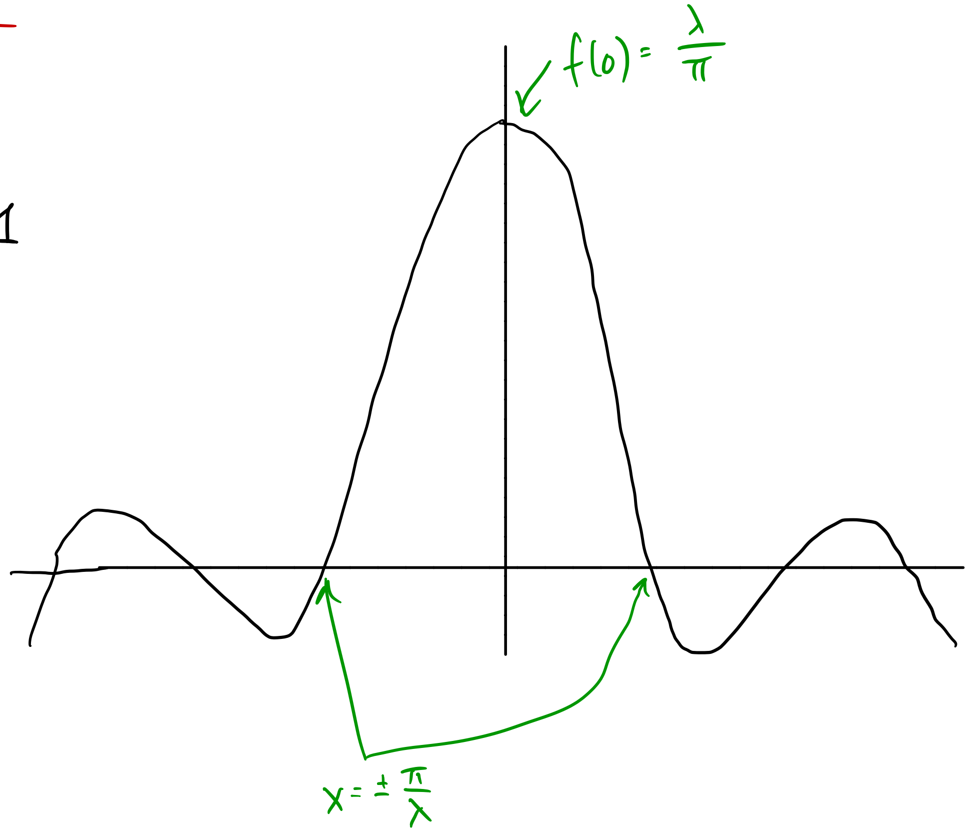
QM Lecture 10

DIRAC δ-FUNCTION

$$f(x) = \frac{\sin(\lambda x)}{\pi x}$$

NORM $\int_{-\infty}^{\infty} f(x)\, dx = 1$

WIDTH $\sim \frac{1}{\lambda}$



$\lambda \to \infty$

LIMIT $\Longrightarrow$ INF TALL & NARROW

REP 1

$$\delta(x) := \lim_{\lambda \to \infty} \frac{\sin \lambda x}{\pi x}$$

REP 2

$$\frac{1}{2} \int_{-\lambda}^{\lambda} dk\, e^{ikx} = \frac{\sin \lambda x}{x}$$

$$\Longrightarrow \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}$$

POSITION & MOMENTUM EIGENSTATES

$$\hat{x}|x\rangle = x|x\rangle \qquad \hat{p}|p\rangle = p|p\rangle$$

$$\langle x|x'\rangle = \delta(x'-x) \qquad \langle p'|p\rangle = \delta(p'-p)$$

IDENTITY $\quad \mathbb{1} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|$

TRANSLATIONS IN SPACE

$$\hat{T}(a)|x\rangle = |x+a\rangle$$

TRANSLATE $|\psi\rangle$

$$|\psi'\rangle = \hat{T}(a)|\psi\rangle = \hat{T}(a)\int dx'\, |x'\rangle\langle x'|\psi\rangle$$
$$= \int dx'\, \hat{T}(a)\,|x'\rangle\langle x'|\psi\rangle = \int dx'\, |x'+a\rangle\langle x'|\psi\rangle$$

$$\Longrightarrow \langle x|\psi'\rangle = \int dx'\, \delta[x-(x'+a)]\,\langle x'|\psi\rangle$$

HOW TO HANDLE? (see APP C)

(1) $\delta(ax) = \frac{1}{|a|}\delta(x)$

(2) $\delta(f(x)) = \dfrac{\delta(x-x_0)}{\left|{}^{df}/{}_{dx}\right|_{x=x_0}}$ WHERE $f(x_0)=0$

