QFT 2

Topic 2: More Condensed Matter

2018: got here in class #8

TOPIC 2: MORE CONDENSED MATTER (ie things in CM systems, but also in HE)

i) ASYMPTOTIC FREEDOM IN NLSM

ii) DEFECTS, CHERN-SIMONS, 2+1 PARTICLE VORTEX-DUALITY

iii) GENERALIZED FERMIONS (ie different types in different dims)

Non-linear σ-model

& Asymptotic Freedom

ASYMPTOTIC FREEDOM IN NLSM & FERROMAGNETS

2D NLSM

ASIDE: SCALAR FIELD DIMENSION

$$[S_\phi] = 0 = \left[\int d^dx\ \partial_\mu\phi\partial^\mu\phi + \cdots\right] = d[dx] + 2[\partial_\mu] + 2[\phi]$$

$$[\partial_\mu] = 1 \qquad [dx] = -1 \qquad \left(M \sim \frac{1}{L}\right)$$

$$\Rightarrow -d + 2 + 2[\phi] = 0 \qquad \text{SO} \qquad [\phi] = \frac{d-2}{2}$$

2D SCALAR FIELDS (1+1)

$$\mathcal{L} = \underbrace{f_{ij}(\phi^i)}_{\text{ANY FUNCTION}}\ \partial_\mu\phi^i\partial^\mu\phi^j$$

ANY FUNCTION $\Rightarrow [f_{ij}\,\partial\phi\,\partial\phi]$ (use symmetry to constrain)

O(N) SYMMETRY $\quad \phi_i \longrightarrow n_i(x) \mid n^i n^i = 1$

$$\xRightarrow{\text{restrict } \mathcal{L}} \mathcal{L} = \frac{1}{2g^2}|\partial_\mu \vec{n}|^2 \quad \text{S.T.} \quad n^i n^i = 1$$

IMPOSE CONSTRAINT $\quad n^i = (\pi^1, \ldots, \pi^{N-1}, \sigma) \xRightarrow{n^in^i=1} \sigma = (1-\pi^2)^{\frac{1}{2}}$

$$|\partial_\mu n^i|^2 \supset \partial_\mu\sigma\partial^\mu\sigma = \partial_\mu\left[(1-\vec{\pi}^2)^{\frac{1}{2}}\right]\partial^\mu\left[(1-\vec{\pi}^2)^{\frac{1}{2}}\right] = \left(\frac{-2\left(\frac{1}{2}\right)\vec{\pi}\cdot\partial_\mu\vec{\pi}}{\sqrt{1-\vec{\pi}^2}}\right)^2 = \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{1-\vec{\pi}^2}$$

$$\mathcal{L} = \frac{1}{2g^2}\left[|\partial_\mu\vec{\pi}|^2 + \frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{1-\pi^2}\right]$$

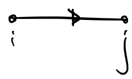
NOTE: $\vec{\pi}$ MASSLESS!
CONSISTENT W/ GOLDSTONE OF O(N) BREAKING

expand in π^k

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu\vec{\pi}|^2 + \frac{1}{2g^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 + \cdots$$

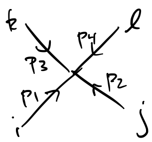
higher point interactions there but not important for us

PROPAGATOR $\quad \frac{1}{2g^2}|\partial_\mu \vec{\pi}|^2 \overset{\text{IBP}}{=} \vec{\pi}\cdot\underbrace{\left(-\frac{1}{2g^2}\square\right)}_{\text{invert for prop}}\cdot\vec{\pi}$

$\Longrightarrow$  $= \frac{i g^2}{p^2}\delta_{ij}$

INTERACTION $\quad \frac{1}{2g^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2$

PROPERTIES: 4PTS, 2 p's, $\sim \frac{1}{g^2}$



$= \frac{-i}{g^2}\underbrace{\left[(p_1+p_2)\cdot(p_3+p_4)\,\delta^{ij}\delta^{kl} + (p_1+p_3)\cdot(p_2+p_4)\,\delta^{ik}\delta^{jl} + (p_1+p_4)\cdot(p_2+p_3)\,\delta^{il}\delta^{jk}\right]}_{\text{all possible index pairings}}$

CALLAN SYMANZIK

RECALL: CONSTRAINTS ON GREEN'S FUNCTIONS

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + n\gamma(g)\right]G^{(n)} = 0$$

NEED: SOME $G^{(n)}$'s (enough to determine β, γ fcns)

FIRST $\quad G^{(1)}_{(x)} = \langle\sigma(x)\rangle = \langle 0|T(\sigma(x))|0\rangle$

BUT $\quad \sigma(x) = (1-\pi^2)^{\frac{1}{2}} = 1 - \frac{1}{2}\pi^2 + \dots$

$\Longrightarrow G^{(1)}(0) = 1 - \frac{1}{2}\langle\pi^2(0)\rangle + \dots = 1 - \frac{1}{2}$ 

$\langle\pi^k(0)\,\pi^l(0)\rangle =$  $= \int\frac{d^dk}{(2\pi)^d}\,\frac{i g^2}{k^2-\mu^2}\,\delta^{kl}$ to handle infrared div.

$= \frac{g^2}{(4\pi)^{d/2}}\,\frac{\Gamma(1-\frac{d}{2})}{(\mu^2)^{1-d/2}}\,\delta^{kl}$

STEPS HERE = ?

USE IN $\langle\sigma\rangle$ & RENORMALIZATION COND.

