QFT 1

Part 3: CM Field Theory

COMMENTS

i) WHAT MAKES QFT QFT?

not relativity.

instead Creation & annih. ops, renormalization, phase transitions

ii) MANY BODY PHYSICS $\overset{\text{EMERGENCE}}{\Longrightarrow}$ SIMPLE LOW E PHENOMENA

iii) SYMMETRY BREAKING

Critical Exponents (aside for Matt on Wilson-Fisher)

$$t = \frac{T - T_c}{T_c} \qquad G(x) = \langle s(x) s(0) \rangle \sim \exp\left[-|x|/\xi\right]$$

away from crit

@ CRIT: correlation diverges

$$\xi \sim |t|^{-\nu} \qquad G(x) \sim \frac{1}{|x|^{d-2+\eta}} \qquad C_H \sim |t|^{-\alpha}$$

$$M \sim |t|^{\beta} \qquad \chi \sim |t|^{-\gamma}$$

2-loop O(N) Wilson-Fisher: Peskin table 13.1

N=1 liquid-gas Xe, CO_2 OR Cu-Zn alloy "β-brass"

QFT: $\nu = 0.63$ EXPT $\nu = .625$ (binary liquid)

$.65$ (β-brass)

N=2 SUPERFLUID ^4He QFT: $\nu = .670$

EXPT: $= .674$

N=3 FERRO & ANTI-FERRO QFT $\nu = .705$

EXPT $\nu = .70$ $\underbrace{EuO,\ EuS}_{\text{ferro}}$

$.724$ $RbMnF_3$ ← anti ferro

WILSONIAN RG & NON-REL FT.

$\underbrace{\text{SUPERFLUIDS, SUPERCONDUCTORS, \& SYMMETRY BREAKING}}_{\text{2 LECTURES}}$

TOPOLOGICAL DEFECTS

CHERN-SIMONS, FQHE, & PARTICLE VORTEX DUALITY

FERROMAGNETS & O(3) NLSM

CFT

LIE THEORY

$\boxed{\text{NON-REL F.T.}}$

$$\mathcal{L} = \underbrace{\partial\Phi^\dagger\partial\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2}_{\text{contains NR physics}}$$

CLASSICAL SOLUTIONS

EOM: $(\partial^2 + m^2)\Phi = 0$

Q: what properties does E have in NR limit?

PLANE WAVE SOL'N $\Phi \sim e^{ip\cdot x} \sim e^{-iEt}$

$p^\mu = (E, 0, 0, |\vec{p}|)$ (in some frame)

$p\cdot p = E^2 - (|\vec{p}|)^2 = m^2$ ← c^4 here if desired

$$\Rightarrow \quad E = \sqrt{m^2 + |\vec{p}^2|} = m\sqrt{1 + \frac{|\vec{p}|^2}{m^2}} \overset{\text{NON-REL}}{\simeq} m\left(1 + \frac{|\vec{p}|^2}{2m^2}\right) = m + \frac{|\vec{p}|^2}{2m}$$

$$=: \underbrace{m}_{\text{REST MASS}} + \underbrace{\varepsilon}_{\text{KINETIC ENERGY}}$$

KEY: REQ'D $\varepsilon \ll m$

$$\Phi \sim e^{-iEt} = \underbrace{e^{-imt}}_{\text{FAST OSC}} \underbrace{e^{-i\varepsilon t}}_{\text{SLOW OSC.}}$$

$\Rightarrow$ WRITE $\Phi(x^\mu) = e^{-imt}\,\underbrace{\varphi(x^\mu)}_{\text{SLOW OSC} \sim \varepsilon,}$

$$\varphi = A(\vec{x})\,e^{-i\varepsilon t}$$

$$\Rightarrow 0 = (\partial^2 + m^2)\,\Phi = (\partial_t^2 - \nabla^2)\, e^{-imt}\varphi + m^2\Phi$$

$$= \partial_t\left[e^{-imt}(-im\varphi + \partial_t\varphi)\right] - \nabla^2\varphi e^{-imt} + m^2\Phi$$

$$= \left[-im\, e^{-imt}(-im\varphi + \partial_t\varphi) + e^{-imt}(-im\,\partial_t\varphi + \partial_t^2\varphi)\right] - \nabla^2\varphi e^{-imt} + m^2\Phi$$

$$= \cancel{-m^2\Phi} - 2im\, e^{-imt}\partial_t\varphi + \partial_t^2\varphi - \nabla^2\varphi e^{-imt} \cancel{+ m^2\Phi}$$

$\frac{1}{2m}\cdot$

$$= -i e^{-imt}\partial_t\varphi + \frac{\partial_t^2\varphi}{m} e^{-imt} - \frac{\nabla^2}{2m}\varphi e^{-imt}$$

$$\left.\begin{array}{l}\partial_t\varphi \sim \varepsilon\varphi \\ \dfrac{\partial_t^2\varphi}{m} \sim \dfrac{\varepsilon^2}{m}\varphi\end{array}\right\} \text{REL. } \frac{\varepsilon}{m} \Rightarrow \text{drop } \partial_t^2 \text{ TERM!}$$

$$\Rightarrow 0 \simeq \left[-i\partial_t\varphi - \frac{\nabla^2}{2m}\varphi\right] e^{-imt}$$

