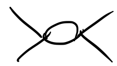
QFT 1

Part 2: Renormalization

PART 2: RENORMALIZATION

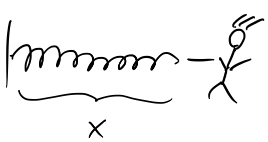
Lectures 1-3:

1. Motivation & 
2. Pauli-Villars & Dim. Reg.
3. Renormalizable & Non-Renormalizable
4. Physical PT
5. RG. Flow

Lectures 4-5: Renormalization of QED

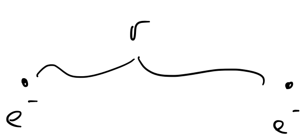
Schematic

POINT 1: COMPARE TO SOMETHING



1. F_0 @ $x_0 \Longrightarrow$ NOTHING
2. & $F_0 + \delta F$ @ $x_0 + \varepsilon \Longrightarrow$ "
3. & $F_0 + n\,\delta F$ @ $x_0 + n\,\varepsilon \Longrightarrow$ LINEAR $F = -kx$

POINT 2: REGIME OF VALIDITY



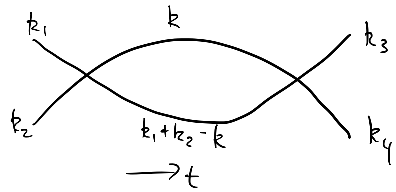
$F \sim \frac{\alpha}{r^2}$

Q: FOR WHAT r? 1 cm? 1 mile?

PT: $r = \frac{1}{M_{pl}}$ CRAZY TO EXPECT

RENORMALIZATION INTRO (Zee III. 1)

RECALL $2 \to 2$



$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} \frac{1}{(K-k)^2-m^2+i\varepsilon} \qquad \omega \quad K = k_1+k_2$$

$$\lim_{k\to\infty} \mathcal{M} \sim \int \frac{d^4k}{k^4} \qquad \text{"LOG DIVERGENT"}$$

PHYSICS: $k \to \infty$ NOT JUSTIFIED!

(eg. QED replaced by standard model)

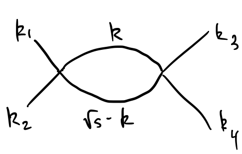
(in general $k >> M_{pl}$ shouldn't trust theory)

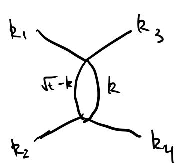
TWO STEPS

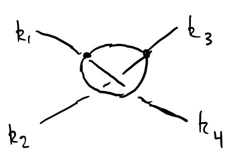
① REGULARIZATION procedure to make $\int$ finite

② RENORMALIZATION from reg'd $\int$, compare to expt, see how couplings depend on scale.

$\exists$ s,t,u CHANNELS $\quad s := (k_1+k_2)^2 \quad t := (k_1-k_3)^2 \quad u := (k_1-k_4)^2$







PAULI-VILLARS REGULARIZATION   See III.1 Appendix 1

↳ one popular way of making $\int$ finite

some calculable #, see appendix

$$\mathcal{M} = -i\lambda + iC\lambda^2 \underbrace{\left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)\right]}_{L} + O(\lambda^3) \qquad (*)$$

Λ = "CUTOFF"

INDEED: $\Lambda^2 \to \infty$, $\mathcal{M}$ LOG. DIV.   (language diverge div. as $\Lambda \to \infty$)

$\mathcal{M}$ FINITE, BUT:

1. WHAT IS MEANING OF Λ?
2. WHAT " " " λ?

RENORMALIZATION:

Q: WHAT IS PHYSICAL?

A: @ s_0, t_0, u_0, EXPT MEASURES $\mathcal{M}$

$$\mathcal{M}\Big|_{s_0,t_0,u_0} =: -i\lambda_p \qquad \lambda_p \text{ "PHYSICAL COUPLING"}$$

$$\overset{(*)}{=} -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right)\right] + \dots$$

DEFINE $\log_0$ SO

$$\mathcal{M} = -i\lambda + iC\lambda^2 L + O(\lambda^3)$$

$$\mathcal{M}\Big|_{s_0,t_0,u_0} = -i\lambda + iC\lambda^2 L_0 + O(\lambda^3) = -i\lambda_p \qquad (*_2)$$

GOAL: EXPRESS $\mathcal{M}$ IN PHYSICAL QUANITIES IE. $\mathcal{M}(\lambda_p)$

$$(*_2) \Rightarrow \quad -i\lambda = -i\lambda_p - iC\lambda^2 L_0 + O(\lambda^3) = -i\lambda_p - iC\left[\frac{-i\lambda_p}{-i}\right]^2 L_0 + O(\lambda_p^3)$$

$$= -i\lambda_p - iC\lambda_p^2 L_0 + O(\lambda_p^3)$$

SIM TO THIS ORD $\lambda^2 L = \lambda_p^2 L$

$$\Longrightarrow \mathcal{M} = -i\lambda_p - iC\lambda_p^2 L_0 + iC\lambda_p^2 L + O(\lambda_p^3)$$

$$= -i\lambda_p + iC\lambda_p^2(L - L_0)$$

(use log rules)

$$\boxed{\mathcal{M} = -i\lambda_p + iC\lambda_p^2\left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)\right] + \mathcal{O}(\lambda_p^3)}$$

THIS IS THE PHYSICAL, RENORMALIZED $\mathcal{M}$!

NOTE:
* Λ TOTALLY GONE
* ONLY PHYSICAL QUANTITIES
  - λ_p MEAS. @ s_0
  - s, t, u SCATTERING ENERGY
* CRANK s, t, u, $\mathcal{M}$ CHANGES!

REFINE $\quad \mathcal{M} =: -i\lambda_p(s,t,u)$

$$= -i\lambda_p(s_0,t_0,u_0) + iC\lambda_p^2(s_0,t_0,u_0)\left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)\right] + \mathcal{O}(\lambda_p^3)$$

$$\Longrightarrow \boxed{\lambda_p(s,t,u) = \lambda_p(s_0,t_0,u_0) - C\lambda_p^2(s_0,t_0,u_0)\left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)\right] + \mathcal{O}(\lambda_p^3)}$$

λ_p FUNCTION OF SCATTERING ENERGY!

Language "RUNNING COUPLING"

β - FUNCTION $\quad \mu^2 = s_0 = t_0 = u_0 \quad$ (for convenience)

$$\Longrightarrow \mathcal{M} = -i\lambda_p(\mu) + iC\lambda_p(\mu)^2\left[\log\left(\frac{\mu^2}{s}\right) + \log\left(\frac{\mu^2}{t}\right) + \log\left(\frac{\mu^2}{u}\right)\right] + \mathcal{O}(\lambda_p(\mu)^3) \qquad (*_3)$$

COULD CHOOSE DIFF. EXP. REF PT.

$$\mu' = \mu + \delta\mu \quad \Longrightarrow \quad (*_3) \text{ w } \mu \to \mu' =: (*_4)$$

$(*_4) - (*_3)$

$$0 = -i\lambda_p(\mu') + i\lambda_p(\mu) + iC\lambda_p(\mu')^2[\text{logs w/ } \mu'] - iC\lambda_p(\mu)^2[\text{logs w/ } \mu]$$

$$= -i\lambda_p(\mu') + i\lambda_p(\mu) + 3iC\lambda_p(\mu)^2\log\left(\frac{\mu'^2}{\mu^2}\right)$$

$\lambda_p(\mu')^2 \longmapsto \lambda_p(\mu)^2$ fine to this order

$$\Longrightarrow \lambda_p(\mu') = \lambda_p(\mu) + 3C\lambda_p(\mu)^2\log\left(\frac{\mu'^2}{\mu^2}\right) + \mathcal{O}(\lambda_p(\mu)^3)$$

$$= \lambda_p(\mu) + 6C\lambda_p(\mu)^2\log\left(\frac{\mu'}{\mu}\right) + \dots$$

$$= \lambda_p(\mu) + 6C\lambda_p(\mu)^2\log\left(1 + \frac{\delta\mu}{\mu}\right) + \dots$$

$$= \lambda_p(\mu) + 6C\lambda_p(\mu)^2\frac{\delta\mu}{\mu} + \dots$$

$$\Rightarrow \mu \frac{\lambda_p(\mu') - \lambda_p(\mu)}{\delta\mu} = 6C\lambda_p(\mu)^2 + \mathcal{O}(\lambda_p(\mu)^3)$$

$$\mu \frac{\partial \lambda_p(\mu)}{\partial \mu} =: \beta(\lambda_p)$$

Language β-function

$$\boxed{\beta(\lambda_p) := \mu \frac{\partial \lambda_p(\mu)}{\partial \mu} = 6C\lambda_p(\mu)^2 + \mathcal{O}(\lambda_p(\mu)^3)}$$

↳ Could compute higher order corrections, too.

