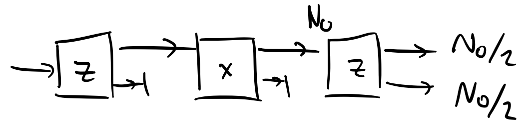
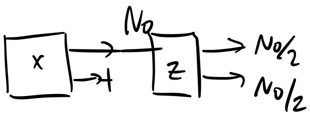
QM Lecture 3

PREVIOUSLY $|x\rangle = \frac{1}{\sqrt{2}}\left(e^{i\delta_+}|{+z}\rangle + e^{i\delta_-}|{-z}\rangle\right)$

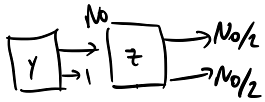


* PHYSICS DOESN'T CARE WHAT YOU CALL AXES!

REP. $\boxed{x}$ BY $\boxed{y}$?



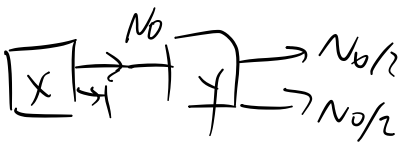
REPLACE $\longmapsto$



STILL 50/50!

$$|{+y}\rangle = \frac{1}{\sqrt{2}}\left(e^{i\gamma_+}|{+z}\rangle + e^{i\gamma_-}|{-z}\rangle\right) = \frac{e^{i\gamma_+}}{\sqrt{2}}\left(|{+z}\rangle + e^{i(\gamma_- - \gamma_+)}|{-z}\rangle\right)$$

SIM.



$$\Rightarrow |\langle +y|x\rangle|^2 = \frac{1}{2}$$

* COMPUTE $\langle +y|+x\rangle = \left(\frac{e^{-i\gamma_+}}{\sqrt{2}}\right)\left(\frac{e^{i\delta_+}}{\sqrt{2}}\right)\left[\langle +z| + e^{-i\gamma}\langle -z|\right]\left[|{+z}\rangle + e^{i\delta}|{-z}\rangle\right]$

$$\delta := \delta_- - \delta_+ \qquad \gamma := \gamma_- - \gamma_+$$

$$\langle +y|+x\rangle = \frac{1}{2}e^{i(\delta_+ - \gamma_+)}\left(1 + e^{i(\delta-\gamma)}\right)$$

$$\Rightarrow |\langle +y|+x\rangle|^2 = \frac{1}{4}\left[1 + e^{i(\delta-\gamma)} + e^{-i(\delta-\gamma)} + 1\right] = \frac{1}{2}\left[1 + \frac{1}{2}\left(e^{i(\delta-\gamma)} + e^{-i(\delta-\gamma)}\right)\right]$$

$$= \frac{1}{2}\left(1 + \cos(\delta-\gamma)\right)$$

$$\therefore \frac{1}{2} = |\langle +y|+x\rangle|^2 = \frac{1}{2}\left(1 + \cos(\delta-\gamma)\right) \Rightarrow \delta - \gamma = \pm\frac{\pi}{2}$$

* NB: (1) CANNOT HAVE SAME PHASES

(2) MUST HAVE AT LEAST ONE COMPLEX COEFF.

EG: CAN TAKE $\delta = 0$. $\Rightarrow |x\rangle = \frac{1}{\sqrt{2}}\left(|{+z}\rangle + |{-z}\rangle\right)$

!!!

$$\Rightarrow \gamma = \pm\frac{\pi}{2} \qquad |{+y}\rangle = \frac{1}{\sqrt{2}}\left(|z\rangle + e^{+i\pi/2}|{-z}\rangle\right) = \frac{1}{\sqrt{2}}\left(|z\rangle + i|{-z}\rangle\right)$$

QUANTUM STATES AS VECTORS

to make a little more obvious, write (for now) $|\psi\rangle$ as col

eg $V=\mathbb{R}^3$ $\vec{A}=A_x\hat{i}+A_y\hat{y}+A_z\hat{z} \longmapsto (A_x, A_y, A_z)$

eg: $|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle \overset{S_z\text{-bas}}{\longmapsto} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$

$$|+z\rangle \overset{S_z\text{-bas}}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-z\rangle \overset{S_z\text{-bas}}{\longmapsto} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

OTHERS $\quad |x\rangle \underset{S_z\text{-bas}}{\longmapsto} \begin{pmatrix} \langle +z|+x\rangle \\ \langle -z|+x\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |y\rangle \underset{S_z\text{-bas}}{\longmapsto} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