$$\Rightarrow \boxed{i\partial_t\varphi = \frac{-\nabla^2\varphi}{2m}}$$

E: KGE $\Rightarrow$ SCHROD. IN NON-REL LIMIT

NON-REL $\mathcal{L}$

convenience

$$\Phi = \frac{1}{\sqrt{2m}} e^{-imt}\varphi$$

& take deriv

$$\underbrace{\frac{\partial\Phi^\dagger}{\partial t}\frac{\partial\Phi}{\partial t} - m^2\Phi^\dagger\Phi}_{\text{APPEARS IN } \mathcal{L}_{\phi^4}} \Rightarrow \frac{1}{2m}\left\{\left[\left(im + \frac{\partial}{\partial t}\right)\varphi^\dagger\right]\left[\left(-im + \frac{\partial}{\partial t}\right)\varphi\right] - m^2\varphi^\dagger\varphi\right\}$$

$$\simeq \frac{1}{2} i\left(\varphi^\dagger\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^\dagger}{\partial t}\varphi\right)$$

$$\text{IBP} \Rightarrow \mathcal{L}_{\phi^4} = i\varphi^\dagger\partial_0\varphi - \frac{1}{2m}\underbrace{\partial_i\varphi^\dagger\partial_i\varphi} - g^2(\varphi^\dagger\varphi)^2$$

$$g^2 = \frac{\lambda}{4m^2}$$

NB:
1. SPACE-TIME SPLIT
2. ROT. INV, NOT LOR.
3. LINEAR IN ∂_t!

NOETHER'S CURRENT

$$J_\mu = i\left(\Phi^* \partial_\mu \Phi - \partial_\mu \Phi^* \Phi\right)$$

$$J_0 = i\left(e^{imt}\varphi^*\left[-imt\,\varphi + \partial_0\varphi\right]e^{-imt} - \left[imt\,\varphi^* + \partial_0\varphi^*\right]e^{imt}e^{-imt}\varphi\right)$$

$$\overset{\text{IBP}}{=} \underbrace{\varphi^*\varphi}_{\text{PROBABILITY DENSITY } \rho} \qquad \& \qquad J_i = \frac{i}{2m}\underbrace{\left(\varphi^*\partial_i\varphi - \partial_i\varphi^*\varphi\right)}_{\text{PROBABILITY CURRENT } \vec{J}}$$

<u>Note</u>: ρ and $\vec{J}$ look very different!

$\therefore$ CONJ. MOM. $\pi = \frac{\delta \mathcal{L}}{\delta \partial_0 \varphi} = i \varphi^\dagger$

$\Rightarrow$ EQ. TIME COMM.

$$[\varphi^\dagger(\vec{x},t), \pi(\vec{x}',t)] = -i\delta^{(D)}(x-x')$$

$$\Rightarrow [\varphi^\dagger(\vec{x},t), \varphi(\vec{x}',t)] = -\delta^{(D)}(x-x')$$

REWRITE $\varphi = \sqrt{\rho}\, e^{i\theta}$

$$\mathcal{L} = \frac{i}{2}\partial_0\rho - \rho\partial_0\theta - \frac{1}{2m}\left[\rho(\partial_i\theta)^2 + \frac{1}{4\rho}(\partial_i\rho)^2\right] - g^2\rho^2$$

$$\Rightarrow \quad \pi_\theta = \frac{\delta\mathcal{L}}{\delta\partial_0\theta} = -\rho \quad \Rightarrow \quad [\rho(\vec{x},t), \theta(\vec{x}',t)] = i\delta^{(D)}(\vec{x}-\vec{x}')$$

$$N := \int d^D x\, \rho(\vec{x},t) = \#\ \text{BOSONS}$$

ABOVE $\Rightarrow [N, \theta] = i$

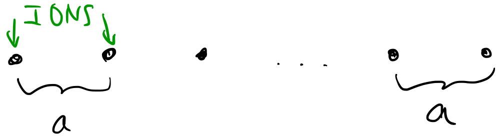
NB: NUMBER CONJ. TO PHASE $\measuredangle$!

(in non-relativistic theory)

FINITE DENSITY:

CONTINUUM LIMITS (already saw, but again)

1D CRYSTAL



ION POSITION R_I

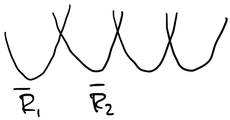
$P_I = M\dot{R}_I$

PHONON HAMILTONIAN 1D $$H = \sum_{I=1}^{N} \frac{P_I^2}{2M} + \frac{k_s}{2}\left(R_{I+1} - R_I - a\right)^2$$

(Hamiltonian of N pt-like particles connected by springs w/ spring const k_s)

LAGRANGIAN: $$L = \underset{\text{kinetic } E}{T} - \underset{\text{potential } E}{U} = \sum_{I=1}^{N}\left[\frac{M\dot{R}_I^2}{2} - \frac{k_s}{2}\left(R_{I+1} - R_I - a\right)^2\right]$$

PERIODIC BOUNDARY $R_{N+1} = R_1$ (B.C. shouldn't matter @ large N)

LOW ENERGIES: $V \sim$ 

$R_I = \bar{R}_I + \phi_I$ $\Rightarrow$ $$L = \sum_{I=1}^{N}\left[\frac{M}{2}\dot{\phi}_I^2 - \frac{k_s}{2}\left(\phi_{I+1} - \phi_I\right)^2\right]$$

↑ FLUCT. ABOUT MIN.

order

Superfluids

Global Symmetry Breaking

Gauge Symmetry Breaking

Superconductors

origins

SUPERFLUIDS $\Longrightarrow$ quantum fluids w/ no viscosity, just keep moving!