DIFF EQ, "FLOW EQN"

Language Renormalization group flow

NOTE: - used "Pauli-Villars" regularization $\Rightarrow \log\left(\frac{\Lambda}{s}\right)$

- $\exists$ "Dimensional Regularization" as well

$d = 4 - \varepsilon$, $\quad \mathcal{M}_{meson} \sim \frac{2}{4-d} = \frac{2}{\varepsilon}$ so divergence in $\varepsilon \to 0$ limit

- pt: divergences drop out when you use physical quantities

MORE λ RENORMALIZATION

$$\mathcal{M} = -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{u}\right) + \log\left(\frac{\Lambda^2}{t}\right)\right] + \dots$$

NB * $\lim_{\Lambda\to\infty} \mathcal{M} = \infty$

* but $\lim_{\Lambda\to\infty} \mathcal{M}(s) - \mathcal{M}(s_0) = -i\left[\lambda(s) - \lambda(s_0)\right] + iC\lambda^2\left[\log\left(\frac{s_0}{s}\right) + \dots\right]$

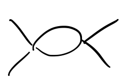
FINITE (*$_2$)

* $\mathcal{M}(s_1) =: -i\lambda_p$ $\mathcal{M}(s)$ AS λ_p FUNC USES COMPARISON LIKE (*$_2$)

ANOTHER WAY: "COUNTERTERMS" (more systematic and general)

$\Rightarrow$ CHANGE @ $\mathcal{L}$ LEVEL, NOT $\mathcal{M}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4!}\phi^4 \xmapsto{\text{redef } \lambda = \lambda_p + \delta_\lambda} \mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda_p}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4$$

 $\Rightarrow$ $\mathcal{M}(s) = -i\lambda_p - i\delta_\lambda + iC\lambda_p^2\left[\log\frac{\Lambda^2}{s} + \dots\right] + iC\delta_\lambda^2\left[\log\frac{\Lambda^2}{s} + \dots\right] + \lambda_p\delta_\lambda\,(\ \ ) + \dots$

Lang: "Renormalization Condition"

COMPARE TO PHYSICAL MEAS

$$-i\lambda_p := \mathcal{M}(s_0) = -i\lambda_p - i\delta_\lambda + iC\lambda_p^2\left[\log\frac{\Lambda^2}{s_0} + \dots\right] + \dots$$

$$\Rightarrow \delta_\lambda = C\lambda_p^2\log\left(\frac{\Lambda^2}{s_0}\right)$$

THEN: $\mathcal{M}(s) = -i\lambda_p - iC\lambda_p^2\left[\log\frac{\Lambda^2}{s_0} + \dots\right] + iC\lambda_p^2\left[\log\frac{\Lambda^2}{s} + \dots\right] + \underbrace{\mathcal{O}(\delta_\lambda^2)}_{\mathcal{O}(\lambda_p^4)} + \mathcal{O}(\lambda_p^3)$

$$\boxed{\mathcal{M}(s) = -i\lambda_p + iC\lambda_p^2\left[\log\left(\frac{s_0}{s}\right) + \dots\right] + \mathcal{O}(\lambda_p)^3}$$

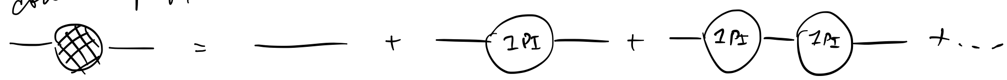
sometimes called Ren P.T.

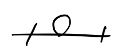
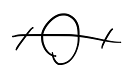
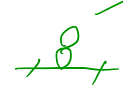
SAME RESULT USED "PHYSICAL PERTURBATION THY"

OTHER RENORMALIZATIONS: MASS & WAVEFUNCTION

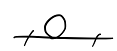
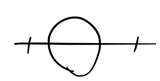
$$\mathcal{L} = \frac{1}{2}\cdot 1\ \partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

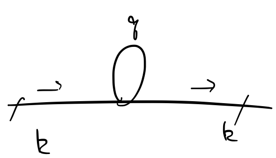
* ONE COEFF REN'D: λ
* m^2 & "1": BOTH IN PROP $\Rightarrow$ CORRECTIONS TO PROP?
* corrected prop, mom space, not amputated



$= D(k) + D(k)\Big($  $+$  $+$  $\Big) D(k) + \dots$

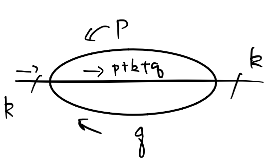
Zee leaves this out, but it won't qualitatively change conclusions

$= D(k)\Big[D(k)^{-1} +$  $+$  $+ \dots \Big] D(k)$



$$= \frac{-i\lambda}{2}\int_0^\Lambda \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m^2+i\varepsilon} =: a\lambda\Lambda^2$$

$\hookrightarrow$ k-INDEP, QUAD DIV



$$= \frac{(-i\lambda)^2}{6}\int^\Lambda\!\!\int^\Lambda \frac{d^4p}{(2\pi)^4}\,\frac{d^4q}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\,\frac{i}{q^2-m^2+i\varepsilon}\,\frac{i}{(p+q+k)^2-m^2+i\varepsilon} =: I$$

$\hookrightarrow$ k-DEP.

- $I = I(k, m, \Lambda, \lambda)$

- LORENTZ $\Longrightarrow$ I A FUNC OF k^2. TAYLOR SERIES IN k^2!

- FIRST $\quad I\big|_{k^2=0} \sim \iint d^4p \; d^4q \; \frac{1}{p^2} \; \frac{1}{q^2} \; \frac{1}{(p+q)^2} \Longrightarrow \Lambda^2$

$$\Longrightarrow \quad I = \lambda^2 \sum_{i=0}^{\infty} b_i \Lambda^{2-2i} (k^2)^i = b_0 \lambda^2 \Lambda^2 + b_1 \lambda^2 k^2 \log\left(\frac{\Lambda^2}{k^2}\right) + \ldots$$

$\lim \Lambda \to \infty$

$\ldots$'s $= 0$

$$\Longrightarrow \quad D_\lambda^{-1}(k) = \Big[\overbrace{k^2 - m^2}^{D^{-1}(k)} + \Lambda^2 (a\lambda + b_0 \lambda^2) + k^2 \lambda^2 b_1 \log \frac{\Lambda^2}{k^2} \Big]$$

$$=: k^2 - m^2 + a + k^2 b$$

$$\Longrightarrow \quad D_\lambda(k) = \frac{1}{(1+b)k^2 - (m^2 - a)} = \frac{(1+b)^{-1}}{k^2 - \frac{m^2 - a}{1+b}}$$

$m_p^2 = \frac{m^2 - a}{1+b}$ POLE IN CORRECTED PROP

Lang "Mass Renorm."

RESIDUE: $\frac{1}{1+b}$

IS INVERSE OF COEFFICIENT OF k^2

FROM COEFF $\partial_\mu \phi \, \partial^\mu \phi$

$\mathcal{O}(\lambda^0)$ COEFF $= 1$

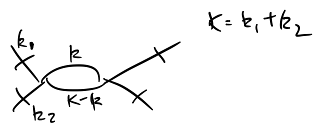
$\mathcal{O}(\lambda^1)$ COEFF $= 1$

$\mathcal{O}(\lambda^2)$ COEFF $= 1 + b$

Language: "Wavefunction Renormalization"

REGULARIZATION IN DETAIL (technicalities, but important)

PAULI-VILLARS



$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(K-k)^2-m^2+i\varepsilon}$$

$$= \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{D}$$

$$\frac{1}{xy} = \int_0^1 d\alpha \frac{1}{[\alpha x + (1-\alpha) y]^2}$$

$$D = [\alpha(K-k)^2 + (1-\alpha)k^2 - m^2 + i\varepsilon]^2$$

$$= [(k-\alpha K)^2 + \alpha(1-\alpha)K^2$$

* NOW SHIFT $k \to k + \alpha K$

$$\Longrightarrow \mathcal{M} \supset \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-c^2+i\varepsilon)^2}}_{\circledast} \quad \text{w/} \quad \underbrace{c^2 = m^2 - \alpha(1-\alpha)K^2}_{\text{doesn't dep on int. var.}}$$

everything else = this is the chg!

