# QFT 1

## PART 1: Foundations

# PART 1 FOUNDATIONS: QFT & SCALARS OUTLINE

## PATH INTEGRALS

- backdrop for everything, developing the theory

## FREE FIELDS

- no interactions
- propagator

## FEYNMAN DIAGRAMS

- including interactions
- doing integrals, $\exists$ integral $\longleftrightarrow$ diagram dictionary

## CANONICAL QUANT.

- another way to develop QFT
- QM more obvious, bras, kets, etc.

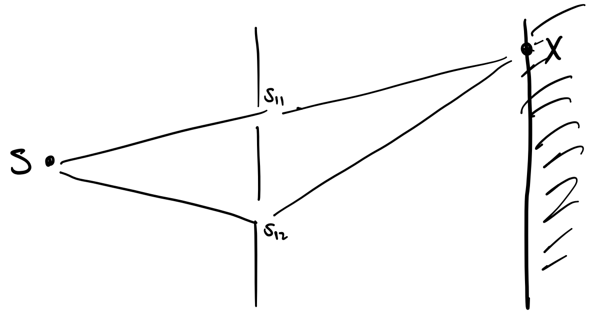
## SYMMETRIES

- Conservation laws when continuous
- determine selection rules in physical processes, eg. decays

# PATH INTEGRALS I.2

## DOUBLE SLIT EXPERIMENT

① FOR US: ONE SCREEN TWO SLIT EXP'T.



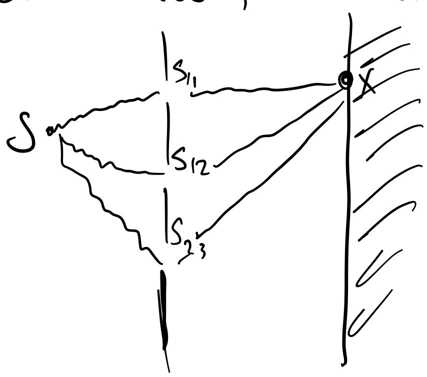

* S: SOURCE OF PARTICLES w/ P WELL-DEFINED

* MOVE X, GET INTERFERENCE PATTERN

$A$ = AMPLITUDE

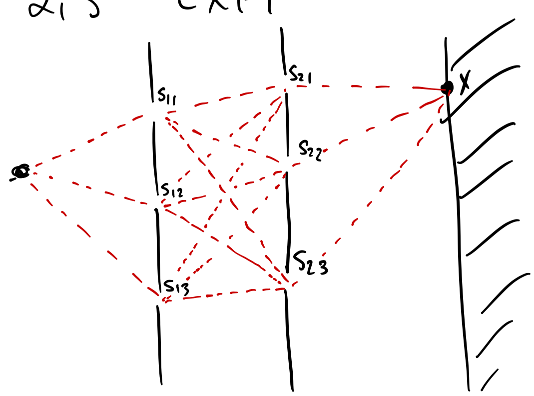
$$A(\text{DETECT } @\, X) = A(S \to S_{11} \to X) + A(S \to S_{12} \to X)$$

② ONE SCREEN, 3 SLIT




$$A(\text{DETECT } @\, X) = \sum_i A(S \to S_{1i} \to X)$$

③ 2, 3 EXPT




$$A(\text{DET } @\, X) = \sum_i \sum_j A(S \to S_{1i} \to S_{2j} \to X)$$

④ N SCREENS, M SLITS, $A(\text{DET } @\, X) = \sum_{\substack{\text{paths} \\ P}} A(S \xrightarrow{\text{via } P} X)$

# DIRAC'S FORMULATION (apparently Dirac beat Feynman to this?)

QM: AMP. TO PROP. FROM $q_I$ TO $q_F$ IN $T$ $= \langle q_F | e^{-iHT} | q_I \rangle$

GOAL: MANY SCREENS $\Rightarrow$ MANY TIME SLICES

$\delta t = T/N$. $\quad \langle q' | q \rangle = \delta(q' - q) \quad \int dq \, |q\rangle\langle q| = \mathbb{1}$

$$= \langle q_F | e^{-iHT} | q_I \rangle = \langle q_F | \overbrace{e^{-iH\delta t} \cdots e^{-iH\delta t}}^{N} | q_I \rangle$$

$$= \left( \prod_{j=1}^{N-1} \int dq_j \right) \langle q_F | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\delta t} | q_{N-2} \rangle \cdots \langle q_1 | e^{-iH\delta t} | q_I \rangle$$

EVAL $\langle q_{j+1} | e^{-iH\delta t} | q_j \rangle$ FREE PART. $H = \frac{\hat{p}^2}{2m}$, $\hat{p}|p\rangle = p|p\rangle$

$$= \langle q_{j+1} | e^{-i\delta t \, \hat{p}^2/2m} | q_j \rangle \overset{\mathbb{1}}{=} \int \frac{dp}{2\pi} \langle q_{j+1} | e^{-i\delta t (\hat{p}^2/2m)} | p \rangle \langle p | q_j \rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\delta t \, p^2/2m} \langle q_{j+1} | p \rangle \langle p | q_j \rangle = \int \frac{dp}{2\pi} e^{-i\delta t \frac{p^2}{2m}} e^{ip(q_{j+1} - q_j)}$$

?

RECALL $\langle q | p \rangle = e^{ipq}$ (position space wavefunction of mom. e-state is plane wave)

$$\text{ABOVE} \overset{\text{gauss. int.}}{=} \left( \frac{-im}{2\pi\delta t} \right)^{\frac{1}{2}} e^{[im(q_{j+1} - q_j)^2]/2\delta t} = \left( \frac{-im}{2\pi\delta t} \right)^{\frac{1}{2}} e^{i\delta t \left(\frac{m}{2}\right) [(q_{j+1} - q_j)/\delta t]^2}$$

$$\Rightarrow \langle q_F | e^{-iHT} | q_I \rangle = \left( \frac{-im}{2\pi\delta t} \right)^{\frac{N}{2}} \left( \prod_{k=1}^{N-1} \int dq_k \right) e^{i\delta t \left(\frac{m}{2}\right) \sum_{j=0}^{N-1} \left[ \frac{q_{j+1} - q_j}{\delta t} \right]^2}$$

CONTINUUM $\quad \delta t \to 0, \quad \left[ \frac{q_{j+1} - q_j}{\delta t} \right]^2 \to \dot{q}^2, \quad \delta t \sum_{j=0}^{N-1} \to \int_0^T dt$

DEFINE "INTEGRAL OVER PATHS" $\displaystyle \int Dq(t) := \lim_{N \to \infty} \left( \frac{-im}{2\pi\delta t} \right)^{\frac{N}{2}} \left( \prod_{k=1}^{N-1} \int dq_k \right)$

$$\Longrightarrow \quad \langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t)\, e^{i\int_0^T dt\, \frac{1}{2} m \dot{q}^2}$$

PHYSICS: AMPLITUDE = $\int$ ALL POSSIBLE PATHS BETW.

$$q(0) = q_I \qquad q(T) = q_F$$

ADD POTENTIAL $\quad H = \frac{\hat{p}^2}{2m} + V(\hat{q})$

same
$$\Longrightarrow \quad \langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t)\, e^{i\int_0^T dt\, [\frac{1}{2} m \dot{q}^2 - V(q)]} = \int Dq(t)\, e^{i\int_0^T dt\, L(\dot{q}, q)}$$

$$= \int Dq(t)\, e^{iS},$$

LAGRANGIAN L FROM CLASSICAL MECH.

$S =$ ACTION $\underbrace{\text{FUNCTIONAL}}$

FUNC. OF FUNCTIONS

## CLASSICAL LIMIT

- RESTORE $\hbar$ $\quad \langle q_F | e^{-iHT/\hbar} | q_I \rangle = \int Dq\, e^{iS/\hbar} = \int Dq\, e^{\frac{i}{\hbar} \int_0^T dt\, L(\dot{q}, q)} \qquad (*)$

- STEEPEST DESCENT (see app. 3)

$$\lim_{\hbar \to 0} (*) = e^{i/\hbar \int_0^T dt\, L(\dot{q}_c, q_c)}$$

$q_c(t) =$ CLASSICAL PATH $:=$ SOL'N TO EULER-LAGRANGE EQN $\left(\frac{d}{dt}\right)\left(\frac{\delta L}{\delta \dot{q}}\right) - \left(\frac{\delta L}{\delta q}\right) = 0$

DERIV.
$$0 = \delta S = \int_0^T dt \left[ L(\dot{q} + \delta\dot{q}, q + \delta q) - L(\dot{q}, q) \right]$$

$$= \int_0^T dt \left[ \delta\dot{q} \frac{\delta L}{\delta \dot{q}} + \delta q \frac{\delta L}{\delta q} \right] \overset{IBP}{=} \int_0^T \delta q \underbrace{\left[ -\frac{d}{dt} \frac{\delta L}{\delta \dot{q}} + \frac{\delta L}{\delta q} \right]}_{=0 \,-\, E\text{-}L}$$

QFT N.B. (1) SOMETIMES $\exists$ MULTIPLE "CL. PATHS"

(eg topological defects)

(cl. paths typically separated in space of paths)

(2) PERTURB. THY = AROUND CL. SOL.

$\Downarrow$

OTHER SOLN'S = "NON-PERT EFFECTS" (N.P.)

IF NOT FEYNMAN DIAG.

$\Longrightarrow$ $\boxed{\text{QFT} \supsetneq \text{FEYNMAN DIAGRAMS}}$

(3) SOMETIMES $\exists$ N.P. EFFECTS THAT ARE LEADING

(e.g. mass of axion particles)

PERT. THY & FEYNMAN DIAG. = PHYSICS ASSOC. w/ GAUSSIAN INTEGRALS

# GAUSSIAN INTEGRALS (I. APP. 2)

(doing this instead of just having you read it for reasons you will see)

**SIMPLEST** $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} = \sqrt{2\pi}$

**IMPLIES** $\displaystyle Z := \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}}$ (by $x \longrightarrow x/\sqrt{a}$)

**COMP.** **THE** **SQ.** $\displaystyle\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 + Jx} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{J^2/2a}$

$$\langle x^{2n} \rangle := \frac{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2}} = \frac{\left(-2\frac{d}{da}\right)^n Z}{Z} = (2n-1)!!\,\frac{1}{a^n}$$

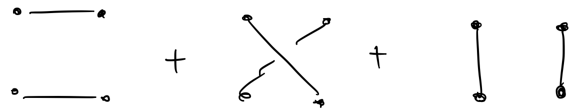
WHERE $(2n-1)!! = (2n-1)(2n-3)\ldots 5\cdot 3\cdot 1$

# WICK'S THEOREM

("theorem" in QFT lit. mountain out of molehill?)

**THM:** $(2N-1)!! = \#$ WAYS TO CONNECT $2N$ POINTS IN PAIRS

EG: $n=2$



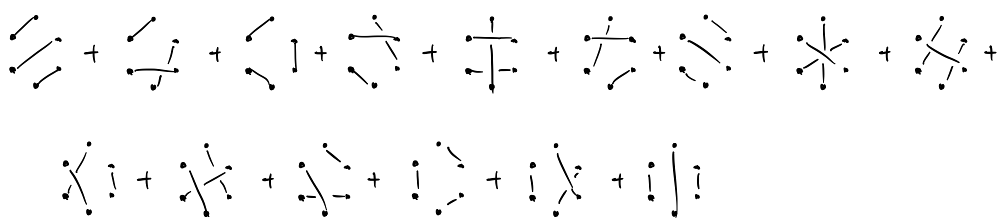
$\overset{?}{=}$ $\langle x^4 \rangle$

HOW!? BY RULE —— $= \frac{1}{a}$

CONJECTURE "$\langle x^{2n} \rangle$ RULES" (a la Feynman)

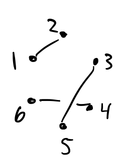
1. CONNECT $2n$ •'s IN ALL WAYS
2. $\frac{1}{a}$ FOR EACH LINE
3. SUM

EG $N=3$



each has 3 —'s $\Rightarrow \frac{1}{a^3}$. CONJ $\Rightarrow \langle x^6 \rangle = \sum_{i=1}^{15} \frac{1}{a^3} = \frac{15}{a^3} = \left.\frac{(2n-1)!!}{a^3}\right|_{n=3}$ ✓

ALT NOTATION:





$= \langle xxxxxx \rangle$

"CONTRACTION"

OR "WICK CONTRACTION"

(Giancarlo Wick)

$\langle x^6 \rangle \longrightarrow \langle xxxxxx \rangle$ & CONNECT THE X'S.

simple ex: each W.C. gives same contrib

- soon: different W.C. different contrib

$\boxed{\text{ESSENCE OF FEYNMAN DIAGRAMS!}}$

# INTERACTIONS (can we see mathematically where they come from?)

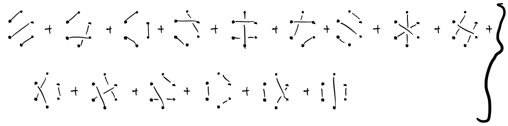
**NOW**
$$\frac{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2+\lambda x^4}\, x^{2m}}{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}} = \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}\, \frac{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}\, x^{4\ell+2m}}{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}}$$

**PERT. THEORY:** $\lambda \ll 1 \Rightarrow$ KEEP FIRST FEW $\ell$'s

**CASE:** $m=1$, $\ell=0,1$

**$\ell=0$**
$$\frac{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}\, x^{2}}{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}} = \frac{1}{a} =$$

**$\ell=1$**
$$\lambda \left\{ \frac{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}\, x^{6}}{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}} \right\} = \lambda \left\{ \vphantom{\int} \right.$$



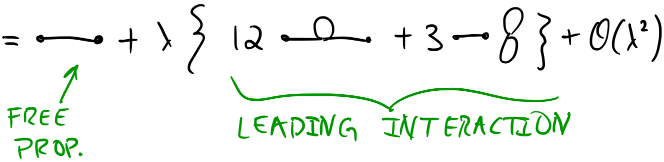
$\left. \vphantom{\int} \right\}$

**REWRITE:** (some x's from upstairs, some downstairs)

$\Rightarrow$ LABEL $\underbrace{u}_{\text{UP}}$, $\underbrace{1, 2}_{\text{DOWN}}$ (same as in x-space, just extra properties)


**ABOVE** =


$$\Longrightarrow \frac{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2+\lambda x^4}\, x^{2}}{\int_{-\infty}^{\infty} dx\; e^{-\frac{1}{2}ax^2}} =$$



# MATRIX GAUSS. INT.

$x \longmapsto x_j \qquad a \longmapsto A_{ij}$

w/ Einstein summation

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 + Jx} \longmapsto \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N\, e^{-\frac{1}{2}\underbrace{x\cdot A\cdot x}_{x_i A_{ij} x_j} + \underbrace{J\cdot x}_{J_i x_i}} = \left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}} e^{\frac{1}{2} J\cdot A^{-1}\cdot J}$$

just diagonalize matrix, do each integral. Possible if $\det A \neq 0$.

$$Z := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N\, e^{-\frac{1}{2} x\cdot A\cdot x}$$

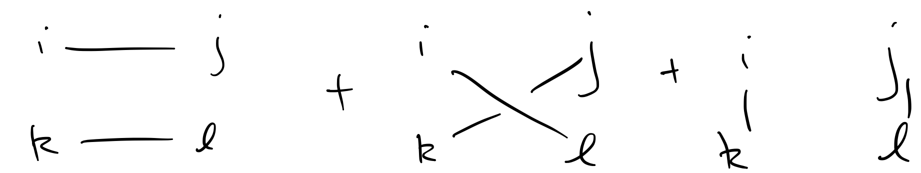
$$\langle x_i x_j \rangle := \frac{-2\frac{d}{dA_{ij}}(Z)}{Z} = \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N\, e^{-\frac{1}{2} x\cdot A\cdot x}\, x_i x_j}{Z} = A^{-1}_{ij}$$

$$\Longrightarrow \langle x_i x_j \cdots x_k x_l \rangle = \frac{\int\int dx_1 \cdots dx_N\, e^{(\;)}\, x_i x_j \cdots x_k x_l}{Z}$$

$$= \sum_{\text{Wick}} (A^{-1})_{ab} \cdots (A^{-1})_{cd}$$

SUM OVER ALL WICK CONTR. = SUM OVER ALL PERMS $(a,b,\ldots,c\,d)$ OF $(i,j,\ldots,k,l)$

EG $\langle x_i x_j x_k x_l \rangle = (A^{-1})_{ij}(A^{-1})_{kl} + (A^{-1})_{il}(A^{-1})_{jk} + (A^{-1})_{ik}(A^{-1})_{jl}$

= 


W/ NEW RULE $\quad i \bullet\!\!-\!\!\!-\!\!\bullet j = (A_{ij})^{-1}$

## DIFFERENT STATES

$$\langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t)\, e^{i\int_0^T dt\, L(\dot{q}, q)}$$

## CHANGE STATES

$$\langle F | e^{-iHt} | I \rangle = \int dq_I \int dq_F \, \langle F | q_F \rangle \langle q_F | e^{-iHT} | q_I \rangle \langle q_I | I \rangle$$

$$= \int dq_I \, dq_F \, \psi_F^*(q_F)\, \psi_I(q_I)\, \langle q_F | e^{-iHT} | q_I \rangle$$

$|0\rangle :=$ GROUND STATE (energy minimizing; in QFT vacuum 0-particle st)

$$Z := \langle 0 | e^{-iHT} | 0 \rangle = \int dq_I \int dq_F \, \langle 0 | q_F \rangle \langle q_I | 0 \rangle \underbrace{\langle q_F | e^{-iHT} | q_I \rangle}$$

Zee I.3 (1)
calls Z, notes "dropped factors"

# DISCRETE & CONTINUUM SYSTEMS

Zee drops factors here, see previous page.

## SINGLE PARTICLE

$$Z := \int Dq(t)\, e^{i\int_0^T dt\,[\frac{1}{2}m\dot{q}^2 - V(q)]}$$

## N-PARTICLE

$\hat{q} \longrightarrow \hat{q}_I$ N POS. OPS

$$H = \sum_a \frac{\hat{p}_a^2}{2m_a} + V(\hat{q}_1, \hat{q}_2, \dots, \hat{q}_N)$$

(same steps) $\Longrightarrow$

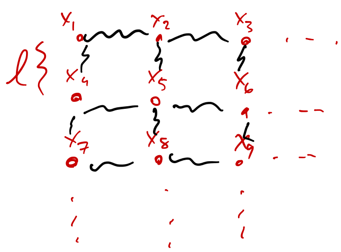
$$Z := \langle 0 | e^{-iHT} | 0 \rangle = \int Dq(t)\, e^{iS(q)}$$

$$S(q) = \int_0^T dt\, \Big( \sum_a \tfrac{1}{2} m_a \dot{q}_a^2 - V[q_1, \dots, q_N] \Big)$$

## "MATTRESS"

- ZEE: $q_a(t)$ HEIGHT OF SITE $a$ @ TIME $t$
- COULD BE OTHER THINGS, EG TEMP, HIGGS VEV, ETC

$$L = \frac{1}{2}\Big( \sum_a m\dot{q}_a^2 - \sum_{a,b} k_{ab}\, q_a q_b - \sum_{abc} g_{abcd}\, q_a q_b q_c q_d \Big)$$




Harmonic approx $g_{abcd} + \text{H.O.T} \Longrightarrow 0$

$$V(q_1, \dots, q_N) = \sum_{ab} \tfrac{1}{2} k_{ab} (q_a - q_b)^2$$

**EOM** $$0 = \frac{d}{dt}\frac{dL}{d\dot{q}_a} - \frac{dL}{dq_a} = m\ddot{q}_a + k_{ab} q_b$$

(solve in eigenbasis)
(sols waves that just pass through each other)

## CONTINUUM MATTRESS

$$\lim_{l\to 0} q_a(t) \longmapsto \varphi(t, \vec{x}), \quad \sum_a \longmapsto \int \frac{d^2x}{l^2}, \quad \sum_a \tfrac{1}{2} m\dot{q}_a^2 \longrightarrow \int d^2x\, \frac{m}{l^2}\, \tfrac{1}{2}\dot{\varphi}^2 =: \int d^2x\, \frac{\sigma}{2}\dot{\varphi}^2$$

$$\lim_{l\to 0} \frac{(q_a - q_b)^2}{l^2} = \Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \Big(\frac{\partial\varphi}{\partial y}\Big)^2$$

$$\Rightarrow S(q) \longrightarrow S(\varphi) := \int_0^T dt \int d^2x\, \mathcal{L}(\varphi)$$

$\rho$ DEP ON $k_{ab}$, $l$, etc

$$= \int_0^T dt \int d^2x\, \tfrac{1}{2}\Big\{ \sigma\Big(\frac{\partial\varphi}{\partial t}\Big)^2 - \rho\Big[\Big(\frac{\partial\varphi}{\partial x}\Big)^2 + \Big(\frac{\partial\varphi}{\partial y}\Big)^2\Big] - \tau\varphi^2 - \varsigma\varphi^4 + \dots \Big\}$$

REWRITE $\rho = \sigma c^2$, $\varphi \longrightarrow \varphi/\sqrt{\sigma}$, $c = 1$ $\Longrightarrow$ LOR. INV.

CONSTS

$$\frac{g}{3!} := \frac{\tau}{\sigma}, \quad \frac{\lambda}{4!} := \frac{\varsigma}{\sigma^2}$$

$$\Rightarrow S = \int d^dx\, \Big[ \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}m^2\varphi^2 - \frac{g}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 + \dots \Big]$$

$$(\partial\varphi)^2 := \partial_\mu\varphi\, \partial^\mu\varphi$$

- $d = D + 1$
  - $\hookrightarrow$ # SP. DIMS

$$\boxed{Z = \int D\varphi \, e^{i\int d^dx \left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right]}}$$

"SCALAR FIELD THEORY"

NOTE $V(\varphi)$ EVEN $\Rightarrow$ $\mathcal{L}$-symm $\varphi \longrightarrow -\varphi$

"D+1 DIM QFT"

$d = D+1$

3+1 PART. PHYS.

2+1 C.M.

