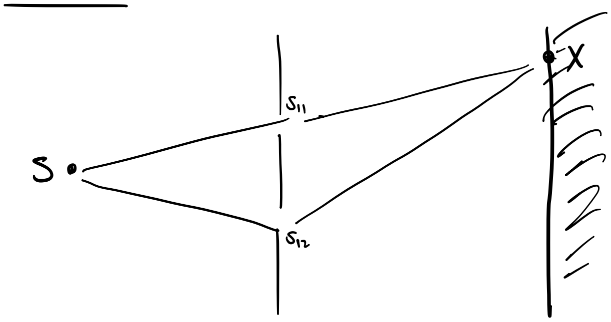
QM Lecture 20

TODAY: DOUBLE SLIT $\Longrightarrow$ PATH INTEGRALS

DOUBLE SLIT EXPERIMENT!

① FOR US: ONE SCREEN TWO SLIT EXP'T.



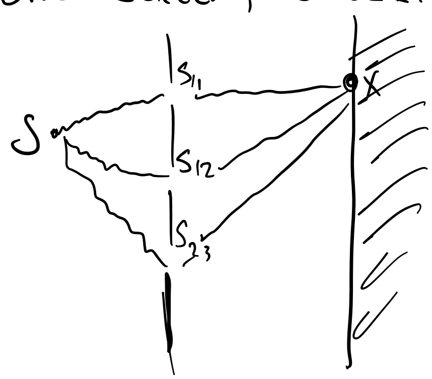
* S: SOURCE OF PARTICLES w/ P WELL-DEFINED

* MOVE X, GET INTERFERENCE PATTERN

$\mathcal{A}$ = AMPLITUDE

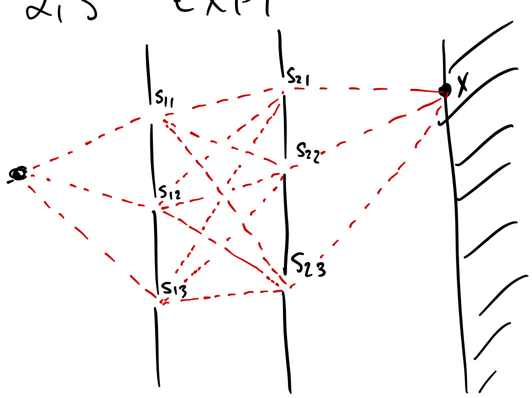
$$\mathcal{A}(\text{DETECT @ } X) = \mathcal{A}(S \to S_{11} \to X) + \mathcal{A}(S \to S_{12} \to X)$$

② ONE SCREEN, 3 SLIT



$$\mathcal{A}(\text{DETECT @ } X) = \sum_i \mathcal{A}(S \to S_{1i} \to X)$$

③ 2, 3 EXPT



$$\mathcal{A}(\text{DET @ } X) = \sum_i \sum_j \mathcal{A}(S \to S_{1i} \to S_{2j} \to X)$$

④ ∞ SCREENS, ∞ SLITS, $\mathcal{A}(\text{DET @ } X) = \sum_{\substack{\text{paths} \\ P}} \mathcal{A}(S \xrightarrow{\text{via } P} X)$

PATH INTEGRALS

$$\langle x', t' | x_0, t_0 \rangle = \langle x' | \mathcal{U}(t'-t_0) | x_0 \rangle = \langle x' | e^{-i\hat{H}(t'-t_0)/\hbar} | x_0 \rangle \qquad \underline{\text{1D}}: \ \hat{H} = \frac{p_x^2}{2m} + V(\hat{x})$$

$$* \quad \psi(x', t') = \langle x' | \psi(t') \rangle = \langle x' | e^{-i\hat{H}(t'-t_0)/\hbar} | \psi(t_0) \rangle = \int_{-\infty}^{\infty} dx_0 \underbrace{\langle x' | e^{-i\hat{H}(t'-t_0)/\hbar} | x_0 \rangle}_{\text{implicitly set our clocks @}} \langle x_0 | \psi(t_0) \rangle$$

$t=0$, so $\langle x' | = \langle x'; 0 |$, eg

$$= \int_{-\infty}^{\infty} dx_0 \ \langle x', t' | x_0, t_0 \rangle \langle x_0 | \psi(t_0) \rangle = \int_{-\infty}^{\infty} dx_0 \ \underbrace{\langle x', t' | x_0, t_0 \rangle}_{\text{"PROPAGATOR"}} \psi(x_0, t_0)$$

$\psi(x_0, t_0)$

use to det how wf. ed. props in time

$*$ EG FREE PARTICLE $\hat{H} = \frac{p_x^2}{2m}$

$$\langle x', t' | x_0, t_0 \rangle = \int_{-\infty}^{\infty} dp \langle x' | e^{-ip_x^2(t'-t_0)/2m\hbar} | p \rangle \langle p | x_0 \rangle = \int_{-\infty}^{\infty} dp \ \langle x' | p \rangle \langle p | x_0 \rangle e^{-ip^2(t'-t_0)/2m\hbar}$$

using $\langle x | p \rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$

$$= \sqrt{\frac{m}{2\pi\hbar i (t'-t_0)}} \ e^{-im(x'-x_0)^2/2\hbar(t'-t_0)}$$

SHORT TIME INT.