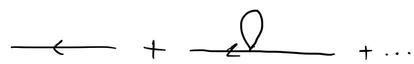
$$\Rightarrow \langle\sigma\rangle = 1 - \tfrac{1}{2}(N-1)\,\frac{g^2}{(4\pi)^{d/2}}\,\underset{\sim\frac{2}{\varepsilon}+\dots}{\Gamma\!\left(1-\tfrac{d}{2}\right)}\left(\underset{(\ )\varepsilon+\dots}{\frac{1}{(\mu^2)^{1-d/2}}} - \underbrace{\frac{1}{(M^2)^{1-d/2}}}_{\text{"subtraction" from ren. P.T., counterterm, etc.}}\right) + \mathcal{O}(g^4)$$

$$\overset{d=2}{=} 1 - \frac{g^2(N-1)}{8\pi}\log\frac{M^2}{\mu^2} + \mathcal{O}(g^4)$$

C-S
$$0 = \left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial g} + \gamma\right)\left(1 - \frac{g^2(N-1)}{8\pi}\log\frac{M^2}{\mu^2} + \dots\right)$$
$$= \left(-\frac{g^2(N-1)}{8\pi}\cdot 2 + \underbrace{\beta\frac{\partial}{\partial g}G^{(1)}}_{\text{H.O.T. IN } g} + \gamma\left(1 - g^2(\ )\dots\right)\right)$$

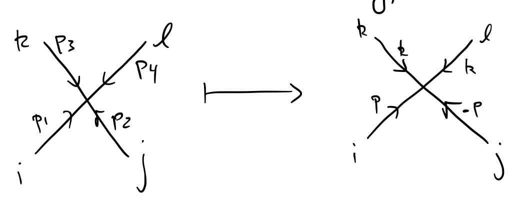
$$\Rightarrow \gamma = \frac{g^2(N-1)}{4\pi} + \mathcal{O}(g^4)$$

SECOND $\langle\pi^k(p)\,\pi^l(-p)\rangle =$



$$= \frac{ig^2}{p^2}\delta^{kl} + \frac{ig^2}{p^2}\underbrace{\left(-i\Pi^{kl}\right)}_{\text{effectively a definition of the thing in the loop}}\frac{ig^2}{p^2}$$

Π^{kl}: TWO CONSTRACTION TYPES @ VERTICES



i) i WITH j (one possibility)

ii) i WITH other two poss)

NEED
$$\langle\partial_\mu\pi^k(0)\,\partial^\mu\pi^l(0)\rangle = \int\frac{d^dk}{(2\pi)^d}\,\frac{ig^2k^2}{k^2-\mu^2}\,\delta^{kl}$$
$$= -\frac{g^2}{(4\pi)^{d/2}}\,\frac{\frac{d}{2}\Gamma\!\left(-\frac{d}{2}\right)}{(\mu^2)^{-d/2}}\,\delta^{kl}$$

$$\Pi^{kl}(p) \overset{\text{HMWK?}}{=} -\delta^{kl}\,p^2\,\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma\!\left(1-\frac{d}{2}\right)}{(\mu^2)^{1-d/2}}$$

$$\Rightarrow \langle\pi^k(p)\,\pi^l(-p)\rangle \overset{\text{W R.C. \& } d=2}{=} \frac{ig^2}{p^2}\delta^{kl} + \frac{ig^2}{p^2}\left(ip^2\,\frac{1}{4\pi}\log\frac{M^2}{\mu^2}\right)\frac{ig^2}{p^2}\delta^{kl} + \dots$$
$$= \frac{i}{p^2}\delta^{kl}\left(g^2 - \frac{g^4}{4\pi}\log\frac{M^2}{\mu^2}\right) + \mathcal{O}(g^6)$$

CALLAN-SYMANZIK

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + 2\gamma\right]\langle \pi^k(p)\,\pi^\ell(-p)\rangle = 0$$

$$\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + 2\gamma\right]\frac{i}{p^2}\,\delta^{k\ell}\left(g^2 - \frac{g^4}{4\pi}\log\frac{M^2}{\mu^2}\right)$$

$$= \frac{i\delta^{k\ell}}{p^2}\cdot\Big[\underbrace{2\left(-\frac{g^4}{4\pi}\right)}_{g^4} + \underbrace{2g\beta}_{\mathcal{O}(g\beta)} + \underbrace{\#_1 g^3\beta}_{\mathcal{O}(g^3\beta)} + \underbrace{2g^2\gamma}_{\mathcal{O}(g^4)} + \underbrace{\#_2 g^4\gamma}_{\mathcal{O}(g^6)}\Big]$$

$\gamma \to \mathcal{O}(g^2)$

NEED $\quad -\dfrac{g^4}{2\pi} + 2g\beta + 2g^2\,\dfrac{g^2(N-1)}{4\pi} = 0$

$$\Longrightarrow \beta = \frac{1}{2g}\left[-2g^2\,\frac{g^2(N-1)}{4\pi} + \frac{g^4}{2\pi}\right]$$

$$= \frac{1}{2g}\left[-2\left[\frac{g^2(N-1)}{4\pi} - \frac{g^2}{4\pi}\right]\right]$$

$$\boxed{\beta = M\frac{\partial g}{\partial M} = -\frac{g^3}{4\pi}(N-2) + \mathcal{O}(g^5)}$$

CASES: i) N=2: $\quad \beta = 0$

CHANGE VARIABLES $\quad \pi' = \sin\theta \quad \sigma = \cos\theta$

$$\Longrightarrow \mathcal{L} = \frac{1}{2g^2}(\partial_\mu\theta)^2$$

Q: WHY $\beta = 0$?

A: FREE THEORY

ii) N>2 $\quad \beta < 0 \Longrightarrow$ ASYMPTOTICALLY FREE (in UV)

see Peskin for another "more physical" derivation using Wilsonian momentum slicing.

$2<D<4$

valid to choose this, soaks dimensionality into g

DIMENSIONS: DEFINE $[\vec{n}]=0$

$$0=[S]=\left[\int d^dx \frac{1}{2g_d^2}(\partial_\mu \vec{n})^2\right] = -d+2-2g_d$$

$$\Rightarrow [g_d] = \frac{2-d}{2} \quad \text{IE} \quad g_d \sim M^{\frac{2-d}{2}}$$

$$T := g_d^2 M^{d-2} \quad \text{IS DIM-LESS}$$

FIXED POINTS

$$d = 2+\varepsilon$$

$$\beta(T) = \varepsilon T - (N-2)\frac{T^2}{2\pi}$$

$$\gamma(T) = (N-1)\frac{T}{4\pi}$$

(this requires some work, see Peskin 13.3. We'll study physics)

F.P #1 $\quad \beta=0 \ @ \ T=0$

$T=0 \Rightarrow g=0 \Rightarrow$ NON-INTERACTING (FREE) F.P.

