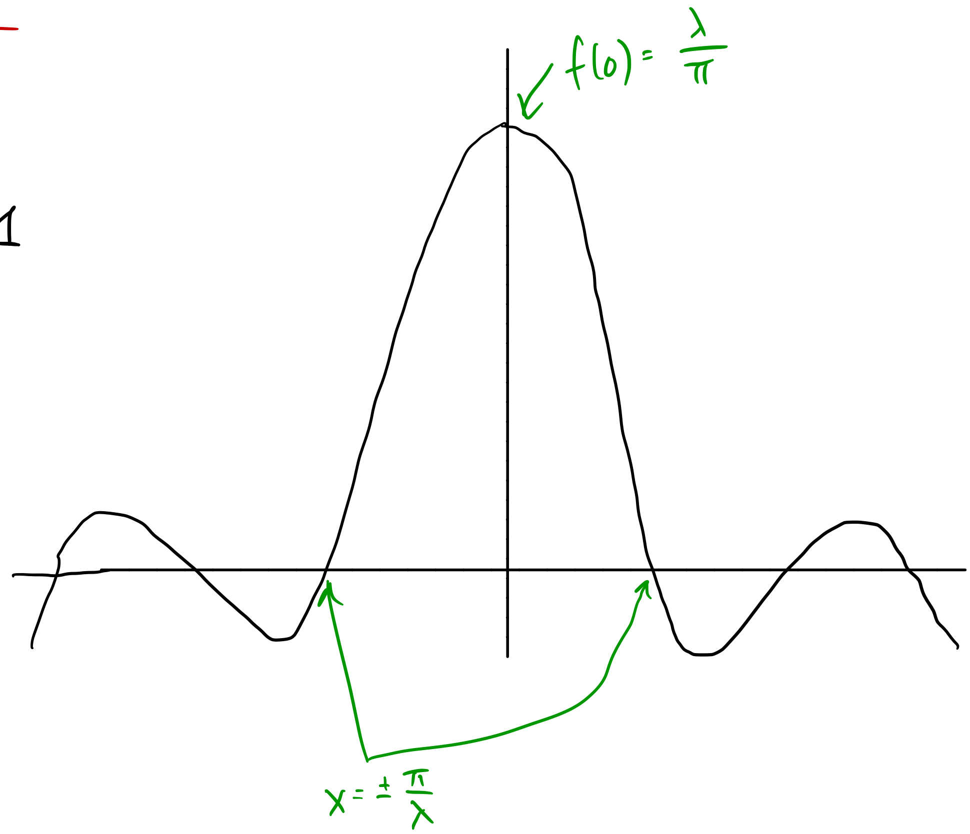
QM Lecture 10

DIRAC δ-FUNCTION

$$f(x) = \frac{\sin(\lambda x)}{\pi x}$$

NORM $\int_{-\infty}^{\infty} f(x)\, dx = 1$

WIDTH $\sim \frac{1}{\lambda}$



$\lambda \to \infty$

LIMIT $\Longrightarrow$ INF TALL & NARROW

REP 1

$$\delta(x) := \lim_{\lambda \to \infty} \frac{\sin \lambda x}{\pi}$$

REP 2

$$\frac{1}{2}\int_{-\lambda}^{\lambda} dk\, e^{ikx} = \frac{\sin \lambda x}{x}$$

$$\Longrightarrow \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}$$

POSITION & MOMENTUM EIGENSTATES

$$\hat{x}|x\rangle = x|x\rangle \qquad \hat{p}|p\rangle = p|p\rangle$$

$$\langle x|x'\rangle = \delta(x'-x) \qquad \langle p'|p\rangle = \delta(p'-p)$$

IDENTITY $\quad \mathbb{1} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x| = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|$

TRANSLATIONS IN SPACE

$$\hat{T}(a)|x\rangle = |x+a\rangle$$

TRANSLATE $|\psi\rangle$

$$|\psi'\rangle = \hat{T}(a)|\psi\rangle = \hat{T}(a)\int dx'\, |x'\rangle\langle x'|\psi\rangle$$
$$= \int dx'\, \hat{T}(a)\, |x'\rangle\langle x'|\psi\rangle = \int dx'\, |x'+a\rangle\langle x'|\psi\rangle$$

$$\Longrightarrow \langle x|\psi'\rangle = \int dx'\, \delta[x-(x'+a)]\, \langle x'|\psi\rangle$$

HOW TO HANDLE? (see APP C)

(1) $\delta(ax) = \frac{1}{|a|}\delta(x)$

(2) $\delta(f(x)) = \dfrac{\delta(x-x_0)}{\left|df/dx\right|_{x=x_0}}$ WHERE $f(x_0)=0$