REGULATE:

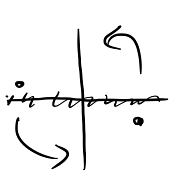
$$\circledast \overset{\Lambda \gg c}{\Longrightarrow} \int \frac{d^4k}{(2\pi)^4} \underbrace{\left[\frac{1}{(k^2-c^2+i\varepsilon)^2} - \frac{1}{(k^2-\Lambda^2+i\varepsilon)^2}\right]}_{\text{APPROACHES 0 FOR } k \gg \Lambda!} =: \circledast_2$$

$\Longrightarrow \circledast_2$ FINITE

TO EVAL: Wick Rotation $\ell^0 =: i\ell_E^0$ $\quad \vec{\ell} = \vec{\ell}_E$ $\quad$ see E.G. Peskin 193

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2-\Delta+i\varepsilon)^m} = i \frac{1}{(2\pi)^4} \int d^4\ell_E \frac{1}{[-\ell_E^{0\,2} - \vec{\ell}_E\cdot\vec{\ell}_E - \Delta + i\varepsilon]^m}$$

from $i\ell_E^0$



$$= \frac{i}{(-1)^m} \frac{1}{(2\pi)^4} \int d^4\ell_E \frac{1}{(\ell_E^2+\Delta-i\varepsilon)^m} \overset{\text{lim } \varepsilon \text{ spherical coords}}{=} \frac{i(-)^m}{(2\pi)^4} \int d\Omega_4 \int d\ell_E \frac{\ell_E^3}{(\ell_E^2+\Delta)^m}$$

Wick Rotation $\Longrightarrow$ $\boxed{\mathcal{M} = \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha \, \log\left(\frac{\Lambda^2}{m^2-\alpha(1-\alpha)K^2-i\varepsilon}\right)}$

DIMENSIONAL REGULARIZATION

SAME = STEPS AS P.V.

Same div integral

Long: "Wick Rotation"

$$\Longrightarrow \quad I = \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-c^2+i\varepsilon)^2} = i\int \frac{d_E^4 k}{(2\pi)^2}\frac{1}{(k^2+c^2)^2}$$

THE CHANGE

$d = 4-\varepsilon$ dims $\varepsilon > 0$ (different ε than above, both conventional)

$$\longmapsto \quad i\int \frac{d_E^d k}{(2\pi)^d}\frac{1}{(k^2+c^2)^2} = i\,\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{1}{(2\pi)^d}\int_0^\infty dk\, k^{d-1}\frac{1}{(k^2+c^2)^2}$$

$k^2+c^2 = \frac{c^2}{x}$

C.O.V do $\int$

$$= i\,\frac{1}{(4\pi)^{\frac{4-\varepsilon}{2}}}\,\Gamma\left(\frac{\varepsilon}{2}\right)c^{-\varepsilon}$$

$$\simeq i\,\frac{1}{(4\pi)^2}\left[\underbrace{\frac{2}{\varepsilon}}_{\text{diverges as } \varepsilon\to 0}\underbrace{-\log c^2 + \log(4\pi) - \overset{0.577\ldots}{\gamma}}_{\text{fixed } \forall \varepsilon} + \underbrace{O(\varepsilon)}_{\text{vanish @ } \varepsilon\to 0}\right]$$

THE POINT:

1. $\varepsilon > 0 \Rightarrow$ FINITE
2. $\varepsilon \to 0$ SEE DIVERGENCE!
3. DROPS OUT WHEN PHYSICAL COUPLINGS ARE USED (counterterms chosen to sop up the ∞)

RENORMALIZABLE vs. NON-RENORMALIZABLE

$\underbrace{\text{RENORMALIZABLE}}_{\text{"R"}}$ vs. $\underbrace{\text{NON-RENORMALIZABLE}}_{\text{"NR"}}$

new generation: nothing wrong with these as effective theories

FACT: Λ-dependence dropped out.

FACT: Doesn't always happen!

Λ DROPS = R-THY

Λ DOESN'T DROP = NR-THY

(one way to characterize the issue)

DIMENSIONAL ANALYSIS

* $e^{iS} \Longrightarrow S$ DIMENSIONLESS

$$S = \int \underbrace{d^4x}_{\text{DIM LENGTH}^4 \text{ OR MASS}^{-4}} \mathcal{L}$$

$[\cdot] :=$ MASS DIM OF $\cdot$ $\Longrightarrow [d^4x] = -4$ THEN $[S] = 0 \Rightarrow [\mathcal{L}] = [S] - [d^4x] = 4$

SO $\boxed{[\mathcal{L}] = 4}$ similarly $[\partial^\mu] = [\partial_\mu] = 1$

* KEY: USE KIN. TERMS TO DET' CLASSICAL FIELD DIM

SCALARS $\mathcal{L} \supset -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \Rightarrow [\phi] = \frac{1}{2}(4 - 2[\partial^\mu]) = 1$

SPINORS $\mathcal{L} \supset i\bar{\psi}\not{\partial}\psi \Longrightarrow [\psi] = [\bar{\psi}] = \frac{1}{2}(4 - [\partial^\mu]) = \frac{3}{2}$

VECTORS $\mathcal{L} \supset -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Rightarrow [A_\mu] = \frac{1}{2}(4 - 2[\partial^\mu]) = 1$

* LORENTZ-INV'T INTERACTIONS

INTERACTION	COUPLING DIMENSION
$m^2\phi\phi$ $m\bar{\psi}\psi$ $m^2A_\mu A^\mu$	$[m] = 1$
$\lambda\phi^4$	$[\lambda] = 0$
$g\phi^3$	$[g] = 1$
$y\phi\bar{\psi}\psi$	$[y] = 0$
$e\bar{\psi}\not{A}\psi$	$[e] = 0$
$G\bar{\psi}\psi\bar{\psi}\psi$	$[G] = -2$ FROM $[G] + 4(\frac{3}{2}) = 4$
$\sqrt{G_N}\, h\partial h\partial h$	$[\sqrt{G_N}] = -1$ $\left(\sqrt{G_N} \sim \frac{1}{M_p}\right)$

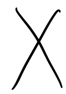
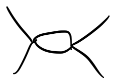
$g^{\mu\nu} := \eta^{\mu\nu} + h^{\mu\nu}$

$S \sim \int d^4x\, (\partial h\, \partial h + \sqrt{G_N}\, h\partial h\partial h$

HEURISTIC IDEA IN EXAMPLES

$\mathcal{M}$ @ INC. PERT ORDERS, DIM ANALYSIS.

$\lambda\phi^4$

$\mathcal{M} \sim$  $+$  $+ \dots$

$\sim A\lambda + B\lambda^2 + \dots$

FACT

$[\lambda]=0 \implies [A\lambda] = [A]+[\lambda] = [A]$ SO $[A]=[B]$

$\quad\quad\quad\quad\quad\quad \shortparallel$

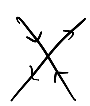
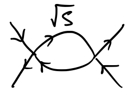
$\quad\quad\quad\quad [B\lambda^2] = [B]+2[\lambda] = [B]$

$\implies$ SCALES $\tilde{\Lambda}$ MUST APPEAR IN DIM'LESS COMBOS

$\hookrightarrow$ EG Λ OR s

$G\bar{\psi}\psi\bar{\psi}\psi$

$[G]=-2$

$\mathcal{M} \sim$  $+$ 

$\sim AG + BG^2$

BUT NOW $[B] = [A]+[2] \implies B \sim C\tilde{\Lambda}^2$ w/ $[C]=[A]$

SO $\mathcal{M} \sim AG + C\tilde{\Lambda}^2 G^2 = G(A + C\tilde{\Lambda}^2 G + \dots)$

H.O.T. IN $\tilde{\Lambda}^2 G$

PHYSICAL THY $\tilde{\Lambda}^2 = s$

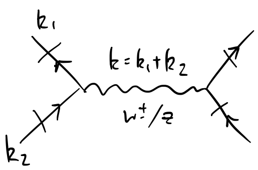
$\hookrightarrow \mathcal{M} =$ PERT IN Gs ONLY GOOD FOR $s \ll G^{-1}$

$\Longrightarrow$ NEW PHYSICS @ $s \simeq \frac{1}{G_F} \simeq (90\,\text{GeV})^2 \simeq M_{W/Z}^2$

$$\boxed{\text{NR THYS ARE LOW ENERGY EFFECTIVE FIELD THEORIES "EFTS"}}$$

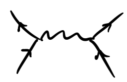
WHAT DOES THIS MEAN? EXAMPLE

WEAK INTERACTIONS



$$= (-ig)^2 \, \bar{u} \gamma^\mu u \, \frac{i}{k^2 - M_{W/Z}^2 + i\varepsilon} \, \bar{u} \gamma_\mu u$$

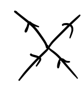
LOW ENERGY $k^2 \ll M_{WZ}^2 \Longrightarrow$



$$\simeq -(-ig)^2 \, \bar{u} \gamma^\mu u \, \frac{i}{M_{W/Z}^2} \, \bar{u} \gamma_\mu u$$

* $\mathcal{L}_{int} = G \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma_\mu \psi$ w/ $G_F \simeq \frac{g^2}{M^2} \Longrightarrow$ SAME $\mathcal{M}$ @ LOW ENERGIES

Language: "Effective Operator"



$$\simeq G_F \, \bar{u} \gamma^\mu u \, \bar{u} \gamma_\mu u$$

* $\mathcal{L}_{int}$ FAILS @ $k^2 \gtrsim M_{W/Z}^2$. UV PHYSICS REPLACES EFFECTIVE INT

CON: MANY UV MODELS COULD GIVE SAME EFF. INT.