- $*$ MANY SCREENS, MANY SLITS, $\Rightarrow$ SHORT PROP.
- $*$ BREAK UP $t' - t_0 \Rightarrow N$ INERVALS w/ $\Delta t = \frac{(t'-t_0)}{N}$

$$e^{-iH\Delta t/\hbar} = \mathbb{1} - \frac{i}{\hbar} \hat{H}(p_x, x) \Delta t + \mathcal{O}(\Delta t^2). \qquad \text{RECALL} \quad V(\hat{x}) | x \rangle = V(x) | x \rangle$$

$$\langle x' | e^{-i\hat{H}\Delta t/\hbar} | x \rangle = \langle x' | [1 - \tfrac{i}{\hbar} H \Delta t] | x \rangle + \mathcal{O}(\Delta t^2) = \langle x' | [1 - \tfrac{i}{\hbar}(\tfrac{p_x^2}{2m} + V(\hat{x}))] \Delta t | x \rangle + \mathcal{O}(\Delta t^2)$$

$$= \int_{-\infty}^{\infty} dp \ \langle x' | p \rangle \langle p | \Big\{ \mathbb{1} - \frac{i}{\hbar} \Big[\frac{p^2}{2m} + \overset{\text{NOT } V(\hat{x})!}{V(x)} \Big] \Delta t \Big\} | x \rangle + \ldots$$

$$= \int_{-\infty}^{\infty} dp \ \langle x' | p \rangle \langle p | [\mathbb{1} - \frac{i}{\hbar} E(p, x) \Delta t] | x \rangle + \ldots$$

$$\text{w/} \quad E(p, x) = \frac{p^2}{2m} + V(x)$$

$$= \int_{-\infty}^{\infty} dp \ \langle x' | p \rangle \langle p | x \rangle e^{-iE\Delta t/\hbar} + \ldots = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \ e^{ip(x'-x)/\hbar} e^{-iE\Delta t/\hbar} + \ldots$$

see book for discussion

$$\boxed{\langle x' | e^{-i\hat{H}\Delta t/\hbar} | x \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \exp\Big\{ \frac{i}{\hbar} \Big[p \frac{(x'-x)}{\Delta t} - E(p, x) \Big] \Delta t \Big\}} \quad \Rightarrow \text{RHS HAS NO OPERATORS AT ALL!}$$

$\underline{\underline{\text{IN}}}$ $\mathcal{O}(\Delta t^2)$ TERMS

THE PATH INTEGRAL

* $\langle x', t' | x_0, t_0 \rangle$ w/ FINITE $t'-t_0$, BROKEN INTO N Δt'S

$$\langle x', t' | x_0, t_0 \rangle = \langle x' | \underbrace{e^{-i\hat{H}\Delta t/\hbar} \cdots e^{-i\hat{H}\Delta t/\hbar}}_{N \text{ TIMES}} | x_0 \rangle$$

* INSERT $\mathbb{1} = \int_{-\infty}^{\infty} dx_i \, |x_i\rangle\langle x_i|$ $\quad i = 1, 2, \dots, N-1$ TIMES

$$\langle x', t' | x_0, t_0 \rangle = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} \, \langle x' | e^{-i\hat{H}\Delta t/\hbar} | x_{N-1} \rangle \langle x_{N-1} | e^{-i\hat{H}\Delta t/\hbar} | x_{N-2} \rangle \cdots \langle x_1 | e^{-i\hat{H}\Delta t/\hbar} | x_0 \rangle$$

$$\langle x', t' | x_0, t_0 \rangle = \lim_{N\to\infty} \int dx_1 \cdots \int dx_N \int \frac{dp_1}{2\pi\hbar} \cdots \int \frac{dp_N}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar} \sum_{i=1}^{N} \left[p_i \frac{(x_i - x_{i-1})}{\Delta t} - E(p_i, x_{i-1}) \right] \Delta t \right\}$$

w/ $x_N = x'$ (just a name)

