QFT 2

Topic 1: Non-Abelian Gauge Theory

NON-ABELIAN GAUGE THEORY

0. LIE GROUPS & ALGEBRAS
1. YANG-MILLS THEORY
2. GAUGE FIXING $\Rightarrow$ FADEEV-POPOV GHOSTS
3. FEYNMAN RULES
4. PHYSICAL STATES: BRST COHOMOLOGY
5. ANOMALIES

LIE GROUPS & ALGEBRAS

DEFN: A GROUP IS A SET G & MAP $\cdot : G \times G \longrightarrow G$ ST

i) CLOSURE: $a \cdot b \in G \quad \forall a, b \in G$

ii) ASSOCIATIVITY: $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in G$

iii) IDENTITY ELEMENT: $\exists e \mid a \cdot e = e \cdot a = a \quad \forall a \in G$

iv) INVERSE $\forall a \in G \ \exists a^{-1} \in G$, IE AN ELEMENT

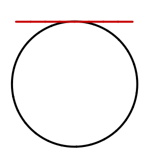
S.T. $a \cdot a^{-1} = a^{-1} \cdot a = e$

EX: $(\mathbb{Q}, +) \quad (\mathbb{Q} \setminus 0, \cdot)$

$(\mathbb{Z}, +) \quad \cancel{(\mathbb{Z} \setminus 0, \cdot)}$ → NO! INVERSE NOT GUARANTEED

ROUGH DEFNS

- A MANIFOLD G IS A SMOOTH CONTINUOUS SPACE
- T_pG IS THE TANGENT SPACE TO G @ PT p.



IT IS A VECTOR SPACE OF SAME DIM AS G

- $\exists \ T_pG \quad \forall p \in G$

Q: WHAT IF G IS BOTH A GROUP & A MANIFOLD?

MORE ROUGH DEF'NS:

- A LIE GROUP G IS A MANIFOLD THAT IS A GROUP.

NB i) $\exists$ IDENTITY e, SPECIAL GP. ELEMENT

$\Downarrow$

e ALSO SPECIAL PT. IN MFLD G

$\Downarrow$

$\exists$ SPECIAL TANGENT SPACE T_eG

ii) GP LAW $\circ$ RELATES PTS IN MFLDS

$\Downarrow$

ADDITIONAL STRUCTURE ON T_eG

- T_eG + STRUCTURE IS LIE ALGEBRA ASSOC. TO G → encodes group els close to 1.

DEF'N A LIE ALGEBRA IS A VECTOR SPACE $\mathfrak{g}$ (over F) w/ AN OPERATION $[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, THE LIE BRACKET ST

I) BILINEARITY

$$[ax+by, z] = a[x,z] + b[y,z], \quad [z, ax+by] = a[z,x] + b[z,y]$$

$$\forall a,b \in F \quad \forall x,y,z \in \mathfrak{g}$$

II) ANTISYMMETRY $[x,y] = -[y,x] \quad \forall x,y \in \mathfrak{g}$

III) JACOBI ID. $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \quad \forall x,y,z \in \mathfrak{g}$

Language: A Lie algebra is just vector space w/ an antisymmetric bilinear form that satisfies the Jacobi identity

DEF'N: LIE ALG $\mathfrak{g}$, T^a $a=1,\ldots,\dim(\mathfrak{g})$ A BASIS, THEN T^a ARE GENERATORS.

DEF'N: $[T^a, T^b] = i f^{abc} T^c$, WITH f^{abc} STRUCTURE CONSTANTS.

COR: JACOBI $\Longrightarrow$ $f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$

DEF'N: IF $\exists\ T^a \in \mathfrak{g} \mid [T^a, T^b] = 0\ \forall\ T^b \in \mathfrak{g}$, T^a GEN'S A $U(1)$ FACTOR.

DEF'N: IF $\mathfrak{g}$ HAS NO $U(1)$ FACTORS, IT IS SEMI-SIMPLE.

DEF'N: IF $\mathfrak{g}$ SEMI-SIMPLE HAS NO MUTUALLY COMMUTING SETS OF GENERATORS, IT IS SIMPLE.

CLASSIFICATION OF $\underbrace{\text{SIMPLE LIE ALGEBRAS}}_{\text{SLA}}$ (Cartan)

note: good to know, but won't be tested

DEF'N: $\mathfrak{g}$ SLA, THE CARTAN SUBALGEBRA IS THE MAXIMAL COMMUTING SUBALGEBRA OF $\mathfrak{g}$.

WRITE ITS GENERATORS H_i $i=1,\ldots,\ell$, & BY DEF $[H_i, H_j] = 0$.

* $\exists$ BASIS | REMAINING GENERATORS E_α SATISFY $[H_i, E_\alpha] = \alpha_i E_\alpha$ (recall $[H_i, \cdot\,]$ a matrix $\Rightarrow$ e-val eqn)

DEF'N: α_i IS A ROOT VECTOR OR ROOT

* ROOTS $\longrightarrow$ + & − ROOTS

DEF'N: $\exists$ SIMPLE ROOTS THAT GEN POS ROOTS AS POS LIN. COMB.

DEF'N: S.R. α_i $\Longrightarrow$ $A_{ij} := \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ CARTAN MATRIX

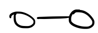
normal dot product

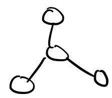
THM: $A_{ij} \Longleftrightarrow$ "Dynkin Diagram" $\Longleftrightarrow$ SLA

(point: classify A_{ij} or Dynkin $\Rightarrow$ classify SLA)

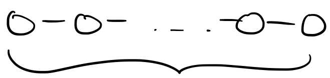
EASY CASES: ("Simply Laced") ○ FOR EACH $A_{ii}=2$ (can only be 2)

— BETW i'th j'th NODE $\Longleftrightarrow A_{ij} \neq 0$

E.G. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \Longleftrightarrow$  $\Longleftrightarrow SU(3)$ AKA A_2

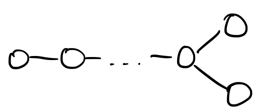
$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & & \\ -1 & & 2 & \\ -1 & & & 2 \end{pmatrix} \Longleftrightarrow$  $\Longleftrightarrow SO(8)$ AKA D_4

SIMPLY LACED CLASS.

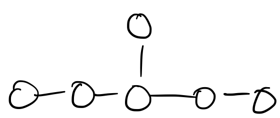
 $\Longleftrightarrow A_n$ AKA $SU(n+1)$

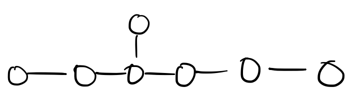
n NODES

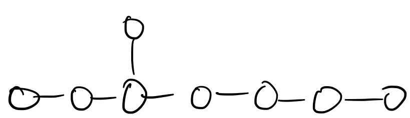
note: 2 infinite series of algebras

 $\Longleftrightarrow D_n$ AKA $SO(2n)$

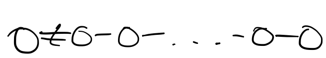
n NODES

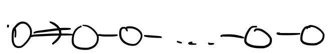
 E_6

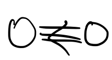
 E_7

 E_8

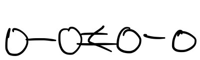
NON-SIMPLY LACED

 B_n AKA $SO(2n+1)$

 C_n AKA $Sp(2n)$

 G_2

THM: A SLA $\in \{ \underbrace{A_n, B_n, C_n, D_n}_{\text{"CLASSICAL"}}, \overbrace{G_2, F_4, E_6, E_7, E_8}^{\text{EXCEPTIONAL}} \}$

 F_4

CLASSICAL GROUPS

① $SU(N)$ SPECIAL ⟹ DET 1 & UNITARY

* $\dim(SU(N)) = N^2 - 1$ (remind me what dim means again!?)
* GENS $N\times N$ HERM MAT T^a

② $SO(N)$ DET 1 & ORTH, IE $O^T O = \mathbb{1}$

* DIM $\frac{N(N-1)}{2}$
* GENS ANTISYMM.

③ $Sp(N)$ Language "Symplectic Group"

* PRESERVES $E = \begin{pmatrix} 0 & \mathbb{I}_{N\times N} \\ -\mathbb{I}_{N\times N} & 0 \end{pmatrix}$ IE $EM + M^T E = 0$

$\Rightarrow -M^T E M^{-1} = E$

* DIM $\frac{N(N+1)}{2}$

REPRESENTATIONS

DEFN $\mathfrak{g}$ w STRUCTURE CONST f^{abc}: A d-dim. REPRESENTATION IS A SET OF $\dim(\mathfrak{g})$ $d\times d$ MAT. t^a | $[t^a, t^b] = i f^{abc} t^c$

ALT: OFTEN REFER TO d-VECTOR ACTED ON BY THESE AS THE REP.

ANY REP CAN BE BLOCK DIAGONALIZED INTO A SET OF IRREDUCIBLE REP'S

Language: "IRREPS"

DEF'N: r AN IRREP OF S.L.A. w t_r^a, $\exists$ BASIS ST. $\mathrm{Tr}(t_r^i t_r^j) = C(r)\,\delta^{ij}$

w/ $C(r) > 0$ THE DYNKIN INDEX OF r.

DEF'N: $\phi \longmapsto (1 + i\alpha^a t_r^a)\,\phi \implies \phi^* \to (1 - i\alpha^a (t_r^a)^*)\,\phi^*$

$(t_r^a)^*$ DEF $\bar{r}$, THE CONJUGATE REPRESENTATION TO r.

some books may have a sign here?

