CFT

Conformal Field Theory

Scale Invariance:

BY EXAMPLE: FREE MASSLESS SCALAR

$$S = \int d^d x \, \phi \Box \phi \qquad \text{SCALE TRAFO} \quad x \longrightarrow \lambda x$$

$$S \longrightarrow \int d^d x \, \lambda^{d-2} \phi(\lambda x) \Box \phi(\lambda x) \qquad \text{(not inv't w/o extra info!)}$$

$$\text{IF} \quad \phi(\lambda x) = \lambda^{-\Delta} \phi(x) \quad (*)$$

$$S \longrightarrow \int d^d x \, \lambda^{d-2\Delta-2} \phi(x) \Box \phi(x) \Longrightarrow \Delta = \frac{d-2}{2} \quad \text{FOR SCALE INVARIANCE}$$

MASS DEFORM $\quad S = \int d^d x \, \phi(\Box + m^2)\phi$

$$\int d^d x \, \phi m^2 \phi \longrightarrow \int d^d x \, \lambda^{d-(d-2)} m^2 \phi^2 \quad \text{NOT INVARIANT}$$

HOW DOES $(*)$ AFFECT CORRELATORS?

$$G^{(n)}(x_1, \ldots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle$$

$$\Downarrow$$

$$G^{(n)}(\lambda x_1, \ldots, \lambda x_n) = \langle \phi(\lambda x_1) \ldots \phi(\lambda x_n) \rangle = \lambda^{-n\Delta} G^{(n)}(x_1, \ldots, x_n)$$

NOTE: SCALE-INV'T THEORIES HAVE CORRS w/ SCALE TRAFOS!

THEORETICAL PICTURE

LOGIC:

1. SCALE INV'T

$$\Downarrow$$

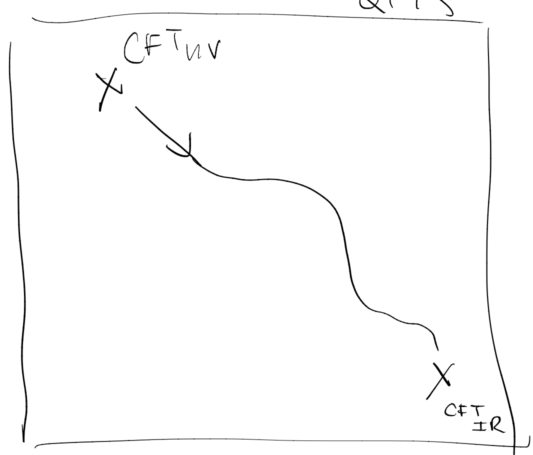
RG FIXED PT

2. SCALE $\overset{\text{OFTEN}}{\Longrightarrow}$ CONFORMAL

IDEA QFT = RG FLOW FROM $CFT_{UV} \Rightarrow CFT_{IR}$

$\Longrightarrow$ ORGANIZING PRINCIPLE!

SPACE OF QFT'S



Q: HOW TO STUDY CFT?

Conformal Group D Dims

CHANGE COORDS $x \to x'$ $\quad g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$

DEF'N: CONFORMAL GROUP IS SUBGROUP OF COORD TRANS. ST

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x)$$

Language: Metric invariant up to local scale factor.

IE $\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} = \delta^\alpha_\mu \delta^\beta_\nu \Omega(x)$

NOTE ① $v^2 = v^\mu g_{\mu\nu} v^\nu \longrightarrow \Omega(x)\, v^2$

But: $\frac{v\cdot w}{|v|\,|w|} \rightarrow \frac{\Omega}{\sqrt{\Omega}\sqrt{\Omega}}\frac{v\cdot w}{|v||w|} = \frac{v\cdot w}{|v||w|} = \cos\theta$, $\measuredangle$'s INV'T!

② POINCARE LEAVES METRIC INVARIANT $g'_{\mu\nu} = g_{\mu\nu}$

$\implies$ POINCARE $\subseteq$ CONFORMAL

INFINITESIMAL TRAFOS AROUND FLAT SPACE $g_{\mu\nu} = \eta_{\mu\nu}$

COORD $x^\mu \to x^\mu + \varepsilon^\mu$

$$\implies g'_{\mu\nu} = \eta_{\mu\nu} + (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) \overset{\text{CONF}}{=} (1+A)\eta_{\mu\nu}$$

$$\therefore \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = A\,\eta_{\mu\nu} \quad (*)$$

TRACE IT (mult by $\eta^{\mu\nu}$)

$$\partial\cdot\varepsilon + \partial\cdot\varepsilon = A\,\eta^{\mu\nu}\eta_{\mu\nu} = A\,\delta^\mu{}_\mu = Ad \implies A = \frac{2}{d}(\partial\cdot\varepsilon)$$

$(*) \implies \partial_\rho \partial_\sigma \partial_{(\mu}\varepsilon_{\nu)} = \partial_\rho\partial_\sigma A \eta_{\mu\nu}$, $\eta^{\rho\mu}$ both sides

HMWK

$$\partial^\mu \partial_\sigma \partial_{(\mu}\varepsilon_{\nu)} = \partial_\sigma[\Box \varepsilon_\nu - \partial_\nu \partial\cdot\varepsilon] = \partial_\nu \partial_\sigma A$$

$$\implies \partial_\sigma \Box \varepsilon_\nu = \partial_\nu \partial_\sigma (A - \partial\cdot\varepsilon)$$

$$\frac{1}{2}[\Box\, \partial_{(\sigma}\varepsilon_{\nu)}] = \frac{1}{2}\Box A \eta_{\nu\sigma}$$

$$\implies \frac{1}{2}\Box \frac{2\,\partial\cdot\varepsilon}{d}\eta_{\nu\sigma} = \partial_\nu\partial_\sigma\left(\frac{2}{d}-1\right)\partial\cdot\varepsilon$$

$$\implies \boxed{\left(\Box\eta_{\mu\nu} + (d-2)\partial_\mu\partial_\nu\right)\partial\cdot\varepsilon = 0} \quad (*_2)$$

$\rightsquigarrow$ something about $d=2$ is special!

HMWK: show 3rd derivative claim

$$(\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu)\partial\cdot\epsilon = 0\,. \qquad (1.3)$$

For $d > 2$, (1.2) and (1.3) require that the third derivatives of ϵ must vanish, so that ϵ is at most quadratic in x. For ϵ zeroth order in x, we have

a) $\epsilon^\mu = a^\mu$, i.e. ordinary translations independent of x.

FACT: $\mathcal{E}$ AT MOST QUADRATIC IN X. MUST SATISFY $(*_1)$ $(*_2)$

CASES

① $\mathcal{E}\sim x^0$ $\quad \mathcal{E}^\mu = a^\mu \quad \partial\cdot\mathcal{E} = 0$ ✓ TRANSLATIONS

② $\mathcal{E}\sim x^1$ i) $\mathcal{E}^\mu = w^\mu{}_\nu x^\nu \Rightarrow w_{\mu\nu} + w_{\nu\mu} = \frac{2}{d} w^\alpha{}_\alpha \eta_{\mu\nu}$ — antisymmetric off-diagonal, on-diag det'd by one #, $w^\alpha{}_\alpha$, just dil

ROTATIONS $\quad (\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu) w^\mu{}_\mu = 0$ ← satisfied b/c d(constant) $= 0$

ii) $\mathcal{E}^\mu = \lambda x^\mu \Rightarrow \partial\cdot\mathcal{E} = \lambda d$

$(*_2)$: DERIVS ON $\partial\cdot\mathcal{E}$, SO ✓

$(*_1)$: $\lambda\,\partial_{(\mu} x_{\nu)} = \frac{2\lambda d}{d}\eta_{\mu\nu}$

$\qquad \lambda\,\eta_{(\mu\nu)} = 2\lambda\eta_{\mu\nu}$ ✓

SCALE TRAFOS AKA DILATATIONS

③ $\mathcal{E}\sim x^2$ $\quad \mathcal{E}^\mu = A^{\mu\nu\rho} x_\nu x_\rho \Rightarrow = b^\mu x^2 - 2x^\mu b\cdot x$ — SPECIAL CONF TRANS