* GAUSSIAN INTEGRALS! (Below is a typical $\int$ from above)

$$\int \frac{dp_i}{2\pi\hbar} \exp\left[-i \frac{p_i^2 \Delta t}{2m\hbar} + i \frac{p_i (x_i - x_{i-1})}{\hbar} \right] = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \exp\left[\frac{i}{\hbar} \left(\frac{m\Delta t}{2} \right) \left(\frac{x_i - x_{i-1}}{\Delta t} \right)^2 \right]$$

$$\Rightarrow \langle x', t' | x_0, t_0 \rangle = \lim_{N\to\infty} \int dx_1 \cdots \int dx_{N-1} \left(\frac{m}{2\pi\hbar i \Delta t} \right)^{N/2} \exp\left\{ \frac{i}{\hbar} \underbrace{\Delta t \sum_{i=1}^{N} \left[\frac{m}{2} \left(\frac{x_i - x_{i-1}}{\Delta t} \right)^2 - V(x_{i-1}) \right]}_{\circledast} \right\}$$

$N \to \infty \Rightarrow \Delta t \to 0 \Rightarrow \circledast$ NOW $\frac{i}{\hbar} \int_{t_0}^{t'} dt \, L(x, \dot{x})$

WITH $L(x, \dot{x}) = \frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) = \frac{1}{2} m \dot{x}^2 - V(x)$

$\Uparrow$ LAGRANGIAN FROM CLASSICAL MECHANICS!

SHORTHAND: $\int_{x_0}^{x'} \mathcal{D}[x(t)] := \lim_{N\to\infty} \int dx_1 \cdots \int dx_{N-1} \left(\frac{m}{2\pi\hbar i \Delta t} \right)^{N/2}$

$$\Rightarrow \boxed{\begin{aligned} &\langle x', t' | x_0, t_0 \rangle = \int_{x_0}^{x'} \mathcal{D}[x(t)] \, e^{iS[x(t)]/\hbar} \\ &\text{w/ ACTION } S[x(t)] = \int_{t_0}^{t'} dt \, L(x, \dot{x}) \end{aligned}}$$

REMEMBER, = TO $\langle x' | e^{-iH(t'-t_0)/\hbar} | x_0 \rangle$

NOTE ① ONE DOESN'T HAVE OPS! BUT DOES HAVE INF # OF PATHS

② P.I.: $e^{iS[x(t)]/\hbar}$ PHASE CONT. FROM EACH PATH.

$\Rightarrow$ SAME MAGNITUDE FROM EACH PATH!

③ CAN BE HARD TO EVALUATE!

EG FREE PARTICLE

* IN GEN DERIV. HAD

$$\langle x't' | x_0 t_0 \rangle = \lim_{N\to\infty} \int dx_1 \ldots \int dx_{N-1} \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{N/2} \exp\left[\frac{i}{\hbar} \Delta t \sum_{i=1}^{N} \frac{m}{2} \left(\frac{x_i - x_{i-1}}{\Delta t}\right)^2\right]$$

* DEF $y_i = x_i \sqrt{\frac{m}{2\hbar\Delta t}}$

$$\Rightarrow \langle x't' | x_0 t_0 \rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{N/2} \left(\frac{2\hbar\Delta t}{m}\right)^{\frac{N-1}{2}} \underbrace{\int_{-\infty}^{\infty} dy_1 \ldots \int_{-\infty}^{\infty} dy_{N-1} \exp\left[i \sum_{i=1}^{N} (y_i - y_{i-1})^2\right]}$$

INDUCTION
& GAUSS. INT. $\Rightarrow = \sqrt{\frac{(i\pi)^{N-1}}{N}}\, e^{i(y_N - y_0)^2/N}$

* $\therefore \langle x't' | x_0 t_0 \rangle = \lim_{N\to\infty} \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{N/2} \left(\frac{2\hbar\Delta t}{m}\right)^{\frac{N-1}{2}} \left(\frac{(i\pi)^{N-1}}{N}\right)^{\frac{1}{2}} e^{i(y_N - y_0)^2/N}$

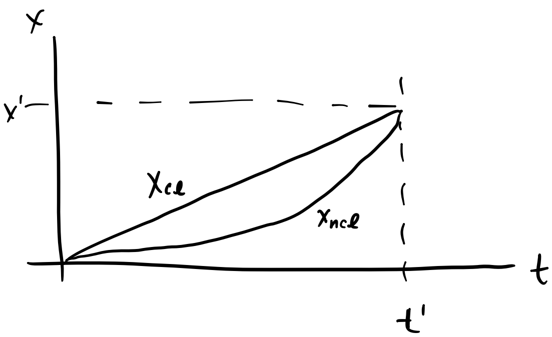
ALG &
$N\Delta t = t' - t_0$ $\rightarrow$ $= \sqrt{\frac{m}{2\pi i\hbar(t'-t_0)}}\, e^{im(x'-x_0)^2/2\hbar(t'-t_0)}$

MATCHES RESULT OBTAINED
BY HAMILTONIAN FORMALISM!

