QM Lecture 12

UNIT SEVEN: QUANTUM HARMONIC OSCILLATOR

(1) SHO has restoring force $\propto$ disp
(2) ie $m\ddot{x} = -kx$ how to get from Hamiltonian?

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$ (take H from classical, put ^'s!)

N.B. $\hat{H} = \hat{H}(\hat{x}, \hat{P}_x) \Rightarrow$ HARD TO SOLVE FOR EITHER $\psi(x)$ OR $\psi(p)$

SOLVE W/ TRICK (rewrite $\hat{H}$!)

$$\boxed{\hat{a} := \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p}_x\right) \Rightarrow \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{P}_x\right)}$$

INVERTING $\hat{a} + \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\, 2\hat{x}$ EG

$$\Rightarrow \boxed{\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^+\right) \\ \hat{P}_x &= -i\sqrt{\frac{m\omega\hbar}{2}} \left(\hat{a} - \hat{a}^+\right) \end{aligned}}$$

$$[\hat{a}, \hat{a}^+] = \frac{m\omega}{2\hbar} \left[\hat{x} + \frac{i}{m\omega} \hat{P}_x,\ \hat{x} - \frac{i}{m\omega} \hat{P}_x\right]$$

$$= \frac{1}{2\hbar}\left([i\hat{p}_x, x] \times 2\right) = \frac{1}{2\hbar}\, i(-i\hbar) \times 2 = \mathbb{1}$$

$$\boxed{[\hat{a}, \hat{a}^+] = 1}$$

REWRITE $\hat{H}$

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = -\frac{m\omega\hbar}{2}\frac{1}{2m}(a-a^+)^2 + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}(a+a^+)^2$$

$$= \frac{+\omega\hbar}{4}\left[-a^2 - a^{+2} + aa^+ + a^+a + a^2 + a^{+2} + aa^+ + a^+a\right]$$

$$\boxed{H = \frac{\hbar\omega}{2}\left[a^+a + aa^+\right] = \hbar\omega\left(a^+a + \frac{1}{2}\right)}$$

(Q: what might it mean to "solve" an $\hat{H}$)

(Q: so how do we do it here?)

FIND E-STATES OF $\hat{N} = a^+a$

$\hat{N}|n\rangle = n|n\rangle$

COOL FACTS $[N, a] = -a$

$[N, a^+] = a^+$

$$\Rightarrow N a^+|n\rangle = (a^+N + a^+)|n\rangle = (a^+n + a)|n\rangle = (n+1)\hat{a}^+|n\rangle$$

$$Na = (n-1)a|n\rangle$$

$\therefore$ $a^+ \longleftrightarrow$ RAISING OP

$a \longleftrightarrow$ LOWERING OP

$\Rightarrow$ $a^+|n\rangle = C_+|n+1\rangle$

$a|n\rangle = C_-|n-1\rangle$

n? $|\psi\rangle := a|n\rangle \Rightarrow 0 \leq \langle\psi|\psi\rangle = \langle n|a^+a|n\rangle = n\langle n|n\rangle$

$\Rightarrow \boxed{n \geq 0}$

$\Rightarrow \; n_{min} = 0$

ENERGY E-STATES

$$\hat{H}|n\rangle = \hbar\omega\left(N+\tfrac{1}{2}\right)|n\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle \Rightarrow \boxed{\begin{array}{l} E_n = \left(n+\tfrac{1}{2}\right)\hbar\omega \\ n = 0,1,2,\ldots \end{array}}$$

C_+? PHYSICAL STATE $\Rightarrow \langle n|n\rangle = 1 \;\; \forall n$

$$a^+|n\rangle = c_+|n+1\rangle \Rightarrow \langle n|a = c_+^*\langle n+1|$$

$$\therefore \langle n|aa^+|n\rangle = |c_+|^2 \langle n+1|n+1\rangle = |C_+|^2$$

$$\overset{\text{(comm.)}}{\shortparallel}$$

$$\langle n|(a^+a+1)|n\rangle = n+1$$

$$\Rightarrow \quad c_+ = \sqrt{n+1} \qquad \boxed{\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

SIM: $\quad c_- = \sqrt{n} \Rightarrow \boxed{\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle}$

MATRIX REP

$$\boxed{\begin{array}{l} \langle n'|\hat{a}^+|n\rangle = \sqrt{n+1}\;\delta_{n',n+1} \\ \langle n'|a|n\rangle = \sqrt{n}\;\delta_{n',n-1} \end{array}}$$

$$a^+ \xrightarrow{|n\rangle} \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & & \end{pmatrix} \qquad a \xrightarrow{|n\rangle} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

(Q: what is dim of these!? A: ∞, unlike rot. ops)

$$(a^+)^n |0\rangle = \sqrt{n!}\, |n\rangle \implies \boxed{|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle}$$

HARM. OSC. WVFN:

WANT $\psi_n(x) \longleftrightarrow |n\rangle$.