HMWK: SHOW SATISFYING $*_{1+2} \Rightarrow A^{\mu\nu\rho}$ S.T. $\mathcal{E}^\mu = b^\mu x^2 - 2x^\mu b\cdot x$

DIM

$$d + \frac{d(d-1)}{2} + 1 + d = \frac{(d+2)(d+1)}{2} = \dim(\underbrace{SO(p+1,q+1)})$$

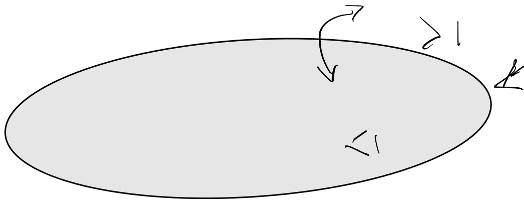
"GLOBAL" CONFORMAL GROUP! (FOR p,q) - SIGNATURE METRIC

FINITE TRAFOS: ① $x'_\mu = x_\mu + a_\mu$

② i) $x'_\mu = \Lambda_\mu{}^\nu x_\nu$, $\Lambda\in SO(p,q)$ $\quad$ ii) $x'_\mu = \lambda x$

③ $x_\mu' = \dfrac{x_\mu + b_\mu x^2}{1 + 2b\cdot x + b^2x^2}$

SCT PRESERVES NORM ON SURFACE $1 = 1 + 2b\cdot x + b^2 x^2$



CONFORMAL ALGEBRA IN 2D

w/ $\eta_{\mu\nu} = \delta_{\mu\nu}$

$$\partial_{(\mu}\varepsilon_{\nu)} = \frac{2}{d}(\partial\cdot\varepsilon) \Rightarrow \cancel{2}\,\partial_1\varepsilon_1 = \cancel{\partial_1\varepsilon_1} + \partial_2\varepsilon_2 \quad \& \quad \partial_1\varepsilon_2 = -\partial_2\varepsilon_1$$

IE ε^μ SATISFIES CAUCHY-RIEMANN EQUATIONS! COMPLEXIFY!

$$z,\bar z = x^1 \pm i x^2 \qquad \partial_{z,\bar z} = \frac{1}{2}\left(\partial_{x^1} \mp i\partial_{x^2}\right) \quad \text{SO} \quad \begin{array}{l} \partial_z z = \partial_{\bar z}\bar z = 1 \\ \partial_{\bar z} z = \partial_z \bar z = 0 \end{array}$$

$\varepsilon,\bar\varepsilon := \varepsilon_1 \pm i\varepsilon_2$ SEE: $\partial_{\bar z}\varepsilon = \frac{1}{2}\left(\overset{\partial_1}{\partial_{x^1}} + i\partial_{x^2}\right)\left(\varepsilon_1 + i\varepsilon_2\right)$

$$= \frac{1}{2}\left(\partial_1\varepsilon_1 - \partial_2\varepsilon_2 + i\left[\partial_1\varepsilon + \partial_2\varepsilon_2\right]\right) \overset{CR}{=} 0$$

SIM $\partial_z\bar\varepsilon = 0$ SO $\varepsilon = \varepsilon(z)$ $\bar\varepsilon = \bar\varepsilon(\bar z)$

SUMMARY IN 2D

$$z \longrightarrow f(z) \qquad \bar z \longrightarrow f(\bar z)$$

$$ds^2 = dz\,d\bar z \longrightarrow \left|\frac{df}{dz}\right|^2 dz\,d\bar z \qquad \Omega = \left|\frac{\partial f}{\partial z}\right|^2$$

INFINITE DIMENSIONAL SYMMETRY

GENERATORS & COMMUTATION RELATIONS

INFTES $\quad z \longrightarrow z' = z + \varepsilon_n(z), \quad \bar z \longrightarrow \bar z' = \bar z + \bar\varepsilon_n(\bar z)$

$$\varepsilon_n(z) = -z^{n+1} \qquad \bar\varepsilon_n(\bar z) = -\bar z^{n+1}$$

GENERATORS $\quad z' = z + \varepsilon_n =: (1 + l_n)\, z$

$$= z - z^{n+1} = (1 - z^{n+1}\partial_z)\, z \qquad \text{SO } l_n = -z^{n+1}\partial_z \quad \text{SIM } \bar l_n = -\bar z^{n+1}\partial_{\bar z}$$

$$[l_n, \bar l_m] = [z^{n+1}\partial_z, \bar z^{m+1}\partial_{\bar z}] \overset{\text{HOL}}{=} 0$$

$$[l_n, l_m] = z^{n+1}\partial_z(z^{m+1})\partial_z - z^{m+1}\partial_z z^{n+1}\partial_z$$

$$= (m+1)\, z^{n+m+1}\partial_z - (n+1)\, z^{n+m+1}\partial_z = (m-n)\, l_{m+n}, \quad \text{SIM FOR BAR}$$

$$\boxed{[l_n, l_m] = (m-n)\, l_{m+n} \qquad [\bar l_n, \bar l_m] = (m-n)\, \bar l_{m+n}}$$

(otherwise zero)

NOTE: NON-SINGULAR l_n NEAR $z = 0 \Rightarrow n \geq -1$

OTOH @ $z = \infty$, $w = -\frac{1}{z} = 0$,

$$l_n = \left(-\frac{1}{w}\right)^{n+1} \frac{\partial w}{\partial z}\, \partial_w = \left(\frac{-1}{w}\right)^{n+1} w^2 \partial_w = -\left(\frac{1}{w}\right)^{n-1}\partial_w \Rightarrow n \leq +1$$

SO $\underbrace{\{l_{-1}, l_0, l_1\} \cup \{\bar l_{-1}, \bar l_0, \bar l_1\}}_{\text{"GLOBAL" GENS}}$ ONLY GENS ON $\mathbb{C} \cup \infty = S^2$

WHAT ARE THEY!?

$\ell_{-1} = -\partial_z \implies$ GEN OF TRANS, SIM FOR $\bar{\ell}_{-1}$

$\ell_0 + \bar{\ell}_0 = -(z\partial_z + \bar{z}\partial_{\bar{z}})$ GENERATOR OF DILATATIONS

BC $z \to (1 - \alpha(z\partial_z + \bar{z}\partial_{\bar{z}}))z = (1-\alpha)z$

$i(\ell_0 - \bar{\ell}_0) \propto x_1\partial_2 - x_2\partial_1$ GEN OF ROTATIONS $\ell_1, \bar{\ell}_1$ GEN SCTS: HMWK

FINITE TRAFOS BECOME $z \to \dfrac{az+b}{cz+d}$ $\bar{z} \to \dfrac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$

WHERE $a,b,c,d \in \mathbb{C}$ & $ad-bc = 1 = \det M$ Ginsparg might have different conv? $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$

PHYSICS FORESHADOW

GLOBAL ALGEBRA $\{\ell_{-1}, \ell_0, \ell_1\} \cup$ BAR

CHARACTERIZES PHYSICAL STATES.

$\ell_0, \bar{\ell}_0$ EIG VALS ARE $h, \bar{h} \in \mathbb{R}$

DILATATION E-VAL $\ell_0 + \bar{\ell}_0 \implies \Delta := h + \bar{h}$

"CONFORMAL DIM"

ROTATION E-VAL $\ell_0 - \bar{\ell}_0 \implies s := h - \bar{h}$

"SPIN"

CONFORMAL INVARIANCE IN D-DIMS

Q: WHAT DOES CFT_D SATISFY?