DEF'N $r, \bar{r}$ EQUIV IF $\exists$ UNITARY U | $t_{\bar{r}}^a = U t_r^a U^+$

$\implies \exists$ MATRIX G_{ab} | $G_{ab}\,\eta_a\,\xi_b$ INV'T FOR η, ξ IN REP r

DEFN: IF r EQUIV TO $\bar{r}$, r IS SELF-CONJUGATE

CASE: $G_{ab} = G_{ba} \implies r$ REAL eg 3 of $SU(2)$

CASE: $G_{ab} = -G_{ba} \implies r$ PSEUDOREAL eg 2 of $SU(2)$

EX - N-DIM REP OF $SU(N)$ HAS $N \neq \bar{N}$

- N-DIM REP OF $SO(N)$ REAL

DEF'N $\exists$ $\dim(\mathfrak{g})$ REP $(t_{\mathfrak{g}}^b)_{ac} := i f^{abc}$ THE ADJOINT REP

crucial!!!

GROUP INV'TS

FACT $T^2 := T^a T^a$ COMMUTES W/ ALL T^b

generalization of J^2 from QM.

DEF'N REP r OF $\mathfrak{g}$, $t_r^a t_r^a = C_2(r) \cdot \mathbb{1}$, $C_2(r)$ THE QUADRATIC CASIMIR

NB: GENERAL YM + MATTER β-FUNC, BOTH DYNK. IND. & QUAD. CAS. APPEAR.

note: see Schwartz for additional useful identities

Yang-Mills Theory

note:
- self interacting spin 1
- aka "non-abelian gauge theory"
- purists: YM = no scalars or fermions
- me: "pure Y-M" means no scalars or fermions

PHILOSOPHY

1. GAUGE INV. FOR CONS. PROP. OF $M=0$ SPIN 1
2. INTERACTIONS EXIST OF STUFF + " "

① + ② $\Longrightarrow$ $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{inv}$ GAUGE INVT

METHOD: ADD LOR. INV. INT, DEMAND GAUGE INVARIANCE, CONSTRAIN $\mathcal{L}$

E.G. $\mathcal{L}_{int} = \overset{\text{const}}{B}\, \phi^* \phi A_\mu A^\mu \Longrightarrow \mathcal{L}$ OF ABELIAN HIGGS MODEL

E.G. $\mathcal{L}_{int} = B \bar{\psi} A^\mu \gamma_\mu \psi = B \bar{\psi} \not{A} \psi \Longrightarrow \mathcal{L}_{QED}$

NOW A_μ SELF-INTERACTIONS

GAUGE INV'T $\mathcal{L}$'S: EXP. MOTIVATED APPROACH

TRY 1: ADD SELF INT, DEMAND G.I.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \underbrace{\underbrace{A\, A_\mu A^\mu A_\nu A^\nu}_{\text{unique 4-pt interaction}} + \underbrace{B\, \partial_\mu A^\mu A_\nu A^\nu}_{\text{unique 3-pt interaction}}}_{\text{must be gauge invariant}} \qquad A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha$$

gauge invariant!

$$A\, A_\mu A^\mu A_\nu A^\nu \longmapsto A\left[A_\mu A^\mu A_\nu A^\nu + \frac{4}{g}\partial_\mu \alpha\, A^\mu A_\nu A^\nu + \mathcal{O}\left(\frac{1}{g^2}\right) + \mathcal{O}\left(\frac{1}{g^3}\right) + \mathcal{O}\left(\frac{1}{g^4}\right)\right]$$

$\boxed{1\ \partial,\ 3A,\ 1\alpha \quad \text{nothing can cancel, can't work!}}$

$$B\, \partial_\mu A^\mu A_\nu A^\nu \longmapsto B\left[\partial_\mu A^\mu A_\nu A^\nu + \partial_\mu\left(\frac{\partial^\mu \alpha}{g}\right)A_\nu A^\nu + 2\,\partial_\mu A^\mu \frac{\partial_\nu \alpha}{g} A^\nu + \underbrace{\cdots}_{\text{sim } \mathcal{O}\text{'s}}\right]$$

TRY 2 ① MORE FIELDS, NOW A_μ^a ← index for diff gauge fields

② TRANSF LAW COULD GENERALIZE

$$A_\mu^a \longrightarrow A_\mu^a + \underbrace{\frac{1}{g_a}\partial_\mu \alpha^a}_{\text{"normal" transf}} + \underbrace{T^{ab} A_\mu^b}_{\text{could mix into other fields}}$$

NB: ② NO MIX POSS, IE $T^{ab} = 0$

③ MOST GENERAL $\mathcal{L}$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + A^{abcd} A_\mu^a A^{\mu b} A_\nu^c A^{\nu d} + \underbrace{B^{abc}\partial_\mu A^{\mu a} A_\nu^b A^{\nu c} + C^{abc}\partial_\mu A_\nu^a A^{\mu b} A^{\nu c}}_{\text{may be able to rewrite by IBP?}}$$

NEED GAUGE INVARIANCE! ORGANIZE #A, #∂, AFTER TRANS

[in fact B^{abc} can be absorbed in C^{abc} because $\partial_\mu A^{\mu a} A_\nu^b A^{\nu c}$]

$$A_\mu^a \longrightarrow A_\mu^a + \frac{1}{g_a}\partial_\mu \alpha^a + T^{ac} A_\mu^c$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + A^{abcd} A_\mu^a A^{\mu b} A_\nu^c A^{\nu d} + B^{abc} \partial_\mu A^{\mu a} A_\nu^b A^{\nu c} + C^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c}$$
$$+ D^{ab} \partial_\mu A^{\mu a} \partial_\nu A^{\nu b}$$

DETERMINE SOL'N FROM KNOWN ANSWER

TRANSF: $A_\mu^a \longmapsto A_\mu^a + \frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A_\mu^c$

$$\mathcal{L}_{YM} = -\frac{1}{4}\Big(\overbrace{[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a]}^{F_{\mu\nu}^a} + g f^{abc} A_\mu^b A_\nu^c\Big)\Big(\overbrace{[\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}]}^{F_{\mu\nu}^a} + g f^{ade} A^{\mu d} A^{\nu e}\Big)$$

Schwartz 25.6

$$= -\frac{1}{4}\Big(F_{\mu\nu}^a F^{\mu\nu a} + \underbrace{2F_{\mu\nu}^a\, g f^{abc} A^{\mu b} A^{\nu c}}_{*_1} + \underbrace{g^2 f^{abc} A_\mu^b A_\nu^c f^{ade} A^{\mu d} A^{\nu e}}_{*_2}\Big)$$

$-\frac{1}{4}(*_1) \Longrightarrow -\frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)\, g f^{abc} A^{\mu b} A^{\nu c} = -\frac{g}{2} f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} + \frac{g}{2} f^{abc} \partial_\nu A_\mu^a A^{\mu b} A^{\nu c}$

$$\overset{\text{ab anti-sym}}{=} -g f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} \Longrightarrow C^{abc} = -g f^{abc}$$

$-\frac{1}{4}(*_2) \Longrightarrow$ $g^2 f^{abc} f^{ade} A_\mu^b A^{\mu d} A_\nu^c A^{\nu e} = g^2 f^{kbc} f^{kde} \times$ ~~same~~

$$= g^2 f^{kac} f^{kbd} A_\mu^a A^{\mu b} A_\nu^c A^{\nu d} \Longrightarrow A^{abcd} = g^2 f^{kac} f^{kbd}$$

SOLUTION

(unique? homework or midterm!)

$g_a = g_b \quad \forall a,b \Longrightarrow g_a =: g$

$T^{ab} = -f^{abc}\alpha^b$ — double check, correct

$C^{abc} = -g f^{abc} \qquad A^{abcd} = g^2 f^{kac} f^{kbd} \qquad B^{abc} = 0$ in hmwk

$$T^{ab} A_\mu^b = -f^{abc}\alpha^b A_\mu^c$$

Q: ONLY SOL'N? HMWK!

$T^{ab} = -f^{acb}\alpha^c \qquad T^{ac} A_\mu^c \Rightarrow T^{ac} = -f^{abc}\alpha^b$

GAUGE INV'T $\mathcal{L}$'S: "MAKE IT LOCAL"

* TAKE ϕ_i $\quad i=1\ldots N$, $\quad \mathcal{L} = \partial\varphi^\dagger\partial\varphi - V(\varphi^\dagger\varphi)$
* $\phi \mapsto U\phi \quad \phi^\dagger \mapsto \phi^\dagger U^\dagger$, $\quad U$ $N\times N$.

THEN $\quad \mathcal{L} \Rightarrow \partial\varphi^\dagger U^\dagger U\partial\varphi - V(\varphi^\dagger U^\dagger U\phi)$

$$= \mathcal{L} \iff U^\dagger U = 1.$$

EG IF $U \in SU(N) \Rightarrow \mathcal{L}$ INV'T!