1+1 C.M. + STRINGS

0+1 QM (ie $Z \longrightarrow Z_{QM}$ for $D=0$)

## CLASSICAL FIELD THEORY

(Q's: how do we take the classical limit of QFT?)

$$\lim_{\hbar\to 0} \int D\varphi \, e^{iS/\hbar} \Longrightarrow \delta S = 0 \text{ Configs Dominate}$$

$$S[\varphi, \partial^\mu\varphi] = \int d^dx \, \mathcal{L}[\varphi, \partial^\mu\varphi]$$

VARY $\varphi$ $\tilde{\varphi} = \varphi + \delta\varphi$ INFINITESIMAL

$\delta(\partial_\mu\phi) = \partial_\mu\delta\phi$

$$\underbrace{S[\tilde{\varphi}, \partial^\mu\tilde{\varphi}]}_{S+\delta S} = \int d^dx \, \mathcal{L}[\tilde{\varphi}, \partial^\mu\tilde{\varphi}] = \int d^dx \left[\mathcal{L}[\varphi, \partial^\mu\varphi] + \delta\phi \frac{\delta\mathcal{L}}{\delta\varphi} + (\partial_\mu\delta\phi)\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\right]$$

$$= \int d^dx \left[\mathcal{L}[\varphi, \partial^\mu\varphi] + \delta\phi\left[\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right]\right]$$

UP TO BOUNDARY

$$\Longrightarrow \delta S = \int d^dx \left(\mathcal{L}[\tilde{\varphi}, \partial_\mu\tilde{\varphi}] - \mathcal{L}[\varphi, \partial_\mu\varphi]\right) = \int d^dx \, \delta\phi\left[\frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\right]$$

$\Downarrow$

CLASSICAL CONFIGS SOLVE $\delta S = \boxed{0 = \frac{\delta\mathcal{L}}{\delta\varphi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}}$

Euler-Lagrange Eqn

EX FREE SCALAR FIELD

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2$$

$$\Rightarrow \underbrace{\partial_\mu\partial^\mu}_{=:\,\Box \text{ "BOX"}}\phi + m^2\phi \Longrightarrow \boxed{(\Box + m^2)\phi = 0}$$

"KLEIN GORDON EQN"

EX $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $A_\mu = A_\mu(x)$ VECTOR FIELD

EXPAND $\Rightarrow$ $\mathcal{L} \propto \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu$

E-L $\Rightarrow$ $0 = -\partial_\rho \frac{\delta \mathcal{L}}{\delta \partial_\rho A_\sigma} + \cancel{\frac{\delta \mathcal{L}}{\delta A_\sigma}}^{\to 0} \propto -\partial_\rho \delta^\rho_\mu \delta^\sigma_\nu \partial^\mu A^\nu + \partial_\rho \delta^\rho_\mu \delta^\sigma_\nu \partial^\nu A^\mu$

$= \partial_\rho [\partial^\rho A^\sigma - \partial^\sigma A^\rho] = \partial_\rho F^{\rho\sigma}$ $\Rightarrow$ $\partial_\mu F^{\mu\nu} = 0$

MODIFY $\mathcal{L} \to \mathcal{L} + A^\mu J_\mu$ $\overset{EL}{\Rightarrow}$ $\boxed{\partial_\mu F^{\mu\nu} = J^\mu}$

"MAXWELL'S EQN'S"

(technically ½ of them)

EXCITATIONS: (EG PARTICLES)

Q: HOW DO WE DISTURB THE VACCUUM $|0\rangle$?

- HILBERT SPACE FORM: $|\psi_{particle}\rangle = a^\dagger |0\rangle$
- PATH INTEGRAL FORM: ADD SOURCE FUNCTION $J$

↳ analogy above: charge densities & currents are photon sources

MATTRESS: WANT: t-DEP ADD. OF EN @ SITE a

- $V(q_1, \dots, q_N) \longrightarrow V + J_a(t) q_a$
- MORE GEN.: $\Delta V = \sum_a J_a(t) q_a$

can be different for different a's!

CONTINUUM: $\Sigma_a J_a(t) q_a \longmapsto \int d^D x \, J(x) \varphi(x)$

so $\boxed{Z[J] = \int D\varphi \, e^{i \int d^4x \left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + J(x)\varphi(x)\right]}}$

function of x, just as $J^\mu = J^\mu(x^\mu) = \begin{pmatrix} \rho(x^\mu) \\ \vec{J}(x^\mu) \end{pmatrix}$

# Free Field Theory

# FREE FIELD THEORY

**FACT:** - $Z[J]$ HAS NO KNOWN EXACT SOL.

**IDEA:** - STUDY CASES WHERE $Z_{solv}[J]$ + PERT

SOLVABLE CASE "FREE FIELDS" $Z = \int D\varphi\, e^{i\int d^4x \left[\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\right]}$

**EOM:** KLEIN-GORDON EQN $(\Box + m^2)\phi = 0$

$\Rightarrow$ CLASSICAL SOL'NS $\supset$ $\varphi(x,t) = e^{i(\omega t - \vec{k}\cdot\vec{x})}$

w/ $\omega^2 = \vec{k}^2 + m^2$

REST FRAME $\omega^2 = m^2 \overset{\text{SINCE } \hbar = c = 1}{\Longleftrightarrow} E = mc^2$

(this is why we write $m^2\varphi^2$ instead of some other coeff. It's the mass!)

ADD SOURCE $\Longrightarrow$ HWK 1: will see $J_\mu$ in EM literally the source

$$Z = \int D\varphi\, e^{i\int d^4x \left\{\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] + J\varphi\right\}} \overset{IBP}{=} \int D\varphi\, e^{i\int d^4x \left[-\frac{1}{2}\varphi\underbrace{(\Box + m^2)}_{:= -A}\varphi + J\varphi\right]}$$

NB: $(\Box + m^2)\phi = J$ (source for $\phi$-particles)

NB: GAUSSIAN + LINEAR $\Rightarrow$ COMPLETE THE SQ.

did discrete version previously:

$$Z = \left(\frac{(2\pi i)^N}{\det[A]}\right)^{\frac{1}{2}} e^{-\frac{i}{2} J\cdot A^{-1}\cdot J}$$

DISCRETE INVERSE $A\cdot A^{-1} = A_{ij} A^{-1}_{jk} = \delta_{ik}$

CONT. INV $A\,A^{-1} = \delta^4(x-y)$

$$\overset{!!}{=} -(\Box + m^2)D(x-y)$$

$$W(J) := -\frac{1}{2}\iint d^4x\, d^4y\, J(x) D(x-y) J(y)$$

$$Z[J] := C e^{iW[J]} = Z(J=0)\, e^{iW[J]}$$

## INTERLUDE

SOURCE $J$, FIELD $F$, OP $\Theta$.

RECALL $\langle x^{2n}\rangle = \dfrac{\int dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int dx\, e^{-\frac{1}{2}ax^2}} = (2n-1)!!\, \dfrac{1}{a^n} = (2n-1)!!\,(a^{-1})^n$

RULE $\bullet\!\!-\!\!\bullet = a^{-1}$

FT WILL HAVE $\sim \dfrac{\int DF\, e^{-\frac{1}{2}F\Theta F} F^{2n}}{\int DF\, e^{-\frac{1}{2}F\Theta F}}$

SIM $\bullet\!\!-\!\!\bullet = \Theta^{-1} \Longrightarrow \Theta^{-1} = \text{PROPAGATOR}$

(other contexts:
- n-point function has only $D(x-y)$ $x \neq y$ factors for free fields)
- canonical quant: literally amplitude for particle to propagate)

## FREE PROPAGATOR

HERE $A^{-1} = D(x) = \text{PROPAGATOR}$

Q: WHAT IS ITS FORM & PHYSICS?
(inverse of diff op, some kind of Green's function!)

PRECISION $Z[J]$ DOESN'T CONVERGE

SOL'N: "$i\varepsilon$ PRESCRIPTION"

1. $m^2 \to m^2 - i\varepsilon \Longrightarrow Z$ GETS $e^{-\frac{\varepsilon}{2}\int d^4x\, \varphi^2}$, $\varepsilon \in \mathbb{R}$ $0 < \varepsilon \lll 1$
2. COMP. $Z$
3. TAKE $\lim_{\varepsilon \to 0}$

PROPAGATOR

CLAIM: $\boxed{D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon}}$ ↗ pole for real $k$ w/o $i\varepsilon$!

CHECK $-(\Box + m^2) D = \int \frac{d^4k}{(2\pi)^4} \frac{k^2 - m^2}{k^2 - m^2 + i\varepsilon} e^{ik\cdot(x-y)} \overset{\lim}{=} \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} = \delta^4(x-y)$

↳ matches definition of $D$ as inv. of $-(\Box + m^2)$

EVALUATE $D(x-y)$ $\int dk_0$ BY CONTOURS

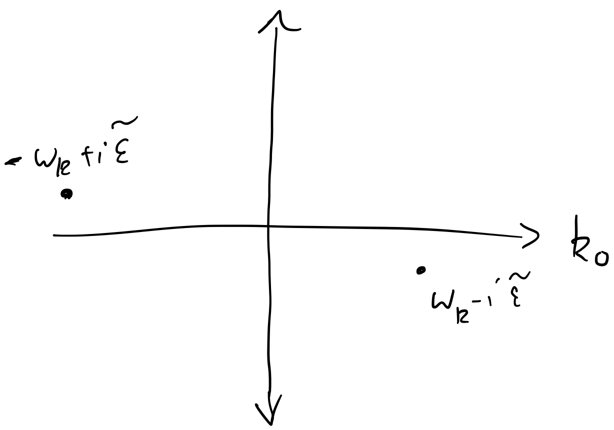
$$\omega_k := +\sqrt{\vec{k}^2 + m^2}$$

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k_0^2 - \omega_k^2 + i\varepsilon} = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk_0}{2\pi} \frac{e^{ik_0(x^0-y^0)}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{k_0^2 - \omega_k^2 + i\varepsilon}$$

POLES @ $k_0 = \pm\sqrt{\omega_k^2 - i\varepsilon} = \pm\sqrt{\omega_k^2\left(1 - \frac{i\varepsilon}{\omega_k^2}\right)}$

$$\overset{\varepsilon\to 0}{=} \pm\,\omega_k\left(1 - \frac{i\varepsilon}{2\omega_k^2}\right) =: \pm\left(\omega_k - i\tilde{\varepsilon}\right)$$

just replace one infinitesimal thing by another




Q: WHICH WAY TO CLOSE CONTOURS?

CASE: $x^0 - y^0 > 0$ $\Rightarrow$ $e^{ik_0(x_0-y_0)}$ EXP. DAMPED FOR $\mathrm{Im}(k_0) > 0$

$y^\mu$ BEFORE $x^\mu$

$\Rightarrow$ CLOSE ABOVE

CASE: $x^0 - y^0 < 0$ $\Rightarrow$ CLOSE BELOW

$x^\mu$ BEFORE $y^\mu$

evaluating the contour integral gives, after taking $\varepsilon \to 0$

$$\boxed{D(x-y) = -i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[e^{-i(\omega_k t - \vec{k}\cdot(\vec{x}-\vec{y}))}\,\Theta(x^0-y^0) + e^{i(\omega_k t + \vec{k}\cdot(\vec{x}-\vec{y}))}\,\Theta(y^0-x^0)\right]}$$

PHYSICS DESC. PROP FROM X TO y OR OPP.

(will see appear again in canonical quantization)

PROPERTIES

TIMELIKE: CASE $\vec{x}-\vec{y}=0$

$$D(x-y)\Big|_{\vec{x}-\vec{y}=0} = -i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\, e^{-\mathrm{sign}(x^0-y^0)\, i\omega_k (\underbrace{|x_0-y_0|}_{\Delta t})}$$

OSC. DIR DEP ON sign $(x^0-y^0)$!

SPACELIKE SEP: $\vec{x}-\vec{y}\neq 0$ $x^0=y^0$

CAUSALITY ✓ (see homework problem)

# THE PATH INTEGRAL ORIGIN OF PARTICLES & FORCE (I.4-I.5)

RECALL $Z(J) = Z(J=0)\, e^{iW[J]}$, $W(J) = -\frac{1}{2}\iint d^4x\, d^4y\; J(x)\, D(x-y)\, J(y)$

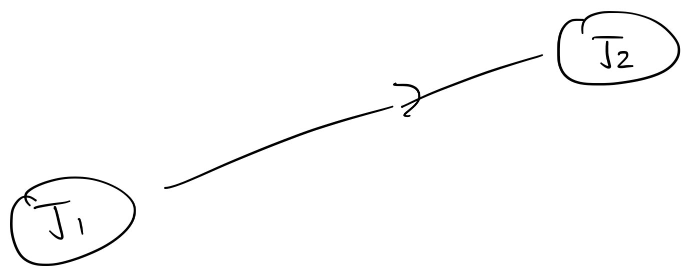
$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2-m^2+i\varepsilon}$$

NOW F.T. $\Rightarrow$ $\tilde{J}(k) := \int d^4x\, e^{-ikx} J(x)$

$$W(J) = -\frac{1}{2}\int d^4x\, d^4y\, d^4k\; J(x) \frac{e^{ik\cdot(x-y)}}{(2\pi)^4[k^2-m^2+i\varepsilon]} J(y)$$

$$= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{J}(-k) \frac{1}{k^2-m^2+i\varepsilon} \tilde{J}(k) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{J}(k)^* \frac{1}{k^2-m^2+i\varepsilon} \tilde{J}(k)$$

CAN CHOOSE J! TAKE $J = J_1 + J_2$



(source-sink in position space)

F.T.R. VERSION

LET $W = W_{11} + W_{12} + W_{21} + W_{22}$ $\Longrightarrow$ $W_{21} = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{J}_2^*(k) \frac{1}{k^2-m^2+i\varepsilon} \tilde{J}_1(k)$

$W_{21}$ IS LARGE IF:

a) $J_1(x)$ $J_2(x)$ OVERLAP SIGNIF.

b) IN OVERLAP REGION, $k^2-m^2 \approx 0$

Cor: SPIKE OR "RESONANCE" FOR $k^2 = m^2$

PARTICLE EN-MOM

LANGUAGE ① $k^2 = m^2$ "ON-SHELL" PARTICLE

② $k^2 \neq m^2$ "OFF-SHELL" OR "VIRTUAL"

**FORCE:** WANT $\varphi$ LOCALIZED

$\Rightarrow$ CONSIDER $J = J_1 + J_2 := \delta^{(3)}(\vec{x}-\vec{x}_1) + \delta^{(3)}(\vec{x}-\vec{x}_2)$

AGAIN $W = W_{11} + W_{12} + W_{21} + W_{22}$

$W_{11}, W_{22}$ $\longrightarrow$ EXIST EVEN W/O $J_1$ OR $J_2$

$\Rightarrow$ "PHYSICS OF FORCE" FROM CROSS TERMS

note superscript since no Mink –

$$W_{12} + W_{21} = 2 \cdot \left(-\tfrac{1}{2}\right) \iint d^4x\, d^4y\; \delta^3(\vec{x}-\vec{x}_1)\, \delta^3(\vec{x}_1-\vec{x}_2) \int \frac{dk^0}{2\pi} \frac{d^3k}{(2\pi)^3} \frac{e^{ik^0(x-y)^0}\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{k^2-m^2+i\varepsilon}$$

since $W_{21} = W_{12}$

$$= -\iint dx^0 dy^0 \int \frac{dk^0}{2\pi} e^{ik^0(x^0-y^0)} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{k^2-m^2+i\varepsilon}$$

$$= \int dx^0 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2}$$

eval $x^0$ or $y^0$ integral, sign from West Coast metric

**RECALL** $Z(J) = \langle 0| e^{-iH(J)T} |0\rangle = \langle 0|0\rangle e^{-iE(J)T} = \mathcal{C}\, e^{iW(J)}$

J modified V, and $\therefore$ H

$\Rightarrow$ $iW = -iET$. $\int dx^0$ LIKE $T$

$$\Rightarrow E = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2}$$

$r := |\vec{x}_1 - \vec{x}_2|$ $\Rightarrow$ $E = E(r) = -\dfrac{e^{-mr}}{4\pi r}$ (see derivation in appendix)

"Yukawa Potential" $\Rightarrow$ BET. SOURCES THAT INT. VIA MASSIVE $\varphi$

(proposed to desc. $\pi^0$ interactions, we see from scalar field theory!)

**PHYSICS** 1) $m \neq 0$ FOR $r \gg \frac{1}{m}$, DIES EXP'LY "SHORT RANGE FORCE"

2) $m = 0$ $\Rightarrow$ $F \sim \dfrac{1}{r^2}$ "LONG RANGE FORCE"

(note: $\varphi$ prop & this J gave force)

$\boxed{\text{PARTICLES AS FORCE CARRIERS!}}$

# REPULSION & COULOMB

$$\mathcal{L}_{EM} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\xRightarrow{\text{mass, source}} \mathcal{L} = \mathcal{L}_{EM} + \tfrac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu$$

**NB:** QED = $\mathcal{L}$ w/ $m=0$, $J=$ CHARGED FERM. CURRENT

**ASSUME:** $J_\mu$ SAT. $\partial_\mu J^\mu = 0$ (conserved current, will derive in G.I. lectures)

$$S(A) = \int d^4x\, \mathcal{L} = \int d^4x \left\{ \tfrac{1}{2} A_\mu \underbrace{\left[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu\right]}_{\text{INVERT}} A_\nu + A_\mu J^\mu \right\}$$ ⟸ HMWK: DERIVE

IE FIND $D_{\nu\lambda}$ S.T. $\underbrace{\left[(\partial^2+m^2) g^{\mu\nu} - \partial^\mu \partial^\nu\right]}_{\mathcal{O}} \underset{\mathcal{O}^{-1}}{D_{\nu\lambda}} = \underbrace{\delta^\mu_\lambda\, \delta^4(x)}_{\text{IDENTITY}}$

$$\xRightarrow{FT} D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + k_\nu k_\lambda / m^2}{k^2 - m^2}$$ ⟸ HMWK: DERIVE

**PREV:** SCALAR $\Rightarrow W(J) = -\tfrac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^*(k)\, D(k)\, J(k)$

**NOW:** VECTOR $\Rightarrow W(J) = -\tfrac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k^*)\, D_{\mu\lambda}(k)\, J^\nu(k)$

$$0 = \partial_\mu J^\mu(x) = \int \frac{d^4k}{(2\pi)^4} \partial_\mu \left[J^\mu(k)\, e^{ik\cdot x}\right] \Rightarrow k_\mu J^\mu(k) = 0$$

$\Rightarrow$ $k^\mu k^\nu$ TERMS IN $W = 0$

SO $W = +\tfrac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_\mu(k)$

**COMPARE** SCALAR $W(J) = -\tfrac{1}{2} \int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J(k)$

- AND CALC $\Rightarrow$ ATTRACT

**VECTOR** SOURCE OF "SAME TYPE" = SAME CHARGE

$W \sim +$ AND CALC $\Rightarrow$ REPEL

**FORCE/POT** $E = \frac{1}{4\pi r} e^{-mr} \xrightarrow{m\to 0} \frac{1}{4\pi r} \Rightarrow \boxed{\text{COULOMB'S LAW}}$

# DISCOVERING MAXWELL (I.5 APP 1)

GOAL: DERIVE $\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

INGREDIENTS:

① SPIN 1 & LORENTZ COVARIANT $\Longrightarrow$ 3 DOF. $A_\mu(x)$ SMALLEST LOR. COV. OBJECT w $\geq 3$ DOF

② VECTOR w/ MASS m

$\Longrightarrow (\Box + m^2) A_\mu = 0$ (think: plane wave $\varepsilon_\mu e^{ikx}$ solution $k^2 = -m^2$

③ $A_\mu \Rightarrow$ 4 DOF
SPIN 1 $\Rightarrow$ 3 DOF $\rangle \rightarrow$ CONSTRAINT $\partial_\mu A^\mu = 0$
(only Lorentz covariant poss. linear in $A_\mu$)

④ NEED: SING. EQN THAT $\Rightarrow (\Box + m^2) A^\mu = 0$ $\partial_\nu A^\nu = 0$

COMBINE $\left(g^{\mu\nu} \partial^2 - B \partial^\mu \partial^\nu\right) A_\nu + m^2 A^\mu = 0$ ⊛

NB - $\partial_\mu \cdot \longrightarrow \left(\partial^\nu \partial^2 - B \partial^2 \partial^\nu\right) A_\nu + m^2 \partial_\mu A^\mu = 0$

- GET $\partial_\mu A^\mu = 0$ if $B = 1$
- ⊛ w/ $B = 1$, $\partial^\nu A_\nu = 0 \Rightarrow 0 = g^{\mu\nu} \partial^2 A_\nu + m^2 A^\mu = (\partial^2 + m^2) A^\mu$

⑤ $\frac{1}{2} A_\mu \cdot$ (⊛ w $B=1$)

$= \frac{1}{2} A_\mu \left[ (\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu \right] A_\nu = \frac{1}{2} A_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\mu + \frac{1}{2} m^2 A^2$

$F_{\mu\nu} F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = 2\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right]$

$\overset{IBP}{=} -2 \left[ A_\nu \left(\partial^2 - \partial^\mu \partial^\nu\right) A_\mu \right]$

$\longrightarrow = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + m^2 A^2 =: \mathcal{L}$

$\Longrightarrow \mathcal{L}$ REP $m \neq 0$ EOM & $\partial_\mu A^\mu = 0$

⑥ PHOTON: $m = 0$ $\mathcal{L}_{EM} := \lim_{m \to 0} \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ ✓

# FORCES IN HIGHER DIM

RECALL: PREVIOUSLY $V(r) \sim \int d^3k \, e^{i\vec{k}\cdot\vec{x}} / k^2$. $\quad e^{i\vec{k}\cdot\vec{x}} = e^{ikr\cos\theta} \Rightarrow k \sim \frac{1}{r}$

$$\Rightarrow V(r) \sim [d^3k]\left[\frac{1}{k^2}\right] \sim \left[\frac{1}{r^3}\right][r^2] \sim \frac{1}{r}$$

NOW: $n$ EXTRA DIM $\quad x^4, \dots, x^{n+3}$

$$\Rightarrow V(r) \sim \int d^{n+3}k \, \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2}$$

$$\Rightarrow V(r) \sim [d^{n+3}k]\left[\frac{1}{k^2}\right] \sim \left[\frac{1}{r^{n+3}}\right][r^2] \sim \frac{1}{r^{n+1}}$$

$$\Rightarrow F \sim \frac{1}{r^{n+2}} \quad \text{IN } n+3 \text{ DIM!}$$

# Feynman Diagrams (1.7)

*Behold, the power of the path integral!*

PREVIOUSLY $Z(J) = \int \mathcal{D}\varphi \, e^{i\int d^4x \{\frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2\} + J\varphi}$

- GAUSSIAN $\Rightarrow$ SOLVE EXACTLY $\Rightarrow$ PROPAGATOR

NOW: CHANGE $\mathcal{L}$

$$\Rightarrow Z[J] = \int \mathcal{D}\varphi \, e^{i\int d^4x \{\frac{1}{2}[\partial\varphi^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4 + J\varphi}$$

CAN'T SOLVE? PERTURB IN $\lambda$. (sometimes approximate solutions are good enough!)