A: CERTAIN CONSTRAINTS ON FIELDS & THEIR CORRELATORS

LAST CLASS $\Rightarrow$ $\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{\sqrt{\det g_{\mu\nu}'}} = \Omega^{-d/2} \overset{DIL}{=} \lambda^d$

$\overset{SCT}{=} \frac{1}{(1+2b\cdot x+b^2x^2)^d}$

DEF'NS: A CFT_D HAS A SET OF FIELDS $\{A_i\}$ w/ ALL DERIV OPS, ETC., AN A SUBSET OF $\{\phi_j\} \subseteq \{A_i\}$ S.T.

$$\boxed{\phi_j \longrightarrow \left|\frac{\partial x'}{\partial x}\right|^{\Delta_j/d} \phi_j(x')}$$

WHERE ϕ_j IS A PRIMARY (or quasi-primary, in Ginsparg)

AND Δ_j IS THE DIMENSION OF ϕ_j.

THEN CORRELATORS OF PRIMARIES STSFY

$$\boxed{\langle \phi_1(x_1)\dots\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/d}_{x=x_1}\cdots\left|\frac{\partial x'}{\partial x}\right|^{\Delta_n/d}_{x=x_n}\langle\phi_1(x_1')\dots\phi_n(x_n')\rangle}$$

REST OF $\{A_i\}$'S L.C.'S OF ϕ'S & THEIR DERIVS.

$\exists\ |0\rangle$ INV'T UNDER GLOB. CONF. GROUP.

CFT INVTS: $x_{ij} := |x_i - x_j|$ T, R -INVT-

DIL & SCT $\Longrightarrow$ $\frac{x_{ij}\,x_{kl}}{x_{ik}\,x_{jl}}$ FULLY CONF-INVT.

HMWK

"conformal cross-ratio"

2-PT FUNCTIONS

$$\langle \phi_1(x_1)\phi(x_2)\rangle \overset{\substack{T\text{-invt}\\ R\text{-invt}}}{=} F(x_{12}).$$

SINCE $\phi(\lambda x)=\lambda^{-\Delta}\phi(x)$, $\langle \phi_1(x_1)\phi_2(x_2)\rangle = \dfrac{C_{12}}{x_{12}^{\Delta_1+\Delta_2}}$

correlator condition

OTOH: $$\frac{1}{x_{12}^{\Delta_1+\Delta_2}} \overset{SCT}{=} \left(\frac{1}{(1+2b\cdot x_1+b^2x_1^2)^d}\right)^{\Delta_1/d}\left(\frac{1}{(1+2b\cdot x_2+b^2x_2^2)^d}\right)^{\Delta_2/d}\frac{1}{(x_{12})^{\Delta_1+\Delta_2}}\left(\sqrt{(1+2b\cdot x_1+b^2x_1^2)(x_2\leftrightarrow)}\right)^{\Delta_1+\Delta_2}$$

$$= (1+2b\cdot x_1+b^2x_1^2)^{-\Delta_1}(1+2b\cdot x_2+b^2x_2^2)^{-\Delta_2}\left((1+2b\cdot x_1+b^2x_1^2)(1+2b\cdot x_2+b^2x_2^2)\right)^{\frac{1}{2}(\Delta_1+\Delta_2)}\frac{1}{x_{12}^{\Delta_1+\Delta_2}}$$

$\Rightarrow \Delta_1=\Delta_2$ TO MATCH

$$\boxed{\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases}\dfrac{C_{12}}{x_{12}^{2\Delta}} & \Delta_1=\Delta_2=\Delta\\ 0 & \Delta_1\neq\Delta_2\end{cases}}$$

EXACT 2-PT FUNCTION!

(contrast to normal QFT eg $G^{(2)}$ @ one loop in ϕ^4)

3-PT FUNCTIONS SIMILARLY CONSTRAINED

$$\boxed{\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\,x_{23}^{\Delta_2+\Delta_3-\Delta_1}\,x_{13}^{\Delta_1+\Delta_3-\Delta_2}}}$$

DETERMINED BY SINGLE CONSTANT! C_{123}

4-PT FUNCTIONS

? label by ops in 4-pt function Grinstein just has F.

$$\boxed{\langle \phi_1(x_1)\dots\phi_4(x_4)\rangle = F\left(\frac{x_{12}x_{34}}{x_{13}x_{24}},\frac{x_{12}x_{34}}{x_{23}x_{41}}\right)\prod_{i<j}x_{ij}^{-(\Delta_i+\Delta_j)+\frac{1}{3}\sum_{k=1}^{4}\Delta_k}}$$

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\qquad V=\frac{x_{23}^2x_{14}^2}{x_{13}^2x_{24}^2}$$

note: wrote square of cross ratios from before, matching conventions of eg 1602.07982 in bootstrap community.

Rychkov EPFL 1601.05000

$$\boxed{\langle \phi(x_1)\dots\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}f(u,v)}$$

sometimes $g(u,v)$, following Simmons-Duffin TASI

CROSSING $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ IS S_4-INV'T

$$G^{(4)} = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\, g\left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}\right) \xmapsto{1\leftrightarrow 3} \frac{1}{x_{23}^{2\Delta}x_{14}^{2\Delta}}\, g\left(\frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}, \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\right)$$

i.e. S_4-INVCE $\Longrightarrow$ $\boxed{g(u,v) = \left(\frac{u}{v}\right)^{\Delta} g(v,u)}$ (C1)

$$G^{(4)} \xmapsto{1\leftrightarrow 2} \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\, g\left(\frac{x_{12}^2 x_{34}^2}{x_{23}^2 x_{14}^2}, \frac{x_{13}^2 x_{24}^2}{x_{23}^2 x_{14}^2}\right)$$

i.e. $\boxed{g(u,v) = g\left(\frac{u}{v}, \frac{1}{v}\right)}$ (C2)

$$G^{(4)} \xmapsto{1\leftrightarrow 4} \frac{1}{x_{24}^2 x_{13}^2}\, g\left(\frac{x_{24}^2 x_{13}^2}{x_{34}^2 x_{12}^2}, \frac{x_{23}^2 x_{14}^2}{x_{34}^2 x_{12}^2}\right)$$

i.e. $\boxed{g(u,v) = \left(\frac{1}{u}\right)^{\Delta} g\left(\frac{1}{u}, \frac{v}{u}\right)}$ (C3)

Consequence of C1, C2

(see 0807.0004 4.2 & 4.3)

normalize to 1 wlog

EX: FREE CFT DIM Δ. $G^{(2)}_{12} = \frac{1}{x_{12}^{2\Delta}}$ $G^{(3)} = 0$ $\qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ $\quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$

(x_1, x_2)

$$G^{(4)}(x_1,\dots,x_4)^{\text{free}} = G^{(2)}_{12}G^{(2)}_{34} + G^{(2)}_{13}G^{(2)}_{24} + G^{(2)}_{14}G^{(2)}_{23}$$

$$= \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left(1 + \left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\right)^{\Delta} + \left(\frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\right)^{\Delta}\right) = \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left(1 + u^{\Delta} + \left(\frac{u}{v}\right)^{\Delta}\right)$$

CROSSING?

C1: $\left(\frac{u}{v}\right)^{\Delta} g(v,u) = \left(\frac{u}{v}\right)^{\Delta}\left(1 + v^{\Delta} + \left(\frac{v}{u}\right)^{\Delta}\right) = \left(\left(\frac{u}{v}\right)^{\Delta} + u^{\Delta} + 1\right) = g(u,v)$ ✓

C2: $g\left(\frac{u}{v}, \frac{1}{v}\right) = 1 + \left(\frac{u}{v}\right)^{\Delta} + \left(\frac{u/v}{1/v}\right)^{\Delta} = 1 + u^{\Delta} + \left(\frac{u}{v}\right)^{\Delta} = g(u,v)$ ✓

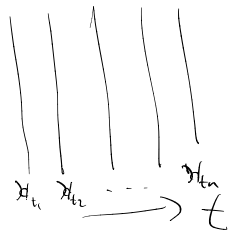
RADIAL QUANTIZATION & OPE (following Rychkov EPFL Ch. 3)

Q: HOW TO PUT ALL THIS INTO HILBERT SPACE?