* "MAKE IT LOCAL" $\quad$ Q: what's not so good about this approach?
  - $\hookrightarrow U$ NOW $U(x)$, $\quad x$ IMPLICIT
  * $V(\phi^\dagger\phi)$ STILL INV'T
  * BUT $\partial_\mu\varphi \mapsto \partial_\mu(U\varphi) = U\partial_\mu\varphi + (\partial_\mu U)\varphi = U[\partial_\mu\varphi + (U^\dagger\partial_\mu U)\varphi]$

    $\Rightarrow$ MUST CANCEL $U^\dagger\partial_\mu U$ TERM
  * SUPPOSE

    $\partial_\mu \mapsto D_\mu = \partial_\mu - iA_\mu$, w/ A_μ TRANSF. S.T. $D_\mu\varphi \to UD_\mu\varphi$

    $\hookrightarrow$ THEN $[D_\mu\varphi]^\dagger \longmapsto [D_\mu\varphi]^\dagger U^\dagger$ SO $[D_\mu\varphi]^\dagger D_\mu\varphi$ INV.T

ANSATZ: $A_\mu \to UA_\mu U^\dagger - i(\partial_\mu U)U^\dagger = UA_\mu U^\dagger - i[\partial_\mu(\underbrace{UU^\dagger}_{=1}) - U\partial_\mu U^\dagger]$ $\quad (\partial_\mu(UU^\dagger) \to 0)$

$$= UA_\mu U^\dagger + iU\partial_\mu U^\dagger$$

THEN: $\quad D_\mu\varphi = \partial_\mu\varphi - iA_\mu\varphi$

$$\mapsto U[\partial_\mu\varphi + U^\dagger\partial_\mu U\varphi] - iUA_\mu U^\dagger U\varphi + U\partial_\mu U^\dagger U\varphi$$

$$= UD_\mu\varphi + \underbrace{\partial_\mu U\varphi + U\partial_\mu U^\dagger U\varphi}$$

wrote this way to pull out overall U

$$\partial_\mu U\, U^\dagger U\varphi + U\partial_\mu U^\dagger U\varphi = \partial_\mu(\overbrace{UU^\dagger}^{1})U\varphi = 0$$

SO $\quad D_\mu\varphi \mapsto UD_\mu\varphi$ ✓

OBSERVATIONS

1) A_μ ARE $N \times N$ MATRICES

$A_\mu = A_\mu^\dagger$ PRESERVED BY GAUGE TRANS $\Longrightarrow$ $\boxed{\text{TAKE } A^\dagger = A}$

2) CAN WRITE $U = e^{i\theta\cdot T}$ w/ T^a GENS OF LIE ALG., $T^\dagger = T$

$$A_\mu \longmapsto U A_\mu U^\dagger + i U \partial_\mu U^\dagger \overset{\text{INFTES}}{\simeq} (1 + i\theta\cdot T) A_\mu (1 - i\theta\cdot T) + i(1 + i\theta\cdot T)(-i\partial_\mu\theta\cdot T)$$

$$\simeq A_\mu + i[\theta\cdot T, A_\mu] + \partial_\mu\theta\cdot T + \mathcal{O}(\theta^2) \qquad (*)$$

3) FACT $\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \Longrightarrow \mathrm{Tr}([A,B]) = 0$

$$\Longrightarrow \mathrm{Tr}(A_\mu) \overset{(*)}{\longmapsto} \mathrm{Tr}(A_\mu) + i\,\cancelto{0}{\mathrm{Tr}([\theta\cdot T, A_\mu])} + \partial_\mu\theta^a\,\cancelto{0}{\mathrm{Tr}(T^a)} = \mathrm{Tr}(A_\mu)$$

$\Longrightarrow$ $\boxed{\text{TAKE } \mathrm{Tr}(A_\mu) = 0}$

$*$ A TR-LESS & HERM $\Longrightarrow$ EXP IN BASIS OF TR-LESS $N\times N$ HERM. MATRICES, PRECISELY T^a!

SO $\boxed{A_\mu = A_\mu^a T^a}$

4) THEN $(*) \simeq A_\mu + i[\theta\cdot T, A_\mu] + \partial_\mu\theta\cdot T = A_\mu + i\theta^a A_\mu^b [T^a, T^b] + \partial_\mu\theta\cdot T$

$$= A_\mu^c T^c + i\theta^a A_\mu^b\, i f^{abc} T^c + \partial_\mu\theta^c\, T^c = [A_\mu^c - \theta^a A_\mu^b f^{abc} + \partial_\mu\theta^c]\, T^c$$

SO $\boxed{A_\mu^c \longmapsto A_\mu^c - \theta^a A_\mu^b f^{abc} + \partial_\mu\theta^c}$

(note: slightly diff conventions rel to Schwartz, $\frac{1}{g}$)

RECALL $(t_g^b)_{ac} := f^{abc}$ SPECIFIC MATRIX REP "ADJOINT"

$\Longrightarrow$ $\boxed{A_\mu^c \text{ IN ADJOINT REP.}}$

5) RECOVER $U(1)$ CASE $U = e^{i\theta(x)} \Longrightarrow$ NO $f^{abc} \Longrightarrow A_\mu \longmapsto A_\mu^c + \partial_\mu\theta^c$

$\Longrightarrow$ NEW TWIST IS THE f^{abc}!

RECAP: $D_\mu \varphi \longmapsto U D_\mu \varphi$

SO NOW $\mathcal{L} = (D_\mu \varphi)^\dagger D^\mu \varphi - V(\varphi^\dagger \varphi)$ GAUGE INV'T.

CAVEAT: A_μ KINETIC TERMS!

NAIVE GUESS $\propto F_{\mu\nu} F^{\mu\nu}$ $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

Q: WHAT'S WRONG W/ THIS?

A1: $A_\mu = A_\mu^a T^a$ IS A MATRIX, $\Rightarrow \mathcal{L}$ A MATRIX! ✗

A2: A_μ TRANS $\supset [\theta \cdot T, A_\mu]$ TERM,
$\Rightarrow$ GENS TERMS IN $F^{\mu\nu}$ TRANS
$\Rightarrow F_{\mu\nu}$ NOT GAUGE INV'T (for non-ab groups)

CORRECT GUESS (will prove on homework!)

$$F_{\mu\nu} = F^a_{\mu\nu} T^a \quad \text{w/} \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$

kinetic term can be written w/o matrices. good, b/c diff reps have diff matrices satisfying same alg!

$$\mathcal{L}_{kin} = -\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu}$$

NORMALIZED | $\operatorname{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

WHY GAUGE INV'T?

$F_{\mu\nu} \longmapsto U F_{\mu\nu} U^\dagger$ (will show on homework)

$$\Rightarrow \operatorname{tr} F_{\mu\nu} F^{\mu\nu} \longmapsto \operatorname{tr}(U F_{\mu\nu} U^\dagger U F^{\mu\nu} U^\dagger) = \operatorname{tr}(U F_{\mu\nu} F^{\mu\nu} U^\dagger) = \operatorname{tr}(F_{\mu\nu} F^{\mu\nu})$$

PHYSICS: WHAT'S DIFFERENT?

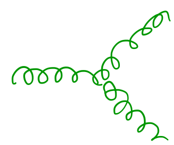
$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$

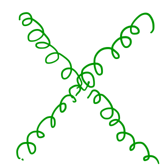
$$F^a_{\mu\nu} F^{\mu\nu a} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + f^{ade} A^{\mu d} A^{\nu e})$$

$$= \underbrace{(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})}_{\text{LOOKS LIKE}} + \underbrace{2 f^{abc} A^b_\mu A^c_\nu (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})}_{\text{3-PT INTERACTION}} + \underbrace{f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}}_{\text{4-PT INTERACTION}}$$

$\mathcal{L}^{ABELIAN}_{kin}$

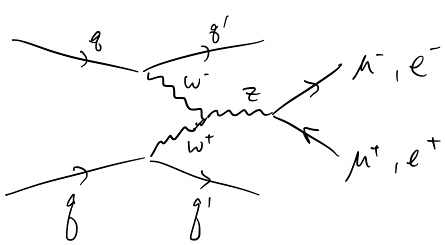
$\Longrightarrow$ 



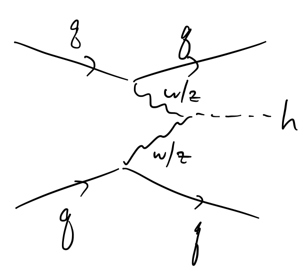


POINT: NON-ABELIAN GAUGE BOSONS HAVE 3-PT & 4-PT SELF-INTERACTIONS

REAL WORLD EG'S VECTOR BOSON FUSION (VBF)



3-PT GB. INT



HIGGS-V-V COUPLING

RECALL $X B_\mu B^\mu$!

GENERAL NON-ABELIAN GAUGE THEORY

(didn't do fermions before, but will add them now)

CAVEAT: – MODULO ANOMALY CONSTRAINTS ON FERM. REPS
– NOT WORRYING ABOUT P_L

$$\mathcal{L} = \left[\sum_{\substack{\text{gauge}\\ \text{bos } i}} -\frac{1}{4}(F_{\mu\nu}^{i})^2\right] + \left[\sum_{\substack{\text{FERM'S}\\ \text{REP } R}} \bar{\psi}_R (i \not{D}_R) \psi_R\right] + \left[\sum_{\substack{\text{BOSONS}\\ \text{REP } r}} \frac{1}{2}(D_\mu \phi_r)^2\right]$$

$$+ \sum \text{OTHER GIOS}$$

$$F_{\mu\nu}^{a} = \partial_\mu A_\nu^{a} - \partial_\nu A_\mu^{a} + g\,\underbrace{(f^{a})^{bc}}_{\text{ADJ. REP}}\, A_\mu^{b} A_\mu^{c}$$

$$D_{\mu_R} = \mathbb{1}_{\dim R \times \dim R}\, \partial_\mu - ig A_\mu^{a} t_R^{a}$$

GENERATORS ASSOC. W/ REP R.

