2} & x_{150,3} & x_{150,4} \end{pmatrix}$$

row

column

The Transposition Matrix $\mathbf{A}^{\mathrm{T}}$ of Matrix $\mathbf{A}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$$

- For example, if $A = \begin{pmatrix} 3 & 4 & 1 \\ 0 & 5 & 2 \end{pmatrix}$, then $A^T = \begin{pmatrix} 3 & 0 \\ 4 & 5 \\ 1 & 2 \end{pmatrix}$

Mathematical Expressions Using Matrices

A matrix allows for a concise mathematical expression.

Ex A Matrix Expression of Polynomials

$$f(x) = f(x1, x2, x3)$$

$$= 2x_1x_1 - 4x_1x_2 + 3x_1x_3 + x_2x_1 + 2x_2x_2 + 6x_2x_3 - 2x_3x_1 + 3x_3x_2 + 2x_1 + 3x_2 - 4x_3 + 5$$

$$= (x_1\ x_2\ x_3)\begin{pmatrix} 2 & -4 & 3 \\ 1 & 2 & 6 \\ -2 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + (2 \quad 3 \quad -4)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 5$$

$$= X^T A_X + bTX + c$$

Special Matrices

Square Matrix, Diagonal Matrix, Unit Matrix, and Symmetric Matrix

Square Matrix $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 21 & 5 \\ 4 & 5 & 12 \end{pmatrix}$, Diagonal Matrix $\begin{pmatrix} 50 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$,

Unit Matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, Symmetric Matrix $\begin{pmatrix} 1 & 2 & 11 \\ 2 & 21 & 5 \\ 11 & 5 & 1 \end{pmatrix}$

Multiplication of Matrices

How to multiply matrices

When $C = AB$, $c_{ij} = \sum_{k=1,s} a_{jk}b_{kj}$

If a $2*3$ Matrix $A = \begin{pmatrix} 3 & 4 & 1 \\ 0 & 5 & 2 \end{pmatrix}$ is multiplied by $3*3$ Matrix $B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 5 \\ 4 & 5 & 0 \end{pmatrix}$,

$2*3$ Matrix $C = AB = \begin{pmatrix} 14 & 5 & 24 \\ 13 & 10 & 27 \end{pmatrix}$.

- The commutative law is not established.
- The distributive law and associative law are established. : $\boldsymbol{A(B+C) = AB + AC}$ *and* $\boldsymbol{A(BC) = (AB)C}$

Addition, Subtraction, and Scalar Multiplication of Matrices

Find $A+B,\ A-\frac{1}{2}B,$ and $5A$ when the matrices A, B are as follows.

$$A = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -8 \\ -8 & 2 \end{bmatrix}$$

In [1]:
```
import numpy as np
A = np.array([[1,-4],[-4,1]])
B = np.array([[2,-8],[-8,2]])
print("A+B=", A+B)
print("A-(1/2)B=", A-(1/2)*B)
print("5A=", 5*A)
```

```
A+B= [[  3 -12]
 [-12   3]]
A-(1/2)B= [[0. 0.]
 [0. 0.]]
5A= [[  5 -20]
 [-20   5]]
```

Find AB and BA when matrices A, B are as follows, respectively.

$$AB = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$$

In [2]:
```
import numpy as np
A = np.array([[2,3,1],[4,5,1]])
B = np.array([[-1,2],[4,-2], [3,6]])
print("AB=", np.dot(A,B))
print("BA=", np.dot(B,A))
```

```
AB= [[13  4]
 [19  4]]
BA= [[ 6  7  1]
 [ 0  2  2]
 [30 39  9]]
```

Unit 4.