PRO: "

IGNORANCE CAN BE BLISS!

(great that predecessors could focus on lower energy physics w/o worrying about SM details)

Systematics of Renormalization

Q: how do we systematically renormalize & deal w/ all divergences, rather than one by one?

STEPS

① DET. ALL POSS. DIVERGENCES

② ELIM SYS. BY RENORMALIZED P.T.

SYSTEMATICS OF RENORMALIZATION in ϕ^4 THY

SUPERFICIAL DEG OF DIVERGENCE $\Rightarrow$ MAX DIV

GOAL GRAPH $\sim \Lambda^D$ ($\log \Lambda$ COUNTS AS $D=0$)

DET. D EASILY AS F(GRAPH PARAMS)

GRAPH PARAMS

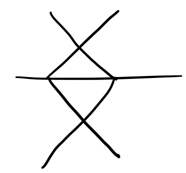
B_E = # EXT. LINES

B_I = # INT. LINES

V = # VERTS $\longrightarrow \phi^4$ VERT

L = # LOOPS

EG



$B_E = 6$
$B_I = 5$
$V = 4$
$L = 2$

$D = -2$

THM: $D = 4 - B_E$

PF: (1) $L = \# \int \frac{d^4k}{(2\pi)^4}$ INTEGRALS = # LOOPS

$= \underbrace{B_I}_{\text{INT. LINE} \Rightarrow \text{MOM. } \int} - (\underbrace{V}_{\substack{\text{VERT} \Rightarrow \text{MOM. CONS.} \Rightarrow \\ \text{COLLAPSE MOM INT}}} - \underbrace{1}_{\text{OVERALL MOM. CONS.}})$

CHECK $L = B_I - (V-1)$
$= 5 - (4-1)$
$= 2$ ✓

(2) $4V = B_E + 2B_I$

4 "CONNECTION PTS" PER VERTEX

EACH EXT LINE HAS ONE END ON CON. PT.

EACH INT LINE HAS 2 ENDS ON CON. PT.

CHECK $4 \cdot 4 = 6 + 2 \cdot 5$
$= 16$ ✓

(3) $D = 4L - 2B_I$

LOOP $\Rightarrow \int d^4k \sim \Lambda^4$

EACH INT LINE $\Rightarrow \frac{1}{k^2}$ DEC. BY 2

CHECK $D = 4 \cdot 2 - 2 \cdot 5 = -2$

THEN: $D = 4(B_I - V + 1) - 2B_I = \underbrace{2B_I - 4V}_{-B_E \text{ FROM } 2} + 4$

$\Rightarrow \boxed{D = 4 - B_E}$

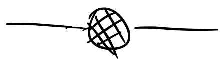
SYSTEMATIC REN. OF ϕ^4

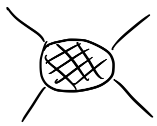
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!}\phi^4$$

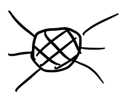
Language: λ_0 m_0 "Bare" mass & coupling
$\rightarrow$ i.e. not physical ones

FACT B_E EVEN

$\Longrightarrow$  $D=4$ (unobservable in QFT)

 $D=2 \Longrightarrow \Lambda^2 + k^2 \log \Lambda + \text{finite}$

 $D=0 \Longrightarrow \log \Lambda + \text{finite}$

 $D=-2 \Longrightarrow$ finite as $\Lambda \to \infty$

HERE: PESKIN 10.2 CONVENTIONS

Peskin 7.1, exact 2-pt func has this form.

$$\int d^4x \, \langle \Omega | T(\phi(x)\phi(0)) | \Omega \rangle e^{ip\cdot x} = \frac{iZ}{p^2 - m^2} + \text{regular @ } p^2 = m^2$$

$\Longrightarrow$ SCALE AWAY Z BY $\phi =: \sqrt{Z}\,\phi_r$

Zee likes P subscripts

$$\Longrightarrow \mathcal{L} = \frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_0^2 Z \phi_r^2 - \frac{\lambda_0}{4!} Z^2 \phi_r^4$$

AS BEFORE: REPLACE BARE COUPLING BY PHYS. THINGS + CT'S

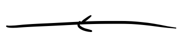
$$Z =: 1 + \delta_Z \qquad m_0^2 Z =: m^2 + \delta m \qquad \lambda_0 Z^2 =: \lambda + \delta_\lambda$$

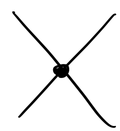
Zee called λ_p.

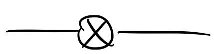
Zee called this λ_p

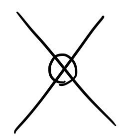
NOW
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4$$ ← Looks like normal φ^4
$$+ \frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4$$ ← Counterterms!

FEYNMAN RULES FOR $\mathcal{L}$

 $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

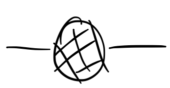
 $= -i\lambda$

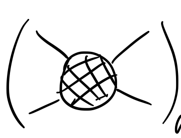
 $= i(p^2\delta_Z - \delta_m)$

 $= -i\delta_\lambda$

PESKIN'S REN. CONDS

really 2 conditions, since both pole and residue are specified

 $= \dfrac{i}{p^2 - m^2} + (\text{reg} \ @ \ p^2 = m^2)$

 $_{\text{amp}} = -i\lambda \quad @ \quad s = 4m^2 \quad t = u = 0$

NOTE: 3 CT'S $\delta_\lambda, \delta_m, \delta_Z$

3 CONDS TO FIX THEM

→ Zee calls "physical"

RENORMALIZED PERTURBATION THEORY STEPS

STEP 1 COMPUTE SOME $\mathcal{M}$ AS SUM OF ALL POSS. DIAG. FROM THESE FEYN. RULES.

STEP 2 IF LOOPS DIVERGE, REGULATE.

RESULT = $F(\delta_Z, \delta_m, \delta_\lambda,$ REG. PARAM$)$

STEP 3 DEMAND REN. CONDS.

$\Rightarrow$ FIXES SOME CT'S

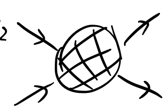
STEP 4: REWRITE $\mathcal{M}$

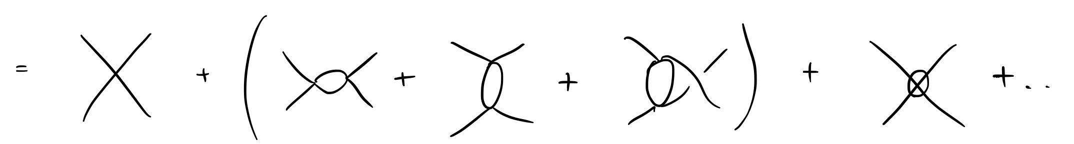
RESULT: FINITE $\mathcal{M}$, INDEPENDENT OF REGULATOR, DEPENDS ONLY ON PHYSICAL QUANTITIES.

ONE-LOOP STRUCTURE OF ϕ^4 THEORY

FIRST  THEN 

:

STEP 1 $i\mathcal{M}(p_1 p_2 \longrightarrow p_3 p_4) =$ 



$$= -i\lambda + (-i\lambda)^2 \left[iV(s) + iV(t) + iV(u) \right] - i\delta_\lambda$$

w/ $$V(p^2) = \frac{-i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

NB: V IS THE HARD PART.