GIO (GAUGE INV. OP) EXAMPLES

(need gauge invariance)
(stopped at ren. ops here)

- MASS $\phi_{r_1}\phi_{r_2}$, $\bar{\psi}_{R_1}\psi_{R_2}$
- YUKAWA $\phi_r \bar{\psi}_{R_1}\psi_{R_2}$
- SCALAR POT. $\phi_{r_1}\phi_{r_2}\phi_{r_3}$, $\phi_{r_1}\phi_{r_2}\phi_{r_3}\phi_{r_4}$

GAUGE FIXING

&

FADEEV-POPOV GHOSTS

ASIDE: ABELIAN FADEEV-POPOV

- what does it do?
- Why does it matter?
- how do we derive the photon propagator?

Recall

PROPAGATORS 4 ARB FIELDS

(point is the schematic form)

SCHEMATIC

$$Z_{free} = \int D\varphi \, e^{-\frac{1}{2}\varphi \cdot K \cdot \varphi + J\varphi} = C \, e^{\frac{1}{2} J \cdot K^{-1} \cdot J}$$

K^{-1} = PROPAGATOR

Q: WHAT IF K HAS NO INVERSE!?

RECALL K^{-1} D.N.E $\Longleftrightarrow \exists$ 0-EIG $\Longleftrightarrow Kv = 0$ FOR SOME v

MAXWELL ACTION

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu$$

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = 2[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu]$$

$$\Longrightarrow \mathcal{L} = -\frac{1}{2}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu] + A_\mu J^\mu = \frac{1}{2} A_\mu [\eta^{\mu\nu}\Box - \partial^\mu \partial^\nu] A_\nu + A_\mu J^\mu$$

HERE K IS $Q^{\mu\nu} := \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu$

NOTE i) $Q^{\mu\nu}\partial_\nu \Lambda(x) = (\eta^{\mu\nu}\partial_\rho\partial^\rho\partial_\nu - \partial^\mu\partial^\nu\partial_\nu)\Lambda = (\partial^\mu\Box - \partial^\mu\Box)\Lambda = 0$

$\Longrightarrow$ $\boxed{Q^{\mu\nu} \text{ HAS NO INVERSE!}}$

ii) $Q_m^{\mu\nu} := Q^{\mu\nu} + m^2 \eta^{\mu\nu}$

HAS AN INVERSE.

$\longrightarrow$ $\boxed{\text{ISSUE IS } m \to 0 \text{ LIMIT}}$

iii) YOU KNOW THIS PHENOMENON

2 MAX. EQN'S $\partial_\mu F^{\mu\nu} = J^\nu$

rename some ind

REWRITE $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box \eta^{\nu\sigma} A_\sigma - \partial^\nu \partial^\sigma A_\sigma = Q^{\mu\nu} A_\mu$

SO MAX. EQN'S: $Q^{\mu\nu} A_\nu = J^\mu \implies A_\sigma = (Q^{-1})_{\sigma\mu} Q^{\mu\nu} A_\nu$

$= Q^{-1}_{\sigma\mu} J^\mu$

PT 1| IF Q^{-1} EXISTED A TOTALLY FIXED

PT 2| E-M: A_μ ISN'T FIXED!

A_μ SOLVES EOM $\implies$ $A_\mu - \partial_\mu \Lambda$ SOLVES EOM

$Q^{\nu\mu} A_\mu = 0$

last page

$Q^{\nu\mu}[A_\mu - \partial_\mu \Lambda] = Q^{\nu\mu} \partial_\mu \Lambda = 0$

GAUGE FIX: IMPOSE EXTRA CONSTRAINT

ASIDE: GLOBAL VS. GAUGE TRANSFORMATIONS

	GAUGE	GLOBAL
Λ:	$\Lambda(x)$	Λ CONST
$A'_\mu = A_\mu - \partial_\mu \Lambda$	A' DIFF.	A' SAME
$\partial_\mu \Lambda$	0-EVEC OF $Q^{\mu\nu}$	$= 0$ $\implies$ 0-VECTOR

POINT: i) $K\vec{0} = \vec{0}$ ALWAYS $\implies$ NO PROB FOR $(Q^{\mu\nu})^{-1}$

i) GAUGE "$K v$" $= Q^{\mu\nu}\partial_\nu \Lambda$ CAUSE Q^{-1} NOT TO EXIST B/C $\Lambda = \Lambda(x)$

SEE: GAUGE NOT GLOBAL HAS PATH INTEGRAL ISSUES

FADEEV-POPOV: PATH INTEGRALS & GAUGE INVARIANCE

(usually introduced for non-abelian gauge symmetry, but it's actually better to start with it here)

(use toy integral to make clear what the issue is)

TOY MODEL $\quad \int_{-\infty}^{\infty} dA \; e^{-A\cdot K\cdot A}$

TAKE $A = \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ ABOVE $= \underbrace{\int_{-\infty}^{\infty} db}_{\text{ONE!}} \int_{-\infty}^{\infty} da \; e^{-a^2}$

$\Rightarrow$ "THY" DOES NOT EXIST, MUST MODIFY!

MODIFICATION $\quad \int_{-\infty}^{\infty} da \; e^{-a^2} \int_{-\infty}^{\infty} db \; \delta(b-\xi) = \underbrace{\int_{-\infty}^{\infty} da \; e^{-a^2}}_{\substack{\text{NO } \xi \text{ DEP!} \\ \text{FINITE!}}}$

RESTRICTING FUNCTIONAL INT.

$I := \int DA \, e^{iS(A)}$

SUPPOSE $A \to A_g \Rightarrow S(A) = S(A_g) \qquad DA = DA_g$

(can act again w/ g', forms group)

GOAL: WRITE $I = \left(\int Dg\right) J$

$\hookrightarrow$ integration over group

TOY EX!

$I = \int dx\,dy \; e^{iS(x^2+y^2)}$ $\longrightarrow$ not a gen func. of x, y. Rot. invariant!

$= \int d\theta \int dr \; r e^{iS(r)} =: \left(\int d\theta\right) J$

FADEEV-POPOV TRICK

some gauge fixing function we choose

$$1 =: \Delta(A) \int dg\, \delta(f(A_g)) \qquad \text{DEFINES } \Delta(A)$$

Language: Fadeev-Popov determinant

$$\Big\downarrow \; A \mapsto A_{g'}$$

group fact $\int Dg'' = \int Dg$

$$1 = \Delta(A_{g'}) \int dg\, \delta(f(A_{g'g})) \overset{g'' := g'g}{=} \Delta(A_{g'}) \int Dg\, \delta(f(A_{g''})) = \Delta(A_{g'}) \underbrace{\int Dg''\, \delta(f(A_{g''}))}_{\Delta(A)^{-1}}$$

$$\Longrightarrow \quad \Delta(A_{g'}) = \Delta(A) \qquad \text{GAUGE INVARIANT!}$$

PATH INTEGRAL:

$$I = \int DA\, e^{iS(A)} = \int DA\, e^{iS(A)}\, \Delta(A) \int Dg\, \delta[f(A_g)]$$

$$= \int Dg \int DA\, e^{iS(A)}\, \Delta(A)\, \delta[f(A_g)]$$

$A \to A_{g^{-1}}$ & use invariance of S, measure, Δ $\longrightarrow$

$$= \left(\int Dg\right) \int DA\, e^{iS(A)}\, \Delta(A)\, \delta[f(A)] =: \left(\int Dg\right) J$$

F-P IN E-M

① GROUP TRANS: $A_\mu \to A_\mu - \partial_\mu \Lambda$

② CHOOSE f: $f(A) = \partial_\mu A^\mu - \sigma$

③ CALC F-P DET: $\Delta(A)^{-1} := \int Dg\, \delta[f(A_g)] = \int D\Lambda\, \delta[\partial_\mu(A^\mu - \partial^\mu \Lambda) - \sigma]$

④ COMPUTE P.I.

$$I = \int D\Lambda \int DA\, e^{iS(A)}\, \delta(\partial A - \sigma) \left[\int D\Lambda\, \delta[\partial A - \sigma - \partial^2 \Lambda]\right]^{-1}$$

$$\xrightarrow{\partial A - \sigma = 0} = \boxed{\frac{\int D\Lambda}{\int D\Lambda\, \delta(\partial^2 \Lambda)}} \int DA\, e^{iS(A)}\, \delta(\partial A - \sigma)$$

drops out of everything so throw away

NB: Δ came outside $\int DA$. Not so for non-abelian case!

⑤ DEFINE $\mathcal{Z}$

- $\mathcal{I}$ IS σ-INDEP, CAN $\int$ w/ ARB FUNCTIONAL.