WARMUP try the ordinary integral

$$Z(J) = \int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq} = \int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 + Jq}\left[1 - \frac{\lambda}{4!}q^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 q^8 + \dots\right]$$

TRICK: $\int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 + Jq} q^{4n} = \left(\frac{d}{dJ}\right)^{4n} \int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 + Jq}$

$$\longrightarrow Z(J) = \left(1 - \frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8 + \dots\right)\int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 + Jq}$$

$$= e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} \int_{-\infty}^{\infty} dq \, e^{-\frac{1}{2}m^2q^2 + Jq}$$

$$= \left(\frac{2\pi}{m^2}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} e^{\frac{J^2}{2m^2}}$$

NOTE: - DOUBLE SERIES, IN $J$ & $\lambda$

- GET RID OF $\left(\frac{2\pi}{m^2}\right)^{\frac{1}{2}} = Z(J=0, \lambda=0) =: Z(0,0)$

BY $\tilde{Z} = \frac{Z(J)}{Z(0,0)}$ (common to all terms, so can divide through)

EXPAND $\lambda^{N_\lambda} J^{N_J}$ TERM $= \frac{1}{N_\lambda!}\left(-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4\right)^{N_\lambda} \frac{1}{\left(\frac{N_J}{2} + 2N_\lambda\right)!}\left(\frac{J^2}{2m^2}\right)^{\frac{N_J}{2} + 2N_\lambda}$

$N_{\frac{1}{2m^2}} := \frac{N_J}{2} + 2N_\lambda$

$$= \frac{1}{N_\lambda!\left(\frac{N_J}{2} + 2N_\lambda\right)!}\left(\frac{-\lambda}{4!}\right)^{N_\lambda}\left(\frac{1}{2m^2}\right)^{\frac{N_J}{2} + 2N_\lambda}\left(\frac{d}{dJ}\right)^{4N_\lambda} J^{N_J + 4N_\lambda}$$

$$= \frac{1}{N_\lambda! \, N_{\frac{1}{2m^2}}!}\left(-\frac{\lambda}{4!}\right)^{N_\lambda}\left(\frac{1}{2m^2}\right)^{N_{\frac{1}{2m^2}}} \frac{(N_J + 4N_\lambda)!}{N_J!} J^{N_J}$$

$P = \frac{1}{2m^2}$

$N_J = 2N_p - 4N_\lambda$

$$\longrightarrow = \frac{(2N_p)!}{N_\lambda! \, N_p! \, N_J!}\left(-\frac{\lambda}{4!}\right)^{N_\lambda}(P)^{N_p} J^{N_J}$$

note structure

$\left(-\frac{\lambda}{4!}\right)^{N_\lambda}(P)^{N_p} J^{N_J}$

repetition!

EXPAND THE OTHER WAY (after all, it's the same thing)

$$Z(J) = \sum_{s=0}^{\infty} \frac{1}{s!} J^s \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2 q^2 - \left(\frac{\lambda}{4!}\right) q^4} q^s \equiv Z(0,0) \sum_{s=0}^{\infty} \frac{1}{s!} J^s G^{(s)}$$

EVALUATE $G^{(s)}$ AS POWER SERIES IN $\lambda$

$$G^{(s)} = \frac{\int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2 q^2 - \left(\frac{\lambda}{4!}\right) q^4} q^s}{Z(0,0)} \qquad Z(0,0) = \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2 q^2}$$

EXAMPLE $\mathcal{O}(\lambda)$ in $G^{(4)}$

$$\frac{-\lambda}{4!\, Z(0,0)} \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2 q^2} q^8 = \left(\frac{1}{m^2}\right)^4 \left(-\frac{\lambda}{4!}\right) 7 \cdot 5 \cdot 3$$

RECALL: "WICK CONTRACTION"

$$\frac{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\sqrt{\frac{2\pi}{a}} \left\{ \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} \right.} = \underset{\substack{\uparrow \\ \text{DIM ANALYSIS}}}{\frac{1}{a^n}} \underbrace{(2n-1)(2n-3)\ldots 5\cdot 3\cdot 1}_{\#\text{ ways } 2n \text{ pts CAN CONNECT}}$$

COMPUTE GRAPHICALLY (construct rules to keep track of $\left(-\frac{\lambda}{4!}\right)$, $\left(\frac{1}{m^2}\right)^4$, too

NOTE $4N_\lambda + s$ $q$'s

$4N_\lambda$: ONES FROM $\lambda$

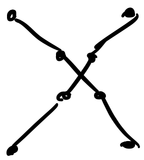
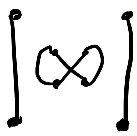
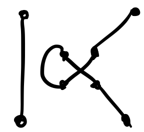
$s$: NOT FROM $\lambda$

$\Longrightarrow 4N_\lambda + s$ PTS

GRAPH | INTEGRAL

RULES:
1. DRAW S EXTERNAL PTS
2. $N_\lambda$ SETS OF 4 INTERNAL PTS
3. CONNECT EACH SET W VERTEX  $\qquad \frac{-\lambda}{4!}$
4. CONNECT $\cdot$'S W LINES $\Rightarrow$ $\qquad \frac{1}{m^2}$ EACH

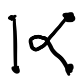
$O(\lambda)$ $G^4$

$4\cdot3\cdot2$  $+\ 3\cdot3$  $+\left(\begin{matrix}3\cdot4\cdot3\\ +\ 4\cdot3\\ +\ 4\cdot2\cdot3\end{matrix}\right)$ 

$$24\left(\frac{1}{m^2}\right)^4\left(\frac{-\lambda}{4!}\right)+9\left(\frac{1}{m^2}\right)^4\left(\frac{-\lambda}{4!}\right)+72\left(\frac{1}{m^2}\right)^4\left(\frac{-\lambda}{4!}\right)=105\left(\frac{-\lambda}{4!}\right)\left(\frac{1}{m^2}\right)^4$$

$$=7\cdot5\cdot3\left(\frac{-\lambda}{4!}\right)\left(\frac{1}{m^2}\right)^4$$

✓

HARDER COMB. 

EP := EXTERNAL PT
VP := VERTEX PT

$3\cdot4\cdot3$
- $1^{ST}$ EP TO ANOTHER EP $\Rightarrow 3$
- NEXT EP TO VP $\Rightarrow 4$
- LAST EP TO VP $\Rightarrow 3$

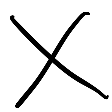
$4\cdot3$
- 1 ST EP TO VP $\Rightarrow 4$
- 2ND EP TO VP $\Rightarrow 3$

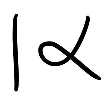
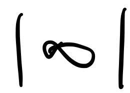
$4\cdot2\cdot3$
- 1ST EP TO VP $\Rightarrow 4$
- NEXT EP TO EP $\Rightarrow 2$
- LAST EP TO VP $\Rightarrow 3$

# ESSENCE OF FEYNMAN DIAGRAMS

(a clever way to compute an integral graphically)

## CONNECTED VS DISCONNECTED

CONNECTED 

DISCONNECTED  AND 

Q: WHAT IF WE WANTED JUST CONNECTED DIAGS?

A: REWRITE Z AGAIN S.T. DISCONNECTED DIAGS. ARE GEN'D BY CONNECTED

$$Z(J,\lambda) =: Z(J=0,\lambda)\, e^{W(J,\lambda)} = Z(J=0,\lambda) \sum_{N=0}^{\infty} \frac{1}{N!} \left[W(J,\lambda)\right]^N$$

(analogy: partition function vs free energy)

WHY? - DISCONN. DIAG. APPEAR IN $Z$

E.G.: $Z \supset A \; X \; p \quad @ \; \mathcal{O}(J^6, \lambda^2)$

- $W(J,\lambda) = \log(Z(J,\lambda)) - \log(Z(J=0,\lambda))$

$$\log(A \; X \; p) \overset{A = A_1 A_2}{=} \log(\underbrace{A_1 \; X}_{\mathcal{O}(J^4,\lambda)}) + \log(\underbrace{A_2 \; p}_{\mathcal{O}(J^2,\lambda)})$$

$\rightsquigarrow$ DISCONN. DIAG SPLIT.

(could Taylor series in log, ie $\ln(x) = x - 1 + \dots$

- ZEE: EXPAND EXPONENTIAL

$$Z(J=0,\lambda) \sum_{N=0}^{\infty} \frac{1}{N!} [iW(J,\lambda)]^N$$

$\frac{1}{N!}$ ACCOUNTS FOR ORDERING

E.G.: $Z(J,\lambda)$

$$A \; X \; p \longmapsto \frac{1}{2!}\left[(A_1 X)(A_2 p) + (A_2 p)(A_1 X)\right]$$

$\mathcal{O}(J^6, \lambda^2)$

LEFT X RIGHT $p$

OPPOSITE

$$= A_1 A_2 \; X \; p$$

$\boxed{W(J,\lambda) \text{ HAS ONLY CONNECTED DIAGRAMS}}$

SECOND WARMUP $\left(\begin{array}{l}\text{"baby problem" = single ordinary integral} \\ \text{"child problem" = multiple ordinary integrals}\end{array}\right)$

KEY: $q \longrightarrow q_i \Longrightarrow$ MULTIPLE LATTICE SITES $\Rightarrow$ PROPAGATION

$$Z(J) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dq_1 dq_2 \ldots dq_N \; e^{-\frac{1}{2} q\cdot A\cdot q - \frac{\lambda}{4!} q^4 + J\cdot q}$$

WHERE $q^4 \equiv \sum_i q_i^4$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dq_1 \ldots dq_N \; e^{-\frac{1}{2} q\cdot A\cdot q + J\cdot q}\left[1 - \frac{\lambda}{4!} q^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 (q^4)^2 + \ldots\right]$$

$$= \left[1 - \frac{\lambda}{4!}\sum_i \left(\frac{d}{dJ_i}\right)^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 \left(\sum_i \left(\frac{d}{dJ_i}\right)^4\right)^2 + \ldots\right] \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dq_1 \ldots dq_N \; e^{-\frac{1}{2} q\cdot A\cdot q + J\cdot q}$$

$$= e^{-\frac{\lambda}{4!}\sum \left(\frac{d}{dJ_i}\right)^4} \int\int \ldots$$

complete square, recall first chapter

$$= \left[\frac{(2\pi)^N}{\det A}\right]^{\frac{1}{2}} e^{-\lambda/4! \sum \left(\frac{d}{dJ_i}\right)^4} e^{\frac{1}{2} J\cdot A^{-1}\cdot J}$$

$$= \sum_{s=0}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_s=1}^{N} \frac{1}{s!} J_{i_1}\cdots J_{i_s} \int_{-\infty}^{\infty}\left(\prod_{\ell} dq_\ell\right) e^{-\frac{1}{2} q\cdot A\cdot q - \left(\frac{\lambda}{4!}\right) q^4} q_{i_1}\cdots q_{i_s}$$

$$=: Z(0,0) \sum_{s=0}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_s=1}^{N} \frac{1}{s!} J_{i_1}\cdots J_{i_s} \underbrace{G^{(s)}_{i_1\cdots i_s}}$$

S-POINT FUNC.

COMPUTE AS P.S. IN $\lambda$

NB: BEFORE: $G^{(s)}$ "baby"

NOW: $G^{(s)}_{i_1\ldots i_s}$ "child" $\longleftarrow$ S-POINT FUNCTION DEP. ON SITES

PROPAGATOR: $Z \sim e^{J\cdot A^{-1}\cdot J} \Longrightarrow \frac{1}{A_{ij}} \Longrightarrow i,j$ DEP IS SITE DEP! PROPAGATION $i \to j$

FROM 2-POINT FN:

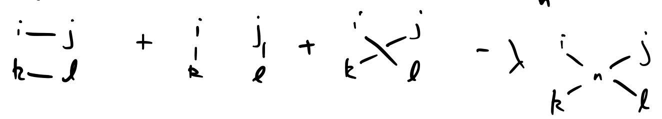
$$G^{(2)}_{ij}(\lambda=0) = \left[\int_{-\infty}^{\infty} dq_1 dq_2 \; e^{-\frac{1}{2} q\cdot A\cdot q} q_i q_j\right] \Big/ Z(0,0) = (A^{-1})_{ij}$$

(see I.2 appendix)

EG

$$G^{(4)}_{ijkl} = \int_{-\infty}^{\infty}\left(\prod_m g_m\right) e^{-\frac{1}{2} g\cdot A \cdot g}\, g_i g_j g_k g_l \left[1 - \frac{\lambda}{4!}\sum_n g_n^4 + \Theta(\lambda^2)\right] \Big/ Z(0,0)$$

$$= (A^{-1})_{ij}(A^{-1})_{kl} + \text{PERM } 1 + \text{PERM } 2 - \lambda \sum_n (A^{-1})_{in}(A^{-1})_{jn}(A^{-1})_{kn}(A^{-1})_{ln} + \text{DISCON.} + \Theta(\lambda^2)$$

$=$  $+ \ldots$

NOTE: $\frac{1}{4!}$ CANCELLED!

$i \to n$ 4 choices $j \to n$ 3, $k \to n$ 2, $l \to n$ 1.

## FEYNMAN DIAGRAMS IN QFT:

(the real problem)

DISCRETE $\longrightarrow$ CONTINUUM. $J = J(x)$ $\varphi = \varphi(x)$

(now depends on spacetime!)

$$Z(J) = \int D\varphi\, e^{i\int d^4x \{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4 + J\varphi\}}$$

$$= \int D\varphi\, e^{i\int d^4x\{\ldots\}}\left[1 - \int d^4w \frac{i\lambda}{4!}\varphi^4(w) + \int d^4w\, d^4y \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \varphi^4(w)\varphi^4(y) + \ldots\right]$$

$$= e^{-\frac{i\lambda}{4!}\int d^4w \left(\frac{\delta}{i\delta J(w)}\right)^4} \int D\varphi\, e^{-i\int d^4x\{\frac{1}{2}\varphi(\Box + m^2)\varphi - J\varphi\}}$$

$$= e^{-i\frac{\lambda}{4!}\int d^4w \left(\frac{\delta}{i\delta J(w)}\right)^4} Z(0,0)\, e^{-\frac{i}{2}\iint d^4x\, d^4y\, J(x) D(x-y) J(y)}$$

I.3.18 already derived

"BABY" $\frac{1}{m^2}$ $\longrightarrow$ "CHILD" $A^{-1}$ $\longrightarrow$ REAL THING $D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 - m^2 + i\varepsilon}$

## EXPAND IN J

$$Z(J) = \sum_{s=0}^{\infty} \frac{i^s}{s!}\int dx_1 \ldots dx_s\, J(x_1)\ldots J(x_s) \int D\varphi\, e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4\}}\varphi(x_1)\ldots\varphi(x_s)$$

$$= Z(0,0)\sum_{s=0}^{\infty}\frac{i^s}{s!}\int dx_1\ldots dx_s\, J(x_1)\ldots J(x_s)\, G^{(s)}(x_1,\ldots,x_s)$$

$G^{(s)}(x_1,\ldots,x_s)$ S-POINT FUNCTION

$$G^2(x_1,x_2)\Big|_{\lambda=0} = \frac{1}{Z(0,0)}\int D\varphi\, e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2\}}\varphi(x_1)\varphi(x_2)$$

$$= \frac{1}{Z(0,0)}\int D\varphi\, e^{i\int d^4x\, \varphi[-(\Box + m^2)]\varphi}\varphi(x_1)\varphi(x_2) = iD_F(x_1 - x_2) = \text{PROP}$$

PHYSICS $\quad G^{(2)}(x_1,x_2) - G^{(2)}(x_1,x_2)\big|_{\lambda=0}$ = CORRECTIONS TO PROP FROM INTERACTIONS

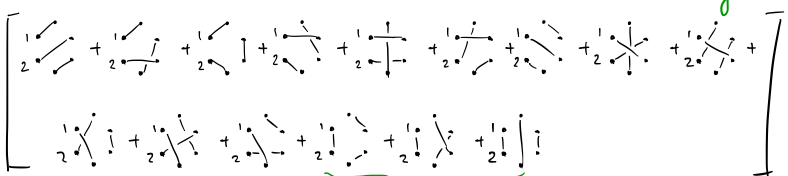
(in $G^{(2)}$, no vac. bubbles when norm.'d)

$$\mathcal{O}(\lambda):\quad G^2(x_1,x_2)\Big|_{\mathcal{O}(\lambda)} = \frac{1}{Z(0,0)}\int D\varphi\, e^{i\int d^4x \{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]\}}\varphi(x_1)\varphi(x_2)\left[-\frac{i\lambda}{4!}\int d^4y\ \varphi(y)\varphi(y)\varphi(y)\varphi(y)\right]$$

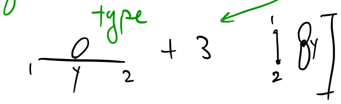
$$= \frac{-i\lambda}{4!}\int d^4y\left[\frac{\int D\varphi\, e^{\frac{1}{2}\int d^4x\, \varphi\overbrace{\{i(\Box+m^2)\}}^{:=K}\varphi}\,\varphi(x_1)\varphi(x_2)\varphi(y)\varphi(y)\varphi(y)\varphi(y)}{\int D\varphi\, e^{\frac{1}{2}\int d^4x\, \varphi\{i(\Box+m^2)\}\varphi}}\right]$$

recall 1st class calc

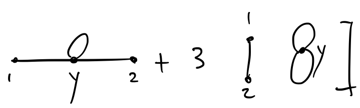
labelled $x_1, x_2$. rest $y$

$$= -\frac{i\lambda}{4!}\int d^4y\ \Big[$$

$$\Big]$$

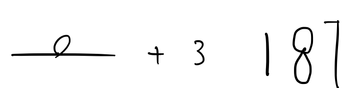
just 12 this type

$$= \frac{-i\lambda}{4!}\int d^4y\Big[\ 12$$

$$\Big]$$

$$= \frac{-i\lambda}{4!}\int d^4y\left[12\cdot D(x_1-y)\,D(x_2-y)\,D(y-y) + 3\,D(x_1-x_2)\,D(y-y)^2\right]$$

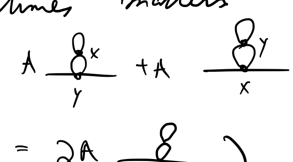
$$= \frac{-i\lambda}{4!}\Big[\ 12$$

$$\Big]$$

w/ RULE $\int d^4y$ IMPLIED FOR VERTICES

$$= -\frac{i\lambda}{4!}\Big[\ 12$$

$$\Big]$$

W RULE PT LABELLINGS IMPLICIT

(sometimes matters

eg

)

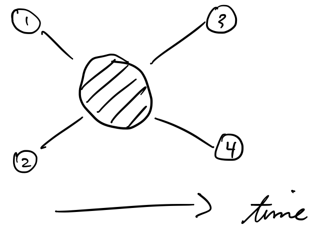
will list canonical ones soon.

**POINT:** FEYNMAN RULES ARE <u>CONVENTIONS</u> FOR A PICTURE $\longleftrightarrow$ EXPR. DICTIONARY

$\rightarrow$ make and use whatever rules you'd like, as long as they correctly reproduce the integral

# COLLISIONS

SCHEMATIC




$\Longrightarrow$ FOUR SOURCES $J(x_i)$

COMPUTE $G^4(x_1, x_2, x_3, x_4)$

$\mathcal{O}(\lambda)$: WICK'S WAY

$$\frac{1}{Z(0,0)}\left(\frac{-i\lambda}{4!}\right)\int d^4w \int D\varphi\, e^{i\int d^4x \{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]\}}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\varphi(w)^4$$

$$= (-i\lambda)\int d^4w\, D(x_1-w)D(x_2-w)D(x_3-w)D(x_4-w)$$ + OTHER WICK CONTR.

