QM Lecture 6

SPIN-$\frac{1}{2}$

notation note: Townsend uses $S_{x,y,z}$ for spin $\frac{1}{2}$, $J_{x,y,z}$ for general
Today I'll use $S_{x,y,z}$ for any spin

NOTATION

$$\vec{S}^2 |s,m\rangle = s(s+1)\hbar^2 |s,m\rangle$$

$$\hat{S}_z |s,m\rangle = m\hbar |s,m\rangle$$

RECALL $s = \frac{1}{2}$ $\quad S_z \xrightarrow{z} \frac{\hbar}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

NOW $\quad S_+ \xrightarrow{z} \begin{pmatrix} \langle z|S_+|z\rangle & \langle z|S_+|-z\rangle \\ \langle -z|S_+|z\rangle & \langle -z|S_+|-z\rangle \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}\left(\frac{3}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}\ \hbar \\ 0 & 0 \end{pmatrix}$

$$\boxed{\begin{aligned} S_+ &\xrightarrow{z} \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ S_- &\xrightarrow{z} \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}}$$

NOTE $\quad S_\pm = S_x \pm i S_y \Rightarrow \boxed{S_x = \frac{1}{2}(S_+ + S_-) \quad S_y = \frac{1}{2i}(S_+ - S_-)}$

$$\Rightarrow \boxed{S_x \xrightarrow{z} \frac{\hbar}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad S_y \xrightarrow{z} \frac{\hbar}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}}$$

FAMOUS PAULI MATRICES $\quad \sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

$$\Rightarrow \boxed{\vec{S} = \frac{\hbar}{2}\vec{\sigma}}$$

SPIN 1 CHECK

ALREADY SAW

$$S_z \xrightarrow{z} \hbar \left.\overbrace{\begin{pmatrix} 1 & & \\ & & \\ & & -1 \end{pmatrix}}^{3}\right\}3$$

$$S_+ \xrightarrow{z} \sqrt{2}\,\hbar \begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix} \qquad S_- \xrightarrow{z} \sqrt{2}\,\hbar \begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix}$$

$$\Rightarrow S_x = \frac{1}{2}(S_+ + S_-) \xrightarrow{z} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) \xrightarrow{z} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}$$

SPIN $\frac{3}{2}$ $\quad S_z \xrightarrow{z} \frac{\hbar}{2} \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{pmatrix}$

NEED $S_\pm$ MAT. REPS.

$$S_+ \left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle = \left(\tfrac{3}{2}\tfrac{5}{2} + \tfrac{3}{2}\left(-\tfrac{1}{2}\right)\right)^{\frac{1}{2}} \hbar \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = \sqrt{3}\,\hbar \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle$$

SIM $\quad S_+\left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle = 2\hbar\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle, \; S_+\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \sqrt{3}\,\hbar\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle, \; S_+\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle = 0$

$$\Rightarrow S_+ \xrightarrow{z} \hbar \overbrace{\begin{pmatrix} & \sqrt{3} & & \\ & & 2 & \\ & & & \sqrt{3} \\ & & & \end{pmatrix}}^{4} \qquad S_- \xrightarrow{z} \hbar \begin{pmatrix} & & & \\ \sqrt{3} & & & \\ & 2 & & \\ & & \sqrt{3} & \end{pmatrix}$$

THEN $\quad S_x \longrightarrow \frac{\hbar}{2} \begin{pmatrix} & \sqrt{3} & & \\ \sqrt{3} & & 2 & \\ & 2 & & \sqrt{3} \\ & & \sqrt{3} & \end{pmatrix} \qquad S_y \longrightarrow \frac{\hbar}{2} \begin{pmatrix} & -\sqrt{3}i & & \\ \sqrt{3}i & & -2i & \\ & 2i & & -\sqrt{3}i \\ & & \sqrt{3}i & \end{pmatrix}$

S_x, S_y HERM? ✓

Spin $\frac{1}{2}$ S_n

$$S_n = \vec{S}\cdot\hat{n} = S_x \cos\phi + S_y \sin\phi \qquad \text{IF } \hat{n} = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$$