F.P #2 $\quad \beta=0 \ @ \ T_* = \dfrac{2\pi\varepsilon}{N-2}$

$\varepsilon \neq 0 \Rightarrow T_* \neq 0 \Rightarrow g \neq 0 \Rightarrow$ INTERACTING F.P.

these are the interesting ones

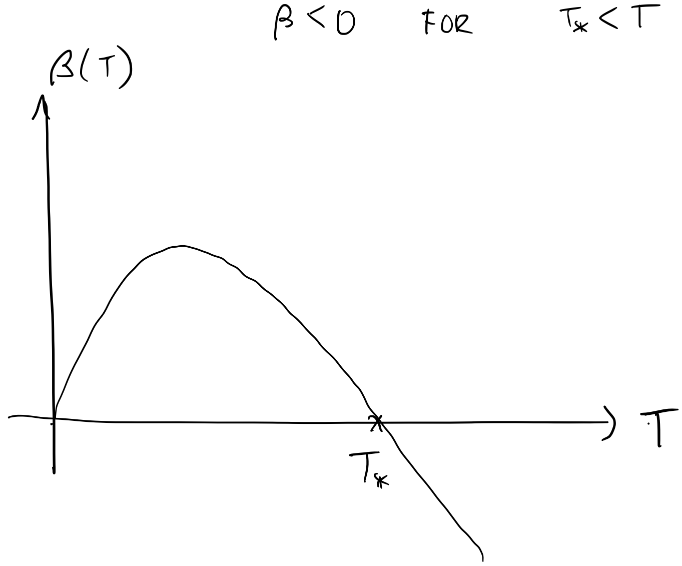
RG FLOW

$\beta>0$ FOR $0<T<T_*$ $\Rightarrow T\uparrow$ AS $M\uparrow$

$\beta<0$ FOR $T_*<T$ $\Rightarrow T\downarrow$ AS $M\uparrow$

$\Rightarrow$ FLOW TO T_* AS $M\uparrow$

Language "UV-stable"



HEURISTIC

- GO TO UV, FLOW INTO F.P.
- GO TO IR, FLOW OUT OF F.P.

NB: NON-TRIVIAL F.P. DISAPPEARS AS $d\to 2$! *(agrees w/ Mermin-Wagner)*

NLSM FROM FERROMAGNETS

spin operators

HEISENBERG MODEL $\quad H = \frac{1}{2} S^2 \sum_{ij} J_{ij} \hat{\Omega}_i \cdot \hat{\Omega}_j$

sometimes restricted to NN's

NB: 2 DOF IN Ω, θ, ϕ

HALDANE'S MAPPING $\quad H \xrightarrow{\text{cont.}} \mathcal{L}_{NLSM}$

(see Auerbach's book "Interacting electrons & quantum magnetism")

$\hat{n}$ $\vec{L}$ decomp:

$$\hat{\Omega}_i = \eta_i \, \hat{n}(x_i) \sqrt{1 - \left|\frac{\vec{L}(x_i)}{S}\right|^2} + \frac{\vec{L}}{S}$$

4 Conditions $\quad |\hat{n}| = 1 \qquad \vec{L} \cdot \hat{n} = 0 \qquad \mathcal{D}\hat{\Omega} = \prod_{|\vec{q}| \leq \Lambda_{BZ}} d\hat{n}_q \, dL_{\hat{q}} \, \delta(L \cdot \hat{n}) \, J(\hat{n}, L)$

$\vec{L}, \hat{n}$ 6 comps $-$ 4 conds. $=$ 2 DOF ✓

EXPAND $\hat{\Omega}_i \cdot \hat{\Omega}_j$ IN $\left|\frac{\vec{L}}{S}\right|$

$$\hat{\Omega}_i \cdot \hat{\Omega}_j \simeq \eta_i \eta_j - \frac{1}{2}\eta_i \eta_j (\hat{n}_i - \hat{n}_j)^2 + \left(\frac{1}{S}\right)^2 \left[\vec{L}_i \vec{L}_j - \frac{1}{2}\eta_i \eta_j (\vec{L}_i^2 + \vec{L}_j^2) \right]$$
$$+ \frac{1}{S}\left(\eta_j \vec{L}_i \hat{n}_j + \eta_i L_j \hat{n}_j \right) + \mathcal{O}\left(|\vec{L}|^2 |\hat{n}_i - \hat{n}_j| \right)$$

double check

CONTINUUM LIMIT

$$\hat{n}_i - \hat{n}_j \simeq \partial_\ell \hat{n}(x_i) x_{ij}^\ell + \frac{1}{2}(\partial_\ell \partial_k \hat{n}) x_{ij}^\ell x_{ij}^k + \cdots \qquad w/ \quad \vec{x}_{ij} = \vec{x}_i - \vec{x}_j$$

* replace $\quad \sum F_i \longrightarrow a^{-d} \int d^d x \sum_i \delta(\vec{x} - \vec{x}_i) F(\vec{x})$

$$\Longrightarrow H \simeq E_0^{cl} + \frac{1}{2}\int d^d x \left[\rho_s |\partial_\ell \hat{n}|^2 + \int d^d x' \left(L_x \chi_{xx'}^{-1} L_{x'} \right) \right]$$

"stiffness constant"

skipping a little ... express in Fourier space & integrate L,

imaginary time

χ variation is small on Λ-scales, effective constant

$$Z \propto \int \mathcal{D}\hat{n}\, e^{i\gamma(\hat{n})} \exp\left[-\int_0^\beta d\tau \int_\Lambda d^dx\, \frac{\rho_s}{2} \sum_{\ell=1}^{d} \left(|\partial_\ell \hat{n}|^2\right) - \frac{1}{2}\chi_0 |\partial_\tau \hat{n}|^2\right]$$

RENAME

$$(x_1, \dots, x_d, c\tau) \longrightarrow (x_1, \dots, x_{d+1})$$

THEN

$$Z \propto \int_\Lambda \mathcal{D}\hat{n}\, e^{i\gamma} \exp\left(-\int d^{d+1}x\, \mathcal{L}^{d+1}_{NLSM}\right)$$

$$\mathcal{L}^{D}_{NLSM} = \frac{\Lambda^{D-2}}{2f_D}(\partial_\mu \hat{n})^2$$

CONCLUSION $\left|\frac{\vec{L}}{S}\right| \ll 1$ & continuum limit

Heisenberg model becomes NLSM in Néel fields!