SCHWINGER'S WAY $\frac{\delta}{\delta J\text{'s}}$ ~~all~~ $Z$ all

FEYNMAN'S WAY: INTEGRALS $\longleftrightarrow$ PICTURES

(Zee: make whatever "picture rules" you want as long as they reproduce the correct results)

SOME RULES: POS SPACE

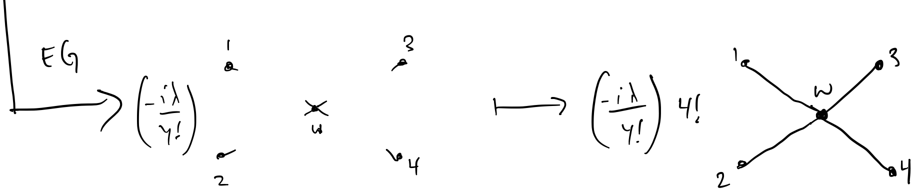
1. $D(x-y) \longleftrightarrow$ 


2. $-\frac{i\lambda}{4!}\int d^4w \longleftrightarrow \times_w$

3. COMBINATORIC COEFF $\longleftrightarrow$ # WAYS OF CONNECTING

4. $\mathcal{O}(\lambda^n)$ GETS $\frac{1}{n!}$

EG




$= (-i\lambda)\int d^4w\, D(x_1-w)D(x_2-w) \times D(x_3-w)D(x_4-w)$

(Compare eg to Peskin: $-i\lambda\int d^4w$ for vertex, divide by "symmetry factor," no $\frac{1}{n!}$)

# ANOTHER NON-TRIVIAL EXAMPLE

$$G^2(x_1,x_2)\Big|_{O(\lambda^2)} = \frac{1}{2!}\frac{1}{Z(0,0)}\int D\varphi\; e^{i\int d^4x\,\{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]\}}\,\varphi(x_1)\varphi(x_2)\times\left[\frac{-i\lambda}{4!}\int d^4x\,\varphi(x)\varphi(x)\varphi(x)\varphi(x)\right]$$

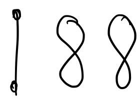
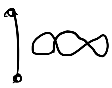
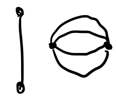
$$\times\left[-\frac{i\lambda}{4!}\int d^4y\,\varphi(y)\varphi(y)\varphi(y)\varphi(y)\right]$$

$$= \left(\frac{-i\lambda}{4!}\right)^2\int d^4x\int d^4y\;\; \text{INTEGRAND} \rightsquigarrow \text{quadratic}$$

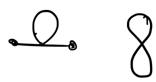
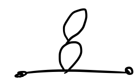
(put here to help vis. $\langle 12\; xxxx\; yyyy\rangle$

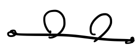
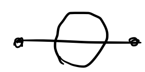
sum to 105! ✓

(point these are $\overline{x\,y}$ diags, leaving $xxxxyyyy$ to contract $\Rightarrow (8-1)!! = 105$ combos

INTEGRAND $= 3\cdot 3\cdot$  $+\ 6\cdot 6\cdot 2$  $+\ 4\cdot 3\cdot 2$ 

must give extra 840

$+\ 2\cdot 4\cdot 3\cdot 3$  $+\ 2\cdot 4\cdot 3\cdot 4\cdot 3$ 

$+\ 2\cdot 4\cdot 4\cdot 3\cdot 3$  $+\ 2\cdot 4\cdot 4\cdot 3\cdot 2$ 

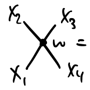
$$105 + 840 = 945 = 9\cdot 7\cdot 5\cdot 3\cdot 1 = (2(5)-1)!!$$

(CM students: compare to Altland-Simons Fig 5.2, same!)

$$\Downarrow$$

# FACTOR OF 2

## MOMENTUM SPACE (this is how experiments are done)

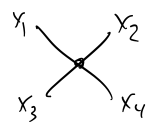
REWRITE 

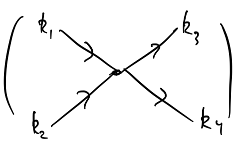

$$= -i\lambda \int d^4w \prod_{i=1}^{4} D(x_i - w) = -i\lambda \int d^4 w \prod_{i=1}^{4} \int \frac{d^4 k_i}{(2\pi)^4} \frac{e^{\pm i k_i (x_i - w)}}{k_i^2 - m^2 + i\varepsilon} \quad (*)$$

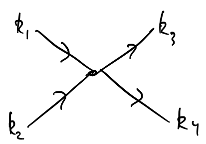
$$= -i\lambda \int d^4 w \left[ \prod_i \frac{d^4 k_i}{(2\pi)^4} \frac{1}{k_i^2 - m^2 + i\varepsilon} \right] e^{-i(k_1 + k_2 - k_3 - k_4) w} \, e^{+i \overbrace{(k_1 x_1 + k_2 x_2 - k_3 x_3 - k_4 x_4)}^{A}}$$

$$= -i\lambda \int \prod_i \frac{d^4 k_i}{(2\pi)^4} e^{+iA} \quad (2\pi)^4 \underbrace{\delta^4(k_1 + k_2 - k_3 - k_4)}_{\text{MOM. CONSERVATION @ VERTEX!}} \prod_i \frac{1}{k_i^2 - m^2 + i\varepsilon}$$

$$=: \underbrace{\int \prod_i \frac{d^4 k}{(2\pi)^4} e^{+iA} \quad (2\pi)^4 \times \delta^4(\ ) \; F(k_1, k_2, k_3, k_4)}_{\text{FOURIER TRANS}}$$




= FOUR TRANS (  )


determine signs of momentum (freedom to choose whatever you want)

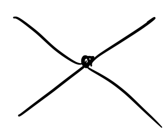
$(*) = 1 \; \mathcal{O}(\lambda)$ PIECE OF $G^{(4)}(x_1, x_2, x_3, x_4)$

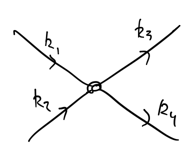
DEFINE MOM. SPACE G's $\underbrace{\text{useful for}}$ scattering amplitudes @ experiments

$$\boxed{G^{(s)}(x_1, \dots, x_s) = \prod_{i=1}^{s} \int \frac{dk_i}{(2\pi)^4} \underbrace{(2\pi)^4 \delta^4(\text{total mom.}) \; F(k_1, \dots, k_s)}_{=: G^{(s)}(k_1, \dots, k_s)} e^{\pm i k_i x_i}}$$

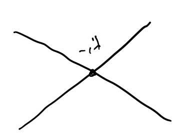
## K-SPACE FEYNMAN RULES

exercise: track down from $G^{(2)}(x_1, x_2)\big|_{\lambda=0} = i D_F(x_1 - x_2)$

① DRAW DIAG 

② LABEL EACH LINE w/ K 


$\longleftrightarrow \prod_j \frac{i}{k_j^2 - m^2 + i\varepsilon}$

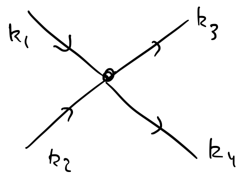
③ FOR EACH VERTEX 


$\longleftrightarrow -i\lambda (2\pi)^4 \times \delta(\sum k_{in} - \sum k_{out})$

④ EACH INTERNAL LINE $\quad \int d^4 k$

⑤ SYMMETRY FACTORS $\quad$ COEFF A

SO




this thing is $G^4(k's)$

$$= \underbrace{-i\lambda\,(2\pi)^4\,\delta^{(4)}(k_1+k_2-k_3-k_4)}_{\text{OVERALL MOM. CONS.}}\ \underbrace{\prod_{j=1}^{4}\left(\frac{i}{k_j^2-m^2+i\varepsilon}\right)}_{\text{EXTERNAL LINES}}$$

## ADDITIONAL SCATTERING AMP. RULES (more later)

PT: PREVIOUS RULES WERE FOR VAC TO VAC NOT ASYMP IN/OUT

6. EACH EXT. LINE $\ \frac{k_j}{\text{EXT}}\ \longleftrightarrow\ (-i)(k_j^2-m^2)$

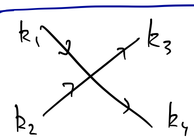
   ↳ "PUT EXT. PART. ON SHELL"
   ie momentum eigenstate

7. PULL OUT OVERALL MOM. CONS $(2\pi)^4\,\delta^4(\text{overall})$,

   LEFTOVER $=: \mathcal{M}$, ie EXPR $= (2\pi)^4\,\delta^4(\text{OVERALL})\,\mathcal{M}$

w/ 6-7

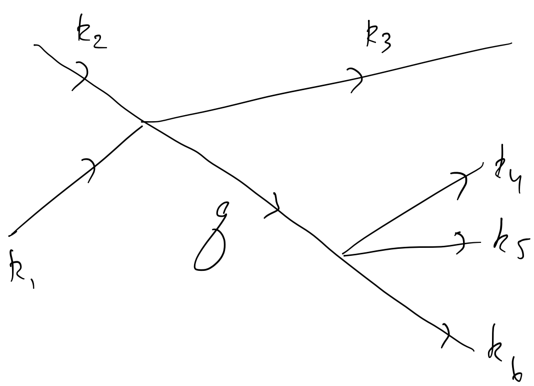
$\Longrightarrow$ $\boxed{\mathcal{M} \text{ FOR }}$




$\boxed{= -i\lambda}$

## THE BIRTH OF PARTICLES

(ie study a process where $\Delta N_{part} \neq 0$)




with some amp. What are they?

$$= (-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4}\ \frac{i}{q^2-m^2+i\varepsilon}$$
$$\times (2\pi)^4\,\delta^4(k_1+k_2-k_3-q)$$
$$\times (2\pi)^4\,\delta^4(q-(k_4+k_5+k_6))$$

$$= (-i\lambda)^2\ \frac{i}{(k_4+k_5+k_6)^2-m^2+i\varepsilon}\ (2\pi)^4\,\delta^4(k_1+k_2-(k_3+k_4+k_5+k_6))$$

rule 7 $\Longrightarrow$ $\boxed{\mathcal{M} = (-i\lambda)^2\ \frac{i}{(k_4+k_5+k_6)^2-m^2+i\varepsilon}}$

(Note: non-zero amplitude for $2\to 4$ process!)

## "VIRTUAL PARTICLES" (the cost of not being real)

NB: INT. LINE MOM $q = k_1 + k_2 + k_3$

$$\mathcal{M} \sim \frac{1}{q^2 - m^2 + i\varepsilon} =: \frac{1}{\Delta + i\varepsilon}$$

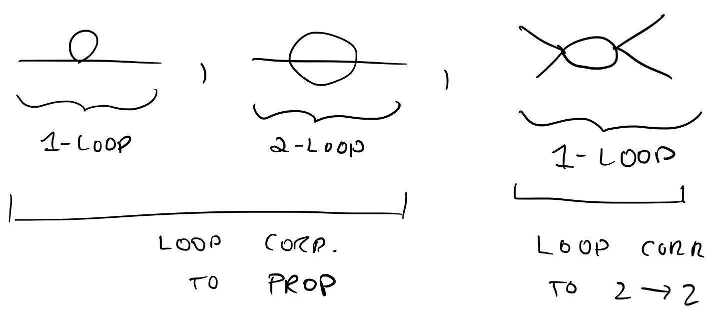
$\Delta = 0$: $q^2 = m^2$ "ON-SHELL" OR "REAL" $\Longrightarrow$ $\mathcal{M}$ LARGE

$\lim \Delta \to \infty$ $q^2 \neq m^2$ "OFF-SHELL" OR "VIRTUAL" $\Longrightarrow$ $\mathcal{M}$ SMALL

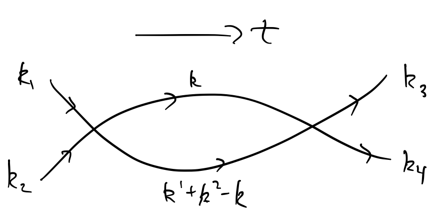
HOMEWORK: OBTAIN (21) STARTING w/ (12), (13)

## FIRST LOOK @ LOOPS

E.G.




## CONCRETE E.G.




HMWK I.7.2

$$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(k_1+k_2-k)^2 - m^2 + i\varepsilon}$$

ASIDE $\int^{\infty} dr\, r^n$

| $n$ | DIV. TYPE |
|---|---|
| 1 | QUAD |
| 0 | LIN. |
| -1 | LOG |

E.G. $\lim_{R^* \to \infty} \int_{R_0}^{R^*} \frac{dr}{r} = \lim_{R^* \to \infty} \ln\left(\frac{R^*}{R_0}\right)$

$= \infty$

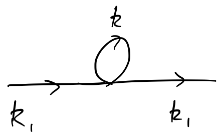
$\Longrightarrow$ ABOVE $\sim \int \frac{d^4k}{k^4}$ $\Longrightarrow$ $\boxed{\text{LOG. DIVERGENCE! } \infty!}$

Comments
- not a sickness, nothing "under the rug"
- leads to deepest aspects of QFT
- "regime of validity" part of the issue

# FIRST LOOK: HIERARCHY PROB.

# HIGGS FIELD = SCALAR FIELD

MASS CORRECTIONS (corrects propagator, specifically mass in prop)



$$= \frac{1}{2}(-i\lambda)\int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}$$

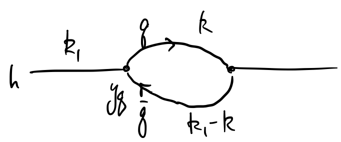
$$\overset{k\to\infty}{\sim} \int \frac{d^4k}{k^2} \sim \text{ANG PART} \int \frac{d|k|}{|k|^2}|k|^3 \sim \int d|k|\,|k| \Longrightarrow \boxed{\text{QUAD. DIV.}}$$

$\uparrow$ analogy $\int dx\,dy\,dz \sim \text{ANG PART} \int r^2 dr$

N.B.: - different severity of divergence

- this is the "naive" degree of divergence. In this case it is also the correct one, but in other cases can be less severe, due to symmetries, e.g.

FERMION LOOPS



$$\sim y_g^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{\not{k}+m_g}\,\frac{i}{(\not{k}_1-\not{k})+m_g} \sim y_g^2 \int \frac{d^4k}{(2\pi)^4 k^2}$$

$\Longrightarrow$ QUAD. DIV. AGAIN.

NB: - for scalars many books have no momentum $\rightarrow$, i.e. prop — w momentum implied.

- follow that convention now bc fermion line $\rightarrow$ has charge flow arrow

FACT $y_t \sim O(1) \Longrightarrow$ largest SM contribution to quadratic divergence of the Higgs mass

# Canonical Quantization

- Nature classicalizes quantum systems, not the other way around.
- "Quantization" is something humans do, a prescription for turning $\overbrace{\text{classical things}}^{\text{fields/vars, etc}}$ into $\underbrace{\text{quantum things}}_{\text{operators, states, etc.}}$.
- Can give rise to quantum systems that make correct predict.

# HEISENBERG QM REVIEW

## SINGLE $m=1$ PART.

CLASSICAL LAGRANGIAN $\quad L = \frac{1}{2}\dot{q}^2 - V(q)$

CANONICAL MOMENTUM $\quad p := \frac{\delta L}{\delta \dot{q}} = \dot{q}$

HAMILTONIAN $\quad H := p\dot{q} - L = \frac{p^2}{2} + V(q)$

EOM $\quad \dot{p} = -\frac{\partial H}{\partial q} \implies \dot{p} = -V'(q)$

$\dot{q} = \frac{\partial H}{\partial p} \implies \dot{q} = \frac{\partial H}{\partial p} = p$

"QUANTIZATION" $\quad p \longmapsto \hat{p} \quad q \longmapsto \hat{q}, \quad [\hat{p}, \hat{q}] = -i$

KEY EQUAL TIME COMMUTATORS $\quad [\hat{p}(t), \hat{q}(t)]$

from $\hat{O}(t) = e^{i\hat{H}t} O(0) e^{-i\hat{H}t}$

THEN

QM

$\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q}) \qquad \dot{\hat{p}} = -V'(\hat{q}) = i[\hat{H}, \hat{p}] \qquad \dot{\hat{q}} = i[\hat{H}, \hat{q}] = \hat{p}$

$\implies \ddot{\hat{q}} = \dot{\hat{p}} = -V'(\hat{q})$

REWRITE $\quad \hat{a} := \frac{1}{\sqrt{2\omega}}(\omega\hat{q} + i\hat{p}), \quad [a, a^\dagger] = 1$ $\quad$ ORDER

$$\frac{d\hat{a}}{dt} = i\left[\hat{H}, \frac{1}{\sqrt{2\omega}}(\omega\hat{q} + i\hat{p})\right] = -i\sqrt{\frac{\omega}{2}}\left(i\hat{p} + \frac{1}{\omega}V'(\hat{q})\right)$$

ALG $\implies \exists$ STATE $|0\rangle$ w/ $a|0\rangle = 0$

$\boxed{\text{CASE}}$ HARMONIC OSC. $\quad V = \frac{1}{2}\omega^2 q^2$

$V'(q) = \omega^2 q \implies \frac{da}{dt} = -i\sqrt{\frac{\omega}{2}}(i\hat{p} + \omega\hat{q}) = -i\omega a$

$H = \omega(a^\dagger a + \frac{1}{2})$ etc

MANY PARTICLES $\quad L = \sum_a \frac{1}{2}\dot{q}_a^2 - V(q_1, \dots, q_N) \quad [\hat{p}_a, \hat{q}_b] = -i\delta_{ab}$

FIELD THEORY $\quad$ spatial coordinates $\quad L = \int d^D x \, \mathcal{L}$

CLASSICAL F.T. LAG.:

interactions, P.T.

$$L = \int d^D x \left\{ \frac{1}{2}\left(\dot{\varphi}^2 - (\vec{\nabla}\varphi)^2 - m^2\varphi^2\right) - u(\varphi) \right\}$$

$$\left( \text{recall} = \text{usual thing} = \int d^D x \left\{ \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - u(\varphi) \right\} \right)$$

CANONICAL MOMENTUM DENS.: $\pi := \frac{\delta L}{\delta \dot{\varphi}} = \dot{\varphi}$

HAMILTONIAN: $H = \int d^D x \left[\pi\dot{\varphi} - \mathcal{L}\right] = \int d^D x \, \underbrace{\left\{ \frac{1}{2}\left[\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\right] + u(\varphi) \right\}}_{\text{what do you notice physically here?}}$

EOM: $(\Box + m^2)\phi + \frac{\partial u}{\partial \phi} = 0$

QUANTIZATION: $\varphi \mapsto \hat{\varphi} \quad \pi \mapsto \hat{\pi} \quad \underbrace{[\hat{\pi}(\vec{x},t), \hat{\varphi}(\vec{x}',t)] = -i\,\delta^D(\vec{x}-\vec{x}')}_{\text{equal time commutators}}$

$\boxed{\text{KEY CASE}}$ $(u=0)$ = HARM. OSC. = FREE FIELD THEORY

EOM = K.G. EQN: $(\Box + m^2)\varphi = 0$

SOLUTIONS: $$\varphi(\vec{x},t) = \int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left[ a(\vec{k})\, e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + a^\dagger(\vec{k})\, e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \right]$$

CHECK: $0 = (\Box + m^2)\, e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} = \left[(-i)^2\omega_k^2 - i^2\vec{k}^2 + m^2\right] e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}$

$$\Longrightarrow \omega_k^2 = \vec{k}^2 + m^2$$

QUANTIZE: $\varphi \mapsto \hat{\varphi} \implies a \mapsto \hat{a}, \quad a^\dagger \longrightarrow \hat{a}^\dagger$

$$-i\,\delta^D(\vec{x}-\vec{x}') \overset{\text{DEMAND}}{=} [\partial_0\hat{\varphi}, \hat{\varphi}] \quad \overset{\text{HMWK}}{\Longrightarrow} \quad \underbrace{[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = \delta^{(D)}(\vec{k}-\vec{k}')}_{[\hat{a},\hat{a}^\dagger] = 1\!\!1 \text{ ANALOG}}$$

$\boxed{\text{DROP HATS FROM NOW ON}}$

ALG. HARM. OSC. $\Rightarrow \exists\ |0\rangle$ S.T.

$$\boxed{a(\vec{k})|0\rangle = 0 \quad \forall \vec{k} \qquad \text{"GROUND" OR "VACUUM" STATE}}$$

GROUND STATE EN.

$$H|0\rangle = E_0|0\rangle$$

EXCITATIONS $|\vec{k}\rangle := a^+(\vec{k})|0\rangle \longleftrightarrow$ SINGLE PARTICLE STATES

$$\underbrace{\langle 0|\varphi(\vec{x},t)|\vec{k}\rangle}_{\text{analog of } \langle x|p\rangle} = \langle 0|\int \frac{d^D k'}{\sqrt{(2\pi)^D 2\omega_{k'}}}[a(\vec{k}'), a^+(\vec{k})]\, e^{-i(\omega_{k'}t - \vec{k}'\cdot\vec{x})}|0\rangle$$

$$= \frac{1}{\sqrt{(2\pi)^D 2\omega_{k'}}} e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} =: \frac{1}{\rho(k)} e^{-ik\cdot x}$$

$\Longrightarrow$ PLANE WAVE w/ MOMENTUM $\vec{k}$

Language:

$a^+$ "CREATION OP"

$a$ "ANNIH. OP."

CONNECTION TO PATH INTEGRAL

$$\langle 0|\varphi(\vec{x},t)\varphi(0,0)|0\rangle = \langle 0|\int \frac{d^D k'}{\rho(k')}\frac{d^D k}{\rho(k)} e^{-ik\cdot x} e^{+ik'\cdot 0}[a(\vec{k}'), a^+(k)]|0\rangle$$

$a^+\cancel{a^+}\ \cancel{aa}$

$aa^+\ \cancel{aa}$

$\| $

$[a, a^+]$

$$= \int \frac{d^D k}{(2\pi)^D 2\omega_k} e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}$$

DEFINE "TIME ORDERED PRODUCT"

$$T[\varphi(x)\varphi(y)] = \Theta(x^0 - y^0)\varphi(x)\varphi(y) + \Theta(y^0 - x^0)\varphi(y)\varphi(x)$$

$$\Longrightarrow \boxed{\langle 0|T[\varphi(\vec{x},t)\varphi(0,0)]|0\rangle = \int \boxed{\frac{d^D k}{(2\pi)^D 2\omega_k}}\left[\Theta(t)e^{-ik\cdot x} + \Theta(-t)e^{ik\cdot x}\right] = iD(\vec{x})}$$

HMWK: Lorentz inv.

RECOVERED THE PROPAGATOR!

(note: now have interpretation in terms of overlaps of particle states)

JUSTIFIES $i\varepsilon$ PRESCRIPTION!

($\Theta(t)$ form of prop arose from closing contour integral with the $i\varepsilon$ presc. Here: create before annihilate explicit)

# PERTURBATION EXPANSION (in interacting theory)

ALREADY KNOW

$$\langle 0| T\phi(x)\phi(y)|0\rangle = iD(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik\cdot(x-y)}}{k^2-m^2+i\varepsilon}$$

free

ie how does prop change in int theory?

WANT E.G. $\langle\Omega| T\phi_I(x)\phi_I(y)|\Omega\rangle$ AS $\lambda$-PERT

$\phi_I(x) =$ INT. THY FIELD $\longrightarrow$ opposite convention rel. Peskin, for whom $\phi_I$ is effectively free

$|\Omega\rangle =$ INT. THY VACUUM

$\hookrightarrow \hat{H}$ changed, expect eigenstates to change!