SO $\delta[x-(x'+a)] = \delta[-x'+(x-a)]$ VAN. @ $x' = x-a$

$\Longrightarrow \langle x|\psi'\rangle = \langle x-a|\psi\rangle$ so "$|\psi'\rangle$ @ x" = "$|\psi\rangle$ @ $x-a$"

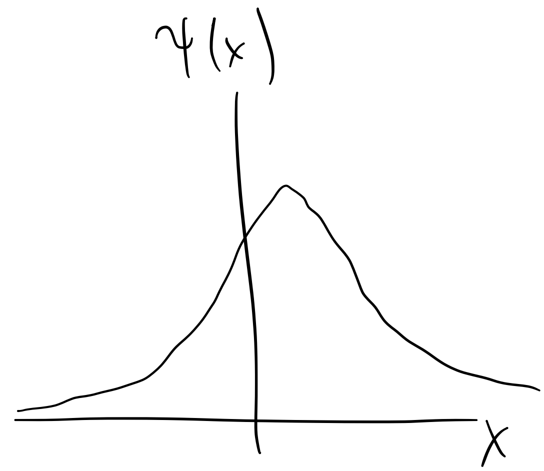
WAVEFUNCTIONS

(prev result somewhat counterintuitive, understand by introduction of function)

$$\boxed{\psi(x) := \langle x | \psi \rangle}$$

Language: "position-space wavefunction", or just "wavefunction"

$$|\psi\rangle \Longrightarrow \psi(x)$$



MATCH MODERN PHYSICS:

$$\langle x \rangle = \langle \psi | \hat{x} | \psi \rangle = \langle \psi | \hat{x} \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x' | \psi \rangle$$

$$= \int_{-\infty}^{\infty} dx' \, \langle \psi | x' | x' \rangle \langle x' | \psi \rangle$$

$$= \int_{-\infty}^{\infty} dx' \; x' \, \langle \psi | x' \rangle \langle x' | \psi \rangle = \int_{-\infty}^{\infty} dx' \; x' \; \psi(x') \psi^*(x')$$

$$= \int_{-\infty}^{\infty} dx' \; x' \; P(x')$$

PREV RESULT $\quad \langle x | \psi' \rangle = \langle x - a | \psi \rangle$

$$\Longrightarrow \quad \psi'(x) = \psi(x-a)$$

GEN OF TRANSLATIONS

$$\hat{T}(\delta x) = \mathbb{1} - \frac{i}{\hbar}\underset{\substack{\uparrow \\ \text{GEN OF TRANS}}}{\hat{A}}\,\delta x \qquad \text{w/} \qquad \hat{T}(\delta x)|x\rangle = |x+\delta x\rangle$$

ANALOGY: $\hat{R}(d\phi\,\hat{k}) = \mathbb{1} - \frac{i}{\hbar}\hat{J}_z\,d\phi \qquad \hat{U}(dt) = \mathbb{1} - \frac{i}{\hbar}\hat{H}\,dt$

NB

1) $T^\dagger T = \mathbb{1} \implies A^\dagger = A$

2) $\hat{T}(a) = \lim_{N\to\infty}\left[\mathbb{1} - \frac{i}{\hbar}\hat{A}\left(\frac{a}{N}\right)\right]^N = e^{-i\hat{A}a/\hbar}$

3) EXPECT NON-COMM. w/ $\hat{x}$

$$\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x} = \frac{-i\delta x}{\hbar}[\hat{x},\hat{A}] + \mathcal{O}(\delta x^2)$$

$$\left(\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x}\right)|\psi\rangle = \text{''}\int dx'|x'\rangle\langle x'|\psi\rangle$$

$$= \hat{x}\int dx'|x'+\delta x\rangle\langle x'|\psi\rangle - \hat{T}(\delta x)\int dx'\,x'|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx'(x'+\delta x)|x'+\delta x\rangle\langle x'|\psi\rangle - \int dx'\,x'|x'+\delta x\rangle\langle x'|\psi\rangle$$

$$= \delta x\int dx'|x'+\delta x\rangle\langle x'|\psi\rangle \underset{\substack{\uparrow \\ \text{TAYLOR}}}{=} \delta x\int dx'\left(|x'\rangle + \mathcal{O}(\delta x')\right)\langle x'|\psi\rangle$$

$$= \delta x\int dx'|x'\rangle\langle x'|\psi\rangle + \dots = \delta x|\psi\rangle + \dots$$

$$\implies \boxed{[\hat{x},\hat{A}] = i\hbar}$$

MOMENTUM GOAL: PHYSICAL IDEA RE: $\hat{A}$

DERIVATION:

i) LEMMA 1: $[\hat{x}^n, \hat{A}] = i\hbar n \hat{x}^{n-1}$

PROOF: (induction)

n=1: TRUE (last class)