CHOOSE $\int D\sigma \, e^{-\frac{i}{2\xi}\int d^4x \, \sigma(x)^2}$, $\xi \neq 0$

- $$Z = \int D\sigma \, e^{-i/2\xi \int d^4x \, \sigma(x)^2} \int DA \, e^{iS(A)} \, \delta(\partial A - \sigma)$$

$$= \int DA \, e^{iS(A) - \frac{i}{2\xi}\int d^4x \, (\partial A)^2}$$

$$=: \int DA \, e^{iS_{eff}}$$

w/ $$S_{eff}(A) = S(A) - \frac{1}{2\xi}\int d^4x \, (\partial A)^2$$

$$= \int d^4x \left\{ \frac{1}{2} A_\mu \underbrace{\left[\partial^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial^\nu\right]}_{=: Q^{\mu\nu}_{eff}} A_\nu + A_\mu J^\mu \right\}$$

THE WHOLE POINT

- $(Q^{\mu\nu})^{-1}$ DNE.
- HAD TO FIX. $(Q^{\mu\nu}_{eff})^{-1}$ EXISTS!

PHOTON PROP: (we can do it now since we solves the inverse issue)

two i's from deriv $\Rightarrow$ sign

k-space $$Q^{\mu\nu}_{eff} = -k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu$$

$$Q^{\mu\nu}_{eff} \left[-g_{\nu\lambda} + (1-\xi)\frac{k_\nu k_\lambda}{k^2} \right] \left(\frac{1}{k^2}\right)$$

$$= \left[k^2 \delta^\mu_\lambda - (1-\xi) k^\mu k_\lambda - \left(1 - \frac{1}{\xi}\right) k^\mu k_\lambda + \left(1 - \frac{1}{\xi} - \xi + 1\right) k^\mu k_\lambda \right] \frac{1}{k^2}$$

$$= \delta^\mu_\lambda =: Q^{\mu\nu}_{eff} D_{\nu\lambda}$$

$\Longrightarrow$ MOM. SPACE PROP.

$$iD_{\nu\lambda} = \frac{i\left[-g_{\nu\lambda} + (1-\xi)\frac{k_\nu k_\lambda}{k^2} \right]}{k^2}$$

compare to (15) of II.7, which we didn't do since it takes a lot of work

NON-ABELIAN CASE: FADEEV-POPOV & GHOSTS

ZEE: SCALE $A \to gA$, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ $\quad \frac{1}{2} A_\mu [\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu] A_\nu$

$$\mathcal{L}_1 = -\frac{1}{2} g (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) f^{abc} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu}$$

derived in abelian notes

$$\mathcal{L}_0 = -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 = \frac{1}{2} A_\mu^a (\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu) A_\nu^a =: \frac{1}{2} A_\mu^a Q^{\mu\nu} A_\nu^a$$

NOTE

$Q^{\mu\nu}\partial_\nu \theta = (\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)\partial_\nu \theta = 0 \implies Q^{\mu\nu}$ HAS NO INVERSE!

RECALL: ALSO IN ABELIAN CASE. FIXED W/ FADEEV-POPOV

FADEEV-POPOV: GIVES $Z = \int DA\, e^{iS(A)} \Delta(A)\, \delta[f(A)]$

W/ F-P DETERMINANT $\Delta(A) := \dfrac{1}{\int Dg\, \delta[f(A_g)]}$

GAUGE TRANS'FD A

CHOOSE $f(A) = \partial A - \sigma$ AS BEFORE

$$\implies \Delta(A) = \frac{1}{\int D\theta\, \delta[\partial A^a - \sigma^a - \partial^\mu (f^{abc}\theta^b A_\mu^c - \partial_\mu \theta^a)]}$$

$$\implies Z = \int DA\, e^{iS(A)} \Delta(A)\, \delta[\partial A - \sigma] = \int DA\, e^{iS(A)} \frac{1}{\int D\theta\, \delta[-\partial^\mu (f^{abc}\theta^b A_\mu^c - \partial_\mu \theta^a)]} \times \delta[\partial A - \sigma]$$

A-DEP

KEY

A-INDEP

COMPARE TO ABELIAN $\quad Z^{AB} = \int DA\, e^{iS^{AB}(A)} \dfrac{1}{\int D\theta\, \delta[\partial^\mu \partial_\mu \theta]} \times \delta[\partial A - \sigma]$

POINT: i) ABELIAN CASE: PULL $\Delta(A)\big|_{\partial A-\sigma=0}$

OUTSIDE $\int DA$, THROW OVERALL FACTOR AWAY

ii) NON-ABELIAN: $\Delta(A)\big|_{\partial A-\sigma=0}$ A-DEP, $\Rightarrow$ REMAIN IN $\int DA$!

$\therefore$ MUST DEAL w/ $\Delta(A)\big|_{\partial A-\sigma=0}$ IN Z

TO DO SO: $K^{ab}(x,y) := \partial^\mu\left(f^{abc}A^c_\mu - \partial_\mu \delta^{ab}\right)\delta^4(x-y)$

$$\Rightarrow \partial^\mu\left(f^{abc}\theta^b A^c_\mu - \partial_\mu \theta^a\right) = \int d^4y \; K^{ab}(x,y)\,\theta^b(y)$$

$$\Rightarrow \int D\theta\, \delta\left[-\partial^\mu\left(f^{abc}\theta^b A^c_\mu - \partial_\mu\theta^a\right)\right] = -\int D\theta\, \delta\left[\int d^4y\, K^{ab}(x,y)\,\theta^b(y)\right]$$

NB: $\int d\theta\, \delta(K\theta) = \frac{1}{K}$ $\quad K, \theta \in \mathbb{R}$

GEN TO $\int d\theta\, \delta(K\theta) = \frac{1}{\det K}$ FOR K MATRIX, θ VECTOR

see e.g. Ryder for functional ident

$$\Rightarrow Z = \int DA\; e^{iS(A)}\, \det\left(\int d^4y\, K^{ab}(x,y)\right)\, \delta[\partial A - \sigma]$$

RECALL $\underbrace{\int d\eta_1 \ldots d\eta_N\, d\bar{\eta}_1 \ldots d\bar{\eta}_N}_{\text{GRASSMANN}}\; e^{\bar{\eta}_i A_{ij} \eta_j} = \det A$

$$\Rightarrow \det\left(\int d^4y\, K^{ab}(x,y)\right) = \int Dc\, Dc^+\; e^{iS_{ghost}(c^+,c)}$$

w/ $S_{ghost}(c^+,c) = \int d^4x \int d^4y\; c^+_a(x)\, K^{ab}(x,y)\, c_b(y)$

$= \int d^4x \int d^4y\, c^+_a(x)\,\delta^4(x-y)\,\partial^\mu\left(f^{abc}A^c_\mu - \partial_\mu \delta^{ab}\right) c_b(y)$

$= \int d^4x\; \partial^\mu c^+_a\, \partial_\mu c_a - \partial^\mu c^+_a\, f^{abc} A^c_\mu\, c_b$ $\leftarrow$ all funcs of x

$= \int d^4x\; \partial c^+_a\, D c_a(x)$ $\quad$ w/ $D_\mu = \partial_\mu - f^{abc}A^c_\mu\, \delta_{ab}$

RECALL CAN $\int$ Z over $\sigma^a(x)$ w/ $e^{-\frac{i}{2\xi}\int d^4x\, \sigma^a(x)^2}$

THEN $\delta[\partial_\mu A^{a\mu} - \sigma^a] \mapsto e^{-\frac{i}{2\xi}\int d^4x\, (\partial A^a)^2}$

FINALLY

$$\boxed{Z = \int DA\, Dc\, Dc^+\, e^{iS(A) + iS_{ghost} - \frac{i}{2\xi}\int d^4x\, (\partial A)^2}}$$

FADEEV-POPOV GHOSTS c, c^+

- GRASSMANN SCALARS
- NOT PHYSICAL STATES
- CRITICAL FOR GAUGE FIXING
- RUN IN LOOPS

$\mathcal{L}$ AFTER F-P

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_0 = \underbrace{-\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2}_{\text{AS BEFORE, WRITE AS } A^a_\mu Q_\xi^{\mu\nu} A^a_\nu,\ \text{INVERT}} + \overbrace{\partial c_a^+ \partial c_a}^{\Rightarrow \text{GHOST PROP}}$$

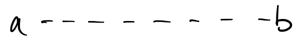
$$\mathcal{L}_1 = \underbrace{-\frac{1}{2} g(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) f^{abc} A^{b\mu} A^{c\nu}}_{a} + \underbrace{\frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu}}_{b} \underbrace{- \partial^\mu c_a^+ g f^{abc} A_\mu^c c_b}_{c}$$

FEYNMAN RULES

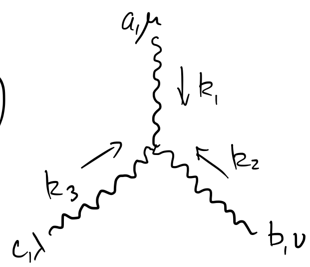
GAUGE PROP: 

SOME 

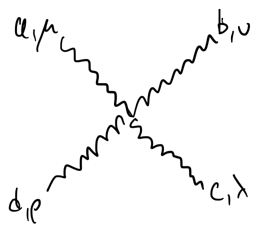
$$\Longleftrightarrow \quad \frac{-i}{k^2}\left[g_{\nu\lambda} - (1-\xi)\frac{k_\nu k_\lambda}{k^2}\right]\delta_{ab}$$

GHOST PROP: 