EASY "DERIVATION"

Heisenberg model SO(3) symmetric w/ 2 massless DOF

or alt. 3 DOF subject to constraint.

$$\Longrightarrow \quad \mathcal{L} = |\partial_\mu \vec{n}|^2 + 0\, \vec{n}\cdot\vec{n} + V(\vec{n}\cdot\vec{n})$$

massless

for O(N) symmetry, but $\vec{n}\cdot\vec{n} = 1$ $\Longrightarrow$ $V(\vec{n}\cdot\vec{n})$ some overall constant can throw away

$$\Longrightarrow \quad \mathcal{L} = |\partial_\mu \vec{n}|^2 = \mathcal{L}_{NLSM}$$

Defects, Chern-Simons, & Particle Vortex Duality

ANYONS

(here path integral makes things more obvious)

2- PARTICLES @ x_1^i, x_2^i EVOLVE TO x_1^f, x_2^f

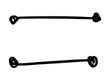
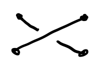
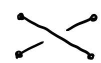
AMPLITUDE:

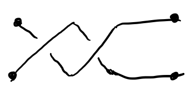
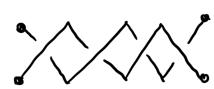
DIRAC: $\langle x_1^f, x_2^f | e^{-iHT} | x_1^i, x_2^i \rangle$

PATH INTEGRAL: $\sum_{\text{histories}}$

2+1 SPECIAL FEATURE: BRAIDING

REP $\begin{matrix} x_1^i & & x_1^f \\ x_2^i & & x_2^f \end{matrix}$ $\xrightarrow{\quad}$ t BY 

"0-PATH"  "1-PATH"  "-1-PATH" 

"2-PATH"  BUT ALSO 

⋮

n-PATH (-n)-PATH

POINT: • IN 2+1, EACH PAIR OF PATHS COMES w/ TOPOLOGICAL # n.

(so $\sum_{\text{hist.}}$ of 2-part paths may be broken into topological classes)

- n-PATH & m-PATH CAN DEFORM INTO EACH OTHER $\iff$ $n = m$
- CAN PICK UP PHASE $e^{-\pi i n \theta}$ ASSOC w/ n-PATH

1-PATH: = PARTICLE SWAP

CASES i) $\theta = 0 \implies$ BOSONS

ii) $\theta = 1 \implies$ FERMIONS

iii) $\theta \in \mathbb{R}$ $\theta \neq 0$ $\theta \neq 1 \implies$ ANYONS

(new type of particle in 2+1 D)

CHERN-SIMONS THEORY

TAKE THY $\mathcal{L}_0$,

FORM NEW THY $\mathcal{L} = \mathcal{L}_0 + \gamma \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + A_\mu J^\mu$, $\quad \lambda \in \mathbb{R}$

GAUGE TRANS: $A_\mu \longmapsto A_\mu + \partial_\mu \Lambda$

$0 = \varepsilon^{\mu\nu\lambda} \partial_\mu \Lambda \partial_\nu \partial_\lambda \Lambda$ why?

$$\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \longmapsto \varepsilon^{\mu\nu\lambda}(A_\mu + \partial_\mu \Lambda)\partial_\nu (A_\lambda + \partial_\lambda \Lambda)$$

$$= \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu \Lambda + \varepsilon^{\mu\nu\lambda}\left[A_\mu \partial_\nu \partial_\lambda \Lambda + \partial_\mu \Lambda \partial_\nu A_\lambda\right] + \dots$$

$$= \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu \Lambda + \partial_\nu (\varepsilon^{\mu\nu\lambda} A_\mu \partial_\lambda \Lambda) \overset{\text{TOT DER.}}{=} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

w/ index rename

EQUATIONS OF MOTION

$$0 = \delta_A \mathcal{L} = \mathcal{L}_{A \to A + \delta A} - \mathcal{L}_A = \gamma \varepsilon^{\mu\nu\lambda}\left[\delta A_\mu \partial_\nu A_\lambda + A_\mu \partial_\nu \delta A_\lambda\right] + \delta A_\mu J^\mu$$

$$= \gamma \varepsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu A_\lambda + \gamma \varepsilon^{\lambda\nu\mu} A_\lambda \partial_\nu \delta A_\mu + \delta A_\mu J^\mu$$

$$= \gamma \varepsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu A_\lambda + \gamma \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \, \delta A_\mu + \delta A_\mu J^\mu = \delta A_\mu \left[2\gamma \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda + J^\mu\right]$$

$\uparrow = (-)(-)$, signs from IBP & rearrange

SO $\boxed{2\gamma \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = -j^\mu}$

CONSERVED CHARGE

$$\partial_\mu J^\mu \sim \varepsilon^{\mu\nu\lambda} \partial_\mu \partial_\nu A_\lambda = 0$$

$Q := \frac{-1}{2\gamma} \int d^2x \; J^0$ HAS $\frac{dQ}{dt} = 0$ (why?)

$$\| $$

$\int d^2x (\partial_1 A_2 - \partial_2 A_1) = \int d^2x \; F_{12} = \int B \implies \exists$ conserved magnetic flux assoc. w Chern-Simons

ANYONS FROM CHERN-SIMONS

$\Theta = \frac{1}{4\gamma}$ FROM AHARANOV-BOHM ASSOC. w/ FLUX

"PURE" CHERN-SIMONS $\quad S = \gamma \int_M d^3x \; \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$

NOTE: • NO $\eta^{\mu\nu}$!

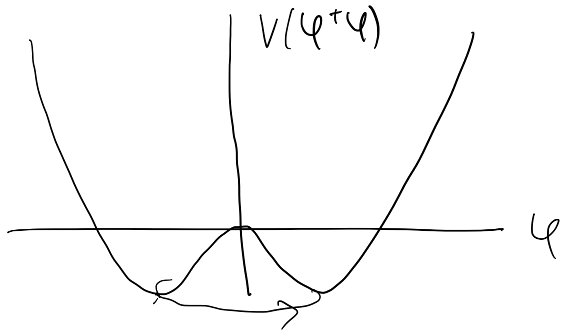
- ON CURVED SPACE, $\sqrt{-\det(g_{\mu\nu})}$ SOMETIMES NEEDED FOR COORD. TRANS. INVARIANCE
- $d^3x \; \varepsilon^{\mu\nu} A_\mu B_\nu C_\lambda$ INV'T $\Rightarrow$ NO $g_{\mu\nu}$ ANYWHERE IN S

TOPOLOGICAL F.T. (depends only on topology of M)

PARTICLE-VORTEX DUALITY

CHECK? NEED MAXWELL?