A: NATURAL QUANTIZATION SCHEME

NORMAL STORY IN QFT:

① FOLIATE IN t



$H = P_0$, generator of translations in time

HILBERT SPACE LEAVES RELATED BY $e^{iH\Delta t}$, $H = P_0$

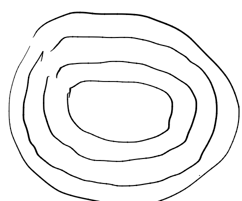
② PHYSICAL STATES ON EACH LEAF, CHARACTERIZE STATES AS H E-STATES, & SIMULTANEOUS E-STATES OF $\mathcal{O}$ ST $[H, \mathcal{O}] = 0$, i.e.

$$P^\mu |k\rangle = k^\mu |k\rangle$$

$\rightsquigarrow$ physical states characterized by energy & momentum

RADIAL QUANTIZATION IN CFT

① FOLIATE IN r (by S^{D-1} in CFT_D)



taste: one might not want to call this H?

Dilatation generator

HILBERT SPACE LEAVES RELATED BY $e^{iH\Delta r}$, $H = D$

② PHYSICAL STATES ON LEAF, H E-STATES

Rychkov has i, TASI Simmons-Duffin doesn't
↳ see pg 16 footnote

$$D|\Delta\rangle = i\Delta|\Delta\rangle$$

label states by spin l, too!

HMWK

FACT: $[M_{\mu\nu}, D] = 0$ $\forall$ ROTATIONS M $\implies$ $|\Delta\rangle \longmapsto |\Delta, l\rangle$

ALGEBRA ACTION ON QUANTUM OPERATORS

Ginsparg calls $\Omega(x)$

definition of primary

$$x \xrightarrow{C.T.} x' \quad \phi(x) \longrightarrow \phi(x') = \frac{1}{b(x)^\Delta}\phi(x) \qquad \text{WHERE} \quad g'_{\mu\nu}(x') = c(x)\, g_{\mu\nu}(x)$$

$$b(x) = \sqrt{c(x)}$$

$$\therefore \langle \phi(x') \dots \phi(y') \rangle = \frac{1}{b(x)^\Delta} \cdots \frac{1}{b(y)^\Delta} \langle \phi(x) \dots \phi(y) \rangle$$

(recall $c(x)$ has two $\frac{\partial x'}{\partial x}$'s)

again similar to what we did earlier.

INFINITESIMAL: $x'^\mu = x^\mu + \varepsilon^\mu(x) \implies b(x) = 1 + \partial_\mu \varepsilon^\mu$

ACTION ON OPERATORS

$$b(x)^\Delta \phi(x') = \phi(x) + \varepsilon\big(G\phi(x)\big) \qquad \text{w/ GENERATOR } G \text{ S.T.}$$

$$(1 + \Delta \partial_\mu \varepsilon^\mu)(\phi(x) + \varepsilon^\mu \partial_\mu \phi) \implies G\phi(x) = \Delta \partial_\mu \varepsilon^\mu \phi(x) + \varepsilon^\mu \partial_\mu \phi$$

FACT: PREVIOUSLY STUDIED $G \longleftrightarrow \varepsilon^\mu$ CORRESP

(again, one of the first things we did, following Ginsparg)

e.g. $\varepsilon^\mu = a^\mu$ CONSTANT $\implies$ TRANSLATIONS,

$$x^\mu \longrightarrow x^\mu + \varepsilon^\mu =: (1 - i\varepsilon^\nu P_\nu)\, x^\mu$$

THEN SEE $P_\mu = i\partial_\mu$ (so $-i\varepsilon^\nu P_\nu x^\mu = \varepsilon^\nu \delta_\nu^\mu = \varepsilon^\mu$)

RYCHKOV'S CONVENTIONS

SCT

$$P_\mu = i\partial_\mu, \quad M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \quad D = i x^\mu \partial_\mu, \quad K_\mu = i(2x_\mu\, x\cdot\partial - x^2 \partial_\mu)$$

$$[P_\mu, \mathcal{O}(x)] = -i\partial_\mu \mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = -i(\Delta + x^\mu \partial_\mu)\mathcal{O}(x)$$

$$[M_{\mu\nu}, \mathcal{O}(x)] = -i(\Sigma_{\mu\nu} + x_\mu \partial_\nu - x_\nu \partial_\mu)\mathcal{O}(x)$$

$$[K_\mu, \mathcal{O}(x)] = -i(2x_\mu \Delta + 2x^\lambda \Sigma_{\lambda\mu} + 2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu)\mathcal{O}(x)$$

FACT: $[K_\mu, \mathcal{O}(0)] = 0$, OPERATOR VERSION OF DEFN OF PRIMARY, SEE BY $b(x)\big|_{x=0}$ FOR SCT'S.

STATE-OPERATOR CORRESPONDENCE

RECALL: VACUUM CONFORMALLY INV'T $\Rightarrow$ $D|0\rangle = 0$

↳ took as axiom in Ginsparg, see Rychkov for argument, "inventing nothing"

OPERATOR $\Longrightarrow$ STATE

CONSIDER $|\Delta\rangle := \mathcal{O}_\Delta(0)|0\rangle$

Rychkov might have a sign issue here?

THEN $D|\Delta\rangle = D\mathcal{O}_\Delta(0)|0\rangle = \left([D,\mathcal{O}_\Delta(0)] + \mathcal{O}_\Delta(0)\cancel{D}\right)|0\rangle = i\Delta|\Delta\rangle$ (0)

RESULT: TO $\mathcal{O}_\Delta(x)$ $\exists$ NATURAL $(H=D)$ E-STATE $|\Delta\rangle$

"OPERATOR $\Longrightarrow$ STATE"

RAISING & LOWERING

WHAT ABOUT $|\Psi\rangle := \mathcal{O}_\Delta(x)|0\rangle$?

$$|\Psi\rangle = e^{iPx}\mathcal{O}(0)e^{-iPx}|0\rangle = e^{iPx}\mathcal{O}(0)|0\rangle = e^{iPx}|\Delta\rangle = \sum_n \frac{1}{n!}(iPx)^n|\Delta\rangle$$

↑ ACTS AS $\mathbb{1}$ on $|0\rangle$

SINCE $[D,P_\mu] = iP_\mu$

$D|\Delta\rangle = i\Delta|\Delta\rangle$

$$DP_\mu|\Delta\rangle = (iP_\mu + P_\mu D)|\Delta\rangle = i(\Delta+1)P_\mu|\Delta\rangle$$

$$\therefore \quad |\Delta\rangle \xrightarrow{P_\mu} |\Delta+1\rangle \xrightarrow{P_\mu} |\Delta+2\rangle \dots$$

$\Longrightarrow$ $\mathcal{O}_\Delta(x)|0\rangle$ SIMPLE L.C. OF EN. E-STATES

→ follows from conformal algebra

OTOH: $[D,K_\mu] = -iK_\mu$ $\Longrightarrow$ $\dots \longleftarrow |\Delta\rangle \xleftarrow{K_\mu} |\Delta+1\rangle \dots$

→ eg. unitarity bound

IF DIMENSIONS HAVE LOWER BOUND, GET 0 EVENTUALLY

$$0 \longleftarrow |\Delta\rangle \xleftarrow{K_\mu} |\Delta+1\rangle \dots$$

THEN EVERY SEQUENCE HAS A PRIMARY, IE

$$|\Delta\rangle \quad \text{S.T.} \quad K_\mu|\Delta\rangle = 0$$

$$\left(K_\mu|\Delta\rangle = K_\mu\mathcal{O}_\Delta(0)|0\rangle = [K_\mu,\mathcal{O}_\Delta(0)]|0\rangle \overset{\text{if primary}}{=} 0|0\rangle = 0\right)$$

(↓ arb over $|\Delta\rangle$)

STATE ⟹ OPERATOR

(so many different approaches)

GIVEN $|\Delta\rangle$, WANT OP $\mathcal{O}_\Delta(0)$

$|\Delta\rangle$ MUST DETERMINE CORRELATORS OF $\mathcal{O}_\Delta(0)$

NATURAL DEFINITION $\langle \ldots \mathcal{O}_\Delta(0)\rangle := \langle 0|\ldots|\Delta\rangle$

ASSUMED COULD GET STATE BY LOC. OP ON VACUUM

$$|\Delta\rangle = \mathcal{O}_\Delta(0)|0\rangle$$

REQUIRES SCALE INVARIANCE

standard argument, e.g. Tong or TASI

STATE - OPERATOR CORRESPONDENCE

$$\boxed{\mathcal{O}(0) \longleftrightarrow \mathcal{O}(0)|0\rangle =: |\mathcal{O}\rangle}$$

NB: IN NORMAL QFT, NOT ALL STATES COME FROM LOCAL OP ACTING ON VACUUM

UNITARITY BOUND FOR SCALAR OP $\mathcal{O}$,

$$\boxed{\Delta_{\mathcal{O}} \geq \frac{D-2}{2}}$$

REQ'D FOR UNITARITY

HWK: Proof!