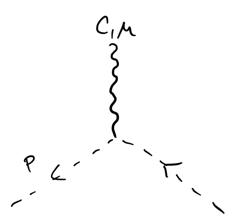
$$\Longleftrightarrow \quad \frac{1}{k^2}\,\delta_{ab}$$

a) 

$$\Longleftrightarrow \quad g f^{abc}\left[g_{\mu\nu}(k_1-k_2)_\lambda + g_{\nu\lambda}(k_2-k_3)_\mu + g_{\lambda\mu}(k_3-k_1)_\nu\right]$$

b) 

$$\Longleftrightarrow \quad -ig^2\Big[f^{abe}f^{cde}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}) + f^{ade}f^{cbe}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\nu}g_{\rho\lambda}) + f^{ace}f^{bde}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\nu\lambda})\Big]$$

c) 

$$\Longleftrightarrow \quad g f^{abc} p^\mu$$

NOTE: THESE INTERACTIONS ARE CRITICAL FOR β_{QCD}

PHYSICAL STATES: BRST COHOMOLOGY

If we were to take gauge-fixed Lagrangian & canonically quantize we'd have creation & annihilation operators for the ghosts. But they're not physical, so what subset of the naive Hilbert space gives rise to the physical Hilbert space? Is there a consistent story?   FOLLOWING PESKIN, SCHWARTZ, WEIGAND

$$\mathcal{L}_{FP} = \mathcal{L}[A_\mu^a, \phi_i] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (\partial_\mu \bar{c}^a)(D_\mu c^a)$$

↳ would work w/ fermions too

Q: GAUGE FIXED, BUT NEW UNPHYSICAL FIELDS

$\Longrightarrow$ WHAT SYMMETRIES FOR $\mathcal{L}_{FP}$?

CLAIM (BRST & HOMEWORK)

$$\phi_i \longmapsto \phi_i + i\theta c^a T^a_{ij} \phi_j \qquad A^a_\mu \longmapsto A^a_\mu + \frac{1}{g}\theta D_\mu c^a$$

$$\bar{c}^a \longmapsto \bar{c}^a - \frac{1}{g}\theta \frac{1}{\xi}\partial_\mu A^a_\mu \qquad c^a \longmapsto c^a - \frac{1}{2}\theta f^{abc} c^b c^c$$

θ GRASSMANN CONSTANT

THEN $\mathcal{L} \mapsto \mathcal{L}$. BRST INV'T (Becchi, Rouet, Stora & Tyutin)

$\Longrightarrow$ GLOBAL SYMMETRY!

$\Longrightarrow \exists$ CONSERVED CURRENT

$\Longrightarrow \exists$ CONSERVED CHARGE Q_{BRST} HMWK: $Q_{BRST}^2 = 0$

Language: "Nilpotent"

MATH DIGRESSION:

TAKE V.S. $\mathcal{H}$, $Q: \mathcal{H} \to \mathcal{H}$ NILPOTENT, IE $Q^2 = 0$

DEF'NS
- $|\psi\rangle \in \mathcal{H} \mid Q|\psi\rangle = 0$, i.e. $|\psi\rangle \in \ker(Q)$, IS Q-CLOSED
- $|\psi\rangle \in \mathcal{H} \mid \exists\ |\chi\rangle \in \mathcal{H} \mid |\psi\rangle = Q|\chi\rangle$, i.e. $|\psi\rangle \in \mathrm{Im}(Q)$ IS Q-EXACT

FACTS
- $\hat{Q}^2 = 0 \Rightarrow \mathrm{Im}(Q) \subset \ker(Q)$

  $\Downarrow$

  $Q|\psi\rangle$ ALSO IN $\ker(Q)$ BC $QQ|\psi\rangle = 0$

- IF $\mathcal{H}$ HAS INNER PRODUCT & $Q = Q^\dagger$

  i) $|\psi\rangle \in \mathrm{Im}(Q)$ HAS $\langle\psi|\psi\rangle = \langle\chi|Q^\dagger Q|\chi\rangle = \langle\chi|Q^2|\chi\rangle = 0$

  ii) $|\psi\rangle \in \mathrm{Im}(Q)$, $|\phi\rangle \in \ker(Q)$ $\Rightarrow \langle\psi|\phi\rangle = \langle\chi|Q^\dagger|\phi\rangle = \langle\chi|Q|\phi\rangle = 0$

SUMMARY: Q-EXACT STATES ARE NULL & ORTHOG. TO Q-CLOSED STATES

- $|\psi_1\rangle, |\psi_2\rangle \in \ker(Q) \mid |\psi_1\rangle - |\psi_2\rangle \in \mathrm{Im}(Q)$
  - HAVE $\langle\psi_1|\psi_1\rangle = \left(\langle\psi_2| + \langle\chi|Q^\dagger\right)\left(|\psi_2\rangle + Q|\chi\rangle\right) = \langle\psi_2|\psi_2\rangle$
  - ALSO, OBSERVABLE $\mathcal{O} = \mathcal{O}^\dagger$

$$\langle\psi_1|\mathcal{O}|\psi_1\rangle = \left(\langle\psi_2| + \langle\chi|Q^\dagger\right)\mathcal{O}\left(|\psi_2\rangle + Q|\chi\rangle\right)$$

$$= \langle\psi_2|\mathcal{O}|\psi_2\rangle + \underbrace{\langle\chi|Q^\dagger\mathcal{O}|\psi_2\rangle + \langle\psi_2|\mathcal{O}Q|\chi\rangle + \langle\chi|Q^\dagger\mathcal{O}Q|\chi\rangle}_{=0 \text{ IF } [Q,\mathcal{O}]=0}$$

$$\Longrightarrow \langle\psi_1|\mathcal{O}|\psi_1\rangle = \langle\psi_2|\mathcal{O}|\psi_2\rangle$$

POINT: MOTIVATES EQUIV RELATION $|\psi_1\rangle \simeq |\psi_2\rangle$ WHEN $|\psi_1\rangle - |\psi_2\rangle \in \mathrm{Im}(Q)$

DEF'N: Q-COHOMOLOGY IS THE QUOTIENT

$$\mathrm{COHOM}(Q) := \frac{\ker(Q)}{\mathrm{im}(Q)} =: \frac{\text{CLOSED } |\psi\rangle}{\text{EXACT } |\psi\rangle}$$

PHYSICAL STATES & COHOM(Q_{BRST})

READ WEIGAND'S NOTES!

NAIVE CANONICAL QUANT: GHOSTS NOT PHYSICAL & $\exists$ NORM ISSUES ON $\mathcal{H}_{NAIVE}$ $\Longrightarrow$ Q: WHAT IS PHYSICAL HILBERT SPACE $\mathcal{H}_{phys}$?

WANT $\mathcal{H}_{phys} \subsetneq \mathcal{H}_{NAIVE}$ S.T.

i) $\exists$ POSITIVE DEFINITE NORM ON $\mathcal{H}_{phys}$

ii) $e^{-iHt}|s\rangle \in \mathcal{H}_{phys}$ $\forall t$, $\forall |s\rangle \in \mathcal{H}_{phys}$

(ie states in $\mathcal{H}_{phys}$ don't time evolve out of it)

(physical states are "protected" from unphysical ones)

FACT: $\boxed{\mathcal{H}_{phys} := \text{COHOM}(Q_{BRST})}$ SATISFIES THESE CONDITIONS!

CHECK ii)

a) BRST AN $\mathcal{L}$-SYMM $\Longrightarrow$ $[Q_{BRST}, H] = 0$

b) TAKE $|s\rangle \in \text{COHOM}(Q_{BRST})$

$$\Longrightarrow |s(t)\rangle = e^{-iHt}|s\rangle \quad \text{HAS} \quad Q_{BRST}|s(t)\rangle = Q_{BRST}\, e^{-iHt}|s\rangle$$
$$= e^{-iHt} Q_{BRST}|s\rangle = 0$$

$\Longrightarrow$ ii) SATISFIED!

$\Downarrow$

see Weigand for proof of these & other facts.

ANOMALIES

Q: CAN QM BREAK CL. SYMMS?

EG. $S[\phi]$ INVT, NOT $\int D\phi\, e^{iS[\phi]}$?

IN PRACTICE: $\partial_\mu J^\mu = 0$ OR NOT?

CHIRAL ANOMALY

$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi$

GLOBAL $U(1)$: $\psi \to e^{i\theta}\psi$ $\quad J^\mu = \bar{\psi}\gamma^\mu\psi$

CHIRAL SYM: $\psi \to e^{i\theta\gamma^5}\psi$ $\quad J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$

NOTE: $A^{\lambda\mu\nu} := \langle 0 | T\, J_5^\lambda(0)\, J^\mu(x_1)\, J^\nu(x_2) | 0 \rangle$

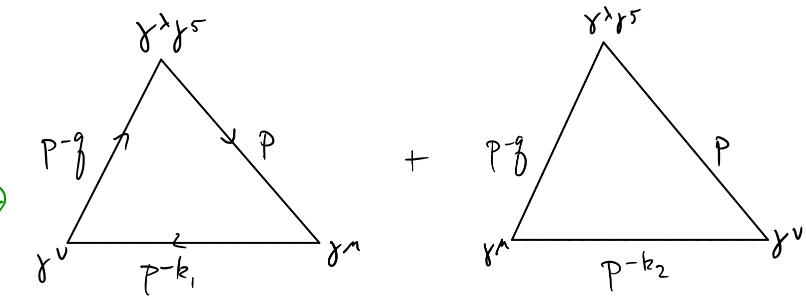
$$\left.\begin{aligned} \partial_\mu J^\mu = 0 &\implies \partial_\mu A^{\lambda\mu\nu} = \partial_\nu A^{\lambda\mu\nu} = 0 \\ \partial_\mu J_5^\mu = 0 &\implies \partial_\lambda A^{\lambda\mu\nu} = 0 \end{aligned}\right\}$$

CONDITIONS TO TEST!