CLASSICAL LIMIT & PATH OF LEAST ACTION

Q: SINCE ALL PATHS CONT. TO P.I., WHY DOES ONE SEEM TO DOM. CLASSICALLY?

CONSIDER TWO PATHS FOR FREE PARTICLE



$x_{cl} = \left(\frac{x'}{t'}\right) t \qquad x_{ncl} = \left(\frac{x'}{t'^2}\right) t^2$

(remember x', t' fixed, not variables)

ACTIONS USE $L = \frac{m\dot{x}^2}{2}$

CL

$$S[x_{cl}(t)] = \int_0^{t'} dt \, \frac{m}{2}\left(\frac{x'^2}{t'^2}\right) = \frac{m x'^2}{2t'}$$

MACROSCOPIC UNITS $m = 1g \quad x' = 1cm \quad t' = 1s$

$\Longrightarrow$ PHASE $S[x_{cl}(t)] \simeq \frac{1}{2} \times 10^{27}$ RADIANS

NCL $\quad S[x_{ncl}(t)] = \int_0^{t'} dt \, \frac{m}{2}\left(\frac{x'}{t'^2}\right)^2 4t^2 = \frac{2m x'^2}{3t'^2} \overset{MAC}{\Longrightarrow} \frac{2}{3} \times 10^{27}$ RAD

PHASES SOMEWHAT CLOSE!

KEY DIFFERENCE x_{cl} PATH OF LEAST ACTION

PROOF $\quad x = x_{LA} + \delta x$

$$L = \frac{1}{2}m(\dot{x})^2 = \frac{1}{2}m(\dot{x}_{LA} + \delta\dot{x})^2 = \frac{1}{2}m\left(\dot{x}_{LA} + 2\dot{x}_{LA}\delta\dot{x}\right) + O(\delta\dot{x}^2)$$

$\Rightarrow \delta L = m\dot{x}_{LA}\delta\dot{x} = 0 \quad \Rightarrow \quad \dot{x}_{LA} = 0 \Rightarrow x_{LA} = At$

$\uparrow$ IF MIN S

$\Rightarrow x_{cl} = \frac{x'}{t'} t$

* WHY DOES THIS MATTER?

* CONSIDER NEARBY PATHS

eg $x = \frac{x'}{t'}\left[t + \varepsilon \frac{t(t-t')}{t'}\right] \simeq x_{cl}$ FOR ε SMALL

$$\Rightarrow S[x] = \int_0^{t'} dt \, \frac{m\dot{x}^2}{2} = \int_0^{t'} dt \, \frac{m}{2}\left(\frac{x'}{t'}\right)^2 \left[1 + \varepsilon \frac{2t - t'}{t'}\right]^2$$

$$= \frac{m}{2}\frac{x'^2}{t'}\left(1 + \frac{\varepsilon^2}{3}\right) = S_{cl}\left(1 + \frac{\varepsilon^2}{3}\right)$$

NB: $\varepsilon^2 > 0 \Rightarrow$ ACTION INCREASES!

$\Rightarrow$ PHASES OF NEARBY PATHS ADD CONSTRUCTIVELY!

OTOH: NON-CLASSICAL

$$x = \frac{x'}{t'^2}\left[t + \varepsilon \frac{t(t-t')}{t'}\right]^2 \Rightarrow S = S[x_{ncl}]\left(1 + \frac{\varepsilon}{2} + \dots\right)$$

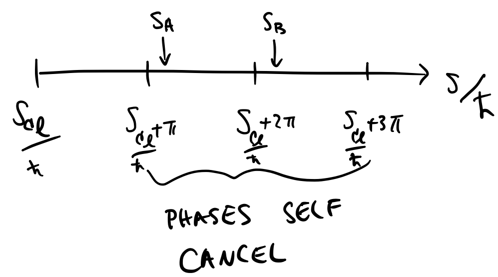
NB: ε AND $-\varepsilon$ PATHS NEAR x_{ncl} INTERFERE DESTRUCTIVELY!

(ie they cancel each other out in the path integral)

(see discussion of "phasons" in book)

REGION OF CONST INT

INCREASE $|\varepsilon| \Rightarrow$ INCREASE $S/\hbar$



$S_A = S[x_A]$

$S_B = S[x_B]$ WITH $\frac{S_A}{\hbar} + \pi = \frac{S_B}{\hbar}$

$$e^{iS_A/\hbar} + e^{iS_B/\hbar} = e^{iS_A/\hbar} - e^{iS_A/\hbar} = 0$$

$\Rightarrow$ IF $S = S_{cl} + \delta S$, CONST. INT. STOPS FOR $\frac{\delta S_{MAX}}{\hbar} \simeq \pi$

CLASSICAL LIMIT:

$\hbar \to 0 \Rightarrow \delta S_{MAX} \to 0 \Rightarrow$ ONLY CLASSICAL PATH CONTRIBUTES!