START w/ $\psi_0(x) = \langle x|0\rangle$

$$a|0\rangle = 0$$

$$\implies \langle x|a|0\rangle = 0$$

$$= \sqrt{\frac{m\omega}{2\hbar}} \langle x|\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)|0\rangle$$

$$\implies x\langle x|0\rangle = \frac{-\hbar}{m\omega} \frac{\partial}{\partial x} \langle x|0\rangle$$

DIFF EQ! $\implies \langle x|0\rangle = N e^{-m\omega x^2/2\hbar}$

GET N BY NORM, GAUSS INT

$$\implies \boxed{\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar}}$$

$\psi_n(x)$: $\langle x|n\rangle = \frac{1}{\sqrt{n!}}\langle x|(a^+)^n|0\rangle = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{\frac{n}{2}}\langle x|\left(\hat{x}-\frac{i}{m\omega}\hat{p}_x\right)^n|0\rangle$

ASIDE: (not in book)

$$\hat{p}_x|0\rangle = \int dx'\,|x'\rangle\frac{\hbar}{i}\frac{\partial}{\partial x'}\langle x'|\psi\rangle,\qquad \hat{x}|0\rangle = \int dx'\,|x'\rangle\langle x'|\hat{x}|0\rangle$$
$$= \int dx'\,|x'\rangle\, x'\langle x'|0\rangle$$

$$\Rightarrow \left(\hat{x}-\frac{i}{m\omega}\hat{p}_x\right)|0\rangle = \int dx'\,|x'\rangle\left(x-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\langle x'|0\rangle$$

$$\Rightarrow \left(\hat{x}-\frac{i}{m\omega}\hat{p}_x\right)^n|0\rangle = \int dx'\,|x'\rangle\left(x-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)^n\langle x'|0\rangle$$

Convince yourself!

THEN ABOVE $= \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{\frac{n}{2}}\int dx'\,\overbrace{\langle x|x'\rangle}^{\delta(x-x')}\left(x-\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)^n\langle x'|0\rangle$

$$\boxed{\psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{\frac{n}{2}}\left(x-\frac{\hbar}{m\omega}\frac{d}{dx}\right)^n\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-m\omega x^2/2\hbar}}$$

DIRECTLY COMP $\psi_1(x) = C_1\, x\, e^{-m\omega x^2/2\hbar}$ $\qquad C_1 = \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{\frac{1}{4}}$

$\psi_2(x) = C_2\left(\frac{2m\omega}{\hbar}x^2-1\right)e^{-m\omega x^2/2\hbar}$ $\qquad C_2 = \left(\frac{m\omega}{4\pi\hbar}\right)^{\frac{1}{4}}$

$\vdots$

CONCLUSION $\psi_n(x) \sim P_n(x)\, e^{-m\omega x^2/2\hbar}$

- GAUSS. w/ POLYNOMIAL PREFACTOR

($\exists$ a lot of math rel. to these poly.)

ENERGY THE HARD WAY:

$$E_n = \langle \hat{H} \rangle_{|n\rangle} = \frac{\langle p_x^2 \rangle}{2m} + \frac{1}{2} m\omega^2 \langle x^2 \rangle$$

(diff from ~~book~~)

$$\langle x^2 \rangle = \langle n | \hat{x}^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | (a + a^\dagger)^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2} | n \rangle$$

$$= \frac{\hbar}{2m\omega} \langle n | (aa^\dagger + a^\dagger a) | n \rangle = \frac{\hbar}{2m\omega} \left(\sqrt{n+1}\sqrt{n+1} + \sqrt{n}\sqrt{n} \right)$$

$$= \frac{\hbar}{2m\omega} (2n+1)$$

a^2 $a^{\dagger 2}$ TERMS $=0$

$$\langle p_x^2 \rangle = -\frac{m\omega\hbar}{2} \langle n | (a - a^\dagger)^2 | n \rangle = -\frac{m\omega\hbar}{2} \langle n | (-)(aa^\dagger + a^\dagger a) | n \rangle$$

$$= \frac{m\omega\hbar}{2} \left(\sqrt{n+1}\sqrt{n+1} + \sqrt{n} + \sqrt{n} \right) = \frac{m\omega\hbar}{2} (2n+1)$$

$$\Rightarrow E_n = \frac{m\omega\hbar}{2}(2n+1)\frac{1}{2m} + \frac{1}{2} m\omega^2 \frac{\hbar}{2m\omega}(2n+1)$$

$$\boxed{E_n = \hbar \quad (n + \tfrac{1}{2})}$$

WHILE WE'RE AT IT:

$$\langle x \rangle = \langle n | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} = \langle n | (a + a^\dagger) | n \rangle = 0$$

OFF DIAG!

SIM: $\langle p_x \rangle = 0$

ABOVE & THESE $\Rightarrow$

$$\boxed{\begin{aligned} \Delta x &= \sqrt{\frac{\hbar}{m\omega}(n+\tfrac{1}{2})} \\ \Delta p_x &= \sqrt{m\omega\hbar(n+\tfrac{1}{2})} \end{aligned} \qquad \Delta x \Delta p_x = (n+\tfrac{1}{2})\hbar}$$