Understanding Data Science: Matrix

Reverse Matrix

Unit Matrix, Identity Matrix

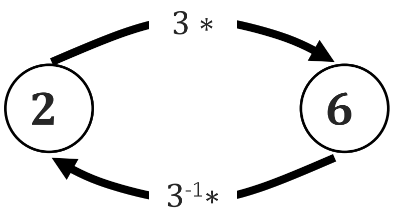
- All of the main diagonal components are square matrices with 1, and all of the remaining components are 0.
- $n \times n$ Unit Matrix ⇨ expressed as In

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

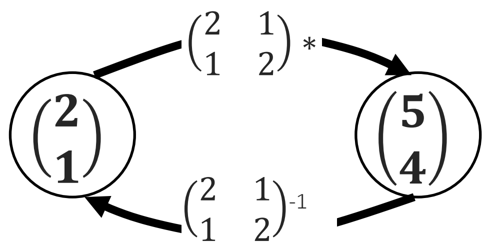
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

A reverse matrix is defined when the product of Matrix A and Reverse Matrix A^{-1} is Unit Matrix E.



$3^{-1} * 3 = 1$

(a) Principle of Reciprocity



$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} * \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

(b) Principle of Reverse Matrix

Reverse Matrix A^{-1} of Square Matrix A

$$\boxed{A^{-1}A = AA^{-1} = I}$$

- For example,

The Reverse Matrix of

$\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} is \begin{pmatrix} 2 & -0.5 \\ -3 & 1 \end{pmatrix}$

The Reverse Matrix A^{-1} of the 2×2 Square Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is as follows.

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, the reverse matrix of the given Matrix A does not exist.

Determinant

- It is the value that is defined only in a square matrix and determines the reversibility of the matrix.
- Historically, the matrix was conceived with the question of “how to solve the simultaneous linear equations.” In his research, Arthur Cayley saw that the solution of the simultaneous equation differed depending on the value of (ad-bc)(ad-bc). This is how the term ‘determinant’ was created, as this value “determines” the existence of the solution (the matrix’s reversibility). In addition to Cayley’s research, William Rowan Hamilton thought, “Well, why don’t we separate the coefficients and the variables of the simultaneous equation?” to create the matrix.

Determinant

- The notation for the determinant is its abbreviation 'det' and the absolute value symbol (| · |).

For the Matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- $|A| = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc$ Etc.

- If the absolute value symbol is double-layered as seen in the latter two equations, it means to find the absolute value after finding the determinant.
- The inner absolute value symbol means "to calculate the matrix equation," and the outer absolute value symbol means "to take the absolute value after the calculation."

$$\|A\| = \left|\begin{vmatrix} ab \\ c\ \mathrm{d} \end{vmatrix}\right| = |ad - bc| = \begin{cases} ad - bc & (ad - bc > 0) \\ -(ad - bc) & (ad - bc < 0) \end{cases}$$

Find the Reverse Matrix of A, B when the 2×2 Matrix A, B are as follows, respectively.

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 \\ -8 & 12 \end{bmatrix}$$

In [3]:

```
import numpy as np
A = np.array([[1,3], [5,7]])
B = np.array([[4,-6],[-8,12]])
detA = np.linalg.det(A)
Ainv = np.linalg.inv(A)
detB = np.linalg.det(B)
print("Determinant of A=", detA)
print("Inverse of A=", Ainv)
print("Determinant of B=", detB)
```

```
Determinant of A= -7.999999999999998
Inverse of A= [[-0.875  0.375]
 [ 0.625 -0.125]]
Determinant of B= 0.0
```

Unit 4.

Understanding Data Science: Matrix

Linear Transformation

Linear transformation is a function that mathematically multiplies a vector by a matrix to create another vector. It is a method of converting from one vector space to another while maintaining the vector's characteristics.

- Transformation: A function in two-dimensions that corresponds $P(x,y)$ to $P'(x', y')$.
- Linear Transformation: A transformation expressed by the linear expression for x, y without a constant term x', y'.

Transformation T: When x', y' in (x, y) → (x', y') is expressed as follows, it is called linear transformation.