OPERATOR PRODUCT EXPANSION:

CONSIDER

$$|\psi\rangle := \phi_i(x)\,\phi_j(0)\,|0\rangle$$

EXPAND

$$|\psi\rangle = \sum_n C_n(x)\,|E_n\rangle$$

PRIMARY OR DESCENDENT (deriv of primary)

STATE OPERATOR CORRESP $\Longrightarrow$ $|E_n\rangle = \tilde{\mathcal{O}}\,|0\rangle$

$$\therefore\quad \phi_i(x)\,\phi_j(0) = \sum_{\phi_k\ \text{prim}} C_{ijk}(x, \partial_y)\,\phi_k(y)\big|_{y=0}\,|0\rangle$$

SO

$$\boxed{\phi_i(x)\,\phi_j(y) = \sum_{\mathcal{O}_k\ \text{prim}} C_{ijk}(x, \partial_y)\,\phi_k(y)} \qquad (*)$$

Q: HOW TO COMPUTE OPE COEFFICIENTS?

A: PUT $(*)$ IN CORRELATOR (following TASI, $\phi \to \mathcal{O}$)

$$\langle \mathcal{O}_i(x_1)\,\mathcal{O}_j(x_2)\,\mathcal{O}_k(x_3)\rangle = \sum_{k'} C_{ijk'}(x_{12}, \partial_2)\underbrace{\langle \mathcal{O}_{k'}(x_2)\,\mathcal{O}_k(x_3)\rangle}_{\delta_{kk'}\, x_{23}^{-\Delta_k}}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{f_{ijk}}{x_{12}^{\Delta_i+\Delta_j-\Delta_k}\, x_{23}^{\Delta_j+\Delta_k-\Delta_i}\, x_{31}^{\Delta_k+\Delta_i-\Delta_j}} \qquad\qquad C_{ijk}(x_{12}, \partial_2)\, x_{23}^{-2\Delta_k}$$

FACT IF $\Delta_i = \Delta_j = \Delta_\phi$, $\Delta_k = \Delta$,

$$C_{ijk}(x,\partial) = f_{ijk}\, x^{\Delta - 2\Delta_\phi}\left(1 + \tfrac{1}{2}x\cdot\partial + \alpha x^\mu x^\nu \partial_\mu\partial_\nu + \beta x^2 \partial^2 + \ldots\right)$$

$$\alpha = \frac{\Delta+2}{8(\Delta+1)} \qquad \beta = -\frac{\Delta}{16\left(\Delta - \frac{d-2}{2}\right)(\Delta+1)}$$

HMWK! EXERCISE 8.2 OF 1602.07982

NB: OPE RELATES $G^{(n)}$'s TO $G^{(n-1)}$'s.

CONFORMAL BOOTSTRAP (see Rychkov & TASI both)

RECAP:

1) SPECTRUM

- PRIMARY OPS. HAVE (Δ, R) LABEL — SO(D) REP
- DESCENDENTS FROM DERIVS

2) STATE / OP CORRESPONDENCE (in radial quantization)

$$|\Delta\rangle \longleftrightarrow \mathcal{O}_\Delta(0)|0\rangle$$

3) UNITARITY BOUND

$$\Delta \geq \Delta_{min}(R, d)$$ (necessary but not sufficient)

4) 2-PT (of scalar primaries)

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{|x-y|^{2\Delta}}$$

suitable normalization

5) 3-PT (of scalar primaries)

matters! can't be rescaled away

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{2\alpha_{123}}|x_{13}|^{2\alpha_{132}}|x_{23}|^{2\alpha_{231}}} \qquad \alpha_{ijk} = \frac{\Delta_i + \Delta_j - \Delta_k}{2}$$

6) OPE:

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O}\ \text{primary}} \lambda_{12\mathcal{O}}\, C_{\mathcal{O}}(x, \partial_y)\mathcal{O}(y)\Big|_{y=0}$$

"OPE coefficients"

7) CFT Data: $(\Delta, R)_i \quad \lambda_{ijk}$

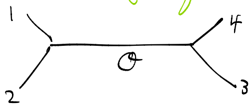
8) CORRELATOR RECURSION $\quad G^{(n)} \longrightarrow G^{(n-1)}$

$$\langle \vdots \ \ \vdots \rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\, C_{\mathcal{O}}(x, \partial_y)\langle \cdot \ \ \vdots \rangle$$

CONSISTENCY OF 4-PT FUNCTION

$$\langle \underbrace{\phi_1(x_1)\,\phi_2(x_2)}_{\text{OPE}}\,\underbrace{\phi_3(x_3)\,\phi_4(x_4)}_{\text{OPE}}\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\,\lambda_{34\mathcal{O}}\,\underbrace{\left[C_{\mathcal{O}}(x_{12},\partial_y)\,C_{\mathcal{O}}(x_{34},\partial_z)\langle \mathcal{O}(y)\,\mathcal{O}(z)\rangle\right]}_{\text{"conformal partial waves" completely fixed by } (\Delta, R) \text{ of } \mathcal{O}}$$

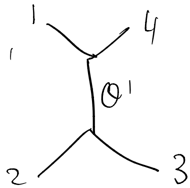
$$= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\,\lambda_{34\mathcal{O}}$$



NB: NOT PERTURBATIVE FEYNMAN DIAGS!

ASSOC. OPE

$$= \langle \phi_1(x_1)\,\overbrace{\phi_2(x_2)\phi_3(x_3)\,\phi_4(x_4)}^{\text{OPE}}\rangle \text{ with } \underbrace{\phi_2(x_2)\phi_3(x_3)}_{\text{OPE}} = \sum_{\mathcal{O}'} \lambda_{14\mathcal{O}'}\,\lambda_{23\mathcal{O}'}$$



"CONFORMAL BOOTSTRAP CONDITION"

FACT: SOLVING 4-PT CONSTRAINT $\Rightarrow$ N-PT CONSISTENCY

(see Rychkov for a picture)

CONVENTION: IMPOSE eg ON $g(u,v)$ OF SCALARS

$$g(u,v) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\,\lambda_{34\mathcal{O}}\,\underbrace{G_{\mathcal{O}}(u,v)}_{\text{CONFORMAL BLOCKS}}$$

BOOTSTRAP PHILOSOPHY

SEARCH FOR CFT DATA SATISFYING CONSTRAINTS!