LANDAU'S ARGUMENT

T=0 QUANTUM FLUID, MASS M

e.g LIQ He^4

Question: can the fluid slow down? ie lose momentum via emission of quanta?

$$Mv = Mv' + \hbar k \qquad \text{(momentum conservation)}$$

$$\tfrac{1}{2}Mv^2 \geq \tfrac{1}{2}Mv'^2 + \hbar\omega \qquad \text{(could lose energy eg to heat)}$$

$$\frac{(Mv'+\hbar k)^2}{2M} \simeq \frac{(Mv')^2\left(1+2\frac{\hbar k}{Mv'}\right)}{2M} = \frac{Mv'^2}{2} + \hbar k v'$$

$$\Longrightarrow \quad \hbar k v' \geq \hbar\omega \qquad v' \geq \frac{\omega}{k} =: V_c$$

$\leftarrow$ critical velocity of fluid

CONCLUSION: FLUID LOSES MOMENTUM UNTIL $V = V_c$, THEN SUPERFLUID

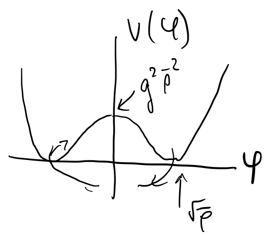
WHEN SUPER FLUID $\exists$ QUANTA S.T $\omega = V_c k$

linear dispersion relation $\longrightarrow$ smoking gun!

F.T. OF SUPER FLUIDS

$$\mathcal{L} = i\varphi^*\partial_0\varphi - \tfrac{1}{2m}\partial_i\varphi^*\partial_i\varphi - g^2(\varphi^*\varphi - \bar\rho)^2$$

$$V(\varphi) = g^2(\varphi^*\varphi - \bar\rho)^2 \Longrightarrow \varphi = \sqrt{\rho}\, e^{i\theta}$$



THEN $\mathcal{L} = -\rho\partial_0\theta - \frac{1}{2m}\left[\frac{1}{4\rho}(\partial_i\rho)^2 + \rho(\partial_i\theta)^2\right] - g^2(\rho-\bar{\rho})^2$

↑ field $\sqrt{\rho(x)}$ wants to be @ value $\sqrt{\bar{\rho}}$

EXPAND AROUND $\bar{\rho}$ $\quad \sqrt{\rho} = \sqrt{\bar{\rho}} + h$

↑ small fluctuation, ie $\frac{h}{\sqrt{\bar{\rho}}} \ll 1$

NOW $\mathcal{L} = -2\sqrt{\bar{\rho}}\,\partial_0\theta - \frac{\bar{\rho}}{2m}(\partial_i\theta)^2 - \frac{1}{2m}(\partial_i h)^2 - 4g^2\bar{\rho}h^2 + \dots$

$$= \bar{\rho}\,\partial_0\theta \frac{1}{4g^2\bar{\rho} - \frac{1}{2m}\partial_i^2}\partial_0\theta - \frac{\bar{\rho}}{2m}(\partial_i\theta)^2 + \dots$$

HMWK

$$= \frac{1}{4g^2}(\partial_0\theta)^2 - \frac{\bar{\rho}}{2m}(\partial_i\theta)^2 + \dots$$

$$4g^2\bar{\rho} \gg \frac{\partial_i^2}{2m} \overset{\text{P-SP.}}{=} \frac{k^2}{2m} \implies k \ll \sqrt{8g^2\bar{\rho}m}$$

EOM FOR θ: $\quad \left[\frac{\partial_0^2}{4g^2} - \frac{\bar{\rho}}{2m}\partial_i^2\right]\theta = 0$

SOL'N: $\quad \theta = e^{-i\omega t}e^{i\vec{k}\cdot\vec{x}} \implies i^2\left[\frac{\omega^2}{4g^2} - \frac{\bar{\rho}\vec{k}^2}{2m}\right] = 0$

$$\implies \omega = \sqrt{\frac{2\bar{\rho}g^2}{m}}\,|\vec{k}| \implies v_c = \sqrt{\frac{2\bar{\rho}g^2}{m}}$$

CONCLUSION: @ CRITICALITY SUPERFLUIDS DESCRIBED BY $\measuredangle$ DOF OF COMPLEX SCALAR

↳ systematic theory?

SPONTANEOUS SYMMETRY BREAKING

"*Let the system break the symmetry itself*"

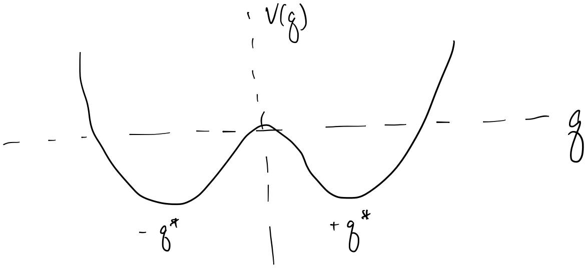
$$\mathcal{L} = \frac{1}{2}\left[(\partial\vec{\varphi})^2 + \mu^2\vec{\varphi}^{\,2}\right] - \frac{\lambda}{4!}(\vec{\varphi}^{\,2})^2 \qquad \vec{\varphi} = (\varphi_1, \dots, \varphi_N)$$

NOTE SIGN DIFFERENCE!