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} (but, \text{ a, b, c, d are constants})$$

4.3. Linear Transformation

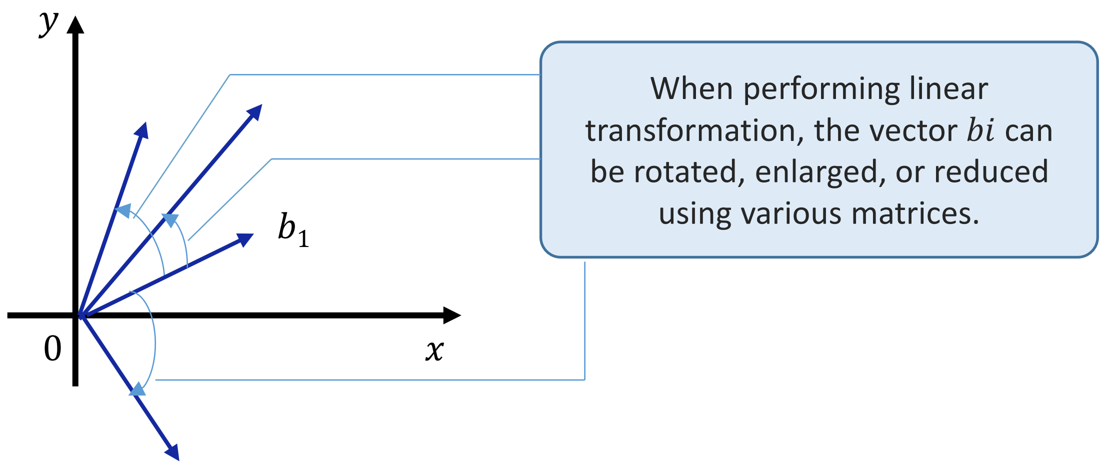
Since linear transformation is in the form of a linear system, it can be expressed in the form of a matrix.

- Transformation T: *When the constants $a, b, c,$ and d of x', y' in* $(x, y) \rightarrow (x', y')$ is expressed as $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$

  Linear Transformation $\boldsymbol{T}$ can be expressed in the form of a matrix as shown below.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Converting a vector in a vector space is called linear transformation.



Unit 4.

Understanding Data Science: Matrix

Eigenvalues and Eigenvectors

Characteristic Polynomials and Eigenvalues

- When A is an n × n matrix and I is an identity matrix,

$$Ax = \lambda x$$

- In the linear system, the necessary and sufficient condition for the existence of a solution for x≠0 is the determinant $|A - \lambda I| = 0$.

Since linear transformation is in the form of a linear system, it can be expressed in the form of a matrix.

- For the following $n \times n$ Matrix A = $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{11} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ $Det\ (A - \lambda E)$ = $\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{11} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$ is called the characteristic polynomial of A.
- $p(\lambda) = |A - \lambda E| = 0$ is called the characteristic equation or determinantal equation.
- In this case, $p(\lambda)$ is the nth degree polynomial of λ. The root of the characteristic equation λ is called the eigenvalue.
- And a vector x is called the eigenvector of the eigenvalue λ.

Definition of Eigenvalues and Eigenvectors

- Suppose that a non-zero vector x goes through a nth-order square matrix A for linear transformation. If the image of x is λx, λ is called the eigenvalue of A, and x is called the eigenvector of λ. In other words, λ and x that satisfy the following relationship are called eigenvalues and eigenvectors, respectively.

$$Ax = \lambda x \ (but, x \neq 0)$$

- In Matrix A = $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, if eigenvalues and eigenvectors are
  - x = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$Ax = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda x$$

  - Therefore, λ=5 becomes the eigenvalue of matrix A, and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ becomes the eigenvector of the matrix A for the eigenvalue 5.

Characteristic Equation (Determinantal Equation)

- The definition of the eigenvalue and characteristic equation is as follow.

$$Ax = \lambda x \; (but, x \neq 0)$$

- If you move the right side of the equation to the left, you can make the following equation where E is a unit matrix.

$$(A - \lambda E)x = 0$$

- If $(A - \lambda E)$ of this equation has the Reverse Matrix $(A - \lambda E)^{-1}$, both sides can be multiplied by the Reverse Matrix as follows.

$$(A - \lambda E)^{-1}(A - \lambda E)x = (A - \lambda E)^{-1}0$$

- Therefore, $x = (A - \lambda E)^{-1}0 = 0$.
- As a result, this equation must have a self-evident solution of $x = 0$. This contradicts the condition that the eigenvector x is initially assumed as $x \neq 0$.
- In order to eliminate this contradiction and to have an eigenvector, $(A - \lambda E)$ must not have a Reverse Matrix $(A - \lambda E)^{-1}$.
- After all, the conditional equation for the existence of an eigenvector is as follows. This equation of λ is called the characteristic equation (or determinantal equation) of Matrix A.

$$\det(A - \lambda E) = 0$$

Solve for eigenvalues and eigenvectors.