BRA STATES: TAKE CONJ. TRANSPOSE

$$\langle\psi| \longmapsto (c_+^*, c_-^*) \quad \therefore \quad \langle +y| \longmapsto \frac{1}{\sqrt{2}}(1, -i)$$

$$\langle +x| = \frac{1}{\sqrt{2}}(1 \;\; 1)$$

CHECK: $\langle +y|+y\rangle = 1 = \frac{1}{\sqrt{2}}(1,-i)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}(1^2 - i^2) = 1$ ✓

Q: WHAT ABOUT $|-x\rangle$? $\qquad$ NB: $\langle x|-x\rangle = 0$

$|+x\rangle \xrightarrow[z\text{-b}]{} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\quad |-x\rangle := \begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \langle x|x\rangle = \frac{1}{\sqrt{2}}(a^* + b^*) = 0$

$$a = -b \Rightarrow |-x\rangle := a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

BUT $\langle -x|-x\rangle = 1 = 2|a|^2 \Rightarrow |a| = \frac{1}{\sqrt{2}}$

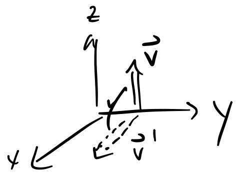
(overall phase isn't fixed, but is unphysical, set to 1)

ROTATIONS & CHANGE OF BASIS

* PHYSICS MUST NOT DEP. ON BASIS!

  (different ways to express the same state)

* ROTATIONS $\hat{R}(\theta \hat{n})$ = ROT BY θ ABOUT $\hat{n}$



$$\Rightarrow \quad \vec{v}\,' = \hat{R}\left(\tfrac{\pi}{2}\,\hat{j}\right)\vec{v}$$

* QM: $|{+x}\rangle = \hat{R}\left(\frac{\pi}{2}\hat{j}\right)|{+z}\rangle$ (see above picture)

* $\hat{R}$ LINEAR OPERATORS IE.

$$\hat{R}(\theta\hat{n})\big(a|\psi\rangle + b|\varphi\rangle\big) = a\big(\hat{R}(\theta\hat{n})|\psi\rangle\big) + b\big(\hat{R}(\theta\hat{n})|\varphi\rangle\big)$$

* Q: Z-BAS. TO X-BAS? $|\psi\rangle = c_+|{+z}\rangle + c_-|{-z}\rangle$

$$\hat{R}\left(\tfrac{\pi}{2}\hat{j}\right)|{\pm z}\rangle = |{\pm x}\rangle \quad \Rightarrow \quad \hat{R}\left(\tfrac{\pi}{2}\hat{j}\right)|\psi\rangle = c_+|{+x}\rangle + c_-|{-x}\rangle$$

||ADJOINT OPERATOR|| TO EACH LIN. OP. $\mathcal{O}: V \to V$
ASSOC. ADJ L.OP. $\mathcal{O}^\dagger: V^* \to V^*$

KEY:

$$|\psi'\rangle = \mathcal{O}|\psi\rangle \implies \langle\psi'| = \langle\psi|\mathcal{O}^\dagger$$

pronounced "O-dagger"

IF: $\mathcal{O}$ a MATRIX, $\mathcal{O}^\dagger = (\mathcal{O}^*)^T$

|UNITARY:| DEF'N ANY OPERATOR $\mathcal{O}$ SATISFYING $\mathcal{O}^\dagger\mathcal{O} = \mathbb{1}$ IS SAID TO BE UNITARY.

PRESERVES PROB: $\langle\psi'|\psi'\rangle = \langle\psi|\mathcal{O}^\dagger\mathcal{O}|\psi\rangle = \langle\psi|\psi\rangle$

E.G CHANGE OF BASIS

IF $|v_i'\rangle = \mathcal{O}|v_i\rangle$, NORM & PROB. PRESERVED

||GENERATOR OF ROTATIONS||

SO(3) special orthog. dim=3
det = 1 $O^T O = \mathbb{1}$ 3x3

IDEA: FOR $\theta \ll 1$, $R \simeq \mathbb{1} - \delta\phi A$

EX $R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ROT ABOUT z-AX

BUT FOR $\theta \ll 1$ WRITE $\theta = \delta\phi$ (some small change)

$\cos\delta\phi \simeq 1 + \ldots \quad \sin\delta\phi \simeq \delta\phi + \ldots$

$$\Rightarrow \quad R = \mathbb{1} - \delta\phi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \mathbb{1} - \delta\phi A$$

SAY: A "GENERATES ROTATION ABOUT z-AX"

QM: (want to rotate in complex QM V.S.)

