PHYS 7326: Running Homework

Professor Halverson

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Homework will be posted on a rolling basis as lectures are developed and given. This is hopefully helpful for you, and it is also more natural for me, as assignment ideas come on a rolling basis.

Instructions: Due dates will be listed throughout the document in bold letters. For the first assignment, complete all problems by the first due date. For the n^{th} assignment, complete all problems between the $(n-1)^{\text{th}}$ and n^{th} due date. I will also organize according homeworks according to topic.

All problems are worth 5 points, unless otherwise noted. (In retrospect, this might happen a lot, sorry.)

1 Conformal Field Theory

1. Derive Ginsparg (1.3).
2. Prove that an infinitesimal conformal transformation quadratic in x must be of the form
$$\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot x. \tag{1}$$
3. Prove the unitarity bound for any scalar operator $\mathcal{O}$, i.e. that $\Delta_{\mathcal{O}} \geq \frac{d-2}{2}$.
4. Do exercise 8.2 of 1602.07982.
5. Consider a field $\phi(z, \bar{z})$ of weight $(h, \bar{h})$, and the state
$$|h, \bar{h}\rangle = \phi(0,0)|0\rangle. \tag{2}$$
Derive the properties analogous to 3.18(a) in Ginsparg that it satisfies.
6. Using the conventions in Ginsparg for the free fermion, compute the $T(z)\psi(w)$ OPE to read off the dimension h, and the $T(z)T(w)$ OPE to read off the central charge c.

Homework 1. Due February 10.

2 Non-abelian Gauge Theory

Here are some problems related to the essence of the subject:

1. 2 pts. Jacobi identity. Prove Schwartz equation (25.11).
2. 3 pts. Non-abelian gauge invariance. Schwartz problem 25.1.
3. 5 pts. Anomaly coefficients. Schwartz problem 25.4.
4. 3 points. Peskin 15.1.
5. 3 points. Peskin 15.2.
6. 5 points. BRST. Prove Peskin equations 16.48-49 and the second BRST variation of the gauge field discussed in between them.

The β-function of QCD.

One of the theoretically deepest things in the Standard Model is the way that renormalization and non-abelian gauge theory come together to explain why the strong interactions become weaker and weaker at higher energy scales. Since $\alpha_s \to 0$ as $\sqrt{s} \to \infty$ we say QCD is "asymptotically free." Your task in this problem to compute the β function of Yang-Mills theories with matter and explain this phenomenon; QCD is one such theory. Zee says something like "now that you know about quantum Yang-Mills theory and renormalization, you can just compute the β-function," and that's true, but we'll follow Peskin Chapter 16.5 where it is spelled out a bit more explicitly.

Let's break this into steps. Here's the logic. First, one could start by defining a Lagrangian for renormalized perturbation theory, as we normally do. This is discussed at the end of 16.5. Then, we could analyze the superficial degrees of divergences to figure out the leading (one-loop) divergent 1PI diagrams. Peskin does this for us. There's a lot of diagrams, and in particular the gauge boson self-energy requires internal ghosts! But it's not Halloween and this isn't scary, because you know what to do: evaluate the diagrams using dimensional regularization. You'll see the requisite $\Gamma(1-\frac{d}{2})$'s and $\Gamma(2-\frac{d}{2})$'s that contain divergences that can be seen by expanding in ϵ, where $d = 4-\epsilon$, and by imposing renormalization conditions the counterterms will cancel these divergences. The n-point functions obey a Callan-Symanzik equation, but for QED we saw that this can be simplified so that the β function is a simple expression in terms of counterterms. We'll use the QCD version of that.

a) **Gauge Boson Self-Energy, 5 pts.** Following Peskin, derive the divergent expressions (16.59), (16.62), (16.65) and the ghostly expression (16.67). Put them together to get (16.71).

b) **Fermion Self-Energy, 3 pts.** Following Peskin, derive the divergent expression (16.76).

c) **Vertex Function, 3 pts.** Following Peskin, derive the divergent expressions (16.81) and (16.83).