VORTICES ABELIAN HIGGS MODEL $\mathcal{L} = \frac{1}{2}|(\partial_\mu - igA_\mu)\varphi|^2 - V(\varphi^\dagger\varphi)$



$V(\varphi) = \mu^2\varphi^2 + \lambda\varphi^4$

$\mu^2 < 0$

DEFINE $v := |\varphi|$ @ $\mathrm{MIN}(V(\varphi^\dagger\varphi))$

EXPAND $\varphi = (v+\chi)e^{i\theta}$

QUIZ: 1) $m_\chi = 0$ OR $m_\chi \neq 0$?

2) $m_\theta = 0$ OR $m_\theta \neq 0$?

(physical args? theorems? how to compute?)

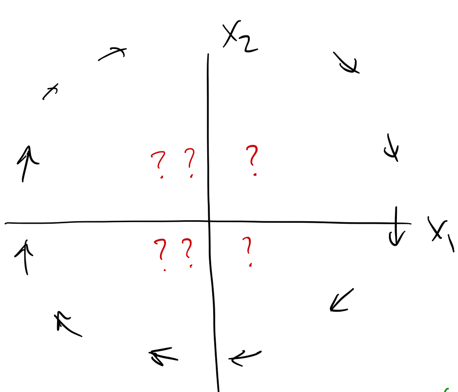
@ LOW EN. $E \ll m_\chi$, χ DECOUPLES

$$\mathcal{L}_{eff} \overset{\downarrow \text{Hmwk}}{=} \frac{1}{2}v^2(\partial_\mu\theta - gA_\mu)^2$$

VACUUM MANIFOLD θ LIVES ON S^1

DEFECTS $\pi_1(S^1) = \mathbb{Z} \Rightarrow \exists$ VORTICES

VORTEX $\lim_{r\to\infty} \varphi = ve^{i\theta} \Rightarrow \varphi$ WHIRLS AROUND @ ∞!



(note: ? can be det, but not worrying about it here)

finite energy req

$$\text{VORTEX FLUX: } gA_i \overset{\downarrow}{\longrightarrow} \partial_i\theta \text{ @ } \infty \Rightarrow \text{FLUX} = \int d^2x\, \varepsilon_{ij}\partial_i A_j = \oint d\vec{x}\cdot\vec{A} = \frac{2\pi n}{g}$$

n = VORTEX CHG.

PHYSICS: 1) @ LONG DIST, VORTEX PT-LIKE

2) VORTEX INTERACTION POTENTIAL $\sim \log \frac{R}{a}$

3) 3D COULOMB " " " "

Q: IS VORTEX CHARGED PARTICLE OF SOME DUAL THEORY?

DUALITY

(different looking descriptions of the same system)

TRICK AUXILIARY FIELD (non-propagating)
AS LAGRANGE MULTIPLIER.

$$\mathcal{L}_\xi = -\frac{1}{2v^2}\xi_\mu^2 + \xi^\mu(\partial_\mu\theta - gA_\mu)$$

ξ^μ "EOM": $\frac{1}{v^2}\xi_\mu = \partial_\mu\theta - gA_\mu$ THEN $\mathcal{L}_\xi\Big|_{\xi^\mu = v^2(\partial_\mu\theta - gA_\mu)} = \mathcal{L}_{eff}$

WRITE $\partial_\mu\theta = \partial_\mu\theta_{smooth} + \partial_\mu\theta_{vortex}$

$$\mathcal{L} = -\frac{1}{2v^2}\xi_\mu^2 + \xi^\mu(\partial_\mu\theta_s + \partial_\mu\theta_v - gA_\mu) = \frac{1}{2v^2}\xi_\mu^2 + \xi^\mu(\partial_\mu\theta_v - gA_\mu) - \theta_s\,\partial_\mu\xi^\mu$$

$$\delta\mathcal{L} = 0 \implies \partial_\mu\xi^\mu = 0 \xrightarrow{\text{one sol}} \xi^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

THEN $\mathcal{L} \overset{\text{HMWK}}{=} -\frac{1}{4v^2}f_{\mu\nu}^2 + \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda(\partial_\mu\theta_{vortex} - gA_\mu)$ w/ $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

E-M current $-A^\mu J_\mu$, $\mathcal{L} \implies J^\mu = +g\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$

WORRY? $\mathcal{L} \supseteq \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda\partial_\mu\theta_{vortex} \overset{IBP}{=} -a_\lambda\varepsilon^{\mu\nu\lambda}\partial_\mu\partial_\nu\theta_{vortex} \overset{?}{=} 0$

NO: $\partial_\mu\partial_\nu$ COMMUTE ON GLOBALLY DEFINED FUNCTIONS $\implies$ NOT θ_v!

VORTEX CURRENT $\quad j^{\lambda}_{vortex} := \frac{1}{2\pi}\varepsilon^{\lambda\mu\nu}\partial_\mu\partial_\nu\theta_{vortex}$

$$Q := \int d^2x\, j^0_{vortex} = \int d^2x\, \frac{1}{2\pi}\varepsilon^{0ij}\partial_i\partial_j\theta_{vort.} = \int \frac{d^2x}{2\pi}\, \vec{\nabla}\times(\vec{\nabla}\theta_{vortex}) = \oint \frac{d\vec{x}}{2\pi}\cdot\vec{\nabla}\theta_{vortex} = 1$$

$$\mathcal{L} = -\frac{1}{4v^2}f_{\mu\nu}^2 + (2\pi)\,a_\mu j^\mu_{vortex} - g A_\mu \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

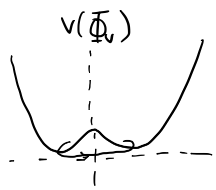
propagating dual U(1) gauge field

vortex gauge field coupling (behaves like charged part!

