

QM Lecture 4



QUANTUM INTERFERENCE

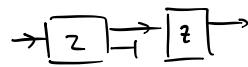
BACK TO T.E.'s



$SG_z + \text{WALL}$ ACTS LIKE \hat{P}_+ IN z -BASIS

SG_z PROJ ONTO ONE OR OTHER

- TE 1:



EXTRA SG_z ? PROJECT AGAIN

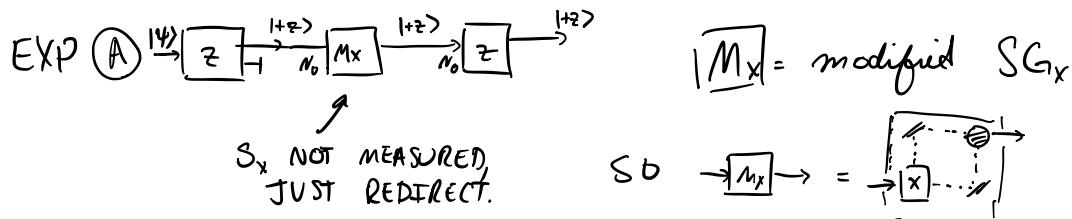
$$(\hat{P}_+)_z \cdot (\hat{P}_+)_z = |+z\rangle \langle +z|_{+z} |+z\rangle \langle +z| = |+z\rangle \langle +z| = (\hat{P}_+)_z$$

$$(\hat{P}_+)_z \cdot (\hat{P}_-)_z = |-z\rangle \underbrace{\langle +z|}_{0} \langle -z| = 0$$

\Rightarrow OUTPUT OF SECOND MATCHES FIRST!

(and this matches the experiment!)

- TE 4.



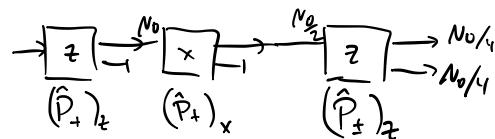
$$|\Psi\rangle = C_+ |+z\rangle + C_- |-z\rangle$$

$$\text{FINAL STATE: } (\hat{P}_+)_z (\boxed{x})_z (\hat{P}_+)_z |\Psi\rangle = (\hat{P}_+)_z |\Psi\rangle = C_+ |+z\rangle$$

$$= C_+ (|+x\rangle \langle +x|_z + |-x\rangle \langle -x|_z) \\ = C_+ (C_+^* |+x\rangle + C_-^* |-x\rangle)$$

X-BAS.

EXP B Q: PUT IN WALL? \Downarrow REPL. BY $(\hat{P}_\pm)_x$ (depend on wall loc.)



$$\text{EXP A} \Rightarrow |(-z)_{+z}\rangle|^2 = 0 \quad (\text{what happens if we exp. same result in } x\text{-basis?})$$

" " "

$$\left. \begin{aligned} & |(-z)_{+x}\rangle|^2 |(+x)_{+z}\rangle|^2 + |(-z)_{-x}\rangle|^2 |(-x)_{+z}\rangle|^2 \\ & + \langle -z|_{+x}\rangle \langle +x|_{-z}\rangle \langle -z|_{-x}\rangle^* \langle -x|_{+z}\rangle^* \\ & + \langle -z|_{+x}\rangle^* \langle +x|_{+z}\rangle^* \langle -z|_{-x}\rangle \langle -x|_{+z}\rangle \end{aligned} \right\} \equiv \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4}$$

MTH: MEASURED TO HAVE

$$\textcircled{1} \quad \text{PROB } |+z\rangle \text{ IN MTH } S_x = \frac{\hbar}{2} \text{ THEN MTH } S_z = -\frac{\hbar}{2}. \stackrel{\text{EXP}}{B} \Rightarrow \textcircled{1} = \frac{1}{4}$$

$$\textcircled{2} \quad \text{PROB } |+x\rangle \text{ IN MTH } S_x = -\frac{\hbar}{2} \quad \text{...} \quad S_z = -\frac{\hbar}{2}. \stackrel{\text{EXP}}{B} \text{ w REPL} \Rightarrow \frac{1}{4}$$

$$\Rightarrow 0 = \frac{1}{4} + \frac{1}{4} + \textcircled{3} + \textcircled{4} \Rightarrow \textcircled{3}, \textcircled{4} \quad \text{CANCEL } \frac{1}{4} + \frac{1}{4}.$$

TAKE AWAY:

- i) $\textcircled{3} + \textcircled{4}$ "INTERFERENCE TERMS"
- ii) AMPLITUDES INTERFERE.