$\hat{R}(\delta\phi\hat{k}) = \mathbb{1} - i\delta\phi\hat{A}$ & $\hat{R}^\dagger(\delta\phi\hat{k}) = \mathbb{1} + i\delta\phi\hat{A}^\dagger$

(free to choose i factor since def $\hat{A}$)

$\hat{R}$ UNITARY $\Rightarrow$ $\mathbb{1} = \underbrace{(\mathbb{1} + i\delta\phi\hat{A}^\dagger)}_{R^\dagger}\underbrace{(\mathbb{1} - i\delta\phi\hat{A})}_{R} = \mathbb{1} + i\delta\phi(\hat{A}^\dagger - \hat{A})$

$\Rightarrow \hat{A}^\dagger = \hat{A}$

DEF'N: AN OPERATOR Θ SATIS. $\Theta^\dagger = \Theta$ IS HERMITIAN

FINITE ROTATIONS: $\hat{R}(\phi\hat{k}) = \lim_{N\to\infty}\left(1 - i\left(\frac{\phi}{N}\right)\hat{A}\right)^N = e^{-i\phi\hat{A}}$

(repeated application of infinitesimal)

EIGENSTATES (recall from lin. alg.)

<u>DEF'N</u> IF $\vec{v} \neq 0 \in V$ AND $A: V \to V$ LIN. OP.

THEN $\vec{v}$ IS AN <u>EIGENVECTOR</u> OF A IF

$$A\vec{v} = \lambda(A,\vec{v})$$

WITH $\lambda(A,\vec{v})$ A #, THE ASSOC. <u>EIGENVALUE</u>

<u>NB</u>: IF $\mathcal{O} = e^{-iA}$ & $A|\psi\rangle = \lambda(A,|\psi\rangle)|\psi\rangle$

THEN $\mathcal{O}|\psi\rangle = \lambda(\mathcal{O},|\psi\rangle)|\psi\rangle$.

<u>PHYSICS EXAMPLE</u>: $\hat{A}$ CORRESP TO SPIN ANG. MOM

* $\hat{A} = \frac{\hat{J}_z}{\hbar}$ $\quad J_z^\dagger = J_z$ $\quad \hat{R}(\phi\hat{k}) = e^{-i\hat{J}_z\phi/\hbar}$

* <u>E-VAL</u>: $\pm\frac{\hbar}{2}$ FOR $|\pm z\rangle$: $\quad \hat{J}_z|\pm z\rangle = \pm\frac{\hbar}{2}|\pm z\rangle$

$$\therefore \hat{R}(\phi\hat{k})|+z\rangle = \left(1 - \frac{i\phi}{2} + \frac{1}{2!}\left(-\frac{i\phi}{2}\right)^2 + \dots\right)|+z\rangle = e^{-i\phi/2}|+z\rangle$$

$$\text{SIM} \quad \hat{R}(\phi\hat{k})|-z\rangle = e^{i\phi/2}|-z\rangle$$

* <u>CRAZY FACT</u>: $\hat{R}(2\pi\hat{k})|+z\rangle = e^{-i\pi}|z\rangle = -|z\rangle$

  " $\quad |-z\rangle = -|-z\rangle$ verified exp'tally in ch 4!

$\Rightarrow$ ROT. BY <u>4π</u> FOR $|z\rangle \mapsto |z\rangle$

* RELATED TO TOPOLOGY: $\pi_1(SO(3)) = \mathbb{Z}_2$

$SU(2)$ is a <u>double</u> <u>cover</u> of $SO(3)$

phone trick: hold forward, turn right. elbow pos = spin!

ARB. STATES   KNOW $\hat{O}|v_i\rangle$, $\{|v_i\rangle\}$ A BASIS? KNOW $\hat{O}|\psi\rangle$

$$|\psi\rangle = \sum c_i |v_i\rangle \longrightarrow \hat{O}|\psi\rangle = \sum c_i \, \hat{O}|v_i\rangle$$

HERE: $|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle \Rightarrow \hat{R}(\phi\hat{k})|\psi\rangle = c_+ e^{-i\phi/2}|+z\rangle + c_- e^{+i\phi/2}|-z\rangle$