– more generally, arb. n-point functions of int. theory

RECALL $H = H_0 + H_{int} = H_{KG} + \int d^3x \frac{\lambda}{4!}\phi(\vec{x})$

$H_{int}$ ENTERS TWICE

(1) $\phi_I(x) = e^{iHt}\phi_I(\vec{x})e^{-iHt}$

(2) $|\Omega\rangle$ G.S. OF $H$

} $H_{int}$ implicitly in both equations

$\Longrightarrow$ GOAL: RELATE TO i) $\phi(x)$, ii) $|0\rangle$

i) $\phi(t,x) := \phi_I(t,x)\big|_{\lambda=0} = e^{iH_0(t-t_0)}\phi_I(t_0,x)e^{-iH_0(t-t_0)}$

$$\Longrightarrow \phi_I(t,x) = e^{iH(t-t_0)}\phi_I(t_0,\vec{x})e^{-iH(t-t_0)} = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi(t,x)e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$$

$$=: U^\dagger(t,t_0)\,\phi(t,x)\,U(t,t_0) \quad (*)$$

NOTE: $i\partial_t U = e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)}$

$$= e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}\underbrace{e^{iH_0(t-t_0)}e^{-iH(t-t_0)}}_{U(t,t_0)}$$

$$=: H_I(t)\,U(t,t_0)$$

$$H_I(t) = e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)} = e^{()}\int d^3x \frac{\lambda}{4!}\phi(\vec{x},t_0)e^{-()} = \int d^3x \frac{\lambda}{4!}\phi^4$$

CALC $\Rightarrow$ $U(t,t_0) = T\{\exp[-i\int_{t_0}^{t} dt'\, H_I(t')]\}$ $\circledast_2$

↳ in real calcs keep only first few terms in series

Summary: due to $\circledast$ & $\circledast_2$, now have control over $\phi_I$.

ii) $|\Omega\rangle$? TAKE $E_0 := \langle\Omega|H|\Omega\rangle$

CALC
$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$
$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

$$E_0 < E_{n>0} \Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_0T}\langle\Omega|0\rangle}$$

SHIFT $T \mapsto T + t_0$ (small shift fine)

CALC $\Rightarrow$
$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{U(t_0,-T)|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle}$$

Summary: $U$ also critical in relating $|\Omega\rangle$ $|0\rangle$

i) & ii) $\Rightarrow$
$$\langle\Omega|T\{\Phi_I(x)\Phi_I(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\{\Phi(x)\Phi(y)\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}$$

GENERALIZATION: INTERACTING S-PT FUNCTION

$$\boxed{\begin{aligned} &G_I^{(s)}(x_1,\dots,x_s) := \\ &\langle\Omega|T\{\phi_I(x_1)\dots\phi_I(x_s)\}|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\{\phi(x_1)\dots\phi(x_s)\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T} dt\, H_I(t)]\}|0\rangle}\end{aligned}}$$

COMPUTE: $G_I^{(2)}(x_1,x_2)$ @ $\mathcal{O}(\lambda)$

NUMERATOR $\langle 0| T(\phi(x_1)\phi(x_2)\, e^{-i\int dt\, H_I(t)})|0\rangle$

$$= \langle 0|T(\phi_1\phi_2 + \phi_1\phi_2\,[(-i)\int d^4z\,\frac{\lambda}{4!}\,\phi_z^4]|0\rangle$$

RECALL $\langle 0|T(\phi_1\phi_2)|0\rangle = i\,D(x_1-x_2)$

$\hookrightarrow$ HOW? $\quad \phi_1\phi_2 \sim (a_{k_1}e^{()} + a^+_{k_1}e^{-()})(a_{k_2}e^{()} + a^+_{k_2}e^{-()})$

$$\sim a_{k_1}a^+_{k_2} \overset{\text{in } \langle 0|0\rangle}{\sim} [a_{k_1}, a^+_{k_2}] \sim \iint d\vec{k}_1 d\vec{k}_2\, \delta^D(\vec{k}_1-\vec{k}_2)$$

KEY COMMUTE $a_{k_1}$ TO RIGHT

THEN: $\langle 0|T(\phi_1\phi_2\phi_z\phi_z\phi_z\phi_z)|0\rangle$

$$\sim \langle 0|T((a_{k_1}+a^+_{k_1})(a_{k_2}+a^+_{k_2})(a_{k_{z_1}}+a^+_{k_{z_1}})\dots(a_{k_{z_4}}+a^+_{k_{z_4}}))|0\rangle$$

$\exists \iint$'s, $e$'s, etc,

focus on operator structure

STRATEGY: JUST TRICKS, NOT PHYSICS $\quad P_{\ell\pm} := e^{\pm i(\omega_\ell t - \vec{\ell}\cdot\vec{x})}$

Language: "normal ordering"

1. USE $aa^+ = [a,a^+] + a^+a$ TO MOVE ALL $a$'s TO RIGHT OF $a^+$'s
2. $a|0\rangle = 0$ & $\langle 0|a^+ = 0$
   $\Rightarrow$ NON-ZERO TERMS $\Longleftrightarrow$ ALL COMM'S, NO "FREE" $a$'s, $a^+$'s
   $\Rightarrow$ EACH $a$ PAIRS w/ AN $a^+$ INTO $[\cdot,\cdot]$. "WICK CONTRACTION"
3. CONTRACTION EXPR.

two time orderings

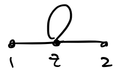
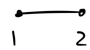
$$\overline{\varphi(x_1)\varphi(x_2)} = \int \frac{d^Dk_1}{\rho(k_1)}\frac{d^Dk_2}{\rho(k_2)}\left\{ P^{x_1}_{1-}P^{x_2}_{2+}[a(\vec{k}_1), a^+(\vec{k}_2)] + P^{x_2}_{2-}P^{x_1}_{1+}[a(\vec{k}_2), a^+(\vec{k}_1)]\right\} = i\,D(x_1-x_2)$$

4. WICK'S THEOREM: (operator version, same basic content)

ANSWER = ALL POSSIBLE CONTR.

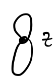
THEN $\mathcal{O}(\lambda)$ IN NUM $= \frac{-i\lambda}{4!}\int d^4z \, \langle 0|T(\phi_1\phi_2\phi_z\phi_z\phi_z\phi_z)|0\rangle$

ALL CONT, EACH GETS PROP

$= \frac{-i\lambda}{4!}\int d^4z \, [12$  $+ 3$   $] = G^{(2)}(x_1,x_2)$ @ $\mathcal{O}(\lambda)$ IN PATH INTEGRAL!

DENOMINATOR @ $\mathcal{O}(\lambda)$

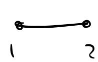
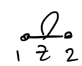
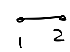
$\langle 0|T(e^{-i\int dt\, H_I(t)})|0\rangle = \langle 0|T(\,1 + [(-i)\int d^4z\, \frac{\lambda}{4!}\phi_z^4]\,|0\rangle$

$= 1 + \underbrace{\left(\frac{-i\lambda}{4!}\right)\int d^4z}_{=:A_z}$ $[3$  $] = G^{(0)}$ @ $\mathcal{O}(\lambda)$ IN PATH INT!

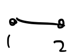
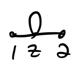
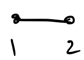
all contractions!

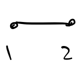
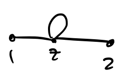
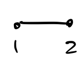
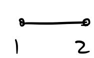
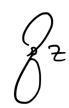
$\hookrightarrow$ PHYSICAL INTERP?

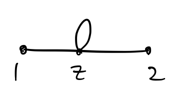
FINALLY

$G_I^{(2)}(x_1,x_2)\Big|_{\mathcal{O}(\lambda)} = \frac{\text{NUM}}{\text{DENOM}} =$ [  $+ A_z[12$  $+ 3$   ] ] / [ $1 + A_z\, 3$  ]

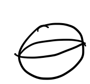
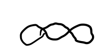
interacting G

$= [$  $+ 12A_z$  $+ 3A_z$   $]\,[1 - 3A_z$  $+ \mathcal{O}(\lambda^2)]$

$=$  $+ 12A_z$  $+ 3A_z$   $- 3A_z$   $(=0)$ $+ \mathcal{O}(\lambda^2)$

$=$  $+ 12\left(\frac{-i\lambda}{4!}\right)\int d^4z$  $+ \mathcal{O}(\lambda^2)$

NOTE:
-  CANCELLED OUT!
- GENERAL $\dfrac{1}{\langle 0|T(e^{-i\int d^4z \frac{\lambda}{4!}\phi_z^4})|0\rangle}$ CANCELS ALL "VACUUM BUBBLES"

e.g. , , 

# SCATTERING AMPLITUDES

## S-MATRIX $\quad 2 \longrightarrow n$ SCAT.

- reference time $t=0$
- "IN" STATE $\quad |k_A, k_B\rangle_{in} = \lim_{T\to\infty} \underbrace{e^{iHT}}_{\text{evolves to } \infty\text{-past}} |k_A, k_B\rangle_{t=0}$
- "OUT" STATE $\quad |p_1, \dots, p_n\rangle_{out} = \lim_{T\to\infty} e^{-iHT} |p_1, \dots, p_n\rangle_{t=0}$
- PHYSICS

$$_{out}\langle p_1, \dots, p_n | k_A, k_B\rangle_{in} = \lim_{T\to\infty} {}_{t=0}\langle p_1, \dots, p_n | e^{\overset{\text{sign rel Peskin}}{+iH(2T)}} | k_A, k_B\rangle_{t=0}$$

$$=: {}_{t=0}\langle p_1, \dots, p_n | \underset{\uparrow \text{ S-matrix}}{S} | k_A k_B\rangle$$

## T-MATRIX $\quad S =: \mathbb{1} + iT$

- T MEAS. INTERACTIONS
- S & T SHOULD RESP. MOM. CONS.

Language: "scattering amplitude" or "invariant matrix element"

$$\overset{\text{MOTIV.}}{\Longrightarrow} \quad \langle p_1, \dots, p_n | iT | k_A k_B\rangle =: (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_f) \cdot \mathcal{M}(k_A, k_B \longrightarrow p_f)$$

$\hookrightarrow$ appears in cross-section; decay rate formulae

## COMPUTING S & T w FEYN. DIAGS.

- GOAL $\quad \langle p_1, \dots, p_n | iT | k_A, k_B\rangle$ = FUNC OF FREE THY STATES, OPS
- RESULT (some work, see eg. Peskin 4.6 for more details)

$$\boxed{\langle p_1, \dots, p_n | iT | k_A, k_B\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left( {}_0\langle p_1, \dots, p_n | T\left(\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]\right) | p_A\, p_B\rangle_0 \right)_{\substack{\text{connected} \\ \text{amputated}}}}$$

- goal for rest of lecture is to see amputation

CASE: $u(\varphi) = \frac{\lambda}{4!}\varphi^4, \quad \lambda \ll 1$

$\boxed{\text{EX 1:}}$ "$1 \to 1$" SCATTERING (a good warmup)

$$\langle \vec{k}_0 | T e^{-iH_{int}T} | \vec{k}_i \rangle = \langle \vec{k}_0 | T e^{-\frac{i\lambda}{4!}\int d^{D+1}x\, \varphi^4(x)} | \vec{k}_i \rangle$$

$$= \langle \vec{k}_0 | \vec{k}_i \rangle + \left(\frac{-i\lambda}{4!}\right) \overbrace{\langle \vec{k}_0 | T \int d^{D+1}x\, \varphi^4(x) | \vec{k}_i \rangle}^{B} + \mathcal{O}(\lambda^2)$$

$$P_{\ell\pm} := e^{\pm(\omega_\ell t - \vec{\ell}\cdot\vec{x})} \implies \varphi = \int \frac{d^D k}{\rho(k)} \left(a(k)\, P_{k-} + a^\dagger(k)\, P_{k+}\right)$$

$\downarrow$ drop $\to$ 'o'

$$B = \langle 0 | T a(k_0) \left[ \int d^{D+1}x \int \frac{d^D k_1\, d^D k_2\, d^D k_3\, d^D k_4}{\rho(k_1)\rho(k_2)\rho(k_3)\rho(k_4)} \left(a(k_1) P_{1-} + a^\dagger(k_1) P_{1+}\right)\left(a(k_2) P_{2-} + a^\dagger(k_2) P_{2+}\right)\right.$$

$$\left. \times \left(a(k_3) P_{3-} + a^\dagger(k_3) P_{3+}\right)\left(a(k_4) P_{4-} + a^\dagger(k_4) P_{4+}\right)\right] a^\dagger(k_i) |0\rangle$$

NEW FEATURE

DIFF CONTRACTION TYPES, ACCORDING TO 0,1,2 INTEGRATION k's

i) $\overline{a(k_0)\, a^\dagger(k_i)} = [a(k_0), a^\dagger(k_i)] = \delta^D(k_0 - k_i)$

$\swarrow$ put $x$ to denote $x$-dep

ii) $\overline{a(k_0)\, \varphi(x)} = \int \frac{d^D k}{\rho(k)} \overset{x}{P}_{k+} [a(k_0), a^\dagger(k)] = \frac{\overset{x}{P}_{k_0+}}{\rho(k_0)}$

iii) $\overline{\varphi(x)\, a^\dagger(k_i)} = \int \frac{d^D k}{\rho(k)} \overset{x}{P}_{k-} [a(k), a^\dagger(k_i)] = \frac{\overset{x}{P}_{k_i-}}{\rho(k_i)}$

iv) $\overbrace{\varphi(x_1)\,\varphi(x_2)} = \int \frac{d^D k_1}{\rho(k_1)} \frac{d^D k_2}{\rho(k_2)} \left\{ P_{1-}^{x_1} P_{2+}^{x_2} [a(k_1), a^\dagger(k_2)] \right\} = \int \frac{d^D k}{2\omega_k (2\pi)^D} \; e^{i\omega_k (t_2 - t_1) - i\vec{k}\cdot(\vec{x}_2 - \vec{x}_1)}$

internal lines!

<u>note</u>: 1) in amplitudes, x's come from interactions, so w $\int d^4 x$.

2) above <u>creates</u> @ $x_2$, <u>annih</u> @ $x_1$, need other possibility ~~too~~

shift $x_2$ to 0

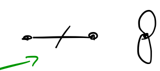
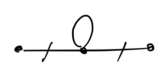
$$\Longrightarrow T[\overbrace{\varphi(x)\varphi(0)}] = \int \frac{d^D k}{2\omega_k (2\pi)^D} \left[ \theta(t) e^{-ik\cdot x} + \theta(-t) e^{ik\cdot x} \right] = i D(x)$$

<u>NOTE:</u> ONLY INT. LINES GET PROPAGATORS!

<u>USING</u> <u>THESE</u> $B = \int d^{D+1}x \, \langle 0 | T a(k_0) \varphi^4(x) \, a^\dagger(k_i) | 0 \rangle$

3 choices for which pairs w/ in

$$= \int d^{D+1}x \left\{ 3 \langle 0 | T a(k_0) \varphi \varphi \varphi \varphi a^\dagger(k_i) | 0 \rangle + 4\cdot 3 \langle 0 | T a(k_0) \varphi \varphi \varphi \varphi a^\dagger(k_i) | 0 \rangle \right.$$

- amputated ~  ~ 

- not connected!

x to x

$$-\frac{i\lambda}{4!} \times \begin{matrix}\text{CONNECTED/INT.}\\ \text{PART OF } B\end{matrix} = -\frac{i\lambda}{2} \int d^{D+1}x \; \frac{P_{k_0 +}^{x}}{\rho(k_0)} \; iD(0) \; \frac{P_{k_i -}^{x}}{\rho(k_i)}$$

$$= \frac{-i\lambda}{2} \frac{1}{\rho(k_0)\rho(k_i)} \int d^{D+1}x \; e^{i(\omega_0 - \omega_i)t - i(\vec{k}_0 - \vec{k}_i)\cdot\vec{x}} \; iD(0)$$

$$= \text{SAME} \times \int d^{D+1}x \; e^{i(\omega_0-\omega_i)t - i(\vec{k}_0 - \vec{k}_i)\cdot\vec{x}} \; i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$= \text{SAME} \times i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\varepsilon} \int d^{D+1}x \; e^{i(k_0 - k_i)\cdot x}$$

$$= \underbrace{\frac{(2\pi)^4 \delta^4(k_0 - k_i)}{\rho(k_0)\,\rho(k_i)} \left[ \frac{-i\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \right]}_{\text{CONTRIB TO } iT_{fi} \text{ @ } O(\lambda)}$$

$$\Longrightarrow \boxed{\mathcal{M} \text{ contrib} = \left[ -\frac{i\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \right]}$$

EX 2 $\quad \langle \vec{k}_3, \vec{k}_4 | e^{-i\int d^{D+1}x \frac{\lambda}{4!}\varphi^4} | \vec{k}_1, \vec{k}_2 \rangle = \langle k_3 k_4 | k_1 k_2 \rangle + \underbrace{\left(\frac{-i\lambda}{4!}\right) \langle k_3 k_4 | \int d^{D+1}x \, \varphi^4(x) | k_1 k_2 \rangle}_{iT(k_3 k_4, k_1 k_2) \ @ \ O(\lambda)}$

$$iT = \left(\frac{-i\lambda}{4!}\right) \int d^{D+1}x \, \underbrace{\langle 0 | a(k_3) a(k_4) \, \varphi(x)\varphi(x)\varphi(x)\varphi(x) \, a^\dagger(k_1) a^\dagger(k_2) | 0 \rangle}$$

USE OUR STRATEGY TO COMPUTE! $\Longrightarrow \boxed{\mathcal{M} = -i\lambda}$

NB we will say more about amplitudes later when studying more complicated theories.

## COMPLEX SCALAR FIELD QUANTIZATION

LAGRANGIAN: $\mathcal{L} = \partial\varphi^* \partial\varphi - m^2 \varphi^*\varphi = \partial_0\varphi^* \partial^0\varphi - \nabla\varphi^* \cdot \nabla\varphi - m^2 \varphi^*\varphi$

EQN'S OF MOT.: $(\Box + m^2)\varphi = 0 \qquad (\Box + m^2)\varphi^* = 0$

CONJUGATE MOMENTA: $\pi = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}} = \partial_0 \varphi^* \qquad \pi^\dagger = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}^*} = \partial_0 \varphi$

QUANTIZATION: $\varphi \longmapsto \hat{\varphi} \qquad \pi \longmapsto \hat{\pi}$

$\varphi^* \longmapsto \hat{\varphi}^* \qquad \pi \longmapsto \hat{\pi}^*$

S.T. $\quad [\hat{\pi}(\vec{x},t), \hat{\varphi}(\vec{x}',t)] = -i\delta^{(D)}(\vec{x}-\vec{x}')$

$[\hat{\pi}^*(\vec{x},t), \hat{\varphi}^*(\vec{x}',t)] = -i\delta^{(D)}(\vec{x}-\vec{x}') \qquad (*)$

FOURIER EXPAND:

$$\hat{\varphi} = \int \frac{d^Dk}{\rho(k)} \left[ \hat{a}(\vec{k}) e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + \hat{b}^\dagger(k) e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \right]$$

$$\hat{\varphi}^* = \int \frac{d^Dk}{\rho(k)} \left[ \hat{b}(k) e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + a^\dagger(k) e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \right]$$

$$(*) \Longrightarrow [\hat{a}(\vec{k}_1), \hat{a}^\dagger(\vec{k}_2)] = [\hat{b}(\vec{k}_1), \hat{b}^\dagger(k_2)] = \delta^{(D)}(\vec{k}_1 - \vec{k}_2)$$

KEY DIFFERENCE: (compared to real scalar field)

$\exists \ \hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$ ! (two sets of creation & annih. ops!)

HOW ARE $\hat{a}^\dagger(k)|0\rangle$ & $\hat{b}^\dagger(k)|0\rangle$ RELATED?

CONSIDER $J_\mu = i(\varphi^* \partial_\mu \varphi - \partial_\mu \varphi^* \varphi)$ Language: "conserved current"

CLAIM 1 $\partial_\mu J^\mu \stackrel{!}{=} 0$ CLASSICALLY (= using EOM here)

$$\partial_\mu J^\mu = i[\varphi^* \Box \varphi - (\Box \varphi^*)\varphi] \underset{\uparrow \text{EOM}}{=} -im^2[\varphi^*\varphi - \varphi^*\varphi] = 0 \checkmark$$

CLAIM 2 $\exists$ CONSERVATION LAW FOR "CHARGE" Q

IE $Q := \int d^D x\, J_0(x)$ SATS. $\frac{dQ}{dt} = 0$

$$\partial_0 Q = \int d^D x\, \partial_0 J_0 = -\int d^D x\, \vec{\nabla}\cdot\vec{J} = 0$$

if J dies off at $\infty$, ie physics of J is local

FACT: ALG $\Longrightarrow$ $\hat{Q} = \int d^D k [a^\dagger(\vec{k}) a(\vec{k}) - b^\dagger(\vec{k}) b(\vec{k})]$

THEN $Q\, a^\dagger(\vec{k})|0\rangle = \int d^D k\, a^\dagger(k) a(k) a^\dagger(k')|0\rangle = \int d^D k\, a^\dagger [a(k), a^\dagger(k')]|0\rangle$

$$= + a^\dagger(k')|0\rangle$$

$$Q\, b^\dagger(\vec{k}')|0\rangle = -b^\dagger(k')|0\rangle$$

NB: PARTICLES OF OPPOSITE CHARGE, SAME MASS.

$b^\dagger(k)|0\rangle$ ANTIPARTICLE OF $a^\dagger(k)|0\rangle$

CONSERVED CHARGE Q $\left(\frac{dQ}{dt} = 0\right)$

FROM CONS. CURR. $J_\mu$ $(\partial_\mu J^\mu = 0)$

$\boxed{\text{Q: WHAT IS THE ORIGIN OF } J_\mu\text{?}}$

# Symmetry

- We will start with simple examples and then be more general

# LAGRANGIAN SYMMETRIES

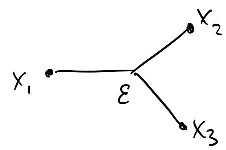
**EXAMPLE:** $\mathbb{R}$ SCALAR FIELD THY

$$\mathcal{L} = \tfrac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

- INV'T UNDER $\varphi \to -\varphi \Rightarrow$ # PARTICLES CONSERVED % 2
- IF $\mathcal{L} \mapsto \mathcal{L} - \varepsilon\varphi^3$, $|\varepsilon| \ll 1$, $\mathcal{L}$ NOT INV'T

*Language:* "SMALL EXPLICIT BREAKING"

% 2 CONS. DNE!