STEP 2 REG., HERE w/ DIM REG

$$V(p^2) \xleftarrow{\text{FEYN. PARAM.}} = \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2+2x\,k\cdot p + xp^2 - m^2]^2}$$

SHIFT $\ell^\mu = k^\mu + x p^\mu$

$$= \frac{i}{2}\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{[\ell^2 + x(1-x)p^2 - m^2]^2}$$

ROT. TO EUC. SP. $\ell_E^0 = -i\ell^0$

$$= -\frac{1}{2}\int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d} \frac{1}{[\ell_E^2 - x(1-x)p^2 + m^2]^2}$$

DO $\int d^d\ell_E$

$$= -\frac{1}{2}\int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \frac{1}{[m^2 - x(1-x)p^2]^{2-d/2}}$$

$d = 4-\varepsilon$ & ε-EXP

$$= -\frac{1}{32\pi^2}\int_0^1 dx \left(\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log[m^2 - x(1-x)p^2] + \mathcal{O}(\varepsilon)\right)$$

STEP 3 ()$_{amp}$ $= -i\lambda$ @ $\underbrace{s = 4m^2,\ t = u = 0}_{\text{"pt"}}$

$$i\mathcal{M}\Big|_{pt} \overset{RC}{=} -i\lambda$$

$$\underset{EXP}{=} -i\lambda + (-i\lambda)^2[iV(4m^2) + 2iV(0)] - i\delta_\lambda + \dots$$

$i \times \frac{1}{i}$ CANCEL

$$\Longrightarrow \delta_\lambda = (-i\lambda)^2[V(4m^2) + 2V(0)]$$

$$= (-i\lambda)^2\Big[-\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log[m^2 - x(1-x)4m^2] + \mathcal{O}(\varepsilon)\right)$$

$$+ 2\left(\frac{-1}{32\pi^2}\right)\int_0^1 dx\left(\text{" \quad " \quad "} \quad - \log[m^2] + \mathcal{O}(\varepsilon)\right)\Big]$$

$$= \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{6}{\varepsilon} - 3\gamma - 3\log 4\pi - \log[m^2 - x(1-x)4m^2] - 2\log[m^2] + \mathcal{O}(\varepsilon)\right)$$

NB: 2 & 3 BOTH DIV. AS $\varepsilon \to 0$!

STEP 4 REWRITE $\mathcal{M}$

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2 \left[iV(s) + iV(t) + iV(u) \right] - i\delta_\lambda$$

$= -\lambda^2$, reason so cancellation does happen

$$= -i\lambda + (-i\lambda)^2$$

$$\times \left[-\frac{i}{32\pi^2} \int_0^1 dx \left(\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left[m^2 - x(1-x)s\right] + \mathcal{O}(\varepsilon) \right) \right.$$

$$- \frac{i}{32\pi^2} \int_0^1 dx \left(\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left[m^2 - x(1-x)t\right] + \mathcal{O}(\varepsilon) \right)$$

$$\left. - \frac{i}{32\pi^2} \int_0^1 dx \left(\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left[m^2 - x(1-x)u\right] + \mathcal{O}(\varepsilon) \right) \right]$$

$$-i\frac{\lambda^2}{32\pi^2} \int_0^1 dx \left(\frac{6}{\varepsilon} - 3\gamma - 3\log 4\pi - \log\left[m^2 - x(1-x)4m^2\right] - 2\log\left[m^2\right] + \mathcal{O}(\varepsilon) \right)$$

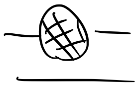
$$= -i\lambda - \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left[\log\left(\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2} \right) + \log\left(\frac{m^2 - x(1-x)t}{m^2} \right) + \log\left(\frac{m^2 - x(1-x)u}{m^2} \right) \right] + \mathcal{O}(\varepsilon)$$

$$\xrightarrow{\varepsilon = 0} \boxed{ i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2} \int_0^1 dx \left[\log\left(\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2} \right) + \log\left(\frac{m^2 - x(1-x)t}{m^2} \right) + \log\left(\frac{m^2 - x(1-x)u}{m^2} \right) \right] }$$

$\hookrightarrow$ RESULT!

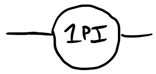
1. FINITE
2. ONLY F(PHYSICAL QUANT)

As of now, we've fixed one counterterm and have a finite physical expression for one of the "divergent amplitudes"

 :

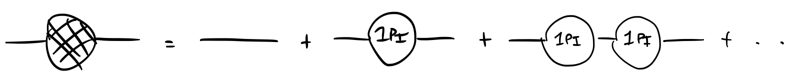
ASIDE: (for more see Peskin section 7.2)

DEF'N: A DIAGRAM IS ONE PARTICLE IRREDUCIBLE IF IT CANNOT BE SPLIT IN TWO BY REMOVING A SINGLE LINE.

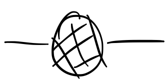

$$= \sum \begin{array}{l}\text{all 1PI insertions}\\ \text{into propagator}\end{array} =: -i M^2(p^2)$$

THEN: 2 PT FN

expand out to check


$$= \frac{i}{p^2 - m^2 - M^2(p^2)}$$

RECALL REN COND

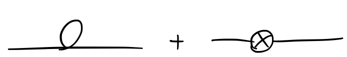

$$= \frac{i}{p^2 - m^2} + (\text{reg @ } p^2 = m^2)$$

as H.O.T in Taylor series in denominator die

IS EQUIV TO

RC1 $\quad M^2(p^2)\Big|_{p^2=m^2} = 0$ & RC2 $\quad \left(\frac{d}{dp^2} M^2(p^2)\right)\Big|_{p^2=m^2} = 0$

end aside, now go through steps using $M(p^2)$ as encoder of  info

STEP 1: $-iM^2(p^2) =$ 

$$= -i\lambda \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} + i(p^2 \delta_z - \delta_m)$$

STEP 2: DIM REG $\Rightarrow$
$$= \frac{-i\lambda}{2} \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2} + i(p^2 \delta_z - \delta_m)$$

?

$$= \frac{-i\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1 - \frac{d}{2})}{(m^2)^{1-d/2}} + i(p^2 \delta_z - \delta_m)$$

STEP 3: IMPOSE REN. COND.

no wave-function renormalization @ one-loop order

RC2 $\quad \frac{d}{dp^2}(M^2) = 0$ ALREADY $\Rightarrow \delta_z = 0$

RC1 $\quad M^2(p)\Big|_{p^2=m^2} = 0 \implies \delta_m = -\frac{\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}}$

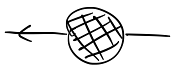
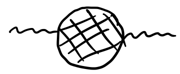
STEP 4: PLUG IN,

$\implies M^2(p^2) = 0$

SYSTEMATIC RENORMALIZATION OF QED

(Using Peskin b/c he does dim reg for QED ch 10)

ORIG: $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\not{\partial} - m_0)\psi - e_0\bar{\psi}\not{A}\psi$

$\Longrightarrow$  $= \dfrac{iZ_2}{\not{p} - m} + \ldots$  $= \dfrac{-iZ_3\, g_{\mu\nu}}{q^2} + \ldots$

RESCALE FIELDS $\quad \psi =: Z_2^{1/2}\psi_r \qquad A^\mu =: Z_3^{1/2}A_r^\mu$

$$\mathcal{L} = -\frac{1}{4}Z_3(F_r^{\mu\nu})^2 + Z_2\bar{\psi}_r(i\not{\partial} - m_0)\psi_r - \underbrace{e_0 Z_2 Z_3^{\frac{1}{2}}}_{=:\, eZ_1}\bar{\psi}_r\not{A}_r\psi_r$$

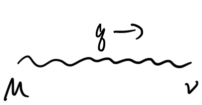
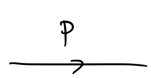
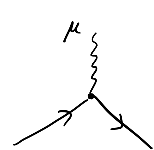
$=: eZ_1$, DEFINES Z_1

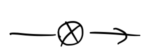
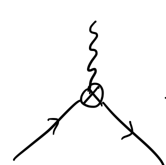
DEFINE CT'S $\quad Z_3 =: 1 + \delta_3 \qquad Z_2 =: 1 + \delta_2 \qquad Z_1 := 1 + \delta_1 \qquad \delta_m = Z_2 m_0 - m$

from above definition $\quad Z_1 = \dfrac{e_0}{e}Z_2 Z_3^{\frac{1}{2}}$

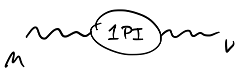
$$\Rightarrow \mathcal{L} = -\frac{1}{4}(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\not{\partial} - m)\psi_r - e\bar{\psi}_r\not{A}_r\psi_r$$
$$-\frac{1}{4}\delta_3(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\delta_2\not{\partial} - \delta_m)\psi_r - e\delta_1\bar{\psi}_r\not{A}_r\psi_r$$

FEYNMAN RULES $\quad$ NOTE: 6 TERMS ABOVE GIVE 6 FR'S BELOW

 $= \dfrac{-ig_{\mu\nu}}{q^2 + i\varepsilon}$  $= \dfrac{i}{\not{p} - m + i\varepsilon}$  $= -ie\gamma^\mu$

 $= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$  $= i(\not{p}\delta_2 - \delta_m)$  $= -ie\gamma^\mu\delta_1$

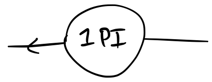
SOME NOTATION: (see Peskin chs 6 & 7)