ASSUME FOR n, PROVE FOR n+1

$$[\hat{x}^n, \hat{A}] = i\hbar n \hat{x}^{n-1}$$

$$[\hat{x}^{n+1}, \hat{A}] = [\hat{x}\hat{x}^n, \hat{A}] = \hat{x}[\hat{x}^n, \hat{A}] + [\hat{x}, \hat{A}]\hat{x}^n$$
$$= i\hbar n \hat{x}\hat{x}^{n-1} + i\hbar \hat{x}^n = i\hbar(n+1)\hat{x}^n \quad \checkmark$$

ii) LEMMA 2: $[f(\hat{x}), \hat{A}] = i\hbar \frac{\partial f}{\partial x}(\hat{x})$

PF: $f(x) = f(0) + \left(\frac{df}{dx}\right)_{x=0} x + \frac{1}{2}\left(\frac{d^2f}{dx^2}\right)_{x=0} x^2 + \ldots + \frac{1}{n!}\left(\frac{d^nf}{dx^n}\right)_{x=0} x^n + \ldots$

$$\Rightarrow [f(x), \hat{A}] = \left[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d^nf}{dx^n}\right)_{x=0} \hat{x}^n, \hat{A}\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d^nf}{dx^n}\right)\Big|_{x=0} i\hbar n \hat{x}^{n-1}$$

$$= i\hbar \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{d^nf}{dx^n}\right)\Big|_{x=0} n \hat{x}^{n-1} = i\hbar \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dx^{n-1}}\left(\frac{df}{dx}\right)\right)_{x=0} \hat{x}^{n-1}$$

$$= i\hbar \sum_{n'=0}^{\infty} \frac{1}{n'!}\left(\frac{d^{n'}}{dx^{n'}}\left(\frac{df}{dx}\right)\right)\Big|_{x=0} \hat{x}^{n'} = i\hbar \frac{\partial f}{\partial x}(\hat{x})$$

iii) PHYSICS 1-PART IN $V(\hat{x})$ POT

$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$ Q: $\langle A \rangle$ - EVOLUTION?

$$\frac{d}{dt}\langle \hat{A} \rangle = \frac{i}{\hbar}\langle \psi | [\hat{H}, \hat{A}] | \psi \rangle = \frac{i}{\hbar}\frac{1}{2m}\langle \psi | [\hat{p}_x^2, \hat{A}] | \psi \rangle + \frac{i}{\hbar}\langle \psi | [V(\hat{x}), \hat{A}] | \psi \rangle$$

$$= \frac{i}{2m\hbar}\langle [\hat{p}_x^2, A] \rangle - \langle \frac{\partial V}{\partial x} \rangle$$

Q: $-\langle \frac{\partial V}{\partial x} \rangle$ LOOK FAMILIAR?

SUPPOSE $\hat{A} = \hat{p}_x$, $\exists$ 2 CONS. CHECKS

1) $\Rightarrow$ $$\frac{d\langle \hat{p}_x \rangle}{dt} = \frac{i}{2m\hbar}\cancelto{0}{\langle [\hat{p}_x^2, \hat{p}_x] \rangle} - \langle \frac{\partial V}{\partial x} \rangle$$

$$\frac{d\langle p_x \rangle}{dt} = -\langle \frac{\partial V}{\partial x} \rangle$$

"Quantum Newton's Law"

EHRENFEST'S THM

2) HEIS. UNC. PRIN. $[x, A] = i\hbar \Rightarrow [x, p_x] = i\hbar$

$\Rightarrow \boxed{\Delta x \Delta p_x \geq \frac{\hbar}{2}}$ BY $[\hat{A}, \hat{B}] = i\hat{C}$ THM.

ALSO

$$\frac{d\langle x \rangle}{dt} = \frac{i}{\hbar}\langle \psi | [\hat{H}, \hat{x}] | \psi \rangle$$

$$= \frac{i}{\hbar}\langle \psi | [\frac{\hat{p}_x^2}{2m}, \hat{x}] | \psi \rangle + \frac{i}{\hbar}\cancelto{0}{\langle \psi | [V(\hat{x}), \hat{x}] | \psi \rangle}$$

$$= \frac{i}{2m\hbar}\langle \psi | (\hat{p}_x [\hat{p}_x, \hat{x}] + [\hat{p}_x, \hat{x}]\hat{p}_x | \psi \rangle$$

$$= \frac{\langle p_x \rangle}{m}$$

Q: why does this make sense?

A: looks like $p = mv$!