VORTEX = CHARGED PARTICLE OF DUAL THEORY

"DUALITY" – CAN WRITE AS $\mathcal{L}[\Phi_v]$

complex scalar field, creates and annihilates vortex & anti vortex

– 

$\Longrightarrow$ GLOBAL U(1) BREAKING

$\Downarrow$

S^1 VAC. MAN

$\Downarrow$

$\exists$ VORTICES OF Φ_V

$\Downarrow$

CHARGED PARTICLE OF DUAL THY

= ORIGINAL φ– QUANTA

MEISSNER – CS – MAXWELL

$$\mathcal{L} = \mathcal{L}(a) + A_\mu \varepsilon^{\mu\nu\lambda}\partial a_\lambda$$ (vector a_μ coupled to external A_μ)

$\mathcal{L}(a) \sim aKa$ & INTEGRATE $\Longrightarrow \mathcal{L}(A) \sim (\varepsilon\partial A)\,\frac{1}{K}\,(\varepsilon\partial A) \sim A\left(\varepsilon\partial\,\frac{1}{K}\,\varepsilon\partial\right)A$

a-DYNAMICAL, A NOT $\qquad$ A-DYNAMICAL, a NOT

a-DYNAMICS	K	$\mathcal{L}_{eff}$ SCHEMATIC	A-DYNAMICS
Meissner $\mathcal{L}(a) \sim a^2$	1	$A(\varepsilon\partial\varepsilon\partial)A \sim A\,\partial^2 A$	Maxwell
Chern-Simons $\sim a\,\varepsilon\partial a$	$\varepsilon\partial$	$A(\varepsilon\partial \frac{1}{\varepsilon\partial}\,\varepsilon\partial)A \sim A\varepsilon\partial A$	Chern-Simons
Maxwell $\sim a\partial^2 a$	∂^2	$A(\varepsilon\partial \frac{1}{\partial^2}\,\varepsilon\partial)A \sim AA$	Meissner A^2

Spinors in Various Dims

(working primarily from Polchinski's "String Theory", vol 2, appendix B)

ASIDE TENSOR PRODUCTS

OF VECTOR SPACES V. SPACES V, W w/ BASES $e_1, \dots, e_m$ & $f_1, \dots, f_n$,

DEFINE $V \otimes W$ AS V.S. w/ BASIS (e_i, f_j), WRITTEN AS $e_i \otimes f_j$.

- $\dim(V \otimes W) = \dim(V) \times \dim(W) = mn$
- IE $v = v_i e_i \in V$ & $w = w_j f_j \in W \implies v \otimes w \in V \otimes W$ IS $\sum_{ij} v_i w_j (e_i \otimes f_j)$
- NOTE $v_1, v_2 \in V$ & $w_1, w_2 \in W$ HAVE

$$(A_1 v_1 + A_2 v_2) \otimes (B_1 w_1 + B_2 w_2) = \sum_{ij} (A_1 v_{1i} + A_2 v_{2i})(B_1 w_{1j} + B_2 w_{2j})(e_i \otimes f_j)$$

$$= \sum_{ij} \left[A_1 B_1 v_{1i} w_{1j} + A_2 B_1 v_{2i} w_{1j} + A_1 B_2 v_{1i} w_{2j} + A_2 B_2 v_{2i} w_{2j} \right] e_i \otimes f_j$$

$$= A_1 B_1 \, v_1 \otimes w_1 + A_2 B_1 \, v_2 \otimes w_1 + A_1 B_2 \, v_1 \otimes w_2 + A_2 B_2 \, v_2 \otimes w_2$$

homework

$\boxed{\text{IE: } \otimes \text{ IS BILINEAR}}$

TENSOR PRODUCTS OF LINEAR MAPS (OF MATRICES)

GIVEN $S: V \to X$ & $T: W \to Y$

THEN $S \otimes T : V \otimes W \to X \otimes Y$ S.T. $(S \otimes T)(v \otimes w) = S(v) \otimes T(w)$

MULTIPLICATION $S, A: V \to V$ $T, B: W \to W$

$$(S \otimes T)(A \otimes B)\overset{\in V\otimes W}{(v \otimes w)} = (S \otimes T)\overset{\in V\otimes W}{(Av \otimes Bw)} = \overbrace{SAv \otimes TBw}^{\in V \otimes W}$$

SO $(S \otimes T)(A \otimes B) = SA \otimes TB$

$$\therefore \quad [S \otimes T, A \otimes B] = (SA \otimes TB) - (AS \otimes BT)$$
$$= SA \otimes TB - AS \otimes BT - AS \otimes TB + AS \otimes TB$$
$$= [S, A] \otimes TB + AS \otimes [T, B]$$

SIM
$$\{S \otimes T, A \otimes B\} = SA \otimes TB + AS \otimes BT$$
$$= SA \otimes TB + AS \otimes BT + AS \otimes TB - AS \otimes TB$$
$$= \{S, A\} \otimes TB + AS \otimes [B, T]$$

will need this!

SIMPLE CASE

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} := \begin{pmatrix} a_{11} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ a_{21} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{22} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{pmatrix}$$

CLIFFORD / DIRAC ALGEBRA

$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ $\quad \eta^{\mu\nu} = \mathrm{diag}(-1,1,\dots,1)$ $\quad$ spacetime dim. d

(recall, gives rise to Lorentz gen.)

EVEN DIMS $\quad d = 2k+2$

$\mu = 0, \dots, 2k+1 \implies$ GROUP Γ^μ's INTO $k+1$ PAIRS

$$\Gamma^{0\pm} = \tfrac{1}{2}(\pm\Gamma^0 + \Gamma^1) \qquad \Gamma^{a\pm} = \tfrac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}) \qquad a = 1,\dots,k$$

$$\{\Gamma^{a+}, \Gamma^{b-}\} = \tfrac{1}{4}\{\Gamma^{2a} + i\Gamma^{2a+1}, \Gamma^{2b} - i\Gamma^{2b+1}\} = \tfrac{1}{4}\left[\{\Gamma^{2a}, \Gamma^{2b}\} + \{\Gamma^{2a+1}, \Gamma^{2b+1}\}\right]$$
$$= \tfrac{1}{4}\left(2\left[\delta_{2a,2b} + \delta_{2a+1,2b+1}\right]\right) = \delta^{ab} \implies \boxed{\{\Gamma^{a+}, \Gamma^{b-}\} = \delta^{ab}}$$

SIM: $\boxed{\{\Gamma^{a+}, \Gamma^{b+}\} = \{\Gamma^{a-}, \Gamma^{b-}\} = 0}$

$$(\Gamma^{a+})^2 = \tfrac{1}{2}(\Gamma^{2a} + i\Gamma^{2a+1})^2 = \tfrac{1}{2}\left[\Gamma^{2a^2} - \Gamma^{2a+1^2} + \cancelto{0}{\{\Gamma^{2a}, \Gamma^{2a+1}\}}\right] = \tfrac{1}{2}\left[\tfrac{1}{2}\underbrace{\{\Gamma^{2a}, \Gamma^{2a}\}}_{2} - \tfrac{1}{2}\underbrace{\{\Gamma^{2a+1}, \Gamma^{2a+1}\}}_{2}\right] = 0$$

SO $\boxed{(\Gamma^{a+})^2 = 0}$ $\qquad$ SIM. $\boxed{(\Gamma^{a-})^2 = 0}$

CONSIDER SPINOR ξ (object acts on by Γ's)

FORM $\displaystyle \varphi = \left(\prod_{\substack{\text{ordered } a \mid \\ \Gamma_a^- \xi \neq 0}} \Gamma^{a-}\right)\xi.$ THEN $\Gamma^{a-}\varphi = 0 \quad \forall a.$ $\quad$ Q: WHY?