2D CFT

CONFORMAL THEORIES IN 2D (again following Ginsparg)

RECALL 2D

i) $l_n = -z^{n+1}\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$

$$\underbrace{[l_m, l_n] = (m-n)\,l_{m+n}}_{=:\,A} \qquad \underbrace{[\bar{l}_m, \bar{l}_n] = (m-n)\,\bar{l}_{m+n}}_{=:\,\bar{A}}$$

$[l, \bar{l}] = 0 \implies$ loc. conf. alg $A \oplus \bar{A}$ (ie the isomorphic pieces don't talk)

ii) $ds^2 \longrightarrow \partial_z f\, \partial_{\bar{z}} \bar{f}\; ds^2$

MOTIVATES

$\bar{h} \in \mathbb{R}$ not c.c. of h

$$\Phi(z,\bar{z}) \longrightarrow \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi\left(f(z), \bar{f}(\bar{z})\right)$$

PRIMARY OF WEIGHT $(h, \bar{h})$

INFINITESIMAL

$$\delta_{\varepsilon,\bar{\varepsilon}} \Phi := \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(z+\varepsilon, \bar{z}+\bar{\varepsilon}) - \Phi(z,\bar{z})$$

$$= (1+\partial_z \varepsilon)^{h} (1+\partial_{\bar{z}}\bar{\varepsilon})^{\bar{h}} \left\{ \Phi(z,\bar{z}) + \varepsilon \partial_z \Phi + \bar{\varepsilon}\partial_{\bar{z}} \Phi + O(\varepsilon^2) \right\} - \Phi(z,\bar{z})$$

$$\implies \boxed{\delta_{\varepsilon,\bar{\varepsilon}} \Phi = \left((h\,\partial\varepsilon + \varepsilon\partial) + (\bar{h}\,\bar{\partial}\bar{\varepsilon} + \bar{\varepsilon}\bar{\partial}) \right) \Phi(z,\bar{z})}$$

SIM $G^{(2)}$ VARIATION GIVES (for $\langle \Phi_1(z_1) \Phi_2(z_2) \rangle$)

$$\left(\left\{ [\varepsilon(z_1)\partial_{z_1} + h_1 \partial \varepsilon(z_1)] + [\varepsilon(z_2)\partial_{z_2} + h_2 \partial\varepsilon(z_2)] \right\} + \left\{ \bar{\varepsilon}, \bar{z} \text{ VERSION} \right\} \right) G^{(2)}(z_i, \bar{z}_i) = 0$$

2-PT

CASE: TRANSLATION $\varepsilon = \bar{\varepsilon} = \text{CONST} \ll 1 \Rightarrow \left(\partial_{z_1} + \partial_{z_2} + \partial_{\bar{z}_1} + \partial_{\bar{z}_2}\right) G^{(2)}(z_i, \bar{z}_i) = 0$

IF $G^{(2)}(z_{12}, \bar{z}_{12})$ EQ SATISFIED!

CASE: DILATATION $\varepsilon(z) = \lambda z \quad \varepsilon(\bar{z}) = \lambda \bar{z} \quad \mathbb{R} \in \lambda \ll 1$

SOLUTION $\Rightarrow$ $G^{(2)}(z_1, z_2) = \dfrac{C_{12}}{z_{12}^{h_1+h_2} \, \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}$

CASE: $\varepsilon(z) = z^2 \quad \bar{\varepsilon}(\bar{z}) = \bar{z}^2$

SOLUTION $\Rightarrow$ $h_1 = h_2 = h \qquad \bar{h}_1 = \bar{h}_2 = \bar{h}$

$$\boxed{G^{(2)}(z_i, \bar{z}_i) = \frac{C_{12}}{z_{12}^{2h} \, \bar{z}_{12}^{2\bar{h}}}}$$

CASE: $s = h - \bar{h} = 0 \Rightarrow \Delta = h + \bar{h} \qquad G^{(2)}(z_i, \bar{z}_i) = \dfrac{C_{12}}{|z_{12}|^{2\Delta}}$

3-PT

$\beta_{ijk} = \dfrac{h_i + h_j - h_k}{2}$

$$\boxed{G^{(3)}(z_i, \bar{z}_i) = C_{123} \cdot \frac{1}{z_{12}^{2\beta_{123}} \, z_{23}^{2\beta_{231}} \, z_{13}^{2\beta_{132}}} \quad \frac{1}{\bar{z}_{12}^{2\bar{\beta}_{123}} \, \bar{z}_{13}^{2\bar{\beta}_{132}} \, \bar{z}_{23}^{2\bar{\beta}_{231}}}}$$

DET'D BY SINGLE CONST!

4-PT

$h = \sum_{i=1}^{4} h_i \qquad \bar{h} = \sum_{i=1}^{4} \bar{h}_i$

$$\boxed{G^{(4)}(z_i, \bar{z}_i) = f(x, \bar{x}) \prod_{i<j} z_{ij}^{(h_i + h_j) + h/3} \prod_{i<j} \bar{z}_{ij}^{(\bar{h}_i + \bar{h}_j) + \bar{h}/3}}$$

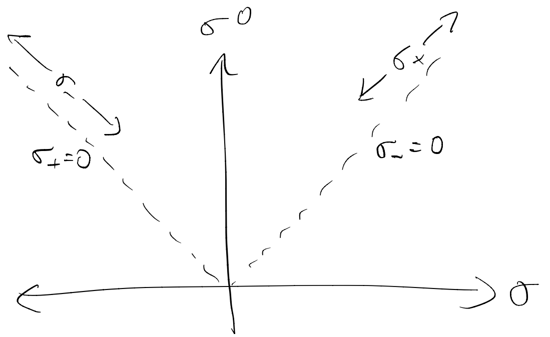
RADIAL QUANTIZATION IN 2D

2D MINKOWSKI (TIME, SPACE) = (σ^0, σ^1)

LIGHT CONE COORDS

$$\sigma^{\pm} = \sigma^0 \pm \sigma^1$$

THEN $\sigma^{\pm} = 0 \Rightarrow$ ON LIGHT CONE



$f(\sigma_+)$ "LEFT-MOVER": $\leftarrow$ AS $\downarrow \sigma_+$

$f(\sigma_-)$ "RIGHT-MOVER": $\rightarrow$ AS $\downarrow \sigma_-$

2D EUCLIDEAN "SPACE" σ^1 & "TIME" σ^0

LIGHT CONE COORDS $\xi, \bar{\xi} = \sigma^0 \pm i\sigma^1$ ($\xi = 0 \Rightarrow \sigma^0 = -i\sigma^1$; $(-i\sigma^1, \sigma^1)^2 = 0$ in Euclidean)

(in Minkowski, would be $\sigma^0 \pm \sigma^1$)

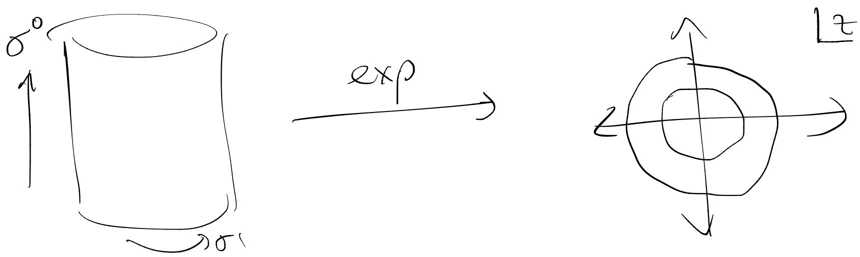
LEFT/RIGHT MOVER IN MINK $\Longrightarrow$ HOL / ANTIHOL IN EUCLIDEAN

$f(\xi)$ $\bar{f}(\bar{\xi})$

CYLINDER MAP

$\xi \to z = \exp(\xi) = \exp(\sigma^0 + i\sigma^1)$

$\mathbb{R} \to S^1$ by exp $\Rightarrow$ cyl



∞ FUTURE $\quad \sigma^0 \to \infty, \quad z \to \infty$

∞ PAST $\quad \sigma^0 \to -\infty, \quad z \to 0$

EQUAL TIME SURFACES $\quad \sigma^0 = \text{CONST} \longrightarrow |z| = r$

FIXED RADIUS!

ADVANTAGE: RQ HAS COMPLEX ANALYSIS!