- In order to obtain eigenvalues and eigenvectors, follow the below procedure.

1. Find det($A - \lambda E$).
2. The root of the obtained characteristic equation becomes the eigenvalue.
3. Obtain the eigenvectors by solving for each corresponding eigenvalue "($A -$ " λ"E)x=0."

Solve for the characteristic equation and the eigenvalue of the given Matrix A.

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

- First, solve for the characteristic equation, then solve for the eigenvalues.

$$A - \lambda E = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda E) = (2-\lambda)(-6-\lambda) - (3)(3) = \lambda^2 + 4\lambda - 21$$

- The characteristic equation is $\lambda^2 + 4\lambda - 21$.
- If you solve $\lambda^2 + 4\lambda - 21$, it becomes $(\lambda-3)(\lambda+7)=0$. Therefore, the eigenvalues of A are 3 and -7.

Solve for the eigenvalue and the corresponding eigenvector of Matrix A.

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

- First, solve for the characteristic equation, then solve for the eigenvalues.

$$A - \lambda E = \begin{bmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{bmatrix} = 0$$

$$\det(A - \lambda E) = (3-\lambda)(-2-\lambda) - (3)(2) = \lambda^2 - \lambda - 12$$

- The characteristic equation is $\lambda^2 + \lambda - 12$.
- If you solve $\lambda^2 + \lambda - 12 = 0$, it becomes ($\lambda$-4)($\lambda$+3)=0. Therefore, the eigenvalues of matrix A are 4 and -3.

Solve for the eigenvalue and the corresponding eigenvector of Matrix A.

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

- 1) In order to obtain the eigenvector corresponding to eigenvalue λ=4, solve for (A-4E)x=0. Then,

$$\begin{bmatrix} 3-4 & 2 \\ 3 & -2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + 2x_2 = 0$

$3x_1 - 6x_2 = 0$

Since $x_1 = 2x_2$, if x_2 is 1, it becomes $x_1 = 2$.

Therefore, the eigenvector corresponding to eigenvalue λ=4 is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Solve for the eigenvalue and the corresponding eigenvector of Matrix A

$$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$$

- 2) In order to obtain the eigenvector corresponding to eigenvalue λ= -3, solve for (A-(-3)E)x=0. Then,

$$\begin{bmatrix} 3-(-3) & 2 \\ 3 & -2-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$6x_1 + 2x_2 = 0$

$3x_1 - x_2 = 0$

$6x_1 = -2x_2$

$x_2 = -3x_1$

If x_1 is 1, it becomes $x_2 = -3$.

Therefore, the eigenvector corresponding to eigenvalue λ=-3 is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

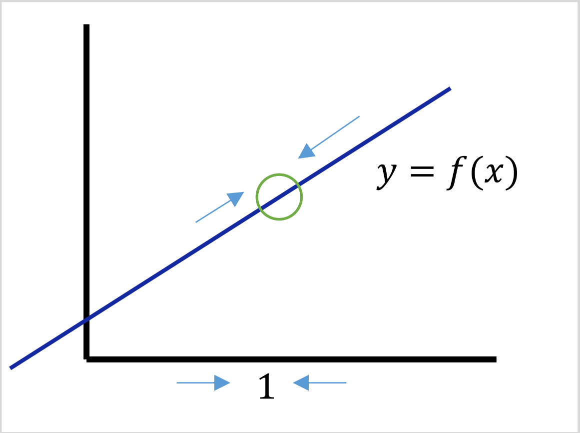
Unit 5.

Understanding Deep Learning: Derivatives

Limits

A limit means that a sequence or function value is infinitely close to a certain value.

- Take a look at the equation $f(x) = \frac{x^2-1}{x-1}$ and the figure below.