$m^2 = -\mu^2 \implies$ "Tachyon" (not a problem if you know where it leads)

ANALOGY — ANHARMONIC OSC.

$$L = \frac{1}{2}(\dot{q}^2 + k q^2) - \frac{\lambda}{4} q^4 \qquad V(q) = -\frac{1}{2}q^2 + \frac{\lambda}{4} q^4$$



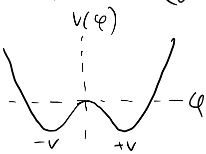
@ LOW ENERGIES, PICK MINIMUM $\pm q^*$, STUDY OSC. ABOUT $\pm q^*$

"PICK MINIMUM" $\implies$ OSC. ABOUT STATE THAT BREAKS $q \to -q$ SYM

SPONTANEOUS SYM BREAKING: L HAS SYM, GROUND STATE BREAKS

N=1 CASE $\quad \mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{\mu^2}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \implies V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4$

$$\frac{\partial V}{\partial\varphi} = \varphi(-\mu^2 + \lambda\varphi^2) \implies \text{min @ } \varphi = 0 \quad \varphi = \pm\sqrt{\frac{\mu}{\lambda}} =: \pm v$$



LOCAL MAX OR MIN? $\quad \frac{\partial^2 V}{\partial \varphi^2} = \frac{\partial}{\partial \varphi}\left(-\mu^2 \varphi + \lambda \varphi^3\right) = -\mu^2 + 3\varphi^2$

$$\left.\frac{\partial^2 V}{\partial \varphi^2}\right|_{\varphi=0} = -m^2 \implies \text{LOCAL MAX}$$

$$\left.\frac{\partial^2 V}{\partial \varphi^2}\right|_{\varphi=\pm v} = -\mu^2 + 3v^2 = -\mu^2 + \frac{\mu^2}{\lambda} = \mu^2\left(\frac{1}{\lambda} - 1\right) \implies \text{LOCAL MIN FOR } \lambda \text{ SMALL}$$

EXPAND ABOUT $\varphi = +v$ GROUND STATE

$$\varphi = v + \delta\varphi =: v + \varphi'$$

$$\implies \mathcal{L} = \frac{\mu^4}{2\lambda} + \frac{1}{2}(\partial \varphi')^2 - \mu^2 \varphi'^2 - \mathcal{O}(\varphi')^3$$

$\implies$ φ' HAS $m = \sqrt{2}\,\mu$

QFT: G.S = VACUUM

HERE: 2 PHYSICALLY EQUIV. VACUA $|\pm v\rangle$.

$\langle \pm v | \hat{\varphi} | \pm v \rangle = \pm v$ Lang: "vacuum expectation value"

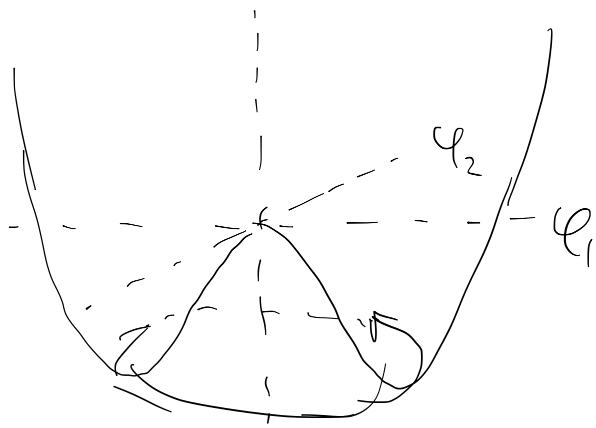
N=2 CASE

$$\mathcal{L} = \frac{1}{2}\left[(\partial\vec{\varphi})^2 + \mu^2 \vec{\varphi}^{\,2}\right] - \frac{\lambda}{4!}(\vec{\varphi}^{\,2})^2 \qquad \vec{\varphi} = (\varphi_1, \varphi_2)$$

$$\frac{\partial V}{\partial \varphi_i} = \frac{\partial}{\partial \varphi_i}\left(-\mu^2 \varphi_j \varphi_j + \frac{\lambda}{4} \varphi_j \varphi_j \varphi_k \varphi_k\right) = -\mu^2 \varphi_i + \frac{\lambda}{2}\varphi_i \varphi_k \varphi_k + \frac{\lambda}{2}\varphi_j \varphi_j \varphi_i$$

$$= \varphi_i(-\mu^2 + \lambda \vec{\varphi}^{\,2}) \Longrightarrow \frac{\partial V}{\partial \varphi_i} = 0 \quad @ \quad \varphi_i = 0 \qquad \vec{\varphi}^{\,2} = \pm\sqrt{\frac{\mu}{\lambda}}$$

$\vec{\varphi}^{\,2}$: ONLY FIXES MAG OF $\vec{\varphi}$! NOT $\measuredangle$



NOTE: $\exists$ "FLAT DIRECTION" @ MIN $V(\varphi) \Rightarrow$ CONTINUUM OF VACUA

Language: "Classical moduli space of vacua"

PICK A VACUUM, COMPUTE $\mathcal{L}_{eff}$

$\varphi_1 = v + \varphi_1'$ $\quad \varphi_2 = \varphi_2'$ $\qquad$ (picked it in φ_1 direction)

$$\mathcal{L}_{eff} = \frac{\mu^4}{4\lambda} + \frac{1}{2}\left[(\partial\varphi_1)^2 + (\partial\varphi_2)^2\right] - \mu^2 \varphi_1^{\,2} + \mathcal{O}(\varphi^3)$$

$m_{\varphi_1'} = \sqrt{2}\,\mu$ $\qquad m_{\varphi_2'} = 0$

φ_1 "climbing the wall"

φ_2 "rolling along the gutter"

Language: "Goldstone Boson"

CRITICAL DIFFERENCE

$N=1$ BROKE DISCRETE SYMMETRY

$N=2$ " CONTIN. " $\Longrightarrow$ MASSLESS BOSON

ANOTHER N=2 VIEW

$\varphi := \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$

$$\Rightarrow \mathcal{L} = \partial\varphi^\dagger\partial\varphi + \mu^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2$$ CMPLX SCALAR!