SPINOR REPRESENTATION

obtain a rep of dim 2^{k+1} by acting on φ with Γ^{a+} in all poss ways. WHY?

$$\varphi^{(\vec{s})} := (\Gamma^{k+})^{s_k + \frac{1}{2}} \cdots (\Gamma^{0+})^{s_0 + \frac{1}{2}}\varphi, \qquad \vec{s} = (s_0, \dots, s_k) \quad \text{w} \quad s_i = \pm\tfrac{1}{2}$$

$\varphi^{(\vec{s})}$ FORM A BASIS FOR REP $\implies$ CAN REP Γ's IN THAT BASIS

MATRIX REPS, ITERATIVELY

$\dim(S^{(s)}) = 2^{k+1}$, $\quad d = 2k+2 \quad$ SO $\quad d \mapsto d+2 \Rightarrow k \mapsto k+1 \Rightarrow \dim(S^{(s)}_{d+2}) = 2\,\dim(S^{(s)}_{d})$

factor of 2 from ↑ d by 2

<u>d=2</u>

$$\Gamma^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \Gamma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$$

<u>d=4</u> OLD $\Gamma^\mu \xrightarrow{\text{CALL}} \gamma^\mu \qquad \mu = 0,1$ (old dimension)

NEW $d=4 \qquad \Gamma^M \qquad M = \underbrace{0,1}_{\mu},2,3$

$$\Gamma^\mu = \gamma^\mu \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mu = 0,1$$

$$\Gamma^2 = \mathbb{1}_{2\times2} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Gamma^3 = \mathbb{1}_{2\times2} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\Gamma^0 = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -1 & \\ & & & 1 \\ \hline +1 & & \multicolumn{2}{c}{0} \\ & -1 & & \end{array}\right) \qquad \Gamma^1 = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -1 & \\ & & & 1 \\ \hline -1 & & \multicolumn{2}{c}{0} \\ & 1 & & \end{array}\right)$$

$$\Gamma^2 = \left(\begin{array}{cc|cc} & 1 & \multicolumn{2}{c}{0} \\ 1 & & & \\ \hline \multicolumn{2}{c|}{0} & & 1 \\ & & 1 & \end{array}\right) \qquad \Gamma^3 = \left(\begin{array}{cc|cc} & -i & \multicolumn{2}{c}{0} \\ i & & & \\ \hline \multicolumn{2}{c|}{0} & & -i \\ & & i & \end{array}\right)$$

<u>d = 2k+2:</u> $\quad \Gamma^\mu = \gamma^\mu \otimes \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \quad \mu = 0, \dots, d-3, \quad \Gamma^{d-2} = \mathbb{1}_{2^k \times 2^k} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Gamma^{d-1} = \mathbb{1}_{2^k \times 2^k} \otimes \begin{bmatrix} & -i \\ i & \end{bmatrix}$

$$\{\Gamma^\mu, \Gamma^\nu\} = \{\gamma^\mu, \gamma^\nu\} \otimes \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}^2 + \gamma^\mu\gamma^\nu \otimes \left[\cancel{\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}}\right]^{\to 0} = 2\eta^{\mu\nu}\, \mathbb{1}_{2^k \times 2^k} \otimes \mathbb{1}_{2\times2} = 2\eta^{\mu\nu}\, \mathbb{1}_{2^{k+1} \times 2^{k+1}} \; ✓$$

⋮

this part excised b/c it's a homework solution.

<u>SUMMARY</u> $\quad \{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\, \mathbb{1}_{2^{k+1} \times 2^{k+1}}$

LORENTZ GENERATORS

$$\Sigma^{\mu\nu} := -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$$

Q: ARE THESE LORENTZ GENERATORS? (what would that mean!?)

$$i[\Sigma^{\mu\nu}, \Sigma^{\sigma\rho}] = \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\sigma}\Sigma^{\nu\rho}$$

(remember: you showed this last semester)

$$[\Sigma^{2a,2a+1}, \Sigma^{2b,2b+1}] = 0$$

$\Longrightarrow$ CAN SIMULT. DIAG!

$\Longrightarrow$ QUANTUM #'S $\quad S_a := i^{\delta_{a,0}}\Sigma^{2a,2a+1} = \Gamma^{a+}\Gamma^{a-} - \frac{1}{2}$

SO $\boxed{S_a \xi^{(\vec{s})} = s_a \xi^{(\vec{s})}}$ (simultaneous eigenstate with eig. s_a)

NOTE:
i) $\xi^{(\vec{s})}$ IS 2^{k+1}-dim "DIRAC" REP OF LORENTZ ALG $SO(2k+1,1)$
ii) $\xi^{(\vec{s})}$ REDUCIBLE

(IE LOR. TRANS DON'T MIX ALL COMPS)

CHIRALITY (one thing that shows $\xi^{(\vec{s})}$ reducible)

$$\Gamma := i^{-k}\Gamma^0\Gamma^1\dots\Gamma^{d-1} \quad \text{HAS} \quad \Gamma^2 = \mathbb{1} \quad \{\Gamma, \Gamma^\mu\} = 0 \quad [\Gamma, \Sigma^{\mu\nu}] = 0$$

$$= 2^{k+1} S_0 \dots S_k$$

$$\Rightarrow \Gamma\xi^{(s)} = 2^{k+1} s_0 \dots s_k \xi^{(\vec{s})} = 2^{k+1}\left(\tfrac{1}{2}\right)^{k+1}(-)^{\#\,-\frac{1}{2}\text{'s IN } \vec{s}}\,\xi^{(\vec{s})}$$