\Rightarrow PROB. AMP'S ADD, SQUARE TO GET PROB,
 \exists INTERFERING CROSS TERMS!

(if you just add probs, get $\textcircled{1} + \textcircled{2}$.
instead, must add amplitudes, then
square to get probability! that
process gives rise to interference terms.)

MATRIX REPS OF BASIS ROTS

(so far: matrix rep of $\mathbb{1}\mathbb{L}$ and \hat{P}_i , z -basis rep of \hat{J}_z .
 now, matrix rep of rot op in quantum state space)

RECALL $|\psi\rangle_{S_z\text{-bas}} \xrightarrow{\quad} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$

\nearrow BRAEV* \searrow KETEV

FACT: OP. $\hat{A}: V \rightarrow V$ ($\text{Q: what maps } V^* \rightarrow V^*$)

ADJ. OP $\hat{A}^\dagger: V^* \rightarrow V^*$

IE $\hat{A}|\psi\rangle = |\psi'\rangle, \langle \psi|\hat{A}^\dagger = \langle \psi'|$

EXAMPLE $\hat{A} = \hat{R}(\theta \hat{n})$.

($\text{Q: what is } (\hat{R}(\theta \hat{n}))^{-1}?$) $\hat{R}(\theta \hat{n})^{-1} = \hat{R}(-\theta \hat{n})$

$\text{ROT } |\psi\rangle \xrightarrow[\text{BUT } S_z = \frac{1}{2}]{-\theta \text{ MEAS}} = \langle +z| \hat{R}(-\theta \hat{n}) |\psi\rangle = \langle +z| \hat{R}(\theta \hat{n})^\dagger |\psi\rangle$

BUT UNITARY $\Rightarrow \hat{R}^\dagger(\theta \hat{n}) \hat{R}(\theta \hat{n}) = \mathbb{1}$

$\therefore \langle +z| \hat{R}(\theta \hat{n})^\dagger |\psi\rangle = \langle z'|\psi\rangle = \text{MEAS. } S_{z'} = \frac{1}{2} \text{ of } |\psi\rangle \text{ IN QOT. } z\text{-BAS. } |+z'\rangle = \hat{R}(\theta \hat{n})|z\rangle$

$\hat{R}^\dagger(\theta \hat{n}) = ?$ GEN: $\underbrace{\langle i|\hat{A}^\dagger|j\rangle}_{\textcircled{*}} = \underbrace{\langle j|\hat{A}|i\rangle^*}_{\textcircled{*}} \Rightarrow \hat{A}_{ij}^\dagger = \hat{A}_{ji}^*$

PROOF: (from above) $\langle x|\hat{A}|\psi\rangle = \langle x|\psi'\rangle \& \langle \psi|\hat{A}^\dagger|x\rangle = \langle \psi'|x\rangle$

THEN $\langle \psi'|x\rangle = \langle x|\psi'\rangle^* \Rightarrow \textcircled{*}$

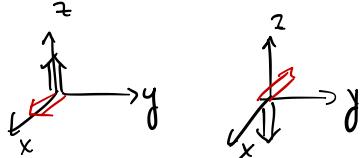
\Rightarrow FOR MATRIX REPS "DAGGER" MEANS CONJ. TRANSPOSE

CHECK $\hat{R}(\theta \hat{n}) = e^{-i\theta \hat{J}_n / \hbar}$ w/ $\hat{J}_n = \hat{J}_n^+$ (i.e., generator is Hermitian)

$$\Rightarrow R^+(\theta \hat{n}) = e^{i\theta \frac{\hat{J}_n^+}{\hbar}} = e^{i\theta \frac{\hat{J}_n}{\hbar}} = \hat{R}(-\theta \hat{n})$$

(matches previous direct result, but here we just took the conjugate transpose!)