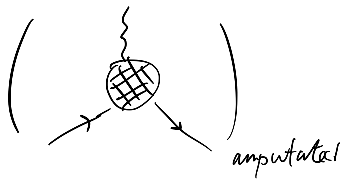
$$= i\Pi^{\mu\nu}(q) = i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\Pi(q^2)$$

This structure is derived, defines remnant $\Pi(q^2)$ function

argue: there's nothing else Lorentz covariant could have written down. Detail: the sign



$$= -i\Sigma(\not{p})$$



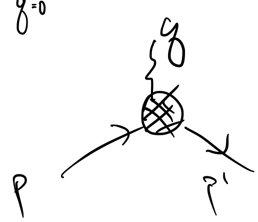
$$= -ie\,\Gamma^\mu(p',p)$$

PESKIN'S RENORMALIZATION CONDITIONS FOR QED

$\Sigma\big|_{\not{p}=m} = 0$ $\Rightarrow$ FIXES PHYSICAL ELECTRON MASS TO BE POLE OF PROP

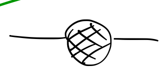
$\frac{d}{d\not{p}}\Sigma(\not{p})\big|_{\not{p}=m} = 0$ $\quad$ $\Pi\big|_{q^2=0} = 0$ $\Longrightarrow$ FIX RES. OF 2 PROPS

$-ie\Gamma^\mu\big|_{p=p'} = -ie\gamma^\mu$



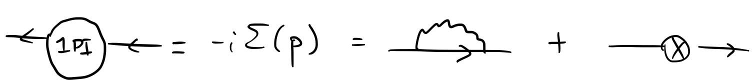
ONE LOOP RENORMALIZATION

FIRST DIVERGENCE



$$= \;\longleftarrow\; + \;\longleftarrow\!\text{1PI}\!\longleftarrow\; + \dots$$

STEP 1



$$= -i\Sigma(p) = \text{(loop)} + \underbrace{\text{(counterterm)}}_{+i(\not{p}\,\delta_2-\delta_m)} =: -i\Sigma_2(p) + i(\not{p}\,\delta_2 - \delta_m)$$

STEP 2

$$\overset{\text{dim reg}}{=} -i\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})}{\left((1-x)m^2 + x\mu^2 - x(1-x)p^2\right)^{2-d/2}} \times \left((4-\varepsilon)m - (2-\varepsilon)x\not{p}\right)$$

HMWK

STEP 3: REN. COND. 1 $\quad \Sigma \Big|_{\not p = m} = 0$

$$\Rightarrow \quad 0 = -i\Sigma_2(m) + i(m\delta_2 - \delta_m)$$

$$\Rightarrow \quad m\delta_2 - \delta_m = \Sigma_2(m) = \frac{e^2 m}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2}) \cdot (4 - 2x - \varepsilon(1-x))}{((1-x)^2 m^2 + x\mu^2)^{2-d/2}}$$

REN. COND 2 $\quad \frac{d}{d\not p}\Sigma(\not p)\Big|_{\not p = m} = 0 \quad \Rightarrow \quad -i\frac{d\Sigma_2}{d\not p}\Big|_{\not p = m} + i\delta_2 = 0$

$$\Rightarrow \quad \delta_2 = \frac{d\Sigma_2}{d\not p}\Big|_{\not p = m} = \frac{-e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{((1-x)^2 m^2 + x\mu^2)^{2-d/2}}$$

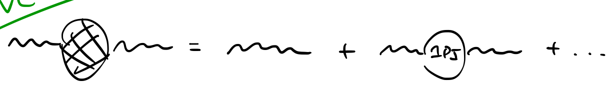
$$\cdot \left[(2-\varepsilon)x - \frac{\varepsilon}{2} \frac{2x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2} (4 - 2x - \varepsilon(1-x)) \right]$$

NB: DIV. ONLY FROM $\Gamma(2-\frac{d}{2})$

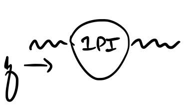
NOTE: NOW ALL CT IN $-i\Sigma(p)$ ARE FIXED! $\Rightarrow$ MOVE ON

why?

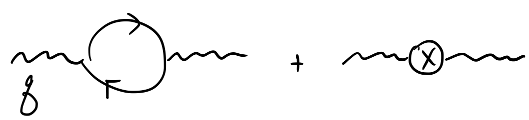
SECOND DIVERGENCE



STEP 1 WRITE DIAGS

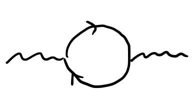


$$= i(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2)$$



$$= i(g^{\mu\nu}q^2 - q^\mu q^\nu)\left[\Pi_2(q^2) - \delta_3\right]$$

STEP 2 REGULATE



$$= i(g^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi_2(q^2)$$

HMWK

$$\text{w/} \quad \Pi_2(q^2) = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{(m^2 - x(1-x)q^2)^{2-d/2}} \left(8x(1-x)\right)$$

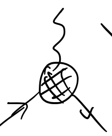
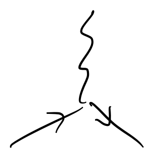
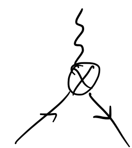
STEP 3: RENORMALIZATION CONDITION

$$\Pi\Big|_{q^2=0} = 0 \qquad \text{w/} \qquad \Pi(q^2) = \Pi_2(q^2) - \delta_3$$

$$\Rightarrow \delta_3 = \Pi_2(0) = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-d/2}} \, 8x(1-x)$$

NOTE: NOW ALL CT IN $\Pi(p)$ ARE FIXED! $\Longrightarrow$ MOVE ON

THIRD DIVERGENCE

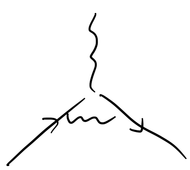
$\Big($  $\Big)_{amp} =: -ie\,\Gamma^\mu(p',p) =$  $+$  $+$ 

HMWK

LOOKS LIKE: $\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + i\,\frac{\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$

Language F_1, F_2 are "Form Factors"

LEADING ORDER: $F_1 = 1 \quad F_2 = 0$



* CORRECTS F_1 & F_2 $\Longrightarrow$ $\delta F_1 \quad \delta F_2$
* FACT δF_2 FINITE
* $F_2 = \underset{\text{ord.}}{\text{leading}} + \delta F_2 = 0 + \delta F_2 = \delta F_2 = \frac{\alpha}{2\pi}$

* MAGNETIC MOMENT $\quad \vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}$

↳ "Landé g-factor"

SOME WORK $\Longrightarrow$ $g = 2[F_1(0) + F_2(0)] = 2\alpha^0[1+0] + \mathcal{O}(\alpha)$

measures deviation from classical result

$$a_e := \frac{g-2}{2} \simeq F_2\Big|_{\mathcal{O}(\alpha)} = \frac{\alpha}{2\pi} = .0011614$$

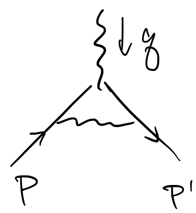
EXPT: $a_e = .0011597$

PRECISION $\quad a_e$ KNOWN TO α^4, 8 SIG. FIGS AGREE w/ a_e^{EXPT}

STEP 1

$$= -ie\left[\gamma^\mu + \gamma^\mu \delta F_1 + i\frac{\sigma^{\mu\nu} q_\nu}{2m}\delta F_2 + \gamma^\mu \delta_1 \right]$$

STEP 2 REGULATE



$p + q = p'$

this is the divergent part, recall F_2 finite @ $\mathcal{O}(\alpha)$!

$$\Rightarrow \delta F_1(q^2) = \frac{e^2}{(4\pi)^{d/2}} \int dx\,dy\,dz\; \delta(x+y+z-1)$$

$$\times \left[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\frac{(2-\varepsilon)}{2} + \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}}\left(q^2\left[2(1-x)(1-y) - \varepsilon xy\right] + m^2\left[2(1-4z+z^2) - \epsilon(1-z)^2\right] \right)\right]$$

WHERE $\Delta = (1-z)^2 m^2 + z\mu^2 - xyq^2$

STEP 3 REN. CONDS. $-ie\Gamma^\mu\big|_{p=p'} = -ie\gamma^\mu$

$$\Rightarrow -ie\left[\gamma^\mu + \gamma^\mu \delta F_1\Big|_{p=p'} + \cancel{\left(\frac{i\sigma^{\mu\nu}q_\nu}{2m}\delta F_2\right)}\Big|_{\substack{p=p' \\ \text{ie } q=0}}^{\to 0} + \gamma^\mu\delta_1 \right] = -ie\gamma^\mu$$

$q=0$ for $p=p'$

$$\Rightarrow \delta_1 = -\delta F_1(q^2)\Big|_{p=p'} = -\delta F_1(0) =$$ PLUG IN 0 FOR q IN ABOVE

HMWK

FACT: $\delta_1 = \delta_2 \Longrightarrow Z_1 = Z_2$

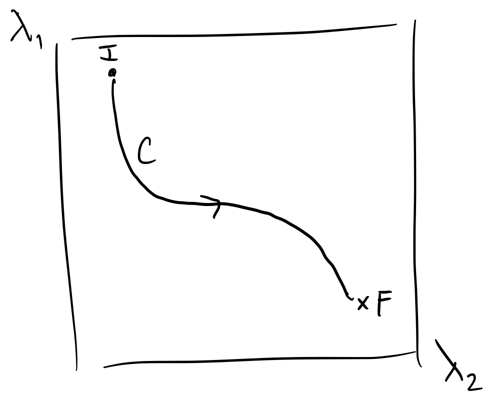
NOTE: All renormalization conditions have been imposed, all counterterms have been fixed, QED is renormalized!