d) **Counterterms, 3 pts.** Following Peskin, derive (16.74), (16.77), (16.84).

e) β **function, 3 pts.** Using (16.73) and your counterterms, compute the β function to get (16.85).

f) **Deep Physics Conclusions, 3 pts.** What is the distinguishing feature in the β function that explains asymptotic freedom at high energies? Why isn't QED asymptotically free? What happens if you add too much matter to Yang-Mills? If $G = SU(3)$ and r is the three-dimensional (fundamental) representation of $SU(3)$ we have QCD. For what values of n_f is QCD asymptotically free?

If you'd done this in the early 70's, you'd be a Nobel prize winner! It went to Gross, Wilczek, and Politzer.

Homework 2. Due February 28.

3 Supersymmetry

One Motivation for Supersymmetry.

There are a number of motivations for supersymmetry (SUSY), which is a symmetry between bosons and fermions. In condensed matter systems they are sometimes studied because they can give rise to interesting properties, such as a large ground state degeneracy. In high energy physics, one of the primary reasons is that SUSY provides a solution to the weak hierarchy problem, which in a broader context (as relevant for condensed matter) can be thought of as softenining the divergence properties of scalars.

Our goal in this problem is to understand how the introduction of new particles can soften the severity of scalar divergences, as measured by the severity of the divergence of certain amplitudes (or, more precisely, the degree of its cutoff dependence).

a) 3 points. Consider the Yukawa interaction

$$\mathcal{L}_y = -y\, h\bar{\psi}\psi \tag{3}$$

where h is a real scalar field, ψ is a Dirac spinor of mass m_f, and y is the Yukawa coupling. The point is to have h be our Higgs boson in this toy model. The interaction gives a one-loop contribution at order y^2 to the 2-point amplitude for h, which is intimately tied to quantum corrections to the Higgs mass. Compute that amplitude contribution, but don't evaluate the momentum integral.

b) 3 points. Now suppose we have some complex scalars ϕ_L and ϕ_R of mass m_L and m_R that interact with h as

$$\mathcal{L}_s = -\lambda\, h^2(\phi_L^*\phi_L + \phi_R^*\phi_R), \tag{4}$$

where λ is a coupling constant. These interactions also gives a loop contribution to the 2-point amplitude for h. Compute it, but don't evaluate the momentum integral.

c) 3 points. Sum the results from the first two parts and study the most divergent pieces of the sum, which are quadratically divergent as discussed in class. Determine a relationship between y and λ under which this quadratically divergent piece is zero, if the masses of all particles in the loops (i.e. ϕ_L, ϕ_R, ψ) are the same. Note: this may require using some fermion identities to simplify the result from a).

d) 3 points. So adding scalars in this special way gets rid of the quadratically divergent piece. Of course, in high energy physics this is ludicrous: we can't just add a new particle of the same mass for every particle that's already been discovered at colliders, since they would have been discovered long ago. But what if the new particles were heavier? Give an argument as to why the divergence problem is still softened even if the new scalars are heavier than the fermions, i.e. there is some mass splitting.

There is much more to the story than this, for example the systematic construction of supersymmetric Lagrangians, as we did in class. But you have now seen the basic idea, and in fact SUSY Lagrangians give nice relationships between y and λ of precisely the required type.

Some problems from Wess and Bagger.

1. 3 points. Wess and Bagger, problem 3.4. transformatons must obey.

2. 3 points. Let Φ be a chiral superfield. Compute the $\theta\theta\overline{\theta\theta}$ component of $\Phi^\dagger\Phi$. Since the latter combination is a vector superfield, its $\theta\theta\overline{\theta\theta}$ transforms into a total spacetime derivative, so that it may be placed in a SUSY invariant Lagrangian. Comment on the physical relevance of these terms.

3. 3 points. From the definition of W_α in equation (6.7), prove that its component form may be written as in equation (6.11). Comment on the physical relevance of these terms.

Homework 3. Due March 28.