P-SPACE: $A^{\lambda\mu\nu}(x_1, x_2) \longrightarrow \Delta^{\lambda\mu\nu}(k_1, k_2) \implies \partial_\lambda A^{\lambda\mu\nu} = 0 \longmapsto q_\lambda \Delta^{\lambda\mu\nu} = 0$

$$\Delta^{\lambda\mu\nu}(k_1,k_2) = (-1)\, i^3 \int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\nu \frac{1}{\not{p}-\not{k}_1}\gamma^\mu \frac{1}{\not{p}} + \gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\mu \frac{1}{\not{p}-\not{k}_2}\gamma^\nu \frac{1}{\not{p}} \right)$$

HAWK



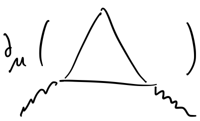
w/ approp shifted momentum $\int$, see Zee

APPROP. CALC: $q_\lambda \Delta^{\lambda\mu\nu} = \frac{i}{(2\pi)^2}\varepsilon^{\mu\nu\lambda\sigma} k_{1\lambda} k_{2\sigma} \neq 0$

$\implies$ QM CAN BREAK CHIRAL SYMM

Language: "Chiral Anomaly" or "Axial Anomaly"

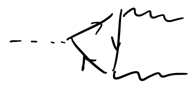
REMARKS:

1) COUPLE TO PHOTON $\Rightarrow e^2 A_\mu A_\nu \cdot \Delta^{\lambda\mu\nu} \Rightarrow e^2 \partial_\mu \Big($  $\Big) \neq 0$

CLASSICAL $\quad \partial_\mu J_5^\mu = 0$

QUANTUM $\quad \partial_\mu J_5^\mu = \frac{e^2}{(4\pi)^2} \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$

$\Rightarrow \partial_\mu J_5^\mu$ PRODUCES $\gamma\gamma$ IN QM!

2) GIVES π^0 DECAY! $\quad \pi^0$ 

3) REPALKAGE R-CHIRAL & L-CHIRAL FERM

$$J^\mu_{L,R} = \bar\psi_{L,R} \gamma^\mu \psi_{L,R} \qquad w/ \qquad \partial_\mu J^\mu_{L,R} = \mp \frac{1}{2} \frac{e^2}{(4\pi)^2} \varepsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$$

NB: L & R HANDED OPP. CONTRIB!

Q: WHAT IF WE HAD AN ANOMALY IN A GAUGE SYMMETRY?
(what do you think?)

GAUGE ANOMALIES (note: Peskin notation slightly diff $\quad \sigma^\mu := (1, \sigma^i)$, $\bar\sigma^\mu = (1, -\sigma^i)$
$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix}$)

G.S. NECESSARY FOR A_μ^a PROP

GAUGE ANOMALY $\Rightarrow$ THEORY D.N.E

CONSIDER NAGT w/ L, R FERMIONS IN G-REPS.

FACT: CAN WRITE R-CHIRAL AS L-CHIRAL

$$\psi'_{Li} = \sigma^2 \psi^*_{Ri} \qquad \psi'^{\dagger}_{Li} = \psi^T_{Ri}\sigma^2$$

r-ferm as anti-ferm
l anti-ferm as new l-ferm species

$$\psi^\dagger_R i\sigma\cdot(\partial - igA^a t^a_r)\psi_R = \psi'^\dagger_L i\bar{\sigma}\cdot(\partial - igA^a t_{\bar{r}}{}^a)\psi'_L$$

NOTE: R in r NOW L in $\bar{r}$!

NON-AB. GS CURR 4-CMPT

GETS ANOMIAL THROUGH AXIAL CURRENT

$$j^{\mu a} = \sum_r \bar{\psi}_r \gamma^\mu \underbrace{\left(\frac{1-\gamma^5}{2}\right)}_{P_L} t^a_r \psi_r$$

$$\langle p,\nu,b;k,\lambda,c|\partial_\mu j^{\mu a}|0\rangle = \frac{g^2}{8\pi^2}\varepsilon^{\alpha\nu\beta\lambda} p_\alpha k_\beta \cdot A^{abc}$$

$$A^{abc} = \mathrm{tr}\left[t^a\{t^b,t^c\}\right] \implies \text{TOTALLY ASYMM}$$

$A^{a,b,c} \neq 0$ FOR ANY $a,b,c \implies$ NO G.S. QM-ICALLY!

Language: "Anomaly Coefficient" only rep-dep here

FACT: FOR SU(N), $A^{abc} = A(R)\, d^{abc}$

SUM OVER FERMIONS $\implies \sum_r A_r^{abc} = \sum_{r_\ell} A_{r_\ell}^{abc} - \sum_{r_r} A_{r_r}^{abc} = \underbrace{\left(\sum_{r_\ell} A(r_\ell) - \sum_{r_r} A(r_r)\right)}_{0 \implies \text{ANOM FREE}} d^{abc}$

Q: DOES SM HAVE GAUGE ANOMALIES?

STANDARD MODEL ANOMALY CANCELLATION

TREAT EVERYTHING AS L-CHIRAL

FERMIONS $(C, L)_Y$

$Q: (3,2)_{\frac{1}{6}}$ $\quad u^c: (\bar{3},1)_{-\frac{2}{3}}$ $\quad D^c: (\bar{3},1)_{\frac{1}{3}}$ $\quad L: (1,2)_{-\frac{1}{2}}$ $\quad E^c: (1,1)_1$

Families

$SU(3)^3$: $\quad$ ANOMALY $\sim (\#3 - \#\bar{3}) = 3[\underset{Q}{2} - \underset{u}{1} - \underset{D}{1}] = 0$

$SU(2)^3$: $\quad$ SELF-CONJUGATE $\Rightarrow$ D.N.E

$U(1)-SU(3)^2$: $\quad$ ANOM $\sim \sum_{\substack{SU(3)\text{-charged} \\ f}} Y_f = 3\left[2\cdot\frac{1}{6} + \frac{1}{3} - \frac{2}{3}\right] = 0$

$U(1)-SU(2)^2$: $\quad$ ANOM $\sim \sum_{\substack{SU(2) \\ \text{charged} \\ \text{terms} \\ f}} Y_f = 3\left[3\cdot\frac{1}{6} + 1\left(-\frac{1}{2}\right)\right] = 0$

$U(1)-G-G$: $\quad$ ANOMALY $\sim \left(\sum_f Y_f\right) = 3\left[6\cdot\frac{1}{6} + 3\cdot\frac{1}{3} + 3\left(-\frac{2}{3}\right) + 2\left(-\frac{1}{2}\right) + 1\right]$

$= 3[1 + 1 - 2 - 1 + 1] = 0$

$U(1)^3$: $\quad$ ANOMALY $\sim \left(\sum_f Y_f^3\right)$

$= 3\left[6\left(\frac{1}{6}\right)^3 + 3\left(\frac{1}{3}\right)^3 + 3\left(-\frac{2}{3}\right)^3 + 2\left(-\frac{1}{2}\right)^3 + 1(1)^3\right]$

$= 3\left[\underbrace{\frac{6}{216} + \frac{24}{216}} - \frac{8\cdot 24}{216} - \frac{54}{216} + \frac{216}{216}\right]$

$= \frac{3}{216}\left[30 - 192 - 54 + 216\right] \sim [246 - 246] = 0$ ✓

WOW!

GRAND UNIFICATION

$$SO(10) \supset SU(5) \supset SU(3)\times SU(2)\times U(1)$$

$16 \ni 15.$ ν_2^c

$$16 = 10+\bar{5} = \begin{array}{ccccc} Q & U & D & L & E \\ 6 & 3 & 3 & 2 & 1 \end{array} = 15$$

PREDICTS HYPERCHARGE!

DEVIL'S ADVOCATE:

Q: BUT WHAT ELSE COULD IT HAVE BEEN?