- $f(x)$ cannot define the value of y because the denominator becomes 0 when x=1.
- Instead, when $x \neq 1$, the value of the function f(x) may be determined.
- So, if we make the value of the variable x as close as possible to 1 but never let x=1, we can find the value of f(x).
- Suppose we make the value of x as close as possible to the value a. We express this shape as "converging," in which the value of the function f(x) gets as close as possible to the value alpha (a).

$$\lim_{x \to \alpha} f(x) = \alpha$$

$f(x) \to \alpha(x \to$a) α is the limit.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1}(x + 1) = 2$$

Unit 5.

Understanding Deep Learning: Derivatives

Basics of Derivatives

Rate of Change

- When you look at the dashboard when the car moves, it tells you the car's speed at every moment.
- Since speed is the "rate of change in distance over time," the car's speed means the "instantaneous rate of change" of the distance traveled at a specific moment.
- As such, the rate of change at some point is called "derivative."
- For the function y=f(x), the amounts of change in x and y are called increments and denoted by Δx and Δy, respectively.
- The ratio of the two increments is called the average rate of change.
- For the function y=f(x), the average rate of change when the value of x changes from a to a+Δx is expressed as $\frac{\Delta y}{\Delta x}$ and defined as follows.

$$\frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{(a+\Delta x) - a} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

Differential Coefficient and Derivatives

- The instantaneous rate of change can naturally represent the "moment" by making Δx close to 0 at the average rate of change $\frac{\Delta y}{\Delta x}$.
- In addition, the instantaneous rate of change can be mathematically defined using limits.
- For the function y=f(x), the instantaneous rate of change in x=a is expressed as $f'(a)$ and defined as follows.

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

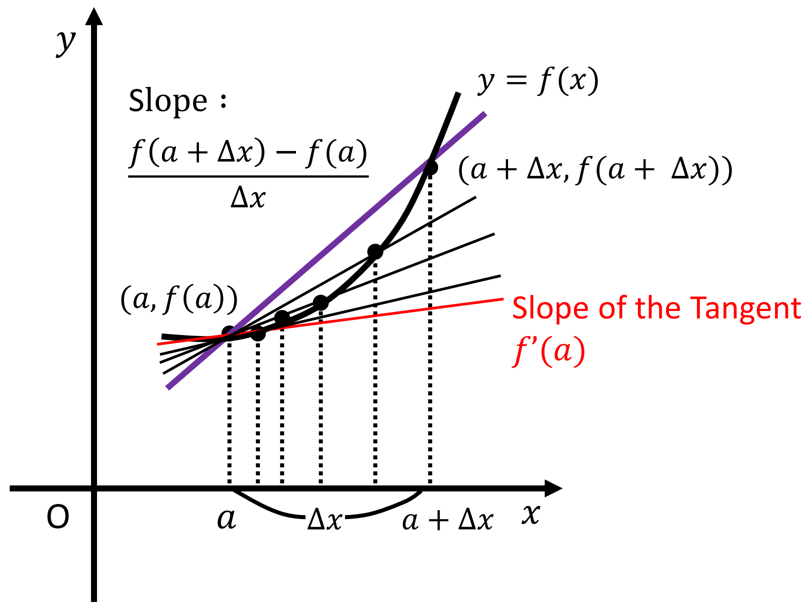
- The rate of instantaneous change for the function is called a differential coefficient.

Differential Coefficient and Derivatives

- Another definition of differential coefficients: $a+\Delta x=x$ (When $\Delta x \to 0$, $x \to a$)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

- Geometric meaning of differential coefficients: The slope of the tangent of the function $y=f(x)$ in $x=a$



Differential Coefficient and Derivatives

- Derivatives: A function representing the “slope of the tangent” at all points above $y=f(x)$
- The derivative of the differentiable function y=f(x) is defined as follows.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Unit 5.

Understanding Deep Learning: Derivatives

Differential Method

We learned that derivatives represent the degree to which a function changes (slope) at a certain time and that we can also use limits to find derivatives.

- However, since it is cumbersome to calculate this way, the following formula is used to calculate differentials.
- The following is true for the two differentiable functions f(x) and g(x).