$\varphi \to e^{i\alpha}\varphi$ INVT, GLOBAL $U(1)$ SYMMETRY

REWRITE $\varphi = \rho e^{i\theta} \Rightarrow \partial_\mu\varphi = (\partial_\mu\rho + i\rho\partial_\mu\theta)e^{i\theta}$

$$\Rightarrow \mathcal{L} = \rho^2(\partial\theta)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4$$

$V(\rho)$ MIN @ $v = \sqrt{\frac{\mu^2}{2\lambda}} \Rightarrow \rho = v + \chi$

$$\mathcal{L}_{eff} = v^2\partial\theta^2 + \left[(\partial\chi)^2 - 2\mu^2\chi^2 - 4\sqrt{\frac{\mu^2\lambda}{2}}\chi^3 - \lambda\chi^4\right] + \left(\sqrt{\frac{2\mu^2}{\lambda}}\chi + \chi^2\right)(\partial\theta)^2$$

$$= \mathcal{L}_{\theta\,kin} + \mathcal{L}_\chi + \mathcal{L}_{\chi,\theta}$$

NB $\theta(x)$ MASSLESS SCALAR!

GOLDSTONE'S THEOREM

* Q CONSERVED, GENS A SYMM $\Rightarrow [H,Q]=0$
* $|0\rangle$ IS VACUUM
* $H \to H + C$ SO $H|0\rangle = 0$
* SYMMETRY BREAKING

SUPPOSE $|0\rangle$ BREAKS SYMM. Q, IE $Q|0\rangle \neq 0$

CONSIDER $|\psi\rangle := Q|0\rangle$

WHAT IS ITS ENERGY?

$$H|\psi\rangle = HQ|\psi\rangle = [H,Q]|\psi\rangle = 0$$

$\Rightarrow \exists$ ANOTHER STATE $|\psi\rangle$ w/ SAME ENERGY AS $|0\rangle$!

QFT $\quad Q = \int d^D x \, J^0(\vec{x}, t)$

CONSIDER $|s\rangle := \int d^D x \, e^{-i\vec{k}\cdot\vec{x}} J^0(\vec{x},t)|0\rangle$

$\lim_{\vec{k}\to 0} |s\rangle = Q|0\rangle = 0$ ENERGY STATE

$\Rightarrow$ BEHAVIOR EXPECTED OF MASSLESS PARTICLE!
(energy $\to 0$ as momentum $\to 0$)

more rigor exists

$\Rightarrow \exists$ massless $\overbrace{\text{spin 0 boson}}^{\text{Language: Goldstone Boson}}$ associated w each broken, continuous, global symmetry

Language: Goldstone's Theorem

HOW MANY GOLDSTONE BOSONS?

SUPPOSE $G \longrightarrow H$ SPONTANEOUSLY

WRITE DIM(G) G-GENS = $\{\underbrace{A_1, A_2, \ldots, A_{\dim(H)}}_{\text{G-GENS \& H-GENS}}, \underbrace{B_1, \ldots B_{\dim(G)-\dim(H)}}_{\text{G-GENS NOT H-GENS}}\}$

$\Rightarrow B_i|0\rangle \neq 0 \quad \forall i = 1, \ldots, \dim(G) - \dim(H)$

$\Rightarrow \exists \dim(G) - \dim(H)$ MASSLESS GOLDSTONE BOSONS.

AKA NAMBU-GOLDSTONE BOSONS

$\exists$ "PSEUDO-NAMBU GOLDSTONE BOSONS" TOO

- MASSIVE DUE TO SPONTANEOUS & EXPLICIT BREAKING
- EG PION: QCD $\Rightarrow$ SSB OF CHIRAL symm $\Rightarrow \pi$ GOLDSTONE

  QUARK MASSES $\Rightarrow$ EXPLICIT CH. S.B. $\Rightarrow$ MASS TO GOLDSTONE π

ANDERSON – HIGGS MECHANISM: GAUGE SYMMETRY BREAKING

WHAT: BREAK GAUGE, NOT GLOBAL, SYMM.