**EXAMPLE:** 2 $\mathbb{R}$ S.F.T.

$$\mathcal{L} = \tfrac{1}{2}(\partial\varphi_1)^2 - \tfrac{1}{2}m_1^2\varphi_1^2 - \frac{\lambda_1}{4}\varphi_1^4 + \tfrac{1}{2}(\partial\varphi_2)^2 - \tfrac{1}{2}m_2^2\varphi_2^2 - \frac{\lambda_2}{4}\varphi_2^4 - \frac{\rho}{2}\varphi_1^2\varphi_2^2$$

| PARAMS | SYMMETRY |
|---|---|
| $m_1, m_2, \lambda_1, \lambda_2, \rho$ | $\varphi_i \longleftrightarrow -\varphi_i$ |
| $m_1 = m_2$, $\lambda_1 = \lambda_2$, $\rho$ | $\varphi_i \longleftrightarrow -\varphi_i$ <br> $\varphi_1 \longleftrightarrow \varphi_2$ |
| $m_1 = m_2$, $\lambda_1 = \lambda_2 = \rho$ | $\varphi_i \longleftrightarrow -\varphi_i$ <br> $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \longleftrightarrow \underbrace{\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}}_{M \in SO(2)}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ |

$\hookrightarrow$ group of $2\times 2$ orthogonal mat. $M^T = M^{-1}$ w $|M| = 1$

NB:
- CONTINUOUS symm. DUE TO $\theta \in [0, 2\pi]$
- Symm. GROUP SINCE $M_1 M_2$ A symm FOR $M_1, M_2 \in SO(2)$ & INVERSES EXIST.

## EXAMPLE: N $\mathbb{R}$ S.F.T.

$$\vec{\varphi} := (\varphi_1, \ldots, \varphi_N)^T$$

$\vec{\varphi} \cdot \vec{\varphi}$ IS SO(N) INV'T.

$$\text{"} (\varphi_1, \ldots, \varphi_N) \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix} = \vec{\varphi}^T \mathbb{1} \vec{\varphi} \xrightarrow{R \in SO(N)} \vec{\varphi}^T R^T \mathbb{1} R \vec{\varphi} = \vec{\varphi} \cdot \vec{\varphi}$$

$$\Longrightarrow \mathcal{L} = \tfrac{1}{2}\left[ (\partial \vec{\varphi})^2 - m^2 \vec{\varphi}^2 \right] - \tfrac{\lambda}{4} (\vec{\varphi}^2)^2 \quad \text{HAS SO(N) SYMMETRY!}$$

## GENERATORS

CONT. SYMM. CAN BE SMALL!

SO(N): $R = e^{\theta \cdot T}$, $\theta \cdot T = \theta^A T^A$ $\quad T^A$ REAL, ANTI-SYMM.

IF $|\theta^A| \ll 1 \ \forall A$, $\quad R \simeq \mathbb{1} + \theta^A T^A$

SO $\quad R_{ab} \varphi_b = (\mathbb{1} + \theta^A T^A)_{ab} \varphi_b =: \varphi_a + \delta \varphi_a$

change in field under symmetry

## $\boxed{\text{NOETHER'S THM}}$

Lang: "Noether's charge"

CONTINUOUS SYMM. GEN $\Longrightarrow$ CONSERVED CURRENT $\Longrightarrow$ CONSERVED CHARGE

PROOF: LAGR. symm. $\Longrightarrow$ $0 = \delta\mathcal{L}$ UNDER $\varphi_a \to \varphi_a + \delta\varphi_a$

$$\Longrightarrow \partial_\mu(\varphi_a) \to \partial_\mu\varphi_a + \underbrace{\partial_\mu\delta\varphi_a}_{=:\delta\partial_\mu\varphi_a}$$

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\varphi_a}\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\partial_\mu\varphi_a$$

$$= \frac{\delta\mathcal{L}}{\delta\varphi_a}\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\partial_\mu\delta\varphi_a \overset{\text{EOM } \frac{\delta\mathcal{L}}{\delta\varphi_a} = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\right)}{=} \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi_a\right)$$

$\Longrightarrow$

$$\boxed{\begin{array}{lll} \text{CONSERVED CURRENT} & J^\mu := \dfrac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi_a & \text{w/ } \partial_\mu J^\mu = 0 \\ \text{CONSERVED CHARGE} & Q := \displaystyle\int d^Dx\, J_0(x) & \text{w/ } \dfrac{dQ}{dt} = 0 \end{array}}$$

same as before

Comment: derivation is similar to the derivation of the Euler-Lagrange equation. This one is about the invariance of the Lagrangian under symmetry transformations, though, whereas the E-L is about finding those configurations that minimize the ~~Lagrangian~~ action. Together, they give Noether's theorem.

We used classical EOM to derive these symmetries, which begs the question, can QM break classically derived symmetries?

Yes, these are called ANOMALIES.

$$i(\varphi^*\partial_\mu\varphi - \partial_\mu\varphi^*\varphi)$$

$\mathbb{C}$ SCALAR CURRENT DERIV. $\mathcal{L} = \partial\varphi^*\partial\varphi - m^2\varphi^*\varphi$

$U(1)$ symmetry: $\varphi \mapsto e^{i\theta}\varphi \qquad \varphi^* \mapsto e^{-i\theta}\varphi$

INF'TES: $\varphi \mapsto \varphi + \underbrace{i\theta\varphi}_{\delta\varphi} \qquad \varphi^* \longmapsto \varphi^* \underbrace{- i\theta\varphi^*}_{\delta\varphi^*}$

$$J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi^a = \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi}\delta\varphi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi^*}\delta\varphi^* = i\theta\varphi\partial^\mu\varphi^* - i\theta\varphi^*\partial^\mu\varphi \quad \checkmark$$

(can drop $\theta$, mult by #, key thing is relative sign!)

# CHARGE AS GENERATORS

NOTE: $$Q := \int d^D x\, J^0(x) = \int d^D x\, \overbrace{\frac{\delta \mathcal{L}}{\delta \partial_0 \varphi_a}}^{\Pi_a} \delta\varphi_a = \int d^D x\, \Pi_a \delta\varphi_a$$

$\therefore$ IN Q. THY

CANON. COMM. RELS

$$i[\hat{Q}, \hat{\varphi}_a] = i\int d^D x\, [\hat{\Pi}_a, \hat{\varphi}_a]\, \delta\hat{\varphi}_a = \delta\hat{\varphi}_a \qquad (*)$$

- MEASURE $\hat{\varphi}$ IN $|S\rangle$ $\qquad \langle S|\hat{\varphi}|S\rangle$
- SYMMETRY $\Longrightarrow$

$$\begin{aligned}
&\overset{!}{=} \langle S|R^\dagger \hat{\varphi} R|S\rangle = \langle S'|\hat{\varphi}|S'\rangle \\
&= \langle S| e^{+i\theta^A \hat{T}_A}\, \hat{\varphi}\, e^{-i\theta^A \hat{T}_A} |S\rangle \\
&\simeq \langle S|\hat{\varphi}|S\rangle + i\theta^A \langle S|[\hat{T}_A, \hat{\varphi}]|S\rangle \\
&=: \langle S|\hat{\varphi}|S\rangle + \langle S|\delta\hat{\varphi}|S\rangle
\end{aligned}$$

HEIS. PICT.

$$\Longrightarrow \delta\hat{\varphi} = +i\theta^A [\hat{T}_A, \hat{\varphi}] \overset{(*)}{\Longrightarrow} \boxed{\hat{Q} = \theta^A \hat{T}_A}$$

IE. **CONSERVED CHARGE GENERATES THE SYMMETRY!**

(fact: if $\hat{Q}$ is the Noether's charge of a gauge symmetry, it acts trivially on H.S.)

# REDUX: what have we learned so far?

1. F.T. INVOLVES DOING A HARD INTEGRAL

$$Z(J) = \int D\varphi \, e^{i\int d^{D+1}x \, \mathcal{L}}$$

2. AMPLITUDES FOR N-PARTICLES, COMPUTE

$$\frac{\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0}}{Z[J]}$$

(brought down $\varphi$'s)

3. PERTURB IN SMALL INTERACTION PARAMS, E.G. $\lambda$

↳ EACH PARAM GIVES VERTEX IN A LITTLE PICTORIAL REP, FEYNMAN DIAGRAMS

4. CAN DO VIA "CANONICAL QUANTIZATION"

$$\varphi \longrightarrow \hat{\varphi} \qquad \pi \longrightarrow \hat{\pi} \qquad [\hat{\pi}(x_1), \hat{\varphi}(x_2)] = -i\,\delta^D(x_1 - x_2)$$

5. SCATTERING AMPLITUDES ENCODED IN S-MATRIX, FUNCTION OF <u>MOMENTA</u>.

6. SYMMETRIES MATTER.

<u>NOETHER</u> CONT. Symm $\Longrightarrow$ CONS. LAW $\Longrightarrow$ CONS CHARGE

# QFT 1

## PART 1: Fondations of Spinors & Vectors

# PART 1 FOUNDATIONS: SPINORS & VECTORS OUTLINE

## Spinors

SPINORS & DIRAC EQN

QUANTIZING DIRAC

PATH INT. & FEYNMAN DIAG. FOR FERMIONS

## Vector Fields

PHYSICAL NEED FOR GAUGE INVARIANCE

FADEEV-POPOV & PROPAGATOR

Q: SPINOR-VECTOR INTERACTIONS?

$\Downarrow$ ← different presentation based on consistency, not "theorist's choice"

↳ derivation of QED Lagrangian

QUANTUM ELECTRODYNAMICS

# Spinors & The Dirac Equation

- so far: scalars fields $\varphi$, spin 0 (no index!)

  vector fields $A_\mu$ spin 1 ($\mu$: Lorentz index)

**Question:** is there some other field w/ some other type of index that gives spin $\frac{1}{2}$?

**STRATEGY:** FIND EQN THAT $\Rightarrow$ K.G. EQN, BUT THAT HAS ITS OWN DIFFERENT SOLN'S.

**LINEAR IN $\partial$?** (haven't tried this yet)

TRY 1: $\partial \sim \sqrt{\partial^2}$. KGE $(\Box + m^2) \Rightarrow$ "$\sqrt{\Box + m^2}$"?

IGNORE MASS FOR NOW.

"$\sqrt{\Box} = \sqrt{\partial_\mu \partial^\mu} = \partial^\mu$" NOT LOR. INV.

TRY 2: SOME LOR. INV. OP $D$ S.T. $D^2 = \Box$

THEN $(D + im)(D - im) = \Box + m^2$ !

TRY 3: GUESS $D = \gamma^\mu \partial_\mu$

$$\Rightarrow D^2 = \gamma^\mu \partial_\mu \gamma^\nu \partial_\nu = \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \tfrac{1}{2} \gamma^\mu \gamma^\nu (\partial_\mu \partial_\nu + \partial_\nu \partial_\mu)$$

$$= \tfrac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu =: \tfrac{1}{2} \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{} \partial_\mu \partial_\nu$$

*Language:* ANTI-COMMUTATOR

THEN $D^2 = \Box$ IF $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$\Rightarrow (D + im)(D - im) = \Box + m^2$

SO $\psi \mid (D + im)\psi = 0$ ALSO SATISFY KGE

*Language:* Clifford Algeb.

$$\boxed{\text{DIRAC EQN: } (i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \text{w/} \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}}$$

↳ <u>note:</u> mult by $i$ relative to above to match convention, perfectly allowed

$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ OBS

1. $(\gamma^0)^2 = 1$
2. $(\gamma^j)^2 = -1 \qquad j = 1,2,3$
3. $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \implies$ NOT NUMBERS!

FACT: $\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \sigma^i$ PAULI MATRICES

SATISFY $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}_{4\times 4}$

Language: $\gamma^\mu$ "gamma matrices." $\psi$ "4-component spinor"

NOTATION: $\not{A} := \gamma^\mu A_\mu \qquad \forall A_\mu$

NEW INDEX!
- $\psi_j, \; j = 1,2,3,4$ ACTED ON BY $\gamma^\mu$
- $j$ DIRAC INDEX, NOT LORENTZ
- $j$ ACTED ON BY $\gamma^\mu$
- SUPPRESS $j$ INDICES COMMON

## LORENTZ TRANSF. OF $\psi$

(rel. between $\gamma^\mu$ & Lorentz trans?)

$A^\mu$ COMPS. TRANS AS 4-VECTORS $A'^\mu = \Lambda^\mu{}_\nu A^\nu$

HOW DO $\psi$ COMPS TRANS? $4\times4$ $S(\Lambda)$

(then rigor. see Zee)

- BUILD $S(\Lambda)$ FROM 16 BASIS ELEMENTS
- $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. $\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
- $\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\} \Rightarrow 1 + 4 + \underset{\uparrow \text{B/C } \sigma^{\mu\nu} = -\sigma^{\nu\mu}}{6} + 4 + 1 = 16$ BASIS
- Lorentz group = 3 ~~boosts~~, 3 rotations $\Rightarrow$ 6-dim

$\Updownarrow$ GEN BY $\sigma^{\mu\nu}$!?

- INTUITION calc. gives

$$\sigma^{0i} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix}$$

DECOMPOSE $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ IN 2-CMPTS

NOTE
$$\Theta_{ij}\sigma^{ij}\psi = 2(\Theta_{12}\sigma^{12} + \Theta_{23}\sigma^{23} + \Theta_{31}\sigma^{31})\psi$$
$$= 2\begin{pmatrix} [\Theta_{12}\sigma^3 + \Theta_{23}\sigma^1 + \Theta_{31}\sigma^2]\,\phi \\ [\qquad " \qquad]\,\chi \end{pmatrix}$$

$\Longrightarrow$ $\Theta_{12}$ ROT $\measuredangle$ ABOUT 3-AX

$\Theta_{23}$ " " 1-AX

$\Theta_{31}$ " " 2-AX

$\phi, \chi$ 2-CMPT SPINORS!

$\sigma^{0k}$'s GEN BOOSTS

RIGOR: depending on conventions, gen might be called $\frac{\sigma}{2} =: J$ (see Peskin; Weinberg)

$\sigma^{\mu\nu}$ GEN. LOR. TRANS ON $\psi$

$\updownarrow$

SAT. LOR. ALG.

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = 2i\left(\eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\mu\rho}\sigma^{\nu\sigma} - \eta^{\nu\sigma}\sigma^{\mu\rho} + \eta^{\mu\sigma}\sigma^{\nu\rho}\right)$$

HMWK: CHECK THIS!

ALGEBRA SUMMARY:

GIVEN OBJECTS $\gamma^\mu$ OF CLIFFORD ALG. $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

THEN $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ SATISFY LORENTZ ALG!!

LORENTZ GROUP (finite, not infinitesimal)

$$S(\Lambda) = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \xrightarrow[\text{TRANS.}]{\text{LOR.}} \psi' = S(\Lambda)\psi$$

BILINEARS HERM $\mathcal{L}$ REQ. LOR. INV. QUAD. TERMS

(remember quad $\Rightarrow$ Gaussian integrals)

FACT: $\gamma^{0\dagger} = \gamma^0 \quad \gamma^{i\dagger} = -\gamma^i$

$$\Rightarrow \sigma^{0i\dagger} = \left(\frac{i}{2}[\gamma^0, \gamma^i]\right)^\dagger = -\frac{i}{2}[\gamma^{i\dagger}, \gamma^{0\dagger}] = -\frac{i}{2}[\gamma^0, \gamma^i] = -\sigma^{0i}$$

SIM $\sigma^{ij\dagger} = \sigma^{ij}$

POINT $S^\dagger = e^{+\frac{i}{4}(2\omega_{0i}\sigma^{0i\dagger} + \omega_{ij}\sigma^{ij\dagger})} = e^{\frac{i}{4}(-2\omega_{0i}\sigma^{0i} + \omega_{ij}\sigma^{ij})} \neq S^{-1}$

$$\Rightarrow (\psi')^\dagger\psi' = \psi^\dagger S^\dagger S\psi \neq \psi^\dagger\psi, \quad \text{NOT L. INV!}$$

INSTEAD: $\bar{\psi}(x) := \psi^\dagger(x)\gamma^0$

$$\bar{\psi}'(x') := \psi(x)^\dagger S^\dagger(\Lambda)\gamma^0 = \bar{\psi}(x)\, e^{\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \implies \boxed{\bar{\psi}\psi \quad \text{LOR. INV.}}$$

GENERAL BILINEARS: $\bar{\psi}\Gamma\psi$, $\Gamma$ ANY $4\times4$ MAT

$\bar{\psi}\psi = \bar{\psi}\,\mathbb{1}\,\psi$ INVT.

Q: HOW DO OTHER $\bar{\psi}\Gamma\psi$'s TRANS.?

$$\Gamma\text{-BASIS}: \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}$$

FACTS

$$[\sigma^{\mu\nu}, \gamma^\lambda] = 2i(\gamma^\mu\eta^{\nu\lambda} - \gamma^\nu\eta^{\mu\lambda})$$

$$\bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}S^{-1}\gamma^\mu S\psi$$

// INFTES

$$\bar{\psi}\left(1 + \tfrac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right)\gamma^\mu \times \left(1 - \tfrac{i}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\right)\psi$$

$$\simeq \bar{\psi}\gamma^\mu\psi + \tfrac{i}{4}\omega_{\rho\sigma}\,\bar{\psi}[\sigma^{\rho\sigma}, \gamma^\mu]\psi$$

$$= \quad '' \quad - \tfrac{1}{2}\bar{\psi}\left(\gamma^\rho\eta^{\sigma\mu} - \gamma^\sigma\eta^{\rho\mu}\right)\omega_{\rho\sigma}\psi$$

$$= \bar{\psi}\left(\gamma^\mu - \tfrac{1}{2}\left[\omega_\rho{}^\mu\gamma^\rho - \gamma^\sigma\omega^\mu{}_\sigma\right]\right)\psi$$

$$= \bar{\psi}\left(\delta^\mu_\sigma + \omega^\mu{}_\sigma\right)\gamma^\sigma\psi$$

NOTE: $$\omega^\mu{}_\sigma = \tfrac{1}{2}[\omega_{\alpha\sigma} - \omega_{\sigma\alpha}]\eta^{\alpha\mu} = \left(\tfrac{-i}{2}\right)\underbrace{\eta^{\alpha\mu}\, i[\delta^\chi_\alpha\delta^\xi_\sigma - \delta^\chi_\sigma\delta^\xi_\alpha]}_{(J^{\chi\xi})^\mu{}_\sigma}\omega_{\chi\xi}$$

GEN LOR ACG TOO!

$$\implies = \bar{\psi}\Big(\underbrace{\delta^\mu{}_\sigma - \tfrac{i}{2}\omega_{\chi\xi}(J^{\chi\xi})^\mu{}_\sigma}_{\text{INFTES OF } e^{-\frac{i}{2}\omega_{\chi\xi}(J^{\chi\xi})^\mu{}_\sigma} = \Lambda^\mu{}_\sigma}\Big)\gamma^\sigma\psi$$

$$\implies \boxed{\begin{gathered}\bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}S^{-1}(\Lambda)\gamma^\mu S\psi = \bar{\psi}\Lambda^\mu{}_\sigma\gamma^\sigma\psi \\ \implies \text{TRANS AS 4-VECTOR!}\end{gathered}}$$

# DIRAC LAGRANGIAN

(we know how to construct Lorentz invts now, so let's write $\mathcal{L}$)

$$\boxed{\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi}$$

CHECK: EOM = DIRAC EQN

check

$$\mathcal{L} \overset{IBP}{=} -i\partial_\mu \bar{\psi}\gamma^\mu\psi - m\bar{\psi}\psi$$

EOM: $0 = \partial_\mu\left(\frac{\delta\mathcal{L}}{\partial_\mu\bar{\psi}}\right) - \frac{\delta\mathcal{L}}{\delta\bar{\psi}} = -i\partial_\mu\gamma^\mu\psi + m\psi$

$$\Rightarrow \quad (i\not{\partial} - m)\psi = 0 \quad \checkmark$$

# PARITY = SPATIAL REFL.

$$x^\mu \to x'^\mu = (x^0, -\vec{x})$$

$$\text{D. EQN} \xrightarrow{\gamma^0\cdot} \gamma^0(i\not{\partial} - m)\psi = 0 \xrightarrow{P} \gamma^0(i\gamma^0\partial_0 \overset{NB}{+} i\gamma^j\partial_j - m)\psi = 0$$

$$= (i\gamma^0\partial_0 - i\gamma^j\partial_j - m)\gamma^0\psi = 0$$

$$\{\gamma^0, \gamma^j\} = 0 \longrightarrow \; = (i\not{\partial}' - m)\gamma^0\psi =: (i\not{\partial}' - m)\psi'$$

so $\quad \psi'(x') = \gamma^0\psi(x)$

$$\Rightarrow \quad \bar{\psi}'\psi' = \bar{\psi}\gamma^{0\dagger}\gamma^0\psi = \bar{\psi}\gamma^{0\,2}\psi = \bar{\psi}\psi \longrightarrow \text{Lang: Parity even}$$

$$\bar{\psi}'\gamma^5\psi' = \bar{\psi}\gamma^0\gamma^5\gamma^0\psi = i\bar{\psi}\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\psi = i(-)^3\bar{\psi}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^{0\,2}\psi$$

$$= -\bar{\psi}\gamma^5\psi \longrightarrow \text{Lang: parity odd}$$

Language:
- Parity even + Lorentz inv = "scalar". eg $\bar{\psi}\psi$
- " odd " " = "pseudoscalar" eg $\bar{\psi}\gamma^5\psi$

# CHIRALITY

PROJECTIONS: $P_L := \frac{1}{2}(1-\gamma^5)$ $\quad P_R := \frac{1}{2}(1+\gamma^5)$

$$\gamma^{5^2} = \mathbb{1} \implies P_L^2 = \frac{1}{4}(1-2\gamma^5+\gamma^{5^2}) = P_L, \quad P_R^2 = P_R, \quad P_L P_R = 0$$

$$\psi_L := P_L\psi \qquad \psi_R := P_R\psi$$

NB: $\gamma^5\psi_L = \frac{1}{2}(\gamma^5 - \gamma^{5^2})\psi = -\psi_L, \qquad \gamma^5\psi_R = \psi_R$

WEYL BASIS $:=$ $\gamma^5$ DIAG, NOT $\gamma^0$

$$\implies \gamma^0 = \begin{pmatrix} & I \\ I & \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -I & \\ & I \end{pmatrix}$$

$$P_L\psi = P_L\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = \begin{pmatrix}\frac{1}{2}(I+I) & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = \begin{pmatrix}\psi_A \\ 0\end{pmatrix} = \psi_L$$

SIM. $\quad P_R\psi = \begin{pmatrix}0 \\ \psi_B\end{pmatrix} = \psi_R \implies \boxed{\psi \overset{\text{Weyl BAS.}}{=} \psi_L + \psi_R}$

$$\implies \mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$$

NB: $m \to 0 \implies \psi_L\ \psi_R$ EOM DECOUPLED!

$$i\not{\partial}\psi_L = 0 \qquad i\not{\partial}\psi_R = 0$$

# PARITY VIOLATION

NB $\quad \psi(x) \overset{P}{\longmapsto} \psi'(x') = \gamma^0\psi(x) \qquad$ SO $\quad \psi_L \overset{P}{\longleftrightarrow} \psi_R$

$\Big\downarrow$ Weyl Bas

$$\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \overset{P}{\longmapsto} \begin{pmatrix} & I \\ I & \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = \begin{pmatrix}\psi_R \\ \psi_L\end{pmatrix}$$

FERMI'S WEAK THY $\quad \mathcal{L} \supset G\,\bar{\psi}_{1L}\gamma^\mu\psi_{2L}\,\bar{\psi}_{3L}\gamma_\mu\psi_{4L} \implies$ P-BREAKING!