SO $\Gamma\xi^{(\vec{s})} = (-)^{\#\,-\frac{1}{2}\text{ IN } \vec{s}}\,\xi^{(\vec{s})} \Rightarrow$ "CHIRALITY"

$\Rightarrow$ 2^k-dim REP w/ + CHIRALITY } 2 inequivalent "Weyl" representations
2^k-dim REP w/ − CHIRALITY }

CASE $d=4=2k+2 \Rightarrow k=1,$

$\Rightarrow$ two 2-dim Weyl spinors

IE $4_{DIRAC} = 2 + 2'$

CASE $d=10=2k+2 \Rightarrow k=4 \Rightarrow$ DIRAC $(2^{4+1}=32)$ - dim REP

Weyl $(2^4=16)$ - dim REP

SO $32_{Dirac} = 16 + 16'$

COMPLEX CONJUGATION Q: in $d=2k+2$, are Weyl & Weyl' comp. conjugates?

FACT: $\Gamma^{\mu *}$, $-\Gamma^{\mu *}$, Γ^{μ} ALL SATISFY DIRAC ALG.

RELATIONSHIP: (via a similarity tran)

$$B_1 := \Gamma^3 \Gamma^5 \dots \Gamma^{d-1} \qquad B_2 = \Gamma B_1$$

THEN $$\boxed{\Gamma^{\mu *} = (-1)^k B_1 \Gamma^{\mu} B_1^{-1} \qquad -\Gamma^{\mu *} = (-1)^k B_2 \Gamma^{\mu} B_2^{-1}}$$

LET $$\Gamma_{*} := i^{-k} \prod_{j=0}^{d-1} (\Gamma^{*})^j$$
$$\Gamma_{-*} := i^{-k} \prod_{j=0}^{d-1} (-\Gamma^{*})^j$$
NOTE $\underbrace{\Gamma_* = \Gamma_{-*}}_{\text{CALL } \Gamma^*}$ (these are the Γ's associated w/ these other reps $\Gamma^{*\mu}$, $-\Gamma^{*\mu}$)

ALSO $$\Sigma^{\mu\nu *} = B_i \Sigma^{\mu\nu} B_i^{-1} \qquad (i=1,2)$$
$$\Gamma^* = (-1)^k B_i \Gamma B_i^{-1}$$

CHIRALITY CHANGE UNDER CONJ? $\quad \xi^{(\vec{s})} \xrightarrow{\text{WRITE}} |\vec{s}\rangle$

$$\langle \vec{s} | \Gamma^* | \vec{s} \rangle = (-1)^k \langle \vec{s} | B_i \Gamma B_i^{-1} | \vec{s} \rangle \overset{\text{chg of basis}}{=} (-)^k \cdot (\text{chirality of } |\vec{s}\rangle)$$

$$\boxed{\text{WEYL REPS SWAP UNDER CONJ} \Leftrightarrow \text{CHIRALITY UNDER CONJ SWAPS} \Leftrightarrow k \text{ odd} \Leftrightarrow d = 2k+2 \;\; 0 \bmod 4 \text{ ie } d = 0, 4, 8, \dots}$$

MAJORANA SPINORS   REALITY CONDITION

$$\xi^{*(\vec{s})} = B_1 \xi^{(\vec{s})} \stackrel{*}{\Longrightarrow} \xi^{(\vec{s})} = B_1^* \xi^{*(\vec{s})} = B_1^* B_1 \xi^{(\vec{s})}$$

NEED $B_1 B_1^* = 1$

BUT $B_1 B_1^* = (-1)^{k(k+1)/2}$

CONDITION CAN BE IMPOSED $\Leftrightarrow$ $\frac{k(k+1)}{2}$ EVEN $\Leftrightarrow$ $k(k+1)$ = mult of 4

$\Updownarrow$

k mult of 4 or $k+1$ mult of 4 $\Leftrightarrow$ $k = 0 \pmod 4$ OR $k = 3 \pmod 4$

SIMILARLY: USE B_2 INSTED!

$$B_2 B_2^* = (-1)^{k(k-1)/2} \Leftrightarrow \frac{k(k-1)}{2} \text{ EVEN} \Leftrightarrow k = 0 \pmod 4 \text{ OR } k = 1 \pmod 4$$

MAJORANA-WEYL   ARE BOTH

NEED SELF CONJUGATE WEYL-SPINORS

& MAJ. COND $\Rightarrow$ $d=2$ OR $d=10$

ODD DIMENSIONS $d = 2k+3$

$$\Gamma^M = \begin{cases} \Gamma^\mu & M = \mu \in \{0, \dots, d-1\} \\ \pm \Gamma & M = d \end{cases}$$

$\Gamma^M \Rightarrow (2^{k+1})$-dim DIRAC REP, IRREDUCIBLE $\Rightarrow$ NO WEYL IN ODD-D

MAJORANA   ONLY B_1 SIMILARITY ALLOWED $\Rightarrow$ $k = 0 \pmod 4$ OR $k = 3 \pmod 4$ nec.

SUMMARY TABLE

Majorana, Weyl: $\dim_{\mathbb{R}} = \frac{\dim_{\mathbb{R}} \text{Dirac}}{2}$

Majorana-Weyl: $\dim_{\mathbb{R}} = \frac{\dim_{\mathbb{R}} \text{Dirac}}{4}$

Dirac: $\dim_{\mathbb{R}} = \underset{\dim_{\mathbb{C}}}{2 \times 2^{k+1}}$

d	k	Dirac	$\dim_{\mathbb{R}} = 2\times 2^{k+1}$	Majorana	Weyl	Majorana-Weyl	minimal rep by $\dim_{\mathbb{R}}$
2	0	Y	4	Y 2	self 2	yes 1	1
3	0	Y	4	Y 2	—	—	2
4	1	Y	8	Y 4	comp 4	—	4
5	1	Y	8	—	—	—	8
6	2	Y	16	—	self 8	—	8
7	2	Y	16	—	—	—	16
8	3	Y	32	Y 16	comp 16	—	16
9	3	Y	32	Y 16	—	—	16
10	4	Y	64	Y 32	self 32	yes 16	16
11	4	Y	64	Y 32	—		32

(slight embellishment on Polchinski)