CONSIDER $|\mu\rangle$ W/ SPIN IN $\hat{n}$ DIR

$$\Rightarrow \quad S_n|\mu\rangle = \frac{\mu\hbar}{2}|\mu\rangle \qquad \boxed{Q:\ \mu = ?}$$

$$\downarrow z$$

$$\frac{\hbar}{2}\left[\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\cos\phi + \begin{pmatrix} & -i \\ i & \end{pmatrix}\sin\phi\right]\begin{pmatrix}\langle z|\mu\rangle \\ \langle -z|\mu\rangle\end{pmatrix} = \frac{\mu\hbar}{2}\begin{pmatrix}\langle z|\mu\rangle \\ \langle -z|\mu\rangle\end{pmatrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} -\mu & e^{-i\phi} \\ e^{i\phi} & -\mu \end{pmatrix}}_{A}\begin{pmatrix}\langle z|\mu\rangle \\ \langle -z|\mu\rangle\end{pmatrix} = 0 \qquad (*)$$

$$\det A = 0 = \mu^2 - e^{-i\phi}e^{i\phi} \implies \mu = \pm 1$$

NOTE $\hat{n}$ ANY-DIR. IN x,y PLANE $\Rightarrow S_n$ E-VALS $\pm\frac{\hbar}{2}$!

SPIN 1 S_n (again want to see e-vals same)

BOOK TAKES $\hat{n} = \hat{\jmath}$. RECALL $S_y \xrightarrow{z} \frac{\hbar}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}$

Spin in y-dir? E-VALS OF S_y

$$\underbrace{S_y |1,\mu\rangle}_{y\ dir} = \mu\hbar|1,\mu\rangle \implies (S_y - \mu\quad)|1,\mu\rangle = 0$$

$$\downarrow z$$

$$\hbar\begin{pmatrix} -\mu & -i/\sqrt{2} & \\ i/\sqrt{2} & -\mu & -i/\sqrt{2} \\ & i/\sqrt{2} & -\mu \end{pmatrix} =: A$$

$$\det A = \mu(\mu^2 - 1) \implies \boxed{\mu = \pm 1, 0 \implies S_y, S_z \text{ SAME E-VALS, NOT E-ST.}}$$

EXPECTATION VALUES & UNCERTAINTIES

* expectation values and uncertainties are statistical object
* They make sense when we talk about an large number of identical systems

SETUP: OPERATOR $\hat{A}$, w/ COMPLETE SET OF E-ST. $|a_i\rangle$, E-VAL a_i

$$\hat{A}|a_i\rangle = a_i |a_i\rangle$$

EXPECTATION VALUE: AVG. OF E-VALS WEIGHTED BY PROB

$$\langle \hat{A} \rangle := \sum_i \underbrace{a_i}_{\text{E-VAL}} \underbrace{|\langle a_i|\psi\rangle|^2}_{\text{PROB.}}$$

NB: SHOULD PROB. WRITE $\langle \hat{A} \rangle_\psi$ TO DENOTE EX. VAL IN $|\psi\rangle$, BUT THIS IS NOT COMMON CONVENTION

TWO WAYS TO COMPUTE

(1) EXPAND $|\psi\rangle$ IN EIG BASIS OF $\hat{A}$ IE. $|\psi\rangle = \sum c_j |a_j\rangle$

(2) "DIAGONALIZE" $\hat{A}$

$$\hat{A} = \sum_i a_i \hat{P}_i = \sum a_i |a_i\rangle\langle a_i|$$

$$\Rightarrow \langle \hat{A} \rangle = \sum a_i \langle \psi|a_i\rangle\langle \psi|a_i\rangle^* = \sum a_i \langle \psi|a_i\rangle\langle a_i|\psi\rangle$$

$$= \langle \psi| \left(\sum a_i |a_i\rangle\langle a_i| \right) |\psi\rangle = \langle \psi|\hat{A}|\psi\rangle$$

$$\boxed{\langle \hat{A} \rangle = \langle \psi|\hat{A}|\psi\rangle}$$

EX $\hat{A} = S_z, \quad |\psi\rangle = |z\rangle\langle z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$

$$\langle S_z \rangle = \frac{\hbar}{2} |\langle z|\psi\rangle|^2 + \left(-\frac{\hbar}{2}\right) |\langle -z|\psi\rangle|^2$$