ROTATE 14)



$$|\pm x\rangle = \hat{R}\left(\frac{\pi}{2} \hat{j}\right) |\pm z\rangle \Rightarrow \langle \pm x | = \langle \pm z | \hat{R}^{\dagger}\left(\frac{\pi}{2} \hat{j}\right)$$

ROTATE:

$$\overline{R^{\dagger}\left(\frac{\pi}{2} \hat{j}\right)} |14\rangle =: |14'\rangle \xrightarrow[S_z-\text{bas}]{ } \begin{pmatrix} \langle +z | 14' \rangle \\ \langle -z | 14' \rangle \end{pmatrix} = \begin{pmatrix} \langle +z | \hat{R}^{\dagger}\left(\frac{\pi}{2} \hat{j}\right) |14\rangle \\ \langle -z | \hat{R}^{\dagger}\left(\frac{\pi}{2} \hat{j}\right) |14\rangle \end{pmatrix} = \begin{pmatrix} \langle +x | 14 \rangle \\ \langle -x | 14 \rangle \end{pmatrix} \xleftarrow[S_x-\text{bas}]{ } |14\rangle$$

ACTIVE: ROTATE STATE BASIS
PASSIVE: ROTATE

(point: this rotated state in unrotated basis and unrotated state in this rotated basis are same thing!)

RELATE TWO COL. VECTORS? (does the above suggest how to do it?)

$$\underbrace{\begin{pmatrix} \langle +x | 14 \rangle \\ \langle -x | 14 \rangle \end{pmatrix}}_{(14)_{S_x}} = \begin{pmatrix} \langle +x | +z \rangle \langle +z | 14 \rangle + \langle +x | -z \rangle \langle -z | 14 \rangle \\ \langle -x | +z \rangle \langle +z | 14 \rangle + \langle -x | -z \rangle \langle -z | 14 \rangle \end{pmatrix} = \underbrace{\begin{pmatrix} \langle +x | +z \rangle & \langle +x | -z \rangle \\ \langle -x | +z \rangle & \langle -x | -z \rangle \end{pmatrix}}_{\underline{S}^{\dagger}\left(\frac{\pi}{2} \hat{j}\right)} \begin{pmatrix} \langle +z | 14 \rangle \\ \langle -z | 14 \rangle \end{pmatrix} \xleftarrow[]{ } (14)_{S_z}$$

so $(14)_{S_x} = \underline{S}^{\dagger}(14)_{S_z}$

S⁺ reps R^+ IN S_z -BAS. i.e. (with rot & $\frac{\pi}{2} \hat{j}$)

$$\begin{pmatrix} \langle +x | +z \rangle & \langle +x | -z \rangle \\ \langle -x | +z \rangle & \langle -x | -z \rangle \end{pmatrix} = \begin{pmatrix} \langle +z | R^{\dagger}\left(\frac{\pi}{2} \hat{j}\right) | +z \rangle & \langle +z | R^{\dagger}\left(\frac{\pi}{2} \hat{j}\right) | -z \rangle \\ \langle -z | R^{\dagger}\left(\frac{\pi}{2} \hat{j}\right) | +z \rangle & \langle -z | R^{\dagger}\left(\frac{\pi}{2} \hat{j}\right) | -z \rangle \end{pmatrix}$$

INVERSE? ① SAME DER. W $(\underline{A})_{S_z}$ ON LHS

$$\Rightarrow (\underline{S})_{S_z} = \begin{pmatrix} \langle +z | \hat{R}(\frac{\pi}{2}) | +z \rangle & \langle +z | \hat{R}(\frac{\pi}{2}) | -z \rangle \\ \langle -z | \hat{R}(\frac{\pi}{2}) | +z \rangle & \langle -z | \hat{R}(\frac{\pi}{2}) | -z \rangle \end{pmatrix}$$

S^+ REPS R^+ , S REPS R !