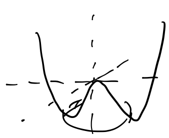
GLOBAL $\Rightarrow$ GOLDSTONE

GAUGE $\Rightarrow$ GOLDSTONE GETS "EATEN" BY VECTOR

ABELIAN HIGGS MODEL

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D\varphi)^\dagger D\varphi + \mu^2 \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2$$

$\varphi^\dagger = \varphi^*$ $\quad D_\mu = \partial_\mu - ieA_\mu$ $\quad \exists\ U(1)$ GAUGE SYMMETRY


$\Rightarrow \langle 0|\phi|0\rangle \neq 0$ SO POLAR ALLOWED

POLAR $\phi = \rho e^{i\theta}$ $\Rightarrow$ $D_\mu \varphi = [\partial_\mu \rho + i\rho(\partial_\mu \theta - eA_\mu)]\, e^{i\theta}$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \rho^2(\partial_\mu \theta - eA_\mu)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4$$

ASIDE $U(1)$ GLOBAL $\Rightarrow$ NO GAUGE FIELD

$$\mathcal{L}_{GLOB} = \rho^2 (\partial_\mu \theta)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4$$

$$\mathcal{L} - \mathcal{L}_{GLOB} = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{PHOTON PROPAGATES}} + \underbrace{\rho^2(\partial_\mu \theta - eA_\mu)^2}_{\text{CRITICAL TERM!}}$$

NB: GAUGE TRANS: $\varphi \mapsto e^{i\alpha}\varphi$ $\Rightarrow$ $\theta + \alpha$ THEN $B_\mu = \partial_\mu\theta - eA_\mu$

$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu \alpha$ $\quad \mapsto \partial_\mu\theta + \partial_\mu\alpha - eA_\mu - \partial_\mu\alpha$

$\Longrightarrow$ B_μ GAUGE INVARIANT!

$$F^B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - \tfrac{1}{e}\partial_\mu\partial_\nu\theta + \tfrac{1}{e}\partial_\nu\partial_\mu\theta = F^A_{\mu\nu}$$

SSB $\quad \rho = \frac{1}{\sqrt{2}}(v + \chi)$ $\quad\quad v = \sqrt{\frac{\mu^2}{\lambda}}$

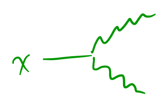
$\hookrightarrow$ VEV $\quad\quad \hookrightarrow V(\varphi)$ MIN

$$\xrightarrow[\text{SSB}]{+B_\mu + \text{ALG}} \mathcal{L} = -\tfrac{1}{4}F^B_{\mu\nu}F^{\mu\nu}_B + \tfrac{1}{2}(\partial\chi)^2 - \mu^2\chi^2 - \sqrt{\lambda}\,\mu\chi^3 - \frac{\lambda}{4!}\chi^4 + \frac{\mu^4}{4\lambda}$$

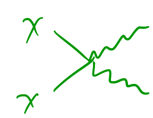
$$+ \tfrac{1}{2}M^2 B_\mu B^\mu + e^2 v \chi B_\mu^2 + \tfrac{1}{2}e\chi^2 B_\mu^2$$

$\Downarrow$ B_μ MASS $\quad M = ev$

$\Downarrow$ 3-PT HIGGS COUPLING



$\Downarrow$ 4-PT HIGGS COUPLING



NB: (1) MASSIVE B_μ, NO A_μ OR θ

A_μ "ATE" θ, GAINED MASS, "IS NOW" B_μ

(2) IF GLOBAL, θ WOULD BE GOLDSTONE BOSON

(3) θ EATEN, BUT MASSIVE SCALAR χ LEFTOVER

Language: "Higgs Boson"

(4) φ 2 DOF = 1 EATEN + 1 HIGGS $\quad 2 = 1 + 1$

REAL WORLD: $SU(2)_L \times U(1)_Y \longrightarrow U(1)_{em}$

(1) - W_+ W_- Z ARE B_μ

- EACH ATE A DOF

(2) 1 MASSIVE SCALAR LEFTOVER $\Longrightarrow$ ACTUAL HIGGS BOSON OF SM

$m = 125$ GeV

(3) $\underbrace{4 \text{ DOF}}_{\text{COMPLEX HIGGS FIELD DOUBLET}} = \underbrace{3 \text{ EATEN}}_{W_\pm\ Z} + \underbrace{1 \text{ HIGGS}}_{\text{REAL ONE}}$

CONDENSED MATTER APPLICATION: SUPERCONDUCTORS

BCS THEORY (see 6.4 of Altland and Simons)

$$\hat{H} = \sum_{\vec{k}\sigma} \epsilon_{\vec{k}}\, \hat{n}_{\vec{k}\sigma} - \frac{g}{L^d} \sum_{k,k',q} \underbrace{c^\dagger_{k+q\uparrow} c^\dagger_{-k\downarrow} c_{-k'+q\downarrow} c_{k'\uparrow}}_{\text{ATTRACTIVE, PHONON-INDUCED INTERACTION}}$$

note: $\exists$ different kinds of superconductors

$T < T_c$: ① ELECTRONS BIND INTO $\underbrace{\text{COOPER PAIRS}}_{\text{SCALAR BOSONS}}$, $Q = 2e$

② RESISTIVITY $\Rightarrow 0$

③ $\vec{B}$ FIELDS EXPELLED

Language "Meissner effect"

FIELD THEORY

ABELIAN HIGGS MODEL w/ CHARGE 2 FIELD φ

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (D\varphi)^\dagger D\varphi + \mu^2 \varphi^\dagger \varphi - \lambda(\varphi^\dagger\varphi)^2$$

Q: RECOVER MEISSNER?