(1) $\{cf(x)\}' = cf'(x)$ (but c is a constant)

(2) $\{f(x) \pm g(x)\}' = f'(x) \pm g'(x)$

(3) $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$ (Product Rule)

(4) $\left\{\frac{f(x)}{g(x)}\right\}' = \frac{f'(x)g(x)-f(x)g'(x)}{\{g(x)\}^2}$ (but $g(x) \neq 0$)(Quotient Rule)

Differential Coefficients and Derivatives

- f(x) = x^n and the derivative of the constant function

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$ (but n is a natural number)

(2) $\frac{d}{dx}(c) = 0$ (but c is a constant)

Find the derivative of the constant function.

(a) $f(x) = x^{2020} + 3x^3 - 1$

In [1]:
```
import sympy as sp
# (a) f(x) = x**2020 + 3*x**3 - 1
x  = sp.Symbol('x') # Use the letter x as a symbol
fx = x**2020+3*x*x*x-1
```

In [2]:
```
fx
```

Out[2]: $x^{2020} + 3x^3 - 1$

In [3]:
```
sp.diff(fx,x)
```

Out[3]: $2020x^{2019} + 9x^2$

Find the derivative of the constant function.

(b) $g(x) = \frac{x}{x+1}$

$$g'(x) = \frac{\frac{d}{dx}(x)\cdot(x+1) - x\cdot\frac{d}{dx}(x+1)}{(x+1)^2} = \frac{1(x+1)-x}{(x+1^2)} = \frac{1}{(x+1)^2}$$

In [4]:
```
# (b) g(x) = x/(x+1)
x  = sp.Symbol('x')
gx = x/(x+1)
print("g'(x)=",sp.simplify(sp.diff(gx,x)))
```

```
g'(x)= (x + 1)**(-2)
```

Unit 5.

Understanding Deep Learning: Derivatives

Difference Between Logarithmic and Exponential

- Natural Log: Log with the irrational number e as its base. log_ex → log$x$ (base e omitted)

(1) $\frac{d}{dx}(e^x) = e^x$

(2) $\frac{d}{dx}(a^x) = a^x \log a$

(3) $\frac{d}{dx}(\log x) = \frac{1}{x}$

(4) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

Find the derivative.

(a) $f(x) = \frac{e^x}{x}$ $(but\ \mathrm{x} \neq 0)$

$$f'(x) = \frac{\frac{d}{dx}(e^x)x - e^x\frac{d}{dx}(x)}{x^2} = \frac{(e^x)x-(e^x)}{x^2} = \frac{e^x(x-1)}{x^2}$$

In [5]:
```
# (a) f(x) = exp(x)/x
x  = sp.Symbol('x')
fx = sp.exp(x)/x
sp.diff(fx,x)
```

Out[5]:

$$\frac{e^x}{x} - \frac{e^x}{x^2}$$

Find the derivative.

(b) $g(x) = 3\log_2 x - x\log x$

$$g'(x) = 3\frac{d}{dx}(log_2 x) - \left\{\frac{d}{dx}(x)\,log\,x + x\frac{d}{dx}(log\,x)\right\}$$
$$= 3\frac{1}{x\,log\,2} - (1 \cdot log\,x + x \cdot \frac{1}{x})$$
$$= \frac{3}{x\,log\,2} - log\,x \text{ -1}$$

In [6]:
```
# (b) g(x) = 3log2(x)-xlogx
x  = sp.Symbol('x')
gx = 3*sp.log(x,2)-x*sp.log(x)
print("g'(x)=",sp.simplify(sp.diff(gx,x)))
```

```
g'(x)= -log(x) - 1 + 3/(x*log(2))
```

Unit 5.

Understanding Deep Learning: Derivatives

Derivatives of Composite Functions

Derivatives of composite functions consist of quadratic functions f(x) and g(x).

$$f(x)=10+x^2$$

$$g(x)=3+x$$

$$f(g(x))=10+g(x)^2=10+(3+x)^2$$

- When the above composite function f(g(x)) is differentiated by x, it is replaced with a variable as follows.

$$y=f(u)$$

$$u=g(x)$$

- In this way, it can be differentiated step by step as follows.

$$\frac{dy}{dx}=\frac{dy}{d\mu}\cdot\frac{d\mu}{dx}$$

Derivatives of composite functions consist of quadratic functions f(x) and g(x).