STRESS TENSOR

LOCAL SYMMETRIES $\Rightarrow \exists\, j^\mu(x)$ S.T. $\partial_\mu j^\mu = 0$

$\Rightarrow Q = \int d^dx\, j_0(x)$ CONSERVED

$\Rightarrow$ GENERATES FIELD VARIATION $\quad \delta_\varepsilon \phi = \varepsilon[Q, \phi]$

(from $e^{+i\varepsilon Q}\phi e^{-i\varepsilon Q} = (1+i\varepsilon Q)\phi(1-i\varepsilon Q) = \phi + i\varepsilon[Q,\phi] + O(\varepsilon)^2$)

CONFORMAL TRAFOS

$T_{\mu\nu}$ IS STRESS TENSOR (is symmetric), $\quad \partial_\mu T^{\mu\nu} = 0$

$x^\mu \xrightarrow{\text{C.T.}} x^{\mu'} = x^\mu + \varepsilon^\mu \quad$ HAS $\quad j^\mu = T^{\mu\nu}\varepsilon_\nu$

$\longrightarrow$ conserved current j^μ for each of d translations, group into E-M tensor $T^{\mu\nu}$ w/ $\partial_\mu T^{\mu\nu} = 0$ see Weinberg v1 pg 310

DILATATIONS $\quad \varepsilon_\nu = \lambda x_\nu$

$$0 = \partial_\mu j^\mu = \lambda\partial_\mu(T^{\mu\nu}\varepsilon_\nu) = \lambda\cancelto{0}{(\partial_\mu T^{\mu\nu})}\, x_\nu + \lambda T^{\mu\nu}\delta_{\mu\nu} = 0$$

$$\Longrightarrow \boxed{T^\mu{}_\mu = 0}$$

stress tensor is traceless!

OTHER C.T'S $\quad 0 = \partial_\mu j^\mu = \cancel{\lambda}(\cancel{\partial_\mu T^{\mu\nu}})^{\to 0}\,\varepsilon_\nu + \lambda\, T^{\mu\nu}\,\partial_\mu \varepsilon_\nu$ $\quad \frac{2}{d}(\partial\cdot\varepsilon)\eta_{\mu\nu}$, first day!

$$\Rightarrow 0 = T^{\mu\nu}\partial_\mu \varepsilon_\nu = \tfrac{1}{2}\left(T^{\mu\nu}+T^{\nu\mu}\right)\partial_\mu \varepsilon_\nu = \tfrac{1}{2}T^{\mu\nu}\left(\partial_\mu\varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) = \tfrac{1}{2}\cancel{T^\mu{}_\mu}^{\;\to 0 \text{ from dil}}\,\partial\cdot\varepsilon = 0$$

$\Longrightarrow$ NO EXTRA CONSTRAINTS!

$\underline{T}$ & COMPLEX ANALYSIS

$$ds^2 = dx^2 + dy^2 = (dx + i dy)(dx - i dy) = dz\, d\bar z$$

$$= (dz \;\; d\bar z)\underbrace{\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0\end{pmatrix}}_{g}\begin{pmatrix} dz \\ d\bar z\end{pmatrix} \Longrightarrow \boxed{g_{z\bar z} = g_{\bar z z} = \tfrac{1}{2}} \quad \text{(rest zero)}$$

COMPLEXIFY $T_{\mu\nu}$

$$(dx \;\; dy)\begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy}\end{pmatrix}\begin{pmatrix} dx \\ dy\end{pmatrix} = T_{xx}\,dx^2 + T_{yy}\,dy^2 + (T_{xy}+T_{yx})\,dx\,dy$$

NB $\quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{i}{2}\end{pmatrix}}_{M}\begin{pmatrix} dz \\ d\bar z\end{pmatrix}$

$$\Longrightarrow \begin{pmatrix} T_{zz} & T_{z\bar z} \\ T_{\bar z z} & T_{\bar z\bar z}\end{pmatrix} = M^T \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy}\end{pmatrix} M$$

SO $\quad T_{zz} = \frac{1}{4}\left(T_{00} - 2i\,T_{10} - T_{11}\right)$

$$T_{\bar z\bar z} = \overline{T_{zz}} \qquad T_{z\bar z} = T_{\bar z z} = \frac{1}{4}T^\mu{}_\mu = 0$$

$$\partial^\mu T_{\mu\nu} = 0 \Longrightarrow \begin{aligned} &\partial_{\bar z} T_{zz} + \partial_z \cancel{T_{\bar z z}}^{\,\to 0} = 0 \\ &\partial_z T_{\bar z\bar z} + \partial_{\bar z} \cancel{T_{z\bar z}}^{\,\to 0} = 0 \end{aligned} \Bigg\} \text{ holo \& anti holo components!}$$

$$\boxed{\begin{aligned} T(z) &:= T_{zz}(z) \\ \bar T(\bar z) &:= T_{\bar z\bar z}(\bar z)\end{aligned}}$$

GENERATOR

$$Q = \frac{1}{2\pi i} \oint \left(dz\, T(z)\, \varepsilon(z) + d\bar z\, \bar T(\bar z)\, \bar\varepsilon(\bar z) \right)$$

(equal time ⟹ equal radius in radial quantization

FIELD VARIATION

$$\delta_{\varepsilon,\bar\varepsilon} \Phi(w,\bar w) = \frac{1}{2\pi i} \oint \left(dz \left[T(z)\,\varepsilon(z), \Phi(w,\bar w) \right] \right) - \left(d\bar z \left[\bar T(\bar z)\,\bar\varepsilon(\bar z), \Phi(w,\bar w) \right] \right)$$

ASIDE IN RAD. QUANT. T-ORDERING ⟼ R-ORDERING

$$R\big(A(z) B(w)\big) := \begin{cases} A(z)\,B(w) & |z| > |w| \\ B(w)\,A(z) & |w| > |z| \end{cases}$$

~ read Ginsparg for elaboration

$$\Rightarrow \delta_{\varepsilon,\bar\varepsilon} \Phi(w,\bar w) = \frac{1}{2\pi i} \oint \left(dz\, \varepsilon(z) R\big(T(z)\Phi(w,\bar w)\big) + d\bar z\, \bar\varepsilon(\bar z) R\big(\bar T(\bar z)\Phi(w,\bar w)\big) \right)$$

$$= h\, \partial\varepsilon(w)\Phi(w,\bar w) + \varepsilon(w)\partial\Phi(w,\bar w) + \bar h\, \bar\partial\bar\varepsilon(\bar w)\, \Phi(w,\bar w) + \bar\varepsilon(\bar w)\, \bar\partial\Phi(w,\bar w)$$

previously derived infinitesimal trafo

key

equality fixes pole structure of radially ordered products!

eg: $\frac{1}{2\pi i} \left(\oint dz\, \frac{\varepsilon(z)}{(z-w)} \right) \partial_w \Phi(w,\bar w) = \varepsilon_w \partial_w \Phi(w,\bar w)$

& $\frac{1}{2\pi i} \left(\oint dz\, \frac{h\varepsilon(z)}{(z-w)^2} \right) \Phi(w,\bar w) = h\, \partial\varepsilon(w)\, \Phi(w,\bar w)$

AND SIMILAR FOR BAR GIVE

"..." = non-singular terms

$$R\big(T(z)\,\Phi(w,\bar w)\big) = \frac{h}{(z-w)^2}\Phi(w,\bar w) + \frac{1}{(z-w)} \partial_w \Phi(w,\bar w) + \dots$$

$$R\big(\bar T(\bar z)\,\Phi(w,\bar w)\big) = \frac{\bar h}{(\bar z-\bar w)^2}\Phi(w,\bar w) + \frac{1}{\bar z-\bar w} \partial_{\bar w} \Phi(w,\bar w) + \dots$$

⟹ SHORT DISTANCE OPE FOR $T, \bar T$ w/ PRIMARY

"Virasoro primary" $(L_n|\phi\rangle = 0 \;\; \forall n > 0)$

stronger condition than previous "primary" of (finite) global group.

OPE $\quad A(x)B(y) \sim \sum_i C_i(x-y)\, \mathcal{O}_i(y)$

"equal up to non-singular terms"

$$C_i \sim \frac{1}{|x-y|^{\Delta_A+\Delta_B-\Delta_{\mathcal{O}_i}}}$$

constant from 3-pt function!