SUPPOSE $U(1)$ w/ UNFIXED CHARGES A, B, C, D, E

$$\boxed{Q\sim(\bar{3},2)_A \quad U\sim(\bar{3},1)_B \quad D\sim(\bar{3},1)_C \quad L\sim(1,2)_D \quad E\sim(1,1)_E} \quad *$$

ANOMALIES CONSTRAIN A, B, C, D, E

$U(1)^3$: $\quad 3\cdot 2A^3+3B^3+3C^3+2D^3+E^3=0$

$U(1)-SU(3)^2$: $\quad 2A+B+C=0$

$U(1)-SU(2)^2$: $\quad 3A+D=0$

$U(1)-G-G$: $\quad 3\cdot 2\cdot A+3B+3C+2D+E=0$

SOLN 1 $\quad B=-C \quad$ NOT CHIRAL $U(1)$, $U(1)$ DOES NOT FORBID UD MASS

SOLN 2 $\quad A=\frac{1}{6} \quad B=-\frac{2}{3} \quad C=\frac{1}{3} \quad D=-\frac{1}{2} \quad E=1$

(up to overall normalization & $B \longleftrightarrow C$)

$\Rightarrow$ FACT

GIVEN (*), $U(1)_Y$ IS THE ONLY CONSISTENT CHIRAL $U(1)$

ANOMALY ARG TOWARD SM

RULES

① SIMPLICITY: MIN DIM $SU(N)$ REPS

② CHIRALITY: NO BARE MASS TERMS ALLOWED BY SYMM

③ CONSISTENT: NO GAUGE ANOMALIES

SIMPLE ANOMALY STATEMENTS

i) $SU(N>2)^3$ REQ $\# N = \# \bar{N}$

ii) WITTEN ANOMALY $\#\, 2$'s OF $SU(2)$ = EVEN

GUESS 1 $G = SU(2)$ $2n_f$ 2's $\mathcal{L} \supset M\, \varepsilon_{ij}\, Q_i Q_j$

Q_i $i = 1, \ldots, 2n_f$ $SU(2)$ INV'T ALLOWED BY SYMM ✗

GUESS 2 $G = SU(N>2)$ $\underbrace{n_f\ 3\text{'s}}_{Q_i}$. ANOM. $\Rightarrow$ $n_f\ \bar{3}$'s $\bar{Q}_i$ $\exists$ MASS TERM ✗

NOTE: NOW WE NEED 2 FACTORS

GUESS 3 $G = SU(2) \times SU(2)$

$\# (1,2) = A$
$\# (2,1) = B$
$\# (2,2) = C$

WITTEN $\Rightarrow$ $A = 2\tilde{A}$
$B = 2\tilde{B}$

AGAIN, MASS! ✗

GUESS 4 $G = SU(3) \times SU(2)$ $\# (3,2) = A$ $\# (3,1) = B$ $\# (1,2) = C$

$\# (\bar{3},2) = A'$ $\# (\bar{3},1) = B'$

i) A & $A' \neq 0 \Rightarrow$ MASS

SO $A \neq 0$ WLOG $A' = 0$

ii) $B' \neq 0$ NEC FOR $SU(3)^3$ A.C.

$\Rightarrow B = 0$ FOR SO NO MASS

iii) $SU(3)^3$

$B' = 2A$

iv) WITTEN ANOM:

CASE A, C EVEN

CASE $A + C$ EVEN

$\underbrace{\hphantom{A+C\ \text{EVEN}}}_{\text{LESS MATTER}}$

v) MIN $A + C$ EVEN

$\Rightarrow A = C = 1$

$\Rightarrow B' = 2$

$(3,2)\ (\bar{3},1)\ (\bar{3},1)\ (1,2)$ $(*_2)$

iv) $\exists$ CHIRAL $U(1)$

$\Rightarrow$ NEED ≥ 1 $(1,1)$

THIS + $(*_2) \Rightarrow (*)$

$\Rightarrow Y \Rightarrow$ SM GEN

SUSY $\Longrightarrow$ 2HDM (MSSM NOT SM + SPARTICLES. WHY?)

Q: WHY DOES MINIMAL SUSY HAVE 5 HIGGS BOSONS?

LOGIC STEPS: (required, not just for fun)

1. SM GEN $\Longrightarrow$ ANOMALY FREE!

2. h IN $(1,2)_{\frac{1}{2}}$

   SUPPOSE SUSY: THEN $\exists$ WEYL FERM $\tilde{h}$ $(1,2)_{\frac{1}{2}}$ $\Rightarrow$ ANOMALY

3. NEED $\tilde{h}'$ TO CANCEL.

   SIMPLEST IS $\tilde{h}' = (1,2)_{-\frac{1}{2}}$

4. $\Longrightarrow \exists$ SCALAR h' $(1,2)_{-\frac{1}{2}}$

   $\Longrightarrow$ 2HDM

5. h, h' 8 D.O.F.

   EWSB: EAT 3

   $\Longrightarrow$ 5 LEFTOVER, HIGGS BOSONS

ASYMPTOTIC FREEDOM

Your midterm! Following Peskin

Q: WHAT IS $\beta(g_{YM}) := \mu \frac{\partial g_{YM}}{\partial \mu}$?

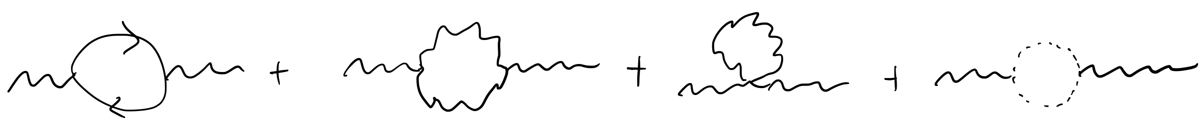
Q: ITS PHYSICS?

Q: DIFFERENCES FROM QED?

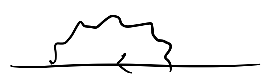
NAGT RENORMALIZATION @ 1-LOOP

DIAGRAMS

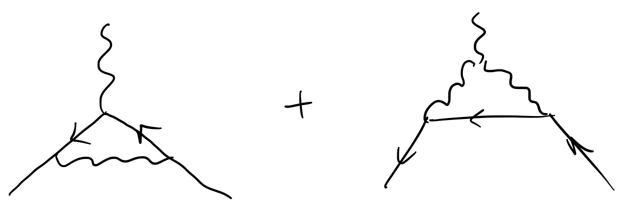
A_μ SELF-ENERGY



ψ SELF-ENERGY



VERTEX CORRECTIONS



Q: ORIGIN OF VERTICES?

PESKIN'S ANSWER: $$\beta(g) = \frac{-g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]$$ ASSUMES A & ψ_{DIRAC}

COMPLETE ANSWER: $$\beta(g) = \frac{-g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \sum_{\substack{\text{Dirac}\\ \text{in } r}}\frac{4}{3}C(r) - \sum_{\substack{\text{Weyl}\\ \text{in } r}}\frac{2}{3}C(r) - \sum_{\substack{\mathbb{C}\text{ sc.}\\ \text{in } r}}\frac{1}{3}C(r) - \sum_{\substack{\mathbb{R}\text{ sc}\\ \text{in } r}}\frac{1}{6}C(r)\right]$$

$\frac{1}{(4\pi)^2}$ "Loop factor"

SOLVE β: $\quad \beta_g = -\frac{g^3}{(4\pi)^2} B$, $\quad$ B GROUP THEORETIC #

$$t = \ln\mu \Rightarrow \beta_g = \mu \frac{\partial g}{\partial \mu} = \frac{\partial g}{\partial \ln\mu} = \frac{\partial g}{\partial t} = -\frac{g^3}{(4\pi)^2} B$$

INTEGRATE $\Rightarrow$ $$\frac{8\pi^2}{g^2(\mu)} = \frac{8\pi^2}{g^2(\Lambda)} - b \log\left(\frac{\Lambda}{\mu}\right)$$

$$g^2(\mu) = \frac{8\pi^2}{\frac{8\pi^2}{g^2(\Lambda)} - b\log\left(\frac{\Lambda}{\mu}\right)} = \frac{g^2(\Lambda)}{1 - b\frac{g^2(\Lambda)}{8\pi^2}\log\frac{\Lambda}{\mu}}$$

IS μ OR Λ THE IR SCALE?

$b > 0 \Rightarrow g^2(\mu_{UV}) < g^2(\mu_{IR})$

$$\frac{\Lambda}{\mu} < 1 \Rightarrow g^2(\mu) = \frac{g^2(\Lambda)}{1 - (+)(+)(-)} < g^2(\Lambda) \checkmark$$

$$\frac{\mu}{\Lambda} < 1 \Rightarrow g^2(\mu) = \frac{g^2(\Lambda)}{1 - (+)(+)(+)} > g^2(\Lambda) \checkmark$$

(equation keeps track of sign through log)

SUSY MODIFICATION

SUSY "MULTIPLETS" HAVE BOSONS & FERMIONS

$m=0$ CHIRAL MULTIPLET = $\mathbb{C}$ sc. + Weyl Ferm = 2 BOS DOF + 2 FERM DOF

$m=0$ VECTOR MULTIPLET = Weyl Ferm + vector = 2 FERM. DOF + 2 BOS DOF

$$B_{\text{NON-SUSY}} = \left[\frac{11}{3} C(\text{Adj}) - \frac{4}{3} C(\text{Dirac}^r) - \frac{2}{3} C(\text{Weyl}^r) - \frac{1}{3} C(\mathbb{C}^r \text{ sc}) - \frac{1}{6} C(\mathbb{R}^r \text{ sc})\right]$$

$\Downarrow$

$$B_{\text{SUSY}} = \left[\left(\frac{11}{3} - \frac{2}{3}\right) C\begin{pmatrix}\text{Adj} \\ m=0 \text{ V}\end{pmatrix} - \left(\frac{2}{3} + \frac{1}{3}\right) C\begin{pmatrix}\text{REP } r \\ m=0 \text{ CH}\end{pmatrix}\right] = 3\, C\begin{pmatrix}\text{Adj} \\ \text{Vector mult}\end{pmatrix} - \sum_r C\begin{pmatrix} r \\ \text{Chiral mult}\end{pmatrix}$$

$\alpha = \frac{g^2}{4\pi}$

$\alpha_1^{-1}(M_Z) = 58.98 \pm .04$

$\alpha_2^{-1}(M_Z) = 29.57 \pm .03$

$\alpha_3^{-1}(M_Z) = 8.4 \pm .14$

<u>MSSM</u> $B_3 = 3$ $B_2 = -1$ $B_Y = -\frac{32}{5}$