- In other words, you can differentiate y by μ and multiply μ by the differentiated x. In fact, the differential is as follows.

$$\frac{dy}{d\mu} = \frac{d}{d\mu}\mathrm{f}(\mu)$$
$$= \frac{d}{d\mu}(10+\mu^2) = 2\mu$$

$$\frac{d\mu}{dx} = \frac{d}{dx}\ g(x)$$
$$= \frac{d}{dx}(3+x) = 1$$

- Each result came out. Now multiply these two results.
- Then, returning μ to g(x) yields the final differential result.

$$\frac{dy}{dx} = \frac{dy}{d\mu} \cdot \frac{d\mu}{dx}$$
$$=2\mu \cdot 1 = 2\mathrm{g(x)} = 2(3+\mathrm{x})$$

Derivatives of Composite Functions

- In summary, it is as follows.
- For two differentiable functions $y = f(u), u = g(x)$, the derivative of the composite function $y = f\big(g(x)\big)$ is as follows.

$$\{f\big(g(x)\big)\}' = f'\big(g(x)\big)g'(x)$$

Find the differential of the function.

(a) $f(x) = (x^2 + 2)^3$

If $\mu=(x^2+2)^2$, then $f(\mu)=\mu^3$,
$u' = 2x,\ and\ f'(\mu) = 3\,u^2.$

- Therefore, the derivative of composite functions is as follows.

$$f'(x) = \{f(\mu)\}' = f'(\mu) \cdot u' = 3\,u^2 \cdot (2\,x) = 6(x^2+2)^2$$

In [1]:
```
import sympy as sp

# (a) f(x) = (x**2+2)**3
x  = sp.Symbol('x')
fx = (x**2+2)**3
sp.diff(fx,x)
```

Out[1]: $6x\left(x^2+2\right)^2$

Find the differential of the function.

(b) $g(x) = \frac{1}{(x^4+1)^2}$

If $\mu=x^4+1$, then $g(\mu)=\frac{1}{\mu^2}$, $u' = 4x^3$, $and\ g'(\mu) = -\frac{2}{\mu^3}$

- Therefore, the derivative of composite functions is as follows.

$$g'(x) = \{g(\mu)\}' = g'(\mu) \cdot u' = -\frac{2}{u^3} \cdot 4x^3 = -\frac{8x^3}{(x^4+1)^3}$$

In [2]:
```
# (b) g(x) = 1/(x**4+1)**2
x  = sp.Symbol('x')
gx = 1/(x**4+1)**2
sp.diff(gx,x)
```

Out[2]:

$$-\frac{8x^3}{(x^4+1)^3}$$

Unit 5.

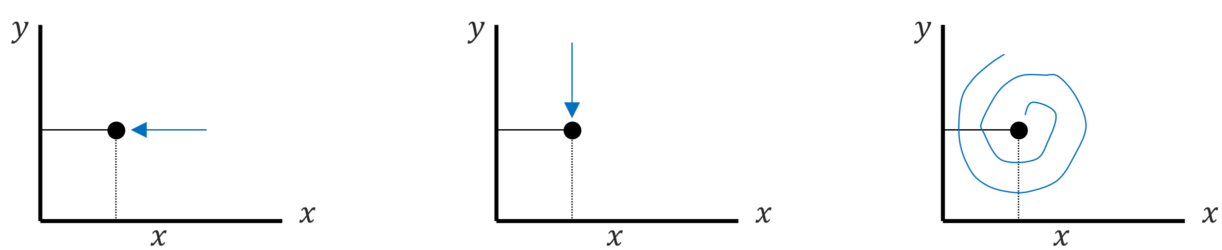
Understanding Deep Learning: Derivatives