$\Longrightarrow$ how to see running coupling?

RENORMALIZATION GROUP

remember already saw on day 1

HEURISTIC: FLOW IN SPACE OF COUPLINGS



* SUPPOSE VAL OF λ_1, λ_2 PUT YOU @ I @ SCALE μ_I
* $\exists$ DIFF EQ'S GOVERNING "FLOW" IN THIS SPACE AS $\mu_I \to \mu$
  $\longrightarrow$ CURVE $C(\mu)$
* FLOW TO "RG FIXED PTS" F

HERE: MASSLESS THYS FOR SIMP.

FIRST: ϕ^4 THEORY

CHANGE NOTATION: (Peskin actually does this, changing from his own notation)

$$\phi \longmapsto \phi_0, \quad \phi_r \longmapsto \phi \qquad \text{SO} \qquad \phi_0 = \sqrt{Z}\,\phi$$

take as effective definition here, since Peskin/Zee conventions a bit different

$$\underbrace{G^{(n)}(x_1,\dots,x_n)}_{\text{renormalized n-pt}} := \langle\Omega|T\,\phi(x_1)\dots\phi(x_n)|\Omega\rangle_{\text{connected}}$$

$$= Z^{-n/2}\langle\Omega|T\phi_0(x_1)\dots\phi_0(x_n)|\Omega\rangle =: Z^{-n/2}\underbrace{G_0^{(n)}(x_1,\dots,x_n)}_{\text{bare n-pt}}$$

(see e.g. Schwartz 23.4.3, Peskin 12.2)

$$\text{SO} \quad \underbrace{G_0^{(n)}(x_1,\dots,x_n)}_{\text{compute in bare p.t. w bare } \lambda_0, m_0} = Z^{\frac{n}{2}}\underbrace{G^{(n)}(x_1,\dots,x_n)}_{\text{computed in ren'd P.T. w/ } \lambda, m \text{ being renormalization conditions}}$$

$\Longrightarrow$ scale μ (e.g. $s=t=u=\mu^2$)

CRITICAL POINT NO μ ON LHS!

$$\Longrightarrow 0 = \mu \frac{\partial}{\partial \mu} G_0^{(n)}(x_1,\dots,x_n) = \mu \frac{\partial}{\partial \mu}\left[Z^{\frac{n}{2}} G^{(n)}(x_1,\dots,x_n)\right]$$

$$= Z^{\frac{n}{2}}\left[\frac{n}{2}\,\frac{1}{Z}\,\mu \frac{\partial Z}{\partial \mu}\, G^{(n)} + \mu \frac{\partial G^{(n)}}{\partial \lambda}\frac{\partial \lambda}{\partial \mu} + \mu \frac{\partial G^{(n)}}{\partial \mu}\right]$$

Language: "β-function" for λ

$$\beta_\lambda := \mu \frac{\partial \lambda}{\partial \mu} \qquad \gamma := \frac{\mu}{Z}\frac{\partial Z}{\partial \mu}$$

$$\Longrightarrow \boxed{\left[\mu \frac{\partial}{\partial \mu} + \beta_\lambda \frac{\partial}{\partial \lambda} + \frac{n}{2}\gamma\right] G^{(n)} = 0}$$

Callan-Symanzik Equation

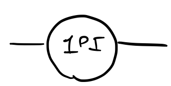
(note: can also do in momentum space)

(note: can also turn on masses)

β_λ @ ONE-LOOP

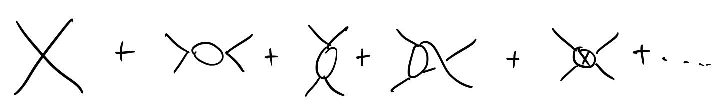
RECALL

$G^2(p) =$  $= \dfrac{i}{p^2 - m^2 - M^2(p^2)}$

w  $= -i M^2(p^2)$

PREV. CALCULATION $\Longrightarrow M^2(p^2) = 0$ @ 1-LOOP

$\Longrightarrow$ NO μ OR λ DEP IN $G^{(2)}$

$G^4(p) =$ 

$$= \Big[-i\lambda + (-i\lambda)^2 \big[\underbrace{i V(s) + i V(t) + i V(u)}_{\text{NO } \lambda,\mu}\big] - \underbrace{i\delta_\lambda}_{\lambda,\mu} \Big] \cdot \prod_{i=1,\dots,4} \frac{1}{p_i^2}$$

$$=: \left[-i\lambda + (-i\lambda)^2 i \tilde{V} - i\delta_\lambda\right] \tilde{\pi}$$

$$\delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2}\left[\frac{1}{2 - d/2} - \log \mu^2 + \underbrace{\text{finite}}_{\lambda\text{-indep}} \text{ \& } \mu\text{-indep}\right]$$

C-S FOR 2-PT FN.

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+\frac{n}{2}\gamma\right]G^{(2)}=0=\frac{n}{2}\gamma G^{(2)}\Longrightarrow\gamma=0$$

C-S FOR 4-PT FN

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+\frac{n}{2}\cancelto{0}{\gamma}\right]G^{(4)}=0 \qquad (*)$$

$$\tilde{\Pi}\left[\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}\right]\left[-i\lambda+(-i\lambda)^2 i\tilde{V}-i\frac{3\lambda^2}{2(4\pi)^2}\left[\frac{1}{2-d/2}-\log\mu^2+\underset{\lambda\text{-indep}}{\text{finite}\ \&\ \mu\text{-indep}}\right]\right]=0$$

↳ only non-zero terms come from explicit μ, $\frac{\partial\lambda}{\partial\mu}\neq 0$ taken into account by β, e.g.

$$0=\tilde{\Pi}\left(\underbrace{\frac{i3\lambda^2}{16\pi^2}}_{\text{from }\frac{\partial}{\partial\mu}}\underbrace{-i\beta_\lambda-2i\lambda\beta_\lambda\tilde{V}-i\beta_\lambda\frac{3\lambda}{16\pi^2}\left[\frac{1}{2-\frac{d}{2}}-\log\mu^2+\dots\right]}_{\text{from }\frac{\partial}{\partial\lambda}}\right)$$

$\beta_\lambda=\frac{3\lambda^2}{16\pi^2}+\mathcal{O}(\lambda^3)$ SOLVES TO $\mathcal{O}(\lambda^2)$.

$$\Longrightarrow\boxed{\beta_\lambda=\mu\frac{\partial\lambda}{\partial\mu}=\frac{\partial\lambda}{\partial\ln\mu}=\frac{3\lambda^2}{16\pi^2}}$$

RUNNING COUPLING

NOTE: β-FUNCTION AROSE FROM μ-DEP OF C.T.

β SYSTEMATICS (any renormalizable massless scalar field theory)

USEFUL SCHEMATIC

$$G^{(2)}(p) = \text{——} + \text{loops} + \text{——}\otimes\text{——} + \dots$$

$$= \frac{i}{p^2} + \frac{i}{p^2}\left(A\log\frac{\Lambda^2}{p^2} + \text{finite}\right) + \frac{i}{p^2}\left(ip^2\delta_z\right)\frac{i}{p^2} + \dots$$

μ-dep only from here

CALLAN-SYMANZIK

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \frac{n}{2}\gamma\right]G^{(n)} = 0$$

(β is H.O.T, neglected) $\implies -\frac{i}{p^2}\mu\frac{\partial}{\partial\mu}\delta_z + 2\gamma\frac{i}{p^2} = 0 \implies \boxed{\gamma = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\delta_z}$ (to lowest order)

then $\delta_z = A\log\frac{\Lambda^2}{\mu^2} + \text{finite} \implies \boxed{\gamma = -A}$

PS: γ is $-$coeff of log in field strength counterterm!