CHECK: $|\psi\rangle = |z\rangle, \quad \langle S_z\rangle = \frac{\hbar}{2} 1^2 + \left(\frac{-\hbar}{2}\right) 0^2 = \frac{\hbar}{2}$ ✓

NOTE: IF $|\langle z|\psi\rangle|^2 = |\langle -z|\psi\rangle|^2$, EQUAL PROB, $\langle S_z\rangle = 0$

↳ $|x\rangle \xrightarrow{z\text{-bas}} \frac{1}{\sqrt{2}}\left(|z\rangle + |-z\rangle\right)$

① $\langle z|x\rangle = \frac{1}{\sqrt{2}} \Rightarrow |\langle z|x\rangle|^2 = \frac{1}{2}$. SIM $|\langle -z|x\rangle|^2 = \frac{1}{2} \Rightarrow \langle S_z\rangle = 0$

② $$\langle S_z\rangle = \langle x|S_z|x\rangle \xrightarrow{z\text{-bas}} \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{(\langle x|)_z} \underbrace{\begin{pmatrix} \frac{\hbar}{2} & \\ & -\frac{\hbar}{2} \end{pmatrix}}_{(S_z)_z} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}}_{(|x\rangle)_z} = 0$$

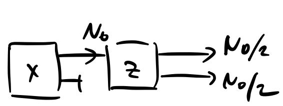
__OR__ IN X-BASIS

$$\langle S_z\rangle = \langle x|S_z|x\rangle = \underbrace{(1, 0)}_{(\langle x|)_x} \underbrace{\begin{pmatrix} & -\frac{\hbar}{2} \\ -\frac{\hbar}{2} & \end{pmatrix}}_{(S_z)_x} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{(|x\rangle)_x} = 0$$

NOTE: ① $\langle A\rangle$ IS PHYSICAL, CANNOT DEPEND ON BASIS CHOICE!

② $|\psi\rangle = |z\rangle \Rightarrow \langle S_z\rangle = \frac{\hbar}{2}$, GET $+\frac{\hbar}{2}$ 100% OF TIME

③ $|\psi\rangle = |x\rangle \Rightarrow$ 50%, 50%



⟶ ∃ DEG OF UNCERTAINTY

* HOW TO QUANTIFY?

$$\underbrace{\langle A \rangle}_{\text{AVG}} \longrightarrow \underbrace{A - \langle A \rangle}_{\text{DIFF OF SING MEASUREMENT FROM AVG}} \longrightarrow \underbrace{(A - \langle A \rangle)^2}_{\text{SQ. OF PREVIOUS}} \longrightarrow \underbrace{\langle (A - \langle A \rangle)^2 \rangle}_{\text{AVG OF (SQ OF DISTANCE FROM AVG)}}$$

*QUANTIFY w/ STANDARD DEV. OR "UNCERTAINTY" OF MEASUREMENT $\hat{A}$

$$\Delta A := \sqrt{\langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle}$$

NB $\Delta A^2 = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle \hat{A}^2 - 2\hat{A}\langle \hat{A} \rangle + \langle \hat{A} \rangle^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$

$$(\Delta A)^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$$

THEOREM: LET $\hat{A}, \hat{B}, \hat{C}$ BE HERM. OPS WITH

$$[\hat{A}, \hat{B}] = iC$$

THEN

$$(\Delta A)(\Delta B) \geq \frac{|\langle C \rangle|}{2}$$

GENERAL UNCERTAINTY PRINCIPLE

CASE: FACT $[\hat{x}, \hat{p}] = i\hbar$

$$\Rightarrow \boxed{(\Delta x)(\Delta p) \geq \frac{\hbar}{2}}$$