FACT: IF 2-PT FN'S NORMALIZED, $\quad \phi_i(z,\bar z)\,\phi_j(w,\bar w) \sim \sum_k C_{ijk} \dfrac{\phi_k(w,\bar w)}{(z-w)^{h_k-h_i-h_j}(\bar z-\bar w)^{\bar h_k-\bar h_i-\bar h_j}}$

CENTRAL CHARGE & VIRASORO ALGEBRA

2 CONFORMAL TRAFO'S $\Rightarrow$ TT OPE

$$\boxed{T(z)\,T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{}{z-w}\partial T(w)}$$

and similar for $\bar T \bar T$

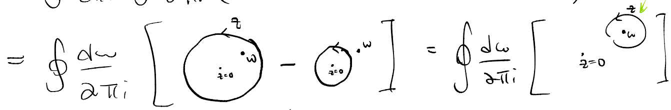
MODE EXPANSION

$\Delta_T = 2$, $(-n-2)$ CHOSEN SO $L_{-n} \to \lambda^n L_{-n}$

$$\boxed{T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad \bar T(\bar z) = \sum_{n\in\mathbb{Z}} \bar z^{-n-2} \bar L_n}$$

TRIVIALLY INVERT $\quad \boxed{L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \qquad \bar L_n = \oint \frac{d\bar z}{2\pi i} \bar z^{n+1} \bar T(\bar z)}$

$$[L_n, L_m] = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\left(z^{n+1} w^{m+1} [T(z), T(w)] \right)$$



z contour integral of $T(z)T(w)$ in z-neighborhood of w

$$= \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} z^{n+1} w^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \right)$$

IBP on last term

$$= \oint \frac{dw}{2\pi i} \left(\frac{c}{12}(n+1)(n)(n-1)\, w^{n-2} w^{m+1} + 2(n+1) w^n w^{m+1} T(w) + w^{n+1} w^{m+1} \partial T(w) \right)$$

just Cauchy, $\dfrac{f^{(n)}(a)}{n!} = \dfrac{1}{2\pi i}\oint \dfrac{f(z)}{(z-a)^{n+1}} dz$, eg $\oint \dfrac{dz}{2\pi i} \dfrac{z^{n+1}\cdot \frac{c}{2}}{(z-w)^{3+1}} = \dfrac{(\partial^3 z^{n+1})|_{z=w}}{3!}\dfrac{c}{2} = \dfrac{c}{12}(n+1)(n)(n-1)\, w^{n-2}$ and similar)

$$\Rightarrow \boxed{\begin{aligned} [L_n, L_m] &= (n-m) L_{n+m} + \frac{c}{12}(n^3-n)\,\delta_{n+m,0} \\ [\bar L_n, \bar L_m] &= (n-m)\bar L_{n+m} + \frac{\bar c}{12}(n^3-n)\,\delta_{n+m,0} \end{aligned}}$$

$\leftarrow$ VIRASORO ALGEBRA

GLOBAL SUBALGEBRA $\{L_{-1}, L_0, L_1\}$ CLOSES $\quad \boxed{[L_{\mp 1}, L_0] = \mp L_{\mp 1} \quad [L_1, L_{-1}] = 2L_0}$

HIGHEST WEIGHT STATES

CONSIDER $|h\rangle := \phi(0)|0\rangle$ CREATED BY HOL ϕ, WEIGHT h

$$[L_n, \phi] = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\phi(w) = \oint \frac{dz}{2\pi i} z^{n+1}\left[\frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial_w \phi(w)\right]$$

$$= h(n+1) w^n \phi(w) + w^{n+1}\partial\phi(w)$$

$\Rightarrow [L_n, \phi(0)] = 0 \quad \forall n > 0 \quad \therefore$

$$\boxed{L_0|h\rangle = h|h\rangle \qquad L_n|h\rangle = 0 \quad \forall n > 0}$$

we say $|h\rangle$ is a highest weight state

HMWK: SAME FOR $\phi(z, \bar{z})$ $(h, \bar{h})$ w/ state $|h, \bar{h}\rangle := \phi(0,0)|0\rangle$

DESCENDENTS: OBTAINED BY $L_{<0}$ ACTIONS ON $|h\rangle$

FREE BOSON (finally an example!)

$S = \frac{1}{2\pi} \int \partial X \bar\partial X$

2-PT $\langle X(z,\bar z) X(w,\bar w) \rangle = -\frac{1}{2} \log |z-w|$

soln of EOM, lowercase x

EOM $\partial \bar\partial X(z,\bar z) = 0 \Rightarrow X(z,\bar z) = \frac{1}{2}(x(z) + \bar x(\bar z))$

$\langle x(z) x(w) \rangle = -\log(z-w) \qquad \langle \bar x(\bar z) \bar x(\bar w) \rangle = -\log(\bar z - \bar w)$

$\Rightarrow \boxed{X,\ x,\ \bar x \quad \text{NOT PRIMARY}}$

OTOH: SEE $\partial x(z) \partial x(w) = -\frac{1}{(z-w)^2} + \ldots$ (by two derivs of $\langle x(z) x(w) \rangle$)

STRESS TENSOR $T(z) = -\frac{1}{2} :\partial x(z) \partial x(z):$

$$T(z)\,\partial x(w) = -\tfrac{1}{2} :\partial x(z) \partial x(z): \partial x(w)$$

2 contractions

$$= -\tfrac{1}{2}\, 2\, \partial x(z) \langle \partial x(z) \partial x(w) \rangle + \ldots$$

$$= \frac{\partial x(z)}{(z-w)^2} + \ldots$$

Taylor $\rightarrow$

$$= \frac{\partial x(w)}{(z-w)^2} + \frac{\partial^2 x(w)}{(z-w)} + \ldots$$

COMPARE TO GENERAL OPE OF VIRASORO PRIMARY $\Rightarrow (h, \bar h) = (1, 0)$

$\partial x,\ \bar\partial \bar x$ ARE PRIMARIES!

FACT: $:e^{i\alpha x(w)}:$ PRIMARY OF $h = \frac{\alpha^2}{2}$ (normal ordering reminds us not to contract x w/ x in expansion of exp)

CENTRAL CHARGE:

$$T(z)T(w) = \tfrac{1}{4}\left(:\partial x(z) \partial x(z)::\partial x(w) \partial x(w): \right) = \tfrac{1}{4}\left(2 \langle \partial x(z) \partial x(w) \rangle^2 + 4\, \partial x(z) \partial x(w) \langle \partial x(z) \partial x(w) \rangle + \ldots \right)$$

$$= \frac{1/2}{(z-w)^4} - \frac{\partial x(z) \partial x(w)}{(z-w)^2} = \frac{1/2}{(z-w)^2} - \frac{\partial x(w)^2 + (z-w)\partial^2 x(w)\, \partial x(w)}{(z-w)^2}$$

$= \frac{1}{2}\partial(\partial x(w) \partial x(w))$

$$= \frac{1/2}{(z-w)^2} + \frac{2}{(z-w)^2}\left(-\tfrac{1}{2}\partial x(w) \partial x(w)\right) + \frac{1}{z-w}\, \partial\left(-\tfrac{1}{2}\partial x(w) \partial x(w)\right) = \frac{1/2}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

$\boxed{c=1}$

FREE FERMION

$$S = \frac{1}{8\pi}\int \left(\psi\bar\partial\psi + \bar\psi\partial\bar\psi\right) \qquad \psi(z)\psi(w) \sim -\frac{1}{(z-w)} \qquad \bar\psi(\bar z)\bar\psi(\bar w) \sim -\frac{1}{\bar z-\bar w}$$

$$T(z) = \frac{1}{2} :\psi(z)\partial\psi(z): \qquad \bar T(\bar z) = \frac{1}{2} :\bar\psi(\bar z)\,\partial\bar\psi(\bar z):$$

HMWK: COMPUTE TT OPE & READ OFF $c = \frac{1}{2}$
$T\psi$ OPE & READ OFF $h = \frac{1}{2}$