SO $\delta[x-(x'+a)] = \delta[-x'+(x-a)]$ VAN. @ $x' = x-a$

$\Longrightarrow \langle x|\psi'\rangle = \langle x-a|\psi\rangle$ so "$|\psi'\rangle$@ x" = "$|\psi\rangle$ @ $x-a$"

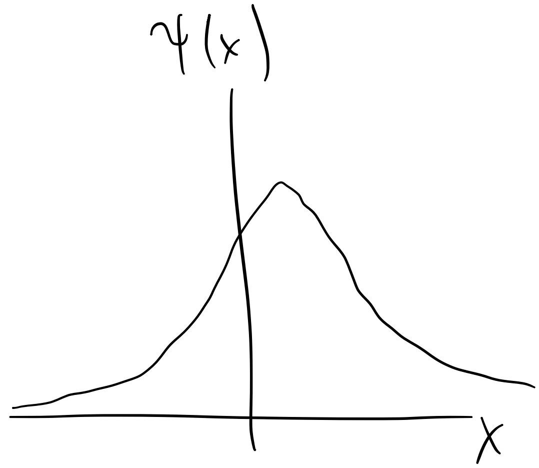
WAVEFUNCTIONS

(prev result somewhat counterintuitive, understand by introduction of function)

$$\boxed{\psi(x) := \langle x|\psi\rangle}$$

Language: "position-space wavefunction", or just "wavefunction"

$$|\psi\rangle \Longrightarrow \psi(x)$$



MATCH MODERN PHYSICS:

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = \langle\psi|\hat{x}\int_{-\infty}^{\infty}dx'\,|x'\rangle\langle x'|\psi\rangle$$

$$= \int_{-\infty}^{\infty}dx'\,\langle\psi|x'|x'\rangle\langle x'|\psi\rangle$$

$$= \int_{-\infty}^{\infty}dx'\;x'\,\langle\psi|x'\rangle\langle x'|\psi\rangle = \int_{-\infty}^{\infty}dx'\;x'\;\psi(x')\psi^*(x')$$

$$= \int_{-\infty}^{\infty}dx'\;x'\;P(x')$$

PREV RESULT $\langle x|\psi'\rangle = \langle x-a|\psi\rangle$

$$\Longrightarrow \quad \psi'(x) = \psi(x-a)$$

GEN OF TRANSLATIONS

$$\hat{T}(\delta x) = \mathbb{1} - \frac{i}{\hbar}\underbrace{\hat{A}}_{\text{GEN OF TRANS}}\delta x \qquad \text{w/} \qquad \hat{T}(\delta x)|x\rangle = |x + \delta x\rangle$$

ANALOGY: $\hat{R}(d\phi\,\hat{k}) = \mathbb{1} - \frac{i}{\hbar}\hat{J}_z\, d\phi \qquad \hat{U}(dt) = \mathbb{1} - \frac{i}{\hbar}\hat{H}\, dt$

NB

1) $T^\dagger T = \mathbb{1} \implies A^\dagger = A$

2) $\hat{T}(a) = \lim_{N\to\infty}\left[\mathbb{1} - \frac{i}{\hbar}\hat{A}\left(\frac{a}{N}\right)\right]^N = e^{-i\hat{A}a/\hbar}$

3) EXPECT NON-COMM. w/ $\hat{x}$

$$\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x} = \frac{-i\delta x}{\hbar}[\hat{x},\hat{A}] + \mathcal{O}(\delta x^2)$$

$$\left(\hat{x}\hat{T}(\delta x) - \hat{T}(\delta x)\hat{x}\right)|\psi\rangle = \text{"} \int dx'|x'\rangle\langle x'|\psi\rangle$$

$$= \hat{x}\int dx'|x'+\delta x\rangle\langle x'|\psi\rangle - \hat{T}(\delta x)\int dx'\, x'|x'\rangle\langle x'|\psi\rangle$$

$$= \int dx'(x'+\delta x)|x'+\delta x\rangle\langle x'|\psi\rangle - \int dx'\, x'|x'+\delta x\rangle\langle x'|\psi\rangle$$

$$= \delta x\int dx'|x'+\delta x\rangle\langle x'|\psi\rangle \underset{\text{TAYLOR}}{=} \delta x\int dx'\left(|x'\rangle + \mathcal{O}(\delta x')\right)\langle x'|\psi\rangle$$

$$= \delta x\int dx'|x'\rangle\langle x'|\psi\rangle + \ldots = \delta x|\psi\rangle + \ldots$$

$$\implies \boxed{[\hat{x},\hat{A}] = i\hbar}$$