## SYMMETRIES

1. GLOBAL $U(1)$

$$\psi \to e^{i\theta}\psi = (1+i\theta)\psi \quad \Rightarrow \quad \delta\psi = i\theta\psi$$

NOETHER $\quad J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta\psi = i\bar{\psi}\gamma^\mu\psi \quad$ (can redefine w/o $\theta$)

$$\partial_\mu J^\mu = 0$$

check signs, i's

2. CHIRAL SYMMETRY:

@ $m=0$, $\quad e^{i\phi\gamma^5}\psi \quad$ A symm $\qquad$ HMWK: CHECK

$$\delta\psi = i\phi\gamma^5\psi \qquad J^{5\mu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta\psi = \bar{\psi}\gamma^\mu\gamma^5\psi$$

## BRIEF INTERACTIONS:

(just to get charge conjugation)

L.INV. COUPLINGS TO DIRAC BILIN.

EG

| | | |
|---|---|---|
| Scalar $\varphi$ | $\varphi\bar{\psi}\psi$ | "Yukawa Coupling" |
| Pseudo-scalar $\theta$ | $\theta\bar{\psi}\gamma^5\psi$ | |
| Vector $A_\mu$ | $\bar{\psi}\gamma^\mu A_\mu\psi = \bar{\psi}\not{A}\psi$ | |
| Axial Vector $B_\mu$ | $\bar{\psi}\gamma^5\gamma^\mu B_\mu\psi = \bar{\psi}\gamma^5\not{B}\psi$ | |

## MASSIVE QED

$$\mathcal{L} = \mathcal{L}_{vec}^{free} + \mathcal{L}_{Dir}^{free} + \mathcal{L}_{int} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu + \bar{\psi}(i\not{\partial} - m)\psi + eA_\mu\bar{\psi}\gamma^\mu\psi$$

$\mu \to 0$: $\quad$ NEW symmetry

QED!

EOM: $\quad \left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi = 0$

## CHARGE CONJ & ANTIMATTER

$(\text{EOM})^* \longrightarrow \left[-i\gamma^{\mu *}(\partial_\mu + ieA_\mu) - m\right]\psi^* = 0 \qquad (*)$

$(\text{Clifford Alg})^* \longrightarrow \{\gamma^{\mu *}, \gamma^{\nu *}\} = 2\eta^{\mu\nu *} = 2\eta^{\mu\nu}$

$\Longrightarrow$ $\boxed{\gamma^{\mu *} \text{ ALSO SAT. CLIFF ALG!}}$

WHY? $\gamma^* = \gamma$ IN DIFF BASIS

$\exists\ M = C\gamma^0$ S.T. $-\gamma^{\mu *} = (M)^{-1}\gamma^\mu (M)$

$$\Longrightarrow \{\gamma^{\mu *}, \gamma^{\nu *}\} = M^{-1}\gamma^\mu M M^{-1}\gamma^\nu M + M^{-1}\gamma^\nu M M^{-1}\gamma^\mu M$$
$$= M\{\gamma^\mu, \gamma^\nu\}M^{-1} = 2\eta^{\mu\nu}\, M \mathbb{1} M^{-1} = 2\eta^{\mu\nu}$$

$$\Longrightarrow \quad (*) = \left[+iM^{-1}\gamma^\mu M(\partial_\mu + ieA_\mu) - m\right]\psi^* = 0$$

$$\overset{M\cdot}{\Longrightarrow} \left[i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right]M\psi^* = 0$$

$\psi_c := M\psi^*$ & $\psi$ SOLVE D. EQN.

SAME MASS, OPPOSITE CHG! $\Longrightarrow$ $\boxed{\text{ANTIMATTER}}$ EG $e^+ e^-$

(see Zee for Majorana, time reversal, CPT)

# Quantizing the Dirac Field

Similar to before...

LAGRANGIAN: $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

EQN'S OF MOT.: $(i\not{\partial} - m)\psi = 0$

CONJUGATE MOMENTA: $\Pi_\alpha = \frac{\delta \mathcal{L}}{\delta \partial_0 \psi_\alpha} = \frac{\delta}{\delta \partial_0 \psi_\alpha}\left(i\psi^\dagger \gamma^0 \gamma^0 \partial_0 \psi_\alpha\right) = i\psi^\dagger$

QUANTIZATION: $\psi_\alpha \longrightarrow \hat{\psi}_\alpha$

$\Pi_\alpha \longrightarrow \hat{\Pi}_\alpha$

S.T. $\{\hat{\psi}_\alpha(\vec{x},t), \hat{\Pi}_\beta(0,t)\} = i\delta^3(x)\delta_{\alpha\beta} \implies \{\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}^\dagger_\beta(0,t)\} = \delta^3(x)\delta_{\alpha\beta}$

(will justify, i) fermi ex / Pauli excl. principle
ii) positive energy)

PLANE WAVE SOLN'S:

$u(p,s)e^{-ip\cdot x} \qquad v(p,s)e^{+ip\cdot x}$

REST FRAME $\overset{\text{EOM}}{(\not{p} - m)u(p,s) = 0} \implies (\gamma^0 - 1)u = 0$

& $\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} \implies$ soln's $\begin{pmatrix}1\\0\\0\\0\end{pmatrix} =: u(p=(m,\vec{0}), s=1) =$

$\begin{pmatrix}0\\1\\0\\0\end{pmatrix} =: u(p=(m,\vec{0}), s=-1) =$

SIM. $v(p=(m,\vec{0}), s=\pm) := \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$\implies \sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = \begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}_{\alpha\beta} = \frac{1}{2}(\gamma^0 + 1)_{\alpha\beta}$$

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = \frac{1}{2}(\gamma^0 - 1)_{\alpha\beta} \qquad \text{REST FRAME}$$

GEN LHS. LINV, p-DEP

$$\implies \boxed{\begin{aligned} \sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) &= \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} \\ \sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) &= \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} \end{aligned}}$$

FOURIER EXPANSION: (now that we have plane waves)

$$\hat{\psi}_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}\left(E_p/m\right)^{\frac{1}{2}}} \sum_s \left[\hat{b}(p,s)\,u_\alpha(p,s)\,e^{-ip\cdot x} + \hat{d}^\dagger(p,s)\,v_\alpha(p,s)\,e^{ip\cdot x}\right]$$

$\hookrightarrow$ (slightly different normalization)

$$\{\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}^\dagger_\beta(0,t)\} = \delta^3(x)\,\delta_{\alpha\beta} \quad \& \quad \text{ORTH OF CLASS. SOL}$$

$$\Longrightarrow \quad \{d(p,s), d^\dagger(p',s')\} = \{b(p,s), b^\dagger(p',s')\} = \delta^3(\vec{p}-\vec{p}\,')\,\delta_{ss'} \qquad \text{REST } 0$$

## HAMILTONIAN & ANTICOMM.

$$\mathcal{H} = \pi\frac{\partial\psi}{\partial t} - \mathcal{L} = (i\psi^\dagger)\gamma^0\gamma^0\partial_t\psi - \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}(i\,\vec{\gamma}\cdot\vec{\nabla} + m)\psi$$

$$H = \int d^3x\,\mathcal{H} = \int d^3x\,\bar{\psi}(i\,\vec{\gamma}\cdot\vec{\nabla} + m)\psi \overset{\text{DIR EQN}}{=} \int d^3x\,\bar{\psi}(i\gamma^0\partial_0 - m + m)\psi = \int d^3x\, i\bar{\psi}\gamma^0\partial_0\psi$$

$\bar{\psi}\partial_t\psi \sim (b^\dagger + d)(b - d^\dagger) = b^\dagger b - dd^\dagger$

$$\overset{\text{CHUG}}{=} \int d^3p \sum_s E_p\left[b^\dagger(p,s)\,b(p,s) - d(p,s)\,d^\dagger(p,s)\right]$$

✗ SUPPOSE COMMUTE $[d(p',s'), d^\dagger(p,s)] = \delta^3(p'-p)\,\delta_{ss'}$

$$H = \int d^3p \sum E_p\left[b^\dagger(p,s)\,b(p,s) - \underbrace{d^\dagger(p,s)\,d(p,s)}_{N_d(p,s)} + \ldots\right.$$

$N_d(p,s)$, # OP FOR $d$-PART.

$\Longrightarrow$ ADDING PARTICLES LOWERS E! SICK

✓ ANTICOMMUTE: $\{d(p',s'), d^\dagger(p,s)\} = \delta^3(p'-p)\,\delta_{s's}$

$$H = \int d^3p \sum E_p\left[\underbrace{b^\dagger(p,s)\,b(p,s)}_{N_b(p,s)} + \underbrace{d^\dagger(p,s)\,d(p,s)}_{N_d(p,s)}\right] - \delta^3(\vec{0})\int d^3p \sum E_p$$

$\Longrightarrow$ BOTH PARTICLE TYPES RAISE E!

# CHARGE & ANTIMATTER

GLOBAL $U(1)$ CURR. $\quad J^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} \delta\psi = i\bar{\psi}\gamma^\mu\psi \qquad \partial_\mu J^\mu = 0$

$$Q := \overset{\text{conv}}{-i} \int d^3x\, J_0 = \int d^3x\, \psi^\dagger(x)\psi(x)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[ b^\dagger(p,s)\, b(p,s) - d^\dagger(p,s)\, d(p,s) \right]$$

$$\Rightarrow \quad Q\, b^\dagger(p',s')|0\rangle = \int \frac{d^3p}{(2\pi)^3} \sum_s b^\dagger(p,s)\, b(p,s)\, b^\dagger(p',s')|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s b^\dagger(p,s) \{ b(p,s), b^\dagger(p',s') \} |0\rangle = \int \frac{d^3p}{(2\pi)^3} b^\dagger(p,s)\, \delta^3(p-p')\, \delta_{ss'} |0\rangle$$

$$= + b^\dagger(p',s')|0\rangle$$

Sim. $\quad Q\, d^\dagger(p',s')|0\rangle = - d^\dagger(p',s')|0\rangle$

$b^\dagger$ CREATES $Q=1$ PARTICLE

$d^\dagger$ CREATES $Q=-1$ ANTIPARTICLE

# PAULI EXCLUSION PRINCIPLE

- TRY TO CREATE 2 FERM w/ SAME $p, s$

$$b^\dagger(p,s)\, b^\dagger(p,s)|0\rangle = \tfrac{1}{2} \{ b^\dagger(p,s), b^\dagger(p,s) \} |0\rangle = 0 \qquad \text{CAN'T} \Rightarrow \text{PEP}$$

NB
- $\{\cdot,\cdot\}$ CRITICAL
- $[\cdot,\cdot]$ WOULDN'T $\Rightarrow$ P.E.P
- $\exists$ POS. SPACE VERSION

## PROPAGATOR

(recall scalar $iD_F(x-y) := \langle 0|T[\phi(x)\,\phi(y)]|0\rangle$)

$$iS_{\alpha\beta}(x-y) := \langle 0|T[\psi_\alpha(x)\bar\psi_\beta(y)]|0\rangle$$

$$T[\psi(x)\bar\psi(0)] := \Theta(x^0)\,\psi(x)\,\bar\psi(0) - \Theta(-x^0)\,\bar\psi(0)\,\psi(x)$$

due to anti comm

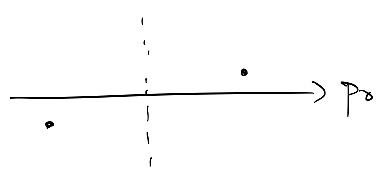
(few lines of alg $\Rightarrow$)

$$iS_{\alpha\beta}(x) = \int \frac{d^3p}{(2\pi)^3\left(E_p/m\right)}\left[\Theta(x^0)\left(\frac{\not p+m}{2m}\right)_{\alpha\beta} e^{-ip\cdot x} - \Theta(-x^0)\left(\frac{\not p-m}{2m}\right)_{\alpha\beta} e^{ip\cdot x}\right] \qquad (*)$$

AS 4-D INT: $\boxed{iS_{\alpha\beta}(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \frac{i}{\not p - m + i\varepsilon}}$

PF: $$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \frac{\not p + m}{p^2-m^2+i\varepsilon} = \int \frac{d^3p}{(2\pi)^4}\, e^{+i\vec p\cdot\vec x} \int dp_0\, e^{-ip_0x_0}\, \frac{\not p+m}{p_0^2-\vec p^{\,2}-m^2+i\varepsilon}$$

$\Longrightarrow$ POLES @ $p_0 = \pm\sqrt{\vec p^{\,2}+m^2-i\varepsilon} = \pm\sqrt{E_p^2-i\varepsilon} \simeq \pm\left(E_p - \frac{i\varepsilon}{2E_p}\right)$




$x^0 > 0 \Longrightarrow e^{-ip_0x_0} \ll 1$ FOR $\mathrm{Im}(p_0) < 0$

$\Longrightarrow$ CLOSE BELOW

$x^0 < 0 \Longrightarrow$ CLOSE ABOVE

CONTOUR + ALG $\Longrightarrow$ (*)

HMWK: CHECK

# MOMENTUM SPACE (see from above)

$$S(p) = \frac{1}{\not{p} - m + i\varepsilon} \Longleftarrow \text{inverse of Dirac op } \not{p} - m$$

COMPARE SCALAR

$$D(k) = \frac{i}{k^2 - m^2 + i\varepsilon} \Longleftarrow \text{inv of mom. space K.G. op } k^2 - m^2$$

Q: how would you compute interactions from here?

# Path Integrals for Fermions

Q: How does anticommutation of fermions manifest itself in the path integral?

Zee has a long discussion of vacuum energy that you may find useful. For brevity we will introduce some formalism & see it reproduces the prop.

# ANTICOMMUTING #'S   Language: Grassmann #'s

IDEA: $\eta\xi = -\xi\eta$   NB $\Rightarrow \eta^2 = 0$

ASSUMPTION: TAYLOR SERIES EXISTS

$$\Rightarrow f(\eta) = a + b\eta + c\cancel{\eta^2}^{\to 0} + d\cancel{\eta^3}^{\to 0} + \dots = a + b\eta$$

DIFFERENTIATION   $\frac{\partial}{\partial\eta}\eta := 1$

$$\Rightarrow \frac{\partial}{\partial\eta}(\xi\eta) = -\frac{\partial}{\partial\eta}(\eta\xi) = -\xi = -\xi\frac{\partial}{\partial\eta}\eta$$

$$\Rightarrow \left(\frac{\partial}{\partial\eta}\xi + \xi\frac{\partial}{\partial\eta}\right)\eta = 0 \quad \forall \eta \quad \text{so} \quad \boxed{\left\{\frac{\partial}{\partial\eta}, \xi\right\} = 0}$$

similarly $\boxed{\left\{\frac{\partial}{\partial\eta}, \frac{\partial}{\partial\xi}\right\} = 0}$

INTEGRATION:

(i) SINCE $\int_{-\infty}^{\infty} dx\, f(x+c) = \int_{-\infty}^{\infty} dx\, f(x)$ DEMAND $\int d\eta\, f(\eta+\xi) = \int d\eta\, f(\eta)$

$$\Rightarrow \int d\eta\,[a + b(\eta+\xi)] = \int d\eta\,[a + b\eta] \Rightarrow \int d\eta\; b\xi = 0$$

TRUE $\forall \xi \Rightarrow \boxed{\int d\eta\; b = 0}$

(ii) $0 \neq \int d\theta\, f(\theta) = \int d\theta\,(a + b\theta) = \cancel{\int d\theta\, a}^{\to 0} + b\int d\theta\,\theta$

$$\Rightarrow \int d\theta\,\theta \neq 0, \quad \text{NORMALIZE} \quad \boxed{\int d\theta\,\theta = 1}$$

NB: DIFF = INT

$$\frac{d}{d\eta} f(\eta) = \frac{d}{d\eta}(a + b\eta) = b \qquad \int d\eta\, f(\eta) = \int d\eta\,(a + b\eta) = b$$

# GRASSMANN GAUSSIANS

$$\int d\eta \int d\bar\eta \; e^{\bar\eta a \eta} = \int d\eta \int d\bar\eta \; (1 + \bar\eta a \eta) = a = e^{\log a}$$

## N GRASSMANNS

need enough $\eta$'s, $\bar\eta$'s to be $\neq 0$!

$$\int d\eta_1 \ldots d\eta_N \, d\bar\eta_1 \ldots d\bar\eta_N \; e^{\bar\eta_i A_{ij} \eta_j} = \frac{1}{n!} \int \text{SAME} \prod_{k=1}^{N} \bar\eta_{i_k} A_{i_k j_k} \eta_{j_k}$$

$$= \frac{1}{n!} \varepsilon_{i_1 \ldots i_N} \varepsilon_{j_1 \ldots j_N} A_{i_1 j_1} \ldots A_{i_N j_N} = \det A$$

PESKIN: DIAG.

$$\left( \prod_i \int d\theta_i^* \, d\theta_i \right) e^{-\theta_i^* B_{ij} \theta_j} = \left( \prod_i \int d\theta_i^* \, d\theta_i \right) e^{-\sum_i \theta_i^* b_i \theta_i}$$

$$= \prod_i b_i = \det B$$

SIM: $\left( \prod_i \int d\theta_i^* \, d\theta_i \right) \theta_k \theta_\ell^* \, e^{-\theta_i^* B_{ij} \theta_j} = \det B \; (B^{-1})_{k\ell}$

$\Uparrow$

need for 2-point functions & eval. in general

Q: having seen scalars, where will these show up?

# GRASSMANN PATH INTEGRAL & DIRAC PROPAGATOR

RECALL SCALAR: $Z[J] = \int D\varphi\, e^{i\int d^4x \frac{1}{2}[(\partial\varphi)^2 - (m^2 - i\varepsilon)\varphi^2 + J\varphi]}$

$\Longrightarrow$ SPINOR ANALOG

$$Z[\eta,\bar\eta] = \int d\psi \int d\bar\psi \; e^{i\int d^4x\, \bar\psi \overbrace{(i\not\partial - m + i\varepsilon)}^{=K}\psi + \bar\eta\psi + \bar\psi\eta}$$

$\eta, \bar\eta$: spinor sources

COMPLETE THE SQUARE

$$\bar\psi K \psi + \bar\eta\psi + \bar\psi\eta = (\bar\psi + \bar\eta K^{-1})\, K\,(\psi + K^{-1}\eta) - \bar\eta K^{-1}\eta$$

$$\Longrightarrow Z(\eta,\bar\eta) = C'' e^{-i\bar\eta \underbrace{(i\not\partial - m)^{-1}}\eta}$$

as w/ scalar, related to propagator. Connects $\eta(x)$ source w/ $\bar\eta(x)$ sink

$$(i\not\partial - m)\underset{\text{prop}}{S(x)} := \delta^{(4)}(x)$$

claim

$$\boxed{S(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{\not p - m + i\varepsilon}}$$

Pf: $(i\not\partial - m)\, S(x) = \int \frac{d^4p}{(2\pi)^4} \frac{(-i)(i)\gamma^\mu p_\mu - m}{\not p - m + i\varepsilon} e^{-ip\cdot x} = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} = \delta^4(x)$ ✓

lim $\varepsilon \to 0$ as w/ scalar prop

# INTERACTING FERMIONS

TAKE $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2}[(\partial\varphi)^2 - \mu^2\varphi^2] - \lambda\varphi^4 + f\varphi\bar{\psi}\psi + \bar{\eta}\psi + \eta\bar{\psi} + J\varphi$

$= \mathcal{L}^f_{free} + \mathcal{L}^s_{free} + \mathcal{L}^s_{int} + \mathcal{L}^{f\text{-}s}_{int}$

## S-POINT FUNCTIONS:

RECALL

$$\langle x^{2n}\rangle := \frac{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2}} = \frac{\left(-2\frac{d}{da}\right)^n Z}{Z} = \underbrace{(2n-1)!!}_{\#\text{ of ways } 2n \text{ points connect}}\frac{1}{a^n}$$

RECALL: SCALAR CASE (just to see this example again)

$$Z(J) = \sum_{s=0}^{\infty} \frac{i^s}{s!}\int dx_1 \dots dx_s\, J(x_1)\dots J(x_s) \int D\varphi\, e^{i\int d^4x \{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] - \frac{\lambda}{4!}\varphi^4\}} \varphi(x_1)\dots\varphi(x_s)$$

$$=: Z(0,0)\sum_{s=0}^{\infty}\frac{i^s}{s!}\int dx_1\dots dx_s\, J(x_1)\dots J(x_s)\, G^{(s)}(x_1,\dots,x_s)$$

CAN PICK OFF $G^{(s)}$

$$G^{(s)}(x_1,\dots,x_s) = \frac{1}{i^s}\frac{\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_s)} Z[J]\Big|_{J=0}}{Z(0,0)}$$

$\lambda$-EXPAND

$$G^{(s)}(x_1,\dots,x_s) := \sum_{k=0}^{\infty}\frac{1}{k!}\, G^{(s)}(x_1,\dots x_s)\Big|_{O(\lambda^k)}$$

$\lambda^k$ factor appears in here

THEN

$$G^2(x_1,x_2)\Big|_{O(\lambda)} = \frac{1}{Z(0,0)}\int D\varphi\, e^{i\int d^4x\{[(\partial\varphi)^2 - m^2\varphi^2]\}}\varphi(x_1)\varphi(x_2)\left[\frac{-i\lambda}{4!}\int d^4y\, \varphi(y)\varphi(y)\varphi(y)\varphi(y)\right]$$

$$= \frac{-i\lambda}{4!}\int d^4y\left[\frac{\int D\varphi\, e^{\frac{1}{2}\int d^4x\, \varphi\overbrace{\{i(\Box+m^2)\}}^{:=K}\varphi}\varphi(x_1)\varphi(x_2)\varphi(y)\varphi(y)\varphi(y)\varphi(y)}{\int D\varphi\, e^{\frac{1}{2}\int d^4x\, \varphi\{i(\Box+m^2)\}\varphi}}\right]$$

- EVALUATE USING GAUSSIAN INT.
- COME UP w/ PICTURES & RULES

FERMIONS EXPAND IN $J, \eta, \bar\eta$

$$Z[J,\eta,\bar\eta] = \int D\varphi D\bar\psi D\psi \; e^{i\int d^4x \mathcal{L} \text{ + sources}}$$

$$=: \sum_{s_\psi=0}^{\infty} \sum_{s_{\bar\psi}=0}^{\infty} \sum_{s_\varphi=0}^{\infty} \frac{(i)^{s_\psi+s_{\bar\psi}+s_\varphi}}{s_\psi!\, s_{\bar\psi}!\, s_\varphi!} \int dx_1 \dots dx_{s_\psi}\, dy_1 \dots dy_{s_{\bar\psi}}\, dz_1 \dots dz_{s_\varphi}$$

$$\times\; \bar\eta(x_1)\dots\bar\eta(x_{s_\psi})\,\eta(y_1)\dots\eta(y_{s_{\bar\psi}})\,J(z_1)\dots J(z_{s_\varphi}) \int D\varphi D\psi D\bar\psi\; e^{i\int d^4x \mathcal{L}}$$

$$\times\; \psi(x_1)\dots\psi(x_{s_\psi})\,\bar\psi(y_1)\dots\bar\psi(y_{s_{\bar\psi}})\,\varphi(z_1)\dots\varphi(z_{s_\varphi})$$

sources, couplings

$$=: Z(0,0) \sum_{s_\psi=0}^{\infty} \sum_{s_{\bar\psi}=0}^{\infty} \sum_{s_\varphi=0}^{\infty} \frac{(i)^{s_\psi+s_{\bar\psi}+s_\varphi}}{s_\psi!\, s_{\bar\psi}!\, s_\varphi!} \int dx_1 \dots dx_{s_\psi}\, dy_1 \dots dy_{s_{\bar\psi}}\, dz_1 \dots dz_{s_\varphi}$$

$$\times\; \bar\eta(x_1)\dots\bar\eta(x_{s_\psi})\,\eta(y_1)\dots\eta(y_{s_{\bar\psi}})\,J(z_1)\dots J(z_{s_\varphi})\; G^{(s_\psi, s_{\bar\psi}, s_\varphi)}(x's, y's, z's)$$

METHOD

① PICK $s_\psi, s_{\bar\psi}, s_\varphi$ FOR PROCESS OF INTEREST

② COMPUTE $G^{(s_\psi, s_{\bar\psi}, s_\varphi)}$ TO DESIRED ORDER IN COUPLINGS

③ IF DESIRED, COME UP W/ FEYNMAN RULES.