ROTATE OPERATORS? (instead of states?)

$$\hat{A} \xrightarrow{S_x \text{-ROT}} \underbrace{\begin{pmatrix} \langle x | \hat{A} | x \rangle & \langle x | \hat{A} | -x \rangle \\ \langle -x | \hat{A} | x \rangle & \langle -x | \hat{A} | -x \rangle \end{pmatrix}}_{= \langle -z | \hat{R}^+(\frac{\pi}{2}) \hat{A} \hat{R}(\frac{\pi}{2}) | z \rangle} = (\underline{A})_{S_x}$$

* MAT. REP OF PRODUCT OF \hat{O} 'S?

$$\langle i | \hat{A} \hat{B} | j \rangle = \langle i | \hat{A} \left(\sum_k | k \rangle \langle k | \right) \hat{B} | j \rangle = \sum_k \langle i | \hat{A} | k \rangle \langle k | \hat{B} | j \rangle = \sum_k A_{ik} B_{kj} = \underbrace{\underline{A} \cdot \underline{B}}_{(\underline{AB})_{ij}}$$

MATRIX
MULT!

SO MATRIX REP OF PRODUCT OF OPS

= PRODUCT OF MAT. REP OF OPS

$$\implies \underline{(\underline{A})_{S_x}} = S^+ (\underline{A})_{S_z} S \quad \text{(linear alg.: just a change of basis!)}$$

APPLICATION COMPUTE $(\hat{T}_z)_{S_x}$ (note: $\hat{z} \neq x!$)

$$\text{ABOVE} \Rightarrow (\hat{T}_z)_{S_x} = S^+ (\hat{T}_z)_{S_z} S \quad \text{NEED } S$$

$$| \pm x \rangle = \frac{1}{\sqrt{2}} (| +z \rangle + | -z \rangle) \quad (\text{wrong choice to make purely real})$$

$$S = \begin{pmatrix} \langle z|x \rangle & \langle z|1-x \rangle \\ \langle -z|x \rangle & \langle -z|1-x \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$S^+ = S^\tau * = S^\tau = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad (\hat{T}_z)_{S_z} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore (\hat{T}_z)_{S_x} = \frac{1}{2} \frac{\hbar}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{\hbar}{4} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

OFF DIAGONAL? $\Rightarrow | \pm x \rangle$ NOT EIGENVECTORS OF T_z (why?)

$$\text{USE IT} \quad | z \rangle \xrightarrow{S_x\text{-BAS}} \begin{pmatrix} | x|z \rangle \\ | -x|z \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

ALT: (express in z -basis and rotate)

$$(| z \rangle)_{S_x} = S^+ (| z \rangle)_{S_z} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \checkmark$$

EIG-STATES ARE BASIS-IND

EG:

$$(\hat{T}_z)_{S_x} (| +z \rangle)_{S_x} = \frac{\hbar}{2} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\hbar}{2} (| +z \rangle)_{S_x} \quad \checkmark$$

ROTATION

NON-COMMUTATION

(R act on $v \in \mathbb{R}^3$)

$$R(\phi_i) = \begin{pmatrix} 1 & & \\ -s\phi & c\phi & \\ s\phi & c\phi & 1 \end{pmatrix}$$

$$R(\phi_j) = \begin{pmatrix} c\phi & s\phi \\ -s\phi & c\phi \end{pmatrix}$$

$$R(\phi_k) = \begin{pmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{pmatrix}$$

$$c_\phi := \cos \phi \quad s_\phi := \sin \phi$$

NOTE: - $\det R = 1\mathbb{L}$

- ϕ small: $c_\phi = 1 - \frac{1}{2}\phi^2 \dots \quad \sin \phi = \phi + \dots$

- ROT. DON'T COMMUTE! (demonstrate)

- NEITHER DO MATRICES $AB \neq BA$ GEN.