NOTE: $\lim_{M\to\infty} \mathcal{L}$, K.E. $\longrightarrow 0$ (particle so massive, doesn't want to prop)

$$\Longrightarrow \text{EOM} \simeq \frac{\delta}{\delta B_\mu} \underbrace{\left[\frac{1}{2} M^2 B_\mu B^\mu + e^2 v \chi B_\mu^2 + \frac{1}{2} e \chi^2 B_\mu^2\right]}_{\text{PART OF } \mathcal{L} \text{ w/ } B_\mu,\ -B_\mu \text{ KIN. TERM.}}$$

$$= B^\mu \left[M^2 + 2e^2 v \chi + e\chi^2 \right] = 0$$

$\Longrightarrow B^\mu = 0$ SOLVES APPROX E.O.M

RECALL $B_\mu = \partial_\mu \theta - e A_\mu$.

THEN $B_\mu = 0 \Rightarrow \partial_\mu \theta = e A_\mu$, $\vec{\nabla}\theta = e\vec{A}$ $\Longrightarrow$ $\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times\frac{\vec{\nabla}\theta}{e} = 0$

normal magnetic field

NO $\vec{B}$-FIELD IN SUPERCONDUCTOR! MEISSNER EFFECT

Solitons & Topological Defects

WHAT? NON-TRIVIAL SOL'NS TO EOM

CM: TONS OF SYSTEMS (superfluids, superconductors, ferromagnets, KT, Top. insulators ...)

H.E.: QCD, COSMOLOGY

1. LOCAL, NOT GLOBAL, ENERGY MIN.
2. TOPOLOGICAL CHARGE
3. TOPOLOGICALLY STABLE
4. ENERGETICS LOWER BOUNDS

(very important in susy gauge theory & string theory)

ORIGIN = Sy~~mmetry~~

KINKS   1+1 DIMS, REAL SCALAR

$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$ $\qquad V(\varphi) = \frac{\lambda}{4!}(\varphi^2 - v^2)^2$

- MINIMA / VACUA AT $\varphi = \pm v$
- $\varphi = v + \chi$, SMALL OSC χ ABOUT v $\overset{\text{HOW!?}}{\Longrightarrow}$ $m_\chi = (\lambda v^2)^{\frac{1}{2}}$

E.O.M. $\quad (\partial_t^2 - \partial_x^2)\varphi + \frac{\lambda}{6}\varphi(\varphi^2 - v^2) = 0$

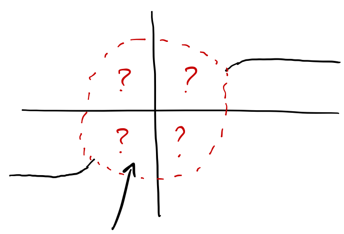
USUAL SOL'NS $\qquad \varphi = \pm v$ (expand as above)

NON-TRIVIAL SOLUTIONS

Q: CAN YOU INTERPOLATE?

$\phi(+\infty) = v$

$\phi(-\infty) = -v$



REGION w/ NON-MIN E!

$\Longrightarrow$ NOT GLOBAL MIN.

$\exists$ LOCAL?

"KINK" SOLUTION

Language: "kink" in 2d, "domain wall" in higher d

$$\boxed{\varphi = v\tanh\left(\frac{\sqrt{\lambda}}{2\sqrt{3}}vx\right)}$$

CHECK: $\quad \frac{d}{dx}\tanh(x) = 1 - \tanh^2 x \qquad A := \frac{\sqrt{\lambda}}{2\sqrt{3}}v$

$$\partial_x^2\phi = \partial_x\left[(1-\tanh^2 Ax)Av\right] = -Av\; 2\tanh Ax\; A\left[1-\tanh^2 Ax\right]$$

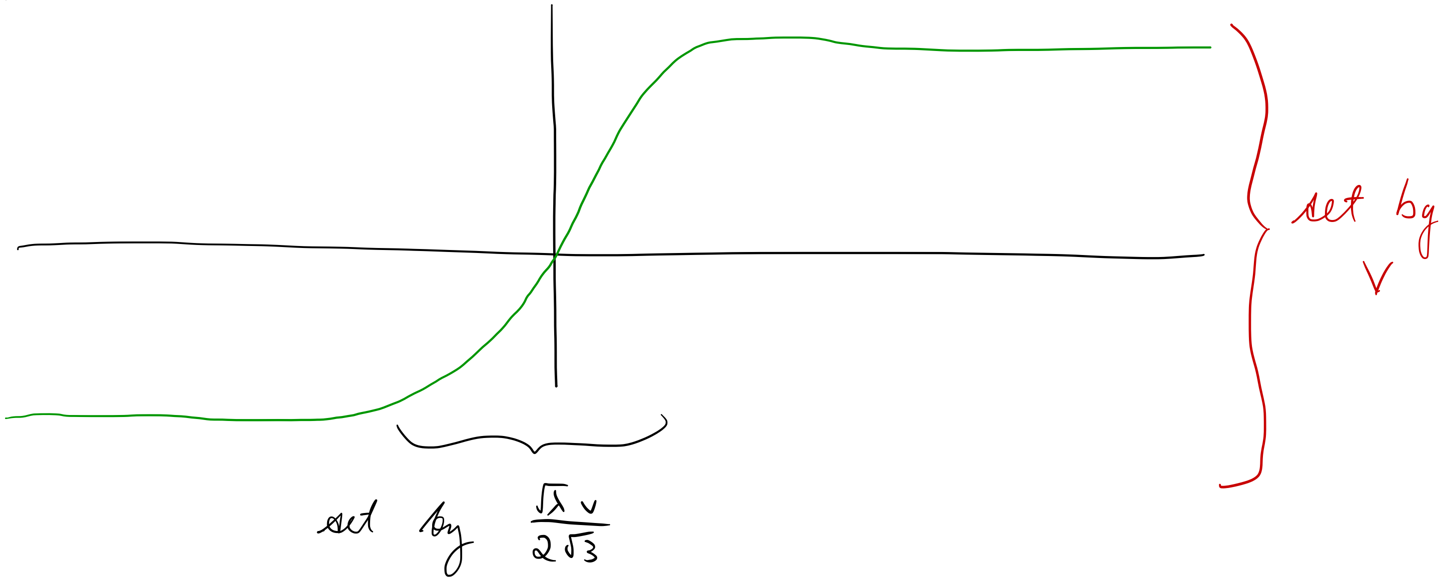
$$= -2A^2 v\tanh(Ax)\left[1-\tanh^2 Ax\right] = -2\left(\frac{\lambda v^2}{12}\right)v\tanh\left(\frac{\sqrt{\lambda}}{2\sqrt{3}}vx\right)\left[1-\tanh^2\left(\frac{\sqrt{\lambda}}{2\sqrt{3}}vx\right)\right]$$