EXAMPLES

$(s_\psi, s_{\bar\psi}, s_\varphi) = (1,1,0)$

$\supset \int d^4z\; i\bar\psi(i\gamma - m)\psi + \dots$

$$\mathcal{O}(f^0):\quad G^{(2)}(x,y)\Big|_{\mathcal{O}(f^0)} = \frac{\int D\varphi D\psi D\bar\psi\; e^{i\int d^4z \mathcal{L}_{free}}\bar\psi(x)\psi(y)}{\int D\varphi D\psi D\bar\psi\; e^{i\int d^4z \mathcal{L}_{free}}} = i\,(i\not\partial - m)^{-1}\Big|_{x-y}$$

$$= i\,S(x-y)$$

IE: FERMION 2-pt FUNCTION @ $\mathcal{O}(f^0)$

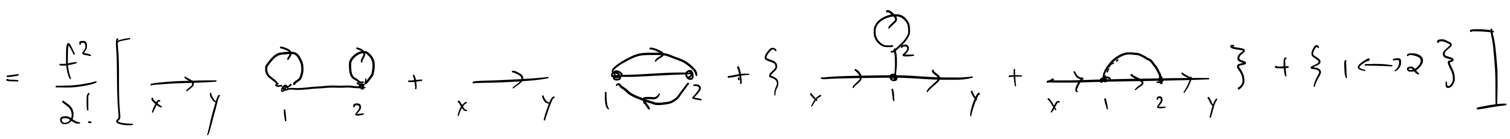
= PROP

$\mathcal{O}(f^2)$: $\quad S_{xy} := iS(x-y) \quad D_{xy} := iD(x-y)$ etc

$$G^{(2)}(x,y)\Big|_{\mathcal{O}(f^2)} = \frac{\int D\varphi\, D\psi\, D\bar\psi\; e^{i\int d^4z\, \mathcal{L}_{free}}\, \bar\psi(x)\,\psi(y)\, \frac{f^2}{2!}\int d^4z_1 \int d^4z_2\; \varphi(z_1)\bar\psi(z_1)\psi(z_1)\varphi(z_2)\bar\psi(z_2)\psi(z_2)}{\int D\varphi\, D\psi\, D\bar\psi\; e^{i\int d^4z\, \mathcal{L}_{free}}}$$

NOTE: not squared! due to direction

$$= \int d^4z_1\, d^4z_2\; \frac{f^2}{2!}\Big[ S_{xy} D_{12} S_{11} S_{22} + S_{xy} D_{12} S_{12} S_{21} + \{ S_{x1} D_{12} S_{1y} S_{22} + S_{x1} D_{12} S_{2y} S_{12} \} + \{ 1 \leftrightarrow 2 \} \Big]$$




$\Longrightarrow\ \exists$ FEYNMAN RULES (see book! did these in EP class)

# <u>Gauge Fields</u>

<u>Here</u>: path integral way more straightforward

# III.4 GAUGE INVARIANCE (note: skipping ahead in Zee)

- what does it do?
- Why does it matter?
- how do we derive the photon propagator?

Recall

## PROPAGATORS $\varphi$ ARB FIELDS (point is the schematic form)

SCHEMATIC

$$Z_{free} = \int D\varphi \, e^{-\frac{1}{2}\varphi \cdot K \cdot \varphi + J\varphi} = C e^{\frac{1}{2} J \cdot K^{-1} \cdot J}$$

$K^{-1}$ = PROPAGATOR

Q: WHAT IF K HAS NO INVERSE!?

RECALL $K^{-1}$ D.N.E $\iff \exists$ 0-EIG $\iff Kv = 0$ FOR SOME $v$

## MAXWELL ACTION

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu$$

$$F_{\mu\nu} F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = 2[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu]$$

$$\Rightarrow \mathcal{L} = -\frac{1}{2}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu] + A_\mu J^\mu = \frac{1}{2} A_\mu [\eta^{\mu\nu}\Box - \partial^\mu \partial^\nu] A_\nu + A_\mu J^\mu$$

HERE K IS $Q^{\mu\nu} := \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu$

NOTE i) $Q^{\mu\nu}\partial_\nu \Lambda(x) = (\eta^{\mu\nu}\partial_\rho\partial^\rho\partial_\nu - \partial^\mu\partial^\nu\partial_\nu)\Lambda = (\partial^\mu\Box - \partial^\mu\Box)\Lambda = 0$

$\Rightarrow$ $\boxed{Q^{\mu\nu} \text{ HAS NO INVERSE!}}$

ii) $Q_m^{\mu\nu} := Q^{\mu\nu} + m^2 \eta^{\mu\nu}$

HAS AN INVERSE.

$\Rightarrow$ $\boxed{\text{ISSUE IS } m \to 0 \text{ LIMIT}}$

iii) YOU KNOW THIS PHENOMENON

2 MAX. EQN'S $\partial_\mu F^{\mu\nu} = J^\nu$

rename some ind

REWRITE $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box \eta^{\nu\sigma} A_\sigma - \partial^\nu \partial^\sigma A_\sigma = Q^{\mu\nu} A_\mu$

SO MAX. EQN'S: $Q^{\mu\nu} A_\nu = J^\mu \implies A_\sigma = (Q^{-1})_{\sigma\mu} Q^{\mu\nu} A_\nu$

$= Q^{-1}_{\sigma\mu} J^\mu$

PT 1| IF $Q^{-1}$ EXISTED $A$ TOTALLY FIXED

PT 2| E-M: $A_\mu$ ISN'T FIXED!

$A_\mu$ SOLVES EOM $\implies$ $A_\mu - \partial_\mu \Lambda$ SOLVES EOM

$Q^{\nu\mu} A_\mu = 0$ $\qquad Q^{\nu\mu}[A_\mu - \partial_\mu \Lambda] = Q^{\nu\mu}\partial_\mu \Lambda = 0$

last page

GAUGE FIX: IMPOSE EXTRA CONSTRAINT

ASIDE: GLOBAL VS. GAUGE TRANSFORMATIONS

| | GAUGE | GLOBAL |
|---|---|---|
| $\Lambda$: | $\Lambda(x)$ | $\Lambda$ CONST |
| $A'_\mu = A_\mu - \partial_\mu \Lambda$ | $A'$ DIFF. | $A'$ SAME |
| $\partial_\mu \Lambda$ | 0-EVEC OF $Q^{\mu\nu}$ | $= 0$ $\implies$ 0-VECTOR |

POINT: i) $K\vec{0} = \vec{0}$ ALWAYS $\implies$ NO PROB FOR $(Q^{\mu\nu})^{-1}$

i) GAUGE "$Kv$" $= Q^{\mu\nu}\partial_\nu \Lambda$ CAUSE $Q^{-1}$ NOT TO EXIST B/C $\Lambda = \Lambda(x)$

SEE: GAUGE NOT GLOBAL HAS PATH INTEGRAL ISSUES

# FADEEV-POPOV: PATH INTEGRALS & GAUGE INVARIANCE

(usually introduced for non-abelian gauge symmetry, but it's actually better to start with it here)

(use toy integral to make clear what the issue is)

**TOY MODEL** $\displaystyle\int_{-\infty}^{\infty} dA\, e^{-A\cdot K\cdot A}$

TAKE $A = \begin{pmatrix} a \\ b \end{pmatrix}$ $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\Longrightarrow$ ABOVE $= \displaystyle\overbrace{\int_{-\infty}^{\infty} db}^{\text{ONE!}} \int_{-\infty}^{\infty} da\, e^{-a^2}$

$\Longrightarrow$ THY DOES NOT EXIST, MUST MODIFY!

**MODIFICATION** $\displaystyle\int_{-\infty}^{\infty} da\, e^{-a^2} \int_{-\infty}^{\infty} db\, \delta(b-\xi) = \underbrace{\int_{-\infty}^{\infty} da\, e^{-a^2}}_{\substack{\text{NO } \xi \text{ DEP!} \\ \text{FINITE!}}}$

## RESTRICTING FUNCTIONAL INT.

$I := \int DA\, e^{iS(A)}$

SUPPOSE $A \rightarrow A_g \Longrightarrow S(A) = S(A_g)$ $\quad DA = DA_g$

(can act again w/ $g'$, forms group)

GOAL: WRITE $I = \left(\int Dg\right) J$

$\rightarrow$ integration over group

TOY EX!

$$I = \int dx\, dy\, e^{iS(x^2+y^2)}$$

$\rightarrow$ not a gen func. of $x, y$. Rot. invariant!

$$= \int d\theta \int dr\, r e^{iS(r)} =: \left(\int d\theta\right) J$$

# FADEEV-POPOV TRICK

some gauge fixing function we choose

$$1 =: \Delta(A) \int dg\, \delta(f(A_g)) \qquad \text{DEFINES } \Delta(A)$$

Language: Fadeev-Popov determinant

$$\Big\downarrow \; A \mapsto A_{g'}$$

group fact $\int Dg'' = \int Dg$

$$1 = \Delta(A_{g'}) \int dg\, \delta(f(A_{g'g})) \overset{g''=g'g}{=:} \Delta(A_{g'}) \int Dg\, \delta(f(A_{g''})) = \Delta(A_{g'}) \underbrace{\int Dg''\, \delta(f(A_{g''}))}_{\Delta(A)^{-1}}$$

$$\Rightarrow \quad \Delta(A_{g'}) = \Delta(A) \qquad \text{GAUGE INVARIANT!}$$

PATH INTEGRAL:

$$I = \int DA\, e^{iS(A)} = \int DA\, e^{iS(A)}\, \Delta(A) \int Dg\, \delta[f(A_g)]$$

$$= \int Dg \int DA\, e^{iS(A)}\, \Delta(A)\, \delta[f(A_g)]$$

$A \to A_{g^{-1}}$ & use invariance of $S$, measure, $\Delta$ $\longrightarrow$

$$= \left(\int Dg\right) \int DA\, e^{iS(A)}\, \Delta(A)\, \delta[f(A)] =: \left(\int Dg\right) J$$

# F-P IN E-M

① GROUP TRANS: $A_\mu \longrightarrow A_\mu - \partial_\mu \Lambda$

② CHOOSE $f$: $f(A) = \partial_\mu A^\mu - \sigma$

③ CALC F-P DET: $\Delta(A)^{-1} := \int Dg\, \delta[f(A_g)] = \int D\Lambda\, \delta[\partial_\mu(A^\mu - \partial^\mu \Lambda) - \sigma]$

④ COMPUTE P.I.

$$I = \int D\Lambda \int DA\, e^{iS(A)}\, \delta(\partial A - \sigma) \left[ \int D\Lambda\, \delta[\partial A - \sigma - \partial^2 \Lambda] \right]^{-1}$$

$\partial A - \sigma = 0$

$$\longrightarrow = \boxed{\frac{\int D\Lambda}{\int D\Lambda\, \delta(\partial^2 \Lambda)}} \int DA\, e^{iS(A)}\, \delta(\partial A - \sigma)$$

drops out of everything so throw away

NB: $\Delta$ came outside $\int DA$. Not so for non-abelian case!

⑤ DEFINE $Z$

- LEFT OVER PIECE HAS $\sigma$-DEP, SO INT. OVER.

- $$Z = \int D\sigma\, e^{-i/2\xi \int d^4x\, \sigma(x)^2} \int DA\, e^{iS(A)}\, \delta(\partial A - \sigma)$$

$$= \int DA\, e^{iS(A) - \frac{i}{2\xi}\int d^4x\, (\partial A)^2}$$

$$= \int DA\, e^{iS_{eff}}$$

w/ $$S_{eff}(A) = S(A) - \frac{1}{2\xi}\int d^4x\, (\partial A)^2$$

$$= \int d^4x \left\{ \frac{1}{2} A_\mu \underbrace{\left[\partial^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu\right]}_{=: Q^{\mu\nu}_{eff}} A_\nu + A_\mu J^\mu \right\}$$

THE WHOLE POINT

- $(Q^{\mu\nu})^{-1}$ DNE.
- HAD TO FIX. $(Q^{\mu\nu}_{eff})^{-1}$ EXISTS!

PHOTON PROP: (we can do it now since we solves the inverse issue)

two i's from deriv ⟹ sign

k-space $$Q^{\mu\nu}_{eff} = -k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu$$

$$Q^{\mu\nu}_{eff}\left[-g_{\nu\lambda} + (1-\xi)\frac{k_\nu k_\lambda}{k^2}\right]\left(\frac{1}{k^2}\right)$$

$$= \left[k^2 \delta^\mu_\lambda - (1-\xi) k^\mu k_\lambda - \left(1 - \frac{1}{\xi}\right) k^\mu k_\lambda + \left(1 - \frac{1}{\xi} - \xi + 1\right) k^\mu k_\lambda\right] \frac{1}{k^2}$$

$$= \delta^\mu_\lambda =: Q^{\mu\nu}_{eff} D_{\nu\lambda}$$

⟹ MOM. SPACE PROP.

$$iD_{\nu\lambda} = \frac{i\left[-g_{\nu\lambda} + (1-\xi)\frac{k_\nu k_\lambda}{k^2}\right]}{k^2}$$

compare to (15) of II.7, which we didn't do since it takes a lot of work

## PHYSICAL NEED FOR GAUGE INVARIANCE

(a photon can find no rest in Zee)

REC: $Q^{\mu\nu}$ HAD NO INV. ONLY WHEN $m \to 0$

(so this is about the difference between massless & massive spin 1)

MASSIVE PARTICLES GO TO REST FRAME $q^\mu = m(1,0,0,0)$

$q^\mu$ NOT BOOST INV

$q^\mu$ IS ROT. INV.

$\Rightarrow q^\mu$ INVARIANT UNDER SUBGROUP $G \subsetneq$ LORENTZ

Language: $G =$ "Little Group"

HERE: $G =$ ROT. GP. $= SO(3)$ (acting on last 3 entries)

(particles in reps of $SO(3)$, called "spin")

dim $-2j+1$ REP. $\Rightarrow$ "spin $j$"

MASSLESS no rest frame $\Rightarrow q^\mu = \omega(1,0,0,1)$

$q^\mu$ $O(2)$ INV. $\Rightarrow$ $G = O(2)$

HELICITY $\pm j \Rightarrow$ 2 STATES

(you know this, photon only has two polarizations)

VECTORS $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + m^2 A_\mu A^\mu$

$\partial_\mu A^\mu = 0 \Rightarrow 4-1=3$ DOF Spin 1

Z, $W_\pm$ massive vectors. Where does extra DOF come from?

$\Big\downarrow m \to 0$

Language: "Gauge Away"

$\mathcal{L}$-symm $A_\mu \to A_\mu + \partial_\mu \alpha$, $4-1-1=2$ DOF

# INTERACTIONS w/ PHOTONS

$A_\mu \to A_\mu + \partial_\mu \alpha$ CRUCIAL (for DOF count and for propagator)
( ∴ interactions must respect it for any of this to work)

## COUPLE TO FERMIONS

$\mathcal{L} \supset \underset{\#}{B}\,\bar\psi \gamma^\mu A_\mu \psi$ (recall this is Lorentz invariant)

$$B\bar\psi\gamma^\mu A_\mu\psi \longmapsto B\bar\psi\gamma^\mu A_\mu\psi + \underbrace{B\bar\psi\gamma^\mu\partial_\mu\alpha\psi}_{\text{EXTRA PIECE } (*)}$$

$\mathcal{L}$-INV $\Rightarrow$ NEED $-(*)$

RECALL $\mathcal{L} \supset \mathcal{L}_{DIRAC} = \bar\psi(i\partial\!\!\!/ - m)\psi$

( $-(*)$ natural from $i\bar\psi\partial\!\!\!/\psi$ part, has derivative)

REQ Language "local" transformation b/c depends on x

$\psi \mapsto e^{ie\alpha(x)}\psi$, $\quad \bar\psi \mapsto e^{-ie\alpha(x)}\bar\psi(x)$

THEN $i\bar\psi\partial\!\!\!/\psi \mapsto i\bar\psi\partial\!\!\!/\psi - e\bar\psi\gamma^\mu\partial_\mu\alpha\psi = -(*)$ IF $B = e$

SO $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar\psi\partial\!\!\!/\psi - m\bar\psi\psi + e\bar\psi A\!\!\!/\psi$ GAUGE INVARIANT

NB $i\bar\psi\partial\!\!\!/\psi + e\bar\psi A\!\!\!/\psi =: i\bar\psi D\!\!\!\!/\psi$ $\quad D_\mu := \partial_\mu - ieA_\mu$

Language "covariant Derivative"

$$\Rightarrow \boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar\psi D\!\!\!\!/\psi + m\bar\psi\psi}$$

$\Downarrow$

FAMOUS! QED Language: Quantum Electrodynamics

KEY! Not because local transformations are cool or more general, needed for massless spin 1 consistency

describes fermion interactions with massless spin 1, gauge invariance crucial for consistency of the theory

# QED PERTURBATION THY

$$Z[\eta,\bar\eta,J_\mu] = \int D\bar\psi D\psi DA_\mu \, e^{i\int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not\partial - m)\psi + e\bar\psi \not A \psi + \overbrace{\bar\eta\psi + \eta\bar\psi + J_\mu A^\mu}^{\text{added sources to } \mathcal{L}}\right]}$$

- EXPAND IN $\bar\eta, \eta, J_\mu$

$$Z = \sum_{s_\psi}\sum_{s_{\bar\psi}}\sum_{s_A} \frac{i^{s_{\bar\psi}+s_\psi+s_A}}{s_{\bar\psi}!\,s_\psi!\,s_A!} \int d^4x_1 \ldots d^4x_{s_{\bar\psi}}\, d^4y_1 \ldots d^4y_{s_\psi}\, d^4z_1 \ldots d^4z_{s_A}\; \eta(x_1)\ldots\eta(x_{s_{\bar\psi}})\,\bar\eta(y_1)\ldots\bar\eta(x_{s_\psi})\, J_{\mu_1}(z_1)\ldots J_{\mu_{s_A}}(z_{s_A})$$

$$\times \int D\bar\psi D\psi DA_\mu\; \bar\psi(x_1)\ldots\bar\psi(x_{s_{\bar\psi}})\,\psi(y_1)\ldots\psi(y_{s_\psi})\,A^{\mu_1}(z_1)\ldots A^{\mu_{s_A}}(z_{s_A})\; e^{i\int d^4x\,\mathcal{L}\big|_{J_\mu=\eta=\bar\eta=0}}$$

$$=: Z(0,0) \sum_{s_\psi}\sum_{s_{\bar\psi}}\sum_{s_A} \frac{i^{s_{\bar\psi}+s_\psi+s_A}}{s_{\bar\psi}!\,s_\psi!\,s_A!} \int d^4x_1 \ldots d^4x_{s_{\bar\psi}}\, d^4y_1 \ldots d^4y_{s_\psi}\, d^4z_1 \ldots d^4z_{s_A}\; \eta(x_1)\ldots\eta(x_{s_{\bar\psi}})\,\bar\eta(y_1)\ldots\bar\eta(x_{s_\psi})\, J_{\mu_1}(z_1)\ldots J_{\mu_{s_A}}(z_{s_A})$$

$$\times\, G^{(s_\psi, s_{\bar\psi}, s_A)}(x\text{'s}, y\text{'s}, z\text{'s})$$

sources, couplings all 0

as usual, compute these!

PER USUAL, $\exists$ FEYNMAN RULES FOR $G^{(s_\psi, s_{\bar\psi}, s_A)}$!