DEFINITION THE EXTENT TO WHICH TWO OPERATORS A, B DO NOT COMMUTE IS MEASURED BY THE COMMUTATOR

$$[A, B] = AB - BA$$

(if matrix reps, just compute!)

$$R(\phi_i) R(\phi_j) = \begin{pmatrix} c\phi_i & 0 & s\phi_i \\ s\phi_i & c\phi_i & -s\phi_i c\phi_i \\ -s\phi_i c\phi_i & s\phi_i & c\phi_i^2 \end{pmatrix} \approx \begin{pmatrix} 1 - \frac{1}{2}\phi_i^2 & 0 & \phi_i \\ \phi_i^2 & 1 - \frac{1}{2}\phi_i^2 & -\phi_i \\ -\phi_i & \phi_i & 1 - \frac{1}{2}\phi_i^2 \end{pmatrix} + O(\phi^3)$$

$$R(\phi_j) R(\phi_i) = \begin{pmatrix} c\phi_j & s\phi_j^2 & s\phi_j c\phi_j \\ s\phi_j & c\phi_j & -s\phi_j c\phi_j \\ -s\phi_j c\phi_j & c\phi_j s\phi_j & c\phi_j^2 \end{pmatrix} \approx \begin{pmatrix} 1 - \frac{1}{2}\phi_j^2 & \phi_j^2 & \phi_j \\ 0 & 1 - \frac{1}{2}\phi_j^2 & -\phi_j \\ -\phi_j & \phi_j & 1 - \frac{1}{2}\phi_j^2 \end{pmatrix}$$

$$[R(\phi_i), R(\phi_j)] \approx \begin{pmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R[\phi^2 \hat{I}] - 1\mathbb{L} \quad (\text{gen. not around third axis!})$$

* NOTE: DON'T COMM. (ω $O(\phi^2)$)



IMPLICATIONS FOR GENERATORS $\hat{J}_x \in \hat{J}_y$

RECALL R IS MATRIX REP OF \hat{R}

$$\hat{R}(\phi) = e^{-i\hat{J}_z \phi/\hbar} \quad \hat{R}(\phi) = e^{-i\hat{J}_x \phi/\hbar} \quad \hat{R}(\phi) = e^{-i\hat{J}_y \phi/\hbar}$$

$$\underline{\phi \text{ small}} \quad \hat{R}(\phi) \approx 1 - i \hat{J}_z \frac{\phi}{\hbar} + \frac{1}{2!} \left(i \hat{J}_z \frac{\phi}{\hbar} \right)^2 + \dots$$

$$\text{comm} \Rightarrow \left(1 - i \frac{\hat{J}_y \phi}{\hbar} - \frac{1}{2} \left(\frac{\hat{J}_x \phi}{\hbar} \right)^2 \right) \left(1 - i \frac{\hat{J}_y \phi}{\hbar} - \frac{1}{2} \left(\frac{\hat{J}_y \phi}{\hbar} \right)^2 \right)$$

- (second)(first) + ...

$$= \left\{ 1 - i \frac{\phi}{\hbar} [J_x + J_y - J_y - J_x] \left(\frac{\phi}{\hbar} \right)^2 \left(J_x J_y + \frac{J_x^2}{2} + \frac{J_y^2}{2} - J_y J_x - \frac{J_x^2}{2} - \frac{J_y^2}{2} \right) \right\} + \dots$$

$$= \left\{ -\frac{1}{2} \left(\frac{\phi}{\hbar} \right)^2 [J_x J_y - J_y J_x] \right\} = -\left(\frac{\phi}{\hbar} \right)^2 [\hat{J}_x, \hat{J}_y]$$

= $\textcircled{*}_2 \text{ LHS}$

$$\textcircled{*}_2 \text{ RHS} = \left\{ 1 - i \frac{\hat{J}_z \phi^2}{\hbar} \right\} - 1 \quad \Rightarrow \underbrace{\left\| [\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \right\|}_{\parallel}$$

$$\text{GEN} \quad \left\| [\hat{J}_i, \hat{J}_j] = i\hbar \epsilon^{ijk} \hat{J}_k \right\| \quad \epsilon^{ijk} = \begin{cases} 0 & \text{odd sign} \\ \pm & \text{even perm } 1, 2, 3 \\ -1 & \text{odd perm } 1, 2, 3 \end{cases}$$

$$\Rightarrow [\hat{J}_z, \hat{J}_z] = 0 \quad [\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_x \quad [\hat{J}_y, \hat{J}_x] = -[\hat{J}_x, \hat{J}_y] = -i\hbar \hat{J}_z$$