$$= -\frac{\lambda}{6}\varphi(v^2 - \varphi^2)$$

NEED $\quad (\partial_t^2 - \partial_x^2)\varphi + \frac{\lambda}{6}\varphi(\varphi^2 - v^2) = 0$

$\partial_t^2\varphi = 0$ & EYEBALL $\Longrightarrow$ ✓

SKETCH



ENERGY OF KINK

CONJ. MOM. $\pi = \frac{\delta \mathcal{L}}{\delta \partial_0 \varphi} = \partial_0 \varphi$

$$\Longrightarrow \mathcal{H}(x,t) = \pi \dot{\varphi} - \mathcal{L} = \frac{1}{2}\left[(\partial_0 \varphi)^2 + (\partial_x \varphi)^2\right] + \frac{\lambda}{4!}(\varphi^2 - v^2)^2$$

KINK EN. PER UNIT LENGTH

$$\mathcal{E}(x) = \mathcal{H}(x,t) - \frac{1}{2}(\partial_0 \varphi)^2 = \frac{1}{2}(\partial_x \varphi)^2 + \frac{\lambda}{4!}(\varphi^2 - v^2)^2$$

$$M := \int dx\, \mathcal{E}(x) = \frac{2}{3\sqrt{3}} \sqrt{\lambda}\, v^3$$

RECALL $m_\chi = (\lambda v^2)^{\frac{1}{2}} = \sqrt{\lambda}\, v$

$$\Longrightarrow M = \frac{2}{3\sqrt{3}} m_\chi v^2 = \frac{2}{3\sqrt{3}} m_\chi \left(\frac{m_\chi^2}{\lambda}\right)$$

(dimensions work b/c $[\lambda] = 2$ in $d=2$)

KEY: χ F.T. PERTURBATIVE $\Longrightarrow \lambda \ll 1 \Longrightarrow M \gg m_\chi$, HEAVY KINKS!

ALSO KEY: P.T. IN λ NEVER FINDS KINK

$\boxed{\text{TOPOLOGICAL CHARGE}}$

Q: DO $\exists$ CONTINUOUS symm?

A: NO $\Rightarrow$ NO NOETHER'S CURRENT

(but Noether $\Rightarrow$ conserved current, not necc other way around)

$$J^\mu = \frac{1}{2v}\varepsilon^{\mu\nu}\partial_\nu\varphi \quad \text{IS CONSERVED!} \qquad \partial_\mu J^\mu = \frac{1}{2v}\varepsilon^{\mu\nu}\partial_\mu\partial_\nu\varphi = 0$$

contracted symm. & anti symm tensors

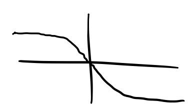
$$\Downarrow$$

$$Q = \int_{-\infty}^{\infty} dx\, J^0(x) \qquad \text{CONSERVED IE} \quad \frac{dQ}{dt} = 0$$

$$\| $$

$$\int_{-\infty}^{\infty} dx\, \frac{1}{2v}\varepsilon^{01}\partial_1\varphi = \int_{-\infty}^{\infty} dx\, \frac{1}{2v}\partial_x\varphi = \frac{1}{2v}\left[\varphi(+\infty) - \varphi(-\infty)\right] = 1$$

ANTI KINK $\quad \varphi_{AK} = -\varphi_K$



$Q = -1$

BOGOMOL'NYI BOUND $\qquad$ USE $a^2 + b^2 \geqslant 2|a||b|$

$$M = \int dx\left[\frac{1}{2}(\partial_x\varphi)^2 + \frac{\lambda}{4!}(\varphi^2 - v^2)^2\right] \geqslant \int dx\, 2\sqrt{\frac{1}{2}(\partial_x\varphi)^2}\sqrt{\frac{\lambda}{4!}(\varphi^2 - v^2)^2}$$

$$= \int dx\, \underbrace{\frac{2\sqrt{\lambda}}{\sqrt{2}\sqrt{4}\sqrt{3}\sqrt{2}}}_{B}\left|(\partial_x\varphi)(\varphi^2 - v^2)\right| = B\int dx\left|\left[\partial_x\left(\frac{\varphi^3}{3}\right) - v^2\partial_x\varphi\right]\right|$$

$$= B\left|\left[\frac{\varphi^3}{3} - v^2\varphi\right]_{-\infty}^{+\infty}\right| \qquad \text{FINISH}$$