High Order Derivatives and Partial Derivatives

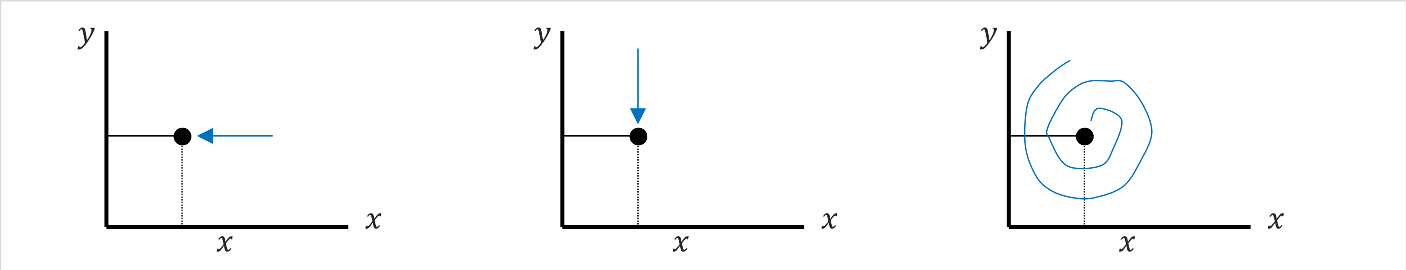
- So far, we have dealt with the function's derivatives with only one variable, x. These derivatives are called ordinary derivatives.
  - Let's learn how to differentiate a multi-variable function with two or more variables, such as $z=f'(x,y)=3x^2 + 2xy + 2y^2$.
  - Applying the method we learned earlier, solve for the rate of change of z, Δz, when the variables x, y change by Δx, Δy.

$$
\begin{aligned}
\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\
&= 3(x+\Delta x)^2 + 2(x+\Delta x)(y+\Delta y) + 2(y+\Delta y)^2 - (3x^2 + 2xy + 2y^2) \\
&= (6x + 2y)\,\Delta x + (2x + 4y)\,\Delta y + 3\Delta x^2 + 2\,\Delta x\,\Delta y + 2\Delta y^2
\end{aligned}
$$

  - Then, we can find the function's derivative f(x,y) by calculating the limit value, such as Δx, Δy →0.
  - In order to obtain the limit value of this function, the point (x+Δx, y+Δy) must be approached as close as possible to point (x,y). There are various approaches as shown below.



High Order Derivatives and Partial Derivatives



- In the case of the figure on the right, the limit value is difficult to calculate because x and y move together.
- Let's fix y as a constant as shown in the figure on the left.
- Changing x when $\Delta y=0$ can be considered as $\Delta x \to 0$, and this approach is called the partial derivative of x.

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} 6x + 2y + 3\Delta x = 6x + 2y$$

- This time, fix x as the constant, and then solve for the partial derivative of y. It will look as follows.

$$\frac{\partial f(x,y)}{\partial y} = 2x + 4y$$

Unit 5.

Understanding Deep Learning: Derivatives

Derivative of the Sigmoid Function

Differentiate the sigmoid function.

- When function $\sigma(x)$ is defined as sigmoid function $\frac{1}{1+e^{-x}}$, the value of the differentiated $\frac{d\sigma(x)}{dx}$ is as follows.

$$\frac{d\sigma(x)}{dx} = \sigma(x) \cdot \left(1 - \sigma(x)\right)$$

How to induce the derivative of the sigmoid function

When $\sigma(x) = \dfrac{1}{1+e^{-x}}$

$$\frac{d}{dx}\sigma(x) = \frac{d}{dx}\left[\frac{1}{1+e^{-x}}\right]$$

$$= \frac{d}{dx}(1+e^{-x})^{-1}$$

$$= -(1+e^{-x})^{-2}(-e^{-x})$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{1}{1+e^{-x}} \cdot \frac{e^{-x}}{1+e^{-x}}$$

$$= \frac{1}{1+e^{-x}} \cdot \frac{(1+e^{-x})-1}{1+e^{-x}}$$

$$= \frac{1}{1+e^{-x}} \cdot \left(1 - \frac{1}{1+e^{-x}}\right)$$

$$= \sigma(x) \cdot (1 - \sigma(x))$$

- Apply the Chain Rule

  1. When $f(x) = x^2(a = natural\ number)$, the differential value is ax^{a-1}.

  2. The differential value of e^{-x} is $-e^{-x}$.

- Proof: $\frac{d}{dx}[e^{-x}]$

$$= e^{-x} \cdot \frac{d}{dx}[-x]$$

$$= \left(-\frac{d}{dx}[x]\right)e^{-x}$$

$$= -1e^{-x}$$

$$= -e^{-x}$$

End of Document

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