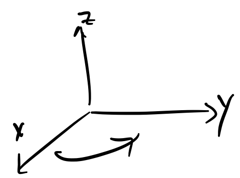
SPECIFIC $|x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)$

$$\mapsto \hat{R}\left(\tfrac{\pi}{2}\hat{k}\right)|+x\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|+z\rangle + e^{i\pi/4}|-z\rangle\right)$$

$$= \frac{e^{-i\pi/4}}{\sqrt{2}}\left(|z\rangle + i|-z\rangle\right)$$

$$= e^{-i\pi/4}\,|+y\rangle$$

INTERP:



x ROT. INTO y!
(up to phase)

MATRIX REPRESENTATIONS

IDEA: REP. OP. BY MATRIX

concrete notions for "adjoint, Hermitian"

CONSIDER $\hat{A}|\psi\rangle = |\varphi\rangle$ (*)

$|\pm z\rangle$ A BAS. $\Rightarrow |\psi\rangle = |+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$

$|\varphi\rangle =$ " " $|\varphi\rangle +$ " " $|\varphi\rangle$

THEN $\hat{A}\left[|+z\rangle\langle +z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle\right] = |+z\rangle\langle +z|\varphi\rangle + |-z\rangle\langle -z|\varphi\rangle$

* ACT W $\langle \pm z| \Rightarrow \langle +z|\hat{A}|z\rangle\langle z|\psi\rangle + \langle z|\hat{A}|-z\rangle\langle -z|\psi\rangle = \langle z|\varphi\rangle$

$\langle -z|\hat{A}|z\rangle\langle z|\psi\rangle + \langle -z|\hat{A}|-z\rangle\langle -z|\psi\rangle = \langle -z|\varphi\rangle$

* EQUIV

$$\begin{pmatrix} \langle z|\hat{A}|z\rangle & \langle z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|z\rangle & \langle -z|\hat{A}|-z\rangle \end{pmatrix} \begin{pmatrix} \langle z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle +z|\varphi\rangle \\ \langle -z|\varphi\rangle \end{pmatrix}$$

(note arb. A encoded in a matrix!)

GEN $\{|v_i\rangle\}$ A BASIS $\underbrace{A_{ij} = \langle v_i|\hat{A}|v_j\rangle}$

DEFN THE MATRIX ELEMENTS OF $\hat{A}$.

||PROJ & IDENT OPS||

$\mathbb{1}$: RECALL $|\psi\rangle = \sum c_i |v_i\rangle$ (O-NORM) BUT $c_i = \langle v_i|\psi\rangle$

SO $|\psi\rangle = \left(\sum_i |v_i\rangle\langle v_i|\right)|\psi\rangle \implies \mathbb{1} = \sum_i |v_i\rangle\langle v_i|$

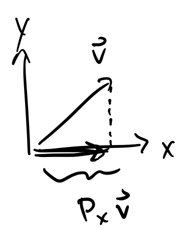
PROJ: WANT P_i ST $P_i|\psi\rangle = c_i|v_i\rangle$

GOOD GUESS: $P_i = |v_i\rangle\langle v_i|$

$P_i|\psi\rangle = |v_i\rangle\langle v_i|\psi\rangle = c_i|\psi\rangle$ ✓

RELATIONSHIP: $\mathbb{1} = \sum_i P_i$

ANALOGY



$\vec{v} = v_x\hat{i} + v_y\hat{j}$

$P_x\vec{v} = v_x\hat{i}$

$P_x = \hat{i}\,(\hat{i}\cdot)$ $\quad P_x\vec{v} = \hat{i}(\hat{i}\cdot(v_x\hat{i} + v_y\hat{j})) = v_x\hat{i}$

||MATRIX REPS OF J_z||

$$\hat{J}_z \xrightarrow[S_z\text{ bas}]{} \begin{pmatrix} \langle z|\hat{J}_z|z\rangle & \langle +z|\hat{J}_z|-z\rangle \\ \langle -z|\hat{J}_z|z\rangle & \langle -z|\hat{J}_z|-z\rangle \end{pmatrix} = \begin{pmatrix} \hbar/2\,\langle z|z\rangle & -\frac{\hbar}{2}\langle +z|-z\rangle \\ \frac{\hbar}{2}\langle -z|z\rangle & -\frac{\hbar}{2}\langle -z|-z\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$$

$$\implies \hat{J}_z \xrightarrow[S_z\text{-bas}]{} \frac{\hbar}{2}\hat{P}_+ - \frac{\hbar}{2}\hat{P}_-$$

(ops. do something to state. Can account for SG?)