CONSIDER g DIM-LESS n-PT COUPLING

$$G^{(n)} = (\text{trees}) + (\text{1PI loops}) + (\text{vertex CT}) + (\text{external leg corrections})$$

→ ext leg momenta can be diff

$$= \prod_{i=1}^{n}\frac{i}{p_i^2}\left[-ig - iB\log\frac{\Lambda^2}{p^2} - i\delta_g + (-ig)\sum_i\left(A_i\log\frac{\Lambda^2}{-p_i^2} - \delta_{z_i}\right)\right] + \text{finite}$$

→ reps invariant built from momenta

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_g\frac{\partial}{\partial g} + \sum_i\gamma_i\right]G^{(n)} = 0$$

$$= \prod_{i=1}^{n}\frac{i}{p_i^2}\left\{\mu\frac{\partial}{\partial\mu}\left[-i\delta_g + ig\sum_i\delta_{z_i}\right] + \beta_g\left[-i\overbrace{-i\sum_i\left(A_i\log\frac{\Lambda^2}{-p_i^2} - \delta_{z_i}\right)}^{\text{subleading in } g}\right]\overbrace{-ig\sum\frac{1}{2}\mu\frac{\partial}{\partial\mu}\delta_{z_i}}^{\text{kept only leading order in } g}\right\}$$

↑ B term H.O. in g

$$\implies \mu\frac{\partial}{\partial\mu}\left[\delta_g - g\sum\delta_{z_i}\right] + \beta_g + g\sum\frac{1}{2}\mu\frac{\partial}{\partial\mu}\delta_{z_i} = 0$$

$$\implies \boxed{\beta_g = \mu\frac{\partial}{\partial\mu}\left[-\delta_g + \frac{g}{2}\sum_i\delta_{z_i}\right]}$$

(to lowest order)

$$\beta_g = \mu \frac{\partial}{\partial \mu}\left[-\delta_g + \frac{g}{2} \sum_i \delta_{z_i} \right]$$

CHECK ϕ^4 @ 1 LOOP $\delta_z = 0$, $\delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2}\left[-\log \mu^2 \right] + \mu\text{-indep}$

$$\Longrightarrow \quad \beta_\lambda = \mu \frac{\partial}{\partial \mu}\left[\frac{+3\lambda^2}{2(4\pi)^2} \log \mu^2 \right] = \frac{3\lambda^2}{16\pi^2} \quad \checkmark$$

matches previous calc!

ANOTHER SIMPLIFICATION

IF $\delta_{z_i} = A_i \log \frac{\Lambda^2}{\mu^2} + \text{finite}$ $\qquad \delta_g = -B \log \frac{\Lambda^2}{\mu^2} + \text{finite}$

$$\beta_g = \mu \frac{\partial}{\partial \mu}\left[-\left(B \log \mu^2 \right) + \frac{g}{2} \sum \left(-A_i \log \mu^2 \right) \right]$$

$$\Longrightarrow \quad \boxed{\beta_g = -2B - g \sum A_i}$$

NB: DEPENDS ONLY ON COEFFICIENTS

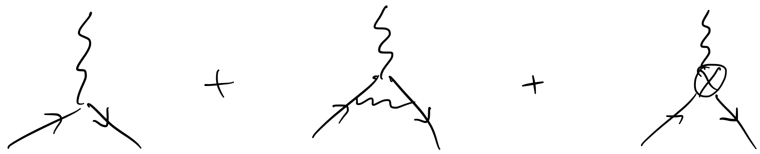
$\hookrightarrow$ extract these, don't have to do nasty integral

QED:

ϕ^4 $\quad \beta_g = \mu \frac{\partial}{\partial \mu}\left[-\delta_g + \frac{g}{2}\sum_i \delta_{Z_i}\right]$ $\qquad G^{(4)} = \prod \frac{i}{(p_i)^2}\left[-ig - i\delta_g + \cdots\right]$

QED $\quad \beta_e = \mu \frac{\partial}{\partial \mu}\left[-\delta_e + \frac{e}{2}\underbrace{\left(\delta_2 + \delta_2 + \delta_3\right)}_{\text{WHY? B/C } \bar{\psi}\psi A}\right]$

DIDN'T DEFINE!



$$-ie\left[\ \gamma^\mu \quad + \quad \gamma^\mu \delta F_1 + i\frac{\sigma^{\mu\nu}q_\nu}{2m}\delta F_2 \quad + \quad \gamma^\mu \delta_1\ \right]$$

$\Longrightarrow$ "δ_e" IS $e\,\delta_1$ TO MATCH PREV. DERIV

$$\Longrightarrow \boxed{\beta_e = \mu \frac{\partial}{\partial \mu}\left[-e\delta_1 + e\,\delta_2 + \frac{e}{2}\delta_3\right]}$$

$$\delta_1 = \delta_2 = \frac{-e^2}{(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{(\mu^2)^{2-d/2}} + \text{finite } \mu\text{-indep}$$

$$\delta_3 = -\frac{e^2}{(4\pi)^2}\,\frac{4}{3}\,\frac{\Gamma(2-\frac{d}{2})}{(\mu^2)^{2-d/2}} + \text{finite } \mu\text{-indep}$$

$$\beta_e = \mu\frac{\partial e}{\partial \mu} = \frac{e\mu}{2}\frac{\partial}{\partial\mu}\left[\delta_3\right] = \frac{e\mu}{2}\left(\frac{-e^2}{(4\pi)^2}\,\frac{4}{3}\,\frac{\Gamma(2-\frac{d}{2})}{(\mu^2)^{3-\frac{d}{2}}}\left[\frac{d}{2}-2\right]2\mu\right)$$

$$= \frac{e\mu}{2}\left(\frac{-e^2}{16\pi^2}\,\frac{4}{3}\right)2\mu\,\frac{1}{\mu^2}\,\underbrace{\frac{\Gamma(2-\frac{d}{2})}{(\mu^2)^{2-d/2}}}\left[\cancel{\frac{4}{2}} - \frac{\varepsilon}{2} - \cancel{2}\right]$$

$$= \frac{e^3}{12\pi^2}(-)\left[\frac{2}{\varepsilon} + \text{finite}\right]\left[-\frac{\varepsilon}{2}\right] = \frac{e^3}{12\pi^2}$$

$$\Longrightarrow \boxed{\beta_e = \mu\frac{\partial e}{\partial\mu} = \frac{e^3}{12\pi^2}} \qquad \alpha = \frac{e^2}{4\pi} \Longrightarrow \boxed{\beta_\alpha = \frac{2\alpha^2}{3\pi}} \ \checkmark$$

FIXED PTS: LEARN BY EX:

WILSON-FISHER

- ϕ^4 TURN ON MASS, STAY IN $4-\varepsilon$

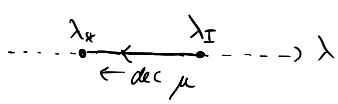
- $$\beta = -(4-d)\lambda + \frac{3\lambda^2}{16\pi^2} = -\varepsilon\lambda + \frac{3\lambda^2}{16\pi^2} = \lambda\left(\frac{3\lambda}{16\pi^2} - \varepsilon\right)$$

- NOTE $\lambda_{WF} = \frac{16\pi^2}{3}\varepsilon$ HAS $\beta = 0$!

Language: "Fixed Pt"
This one: Wilson-Fisher

- RG FLOW SUPPOSE λ_I @ μ_I ST $\left(\frac{3\lambda_I}{16\pi^2} - \varepsilon\right) > 0$ IE $\lambda_I > \lambda_{WF}$

$\beta\big|_{\lambda_I} > 0 \implies$ λ GROWS FOR μ INC
λ SHRINKS FOR μ DEC.



Language: "Flow to a fixed point"

BANKS-ZAKS

YM: $\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a}$ $\quad \beta(g) \sim -c g^3 + \mathcal{O}(g^5)$

YM + MATTER: $SU(N_c)$ QCD w/ N_f FLAVORS $\rightarrow$ N_f Dirac Fermions in Fund. Rep

@ 2-LOOPS

$$\beta = -\beta_0 \frac{g^3}{16\pi^2} + \beta_1 \frac{g^5}{(16\pi^2)^2} + \mathcal{O}(g^7)$$

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \qquad \beta_1 \sim \mathcal{O}(N_c^2, N_c N_f)$$

$N_c, N_f >> 1$ $\implies \frac{\beta_0}{\beta_1} << 1$, trust P.T.

$$\beta(g) = \frac{-g^3}{16\pi^2}\left(1 - \frac{\beta_1}{\beta_0}\frac{g^2}{16\pi^2} + \dots \text{HOT}\right) \qquad g_* = \sqrt{16\pi^2 \frac{\beta_0}{\beta_1}} \implies \beta\big|_{g_*} = 0$$

Language "Banks-Zaks Fixed Pt"