QM Lecture 13

HARMONIC OSC.

$$\Delta x \Delta p_x = (n+\tfrac{1}{2})\hbar$$

SO GROUND ($n=0$) HAS $\Delta x \Delta p_x = \frac{\hbar}{2}$, MIN. ALLOWED!

(Q: why is this not surprising?
A: $\psi_0(x) \sim e^{-2x^2/\hbar}$ GAUSSIAN $\Rightarrow$ MIN. $\Delta x \Delta p_x$)

FULL SOLUTION OF ONE-DIM. SHO

(Q: why do this? A: more gen solu tech)

GOAL SOLVE $\langle x|\hat{H}|E\rangle = E\langle x|E\rangle$ (what does "solve" mean?)

$$\langle x|\left(\frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)|E\rangle = E\langle x|E\rangle$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\langle x|E\rangle + \frac{1}{2}m\omega^2x^2\langle x|E\rangle = E\langle x|E\rangle \qquad \text{S.EQN.}$$

SIMP. $y := \sqrt{\frac{m\omega}{\hbar}}\,x \qquad \varepsilon := \frac{2E}{\hbar\omega}$

$$\langle x|E\rangle = \psi(x) = \psi(y) \qquad \frac{d}{dx}\frac{d}{dx} = \left(\frac{dy}{dx}\right)^2\frac{d}{dy}\frac{d}{dy} = \frac{m\omega}{\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2}{dy^2}\psi(x) + \frac{1}{2}m\omega^2\frac{\hbar}{m\omega}y^2\psi(y) = \frac{\hbar\omega\varepsilon}{2}\psi(y)$$

$$\Rightarrow \left(\frac{d^2}{dy^2} + \varepsilon - y^2\right)\psi(y) = 0 \qquad (*)$$

as $y \to \infty$ HAVE $\frac{d^2\psi}{dy^2} - y^2\psi = 0$

$$\Rightarrow \psi(y)\Big|_{y\to\infty} = A\,e^{-y^2/2}$$ (note diff from book)

$\therefore$ ANSATZ ("educated guess")

$$\psi(y) = A(y)\, e^{-y^2/2} \quad (**)$$

$(**)$ INTO $(*)$

$$\frac{d\psi}{dy} = \frac{dA}{dy} e^{-y^2/2} - yA e^{-y^2/2}$$

$$\frac{d^2}{dy^2}\psi = \frac{d^2}{dy^2} A\, e^{-y^2/2} - y \frac{dA}{dy} e^{-y^2/2} - A e^{-y^2/2} - y\frac{dA}{dy} e^{-y^2/2} + \underbrace{y^2 A e^{-y^2/2}}_{\text{CANC. by } -y^2\psi}$$

$$\therefore \boxed{\frac{d^2 A(y)}{dy^2} - 2y\frac{dA(y)}{dy} + (\varepsilon - 1) A(y) = 0}$$

$\rightarrow$ HERMITE'S DIFF EQ.

TRICK $\quad A(y) = \sum_{k=0}^{\infty} a_k y^k$

$$\Rightarrow \underbrace{\sum_{k=0}^{\infty} k(k-1) a_k y^{k-2}}_{k=0,1 \text{ TERMS VANISH!}} - 2\sum_{k=0}^{\infty} k a_k y^k + (\varepsilon - 1)\sum_{k=0}^{\infty} a_k y^k = 0$$

$$\sum_{k=0}^{\infty} k(k-1) a_k y^{k-2} = \sum_{k'=0}^{\infty} (k'+2)(k'+1) a_{k'+2} y^{k'}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2}\, y^k$$

$$\Rightarrow \sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - 2k a_k + (\varepsilon - 1) a_k\right] y_k = 0$$

$$\Rightarrow \frac{a_{k+2}}{a_k} = \frac{2k+1-\varepsilon}{(k+1)(k+2)}$$

- RECURSION REL.
- SPEC. a_0, a_1 GET ALL OTHERS

ASYMPTOTICS

$$\frac{a_{k+2}}{a_k} \xrightarrow[k\to\infty]{} \frac{2k}{k^2} = \frac{2}{k}$$

COMPARE $\quad e^{y^2} = \sum_{n=0}^{\infty} \frac{y^{2n}}{n!} = \sum_{k=0}^{\infty} b_k y^k$

$$\Rightarrow \frac{b_{k+2}}{b_k} = \frac{\left(\frac{k}{2}\right)!}{\left[\left(k/2\right)+1\right]!} = \frac{1}{\frac{k}{2}+1} \xrightarrow{k\to\infty} \frac{2}{k}$$

SO $\quad A(y) \sim e^{-y^2/2}$ @ LARGE k $\quad$ NOT GOOD!

CUTOFF: IF $\exists$ SOME $k=n$ w/ $a_{n+2}=0$ THEN $a_{n+2+i}=0 \;\forall i>0$

SOLVE $\quad 2n+1-\varepsilon=0 \Rightarrow \varepsilon = 2n+1$

$$\Rightarrow \boxed{\frac{a_{k+2}}{a_k} = \frac{2(k-n)}{(k+2)(k+1)}}$$

* SOL. DEP NOW ON a_0, a_1, n.
* BUT n EVEN REQ $a_1=0 \Rightarrow \psi(y)$ EVEN

  n ODD REQ $a_0=0 \Rightarrow \psi(y)$ ODD

EXAMPLES

$n=0|$ $\quad a_0$ CONST $\quad a_1=0 \quad a_2=0 \ldots$

$n=1|$ $\quad a_0=0 \quad a_1=$ CONST $\quad a_2=0\ldots \quad a_3=0\ldots$

$n=2|$ $\quad a_0=\text{const} \quad a_1=0 \quad a_2=-2a_0 \quad a_3=a_4=\ldots=0$

$n=3|$ $\quad a_1=\text{const} \quad a_0=0 \quad a_2=0 \quad a_3=-\frac{2}{3}a_1 \quad a_4=a_5=\ldots=0$

NORMALIZATION FIXES a_0, a_1

$$\therefore \quad \boxed{\psi_n(x) = \langle x | E_n \rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} A_n(y)\, e^{-y^2/2} \qquad y = \sqrt{\frac{m\omega}{\hbar}}\, x}$$

FIRST FEW A's

$A_0(y) = 1$

$A_1(y) = 2y$

$A_2(y) = 4y^2 - 2$

$A_3(y) = 8y^3 - 12y$

$A_4(y) = 16y^4 - 48y^2 + 12$

$A_5 = 32y^5 - 160y^3 + 120y$

LESSONS

1) a_k TERMIN. $\Rightarrow$ ONLY EVEN OR ODD POWERS $\Rightarrow$ ψ EVEN OR ODD

2) THAT REQ PICKING $n \in \mathbb{Z}$ $n \geq 0$

3) THEN $\varepsilon = \frac{2E}{\hbar\omega} = 2n+1 \Rightarrow E = (n+\frac{1}{2})\hbar\omega$ ✓ QUANTIZED

NOTE 2) 3) ALREADY KNEW FROM $a^\dagger a$ TREATMENT.

(was much easier that way. There are many ways to explain a fact!)