QM Lecture 14

LAST TIME   T-INDEP SCH. EQN FOR HARM. OSC.

$\psi_n(x)$ ODD FOR n ODD

$\psi_n(x)$ EVEN FOR n EVEN

PARITY

DEFINE $\hat{\pi}$ S.T. $\hat{\pi}|x\rangle = |-x\rangle$

NOTE $\hat{\pi}^2 = \mathbb{1}$

$\hat{\pi}$ E-STATES

$$\hat{\pi}|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle \Rightarrow \hat{\pi}^2|\psi_\lambda\rangle = \lambda^2|\psi_\lambda\rangle = \mathbb{1}|\psi_\lambda\rangle = |\psi_\lambda\rangle$$

$$\Longrightarrow \lambda = \pm 1$$

E-STATES
$\lambda = 1$ $|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + |-x\rangle\right)$

$\lambda = -1$ $|\psi_{-1}\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - |-x\rangle\right)$

POSITION SPACE
$\psi_1(x) = \langle x|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(\psi(x) + \psi(-x)\right)$

$\psi_{-1}(x) = \langle x|\psi_{-1}\rangle = \frac{1}{\sqrt{2}}\left(\psi(x) - \psi(-x)\right)$

$\psi(x)$ EVEN $\Rightarrow \psi(x) = \sqrt{2}\,\psi_1(x) \Rightarrow \lambda = 1$

$\psi(x)$ ODD $\Rightarrow \psi(x) = \sqrt{2}\,\psi_{-1}(x) \Rightarrow \lambda = -1$

$\Rightarrow$ $\boxed{\text{H.O. E-STATES ARE } \hat{\pi} \text{ (PARITY) E-STATES}}$

HOW? HAPPENS IF $[\hat{\Pi}, \hat{H}] = 0$

$$\langle x | \Pi H | \psi \rangle = \langle -x | \hat{H} | \psi \rangle = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(-x) \right] \psi(x) \qquad (*)$$

$$\langle x | H \Pi | \psi \rangle = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(-x) \qquad (**)$$

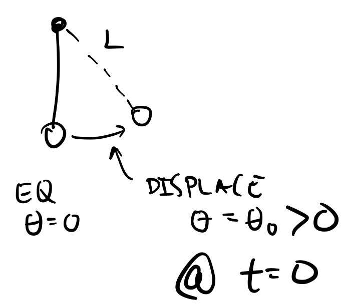
$(*) - (**) = 0$ REQ $V(x) = V(-x)$ — SIGN?

norm in case $x=0$

SATISFIED BY $V(x) = \frac{1}{2} m\omega^2 x^2$!

& SQUARE WELL!

TIME DEP. AND S.H.O.



Q: WHAT CORRESPONDS TO MOTION OF DISPLACED OSC?

- NOT $\langle n | \hat{x} | n \rangle$, WHICH $= 0$. NOR $\langle n | \hat{p}_x | n \rangle = 0$.
- EXPECTED SYMMETRY?

$t = 0$ PROB $P(\theta, t=0)\, d\theta$ SHOULD FAVOR $\theta > 0$

$t = \frac{T}{2} = \pi \sqrt{\frac{L}{g}}$ $P(\theta, t = \frac{T}{2})\, d\theta$ " " $\theta < 0$

NOTE $|\psi_n(x)|^2 dx$ SYMMETRIC ✗

A: A COHERENT STATE, $|\psi_{coh}\rangle = \sum c_n |n\rangle$ — all $\neq 0$

WARM UP: $|\psi\rangle_{t=0} = |\psi(0)\rangle = C_n|n\rangle + C_{n+1}|n+1\rangle$

(very special superposition, but a good start.)

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle \qquad \hat{H} = (\hat{N}+\tfrac{1}{2})\hbar\omega$$

$$= C_n e^{-i(n+\frac{1}{2})\omega t}|n\rangle + C_{n+1}e^{-i(n+\frac{3}{2})\omega t}|n+1\rangle$$

$$= e^{-i(n+\frac{1}{2})\omega t}\left(C_n|n\rangle + C_{n+1}e^{-i\omega t}|n+1\rangle\right)$$

OSCILLATE?

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[C_n^*\langle n| + C_{n+1}^* e^{i\omega t}\langle n+1|\right](\hat{a}+\hat{a}^\dagger)\left[C_n|n\rangle + C_{n+1}e^{-i\omega t}|n+1\rangle\right]$$

⋮

USE a, $a^\dagger$ RELS.

⋮

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(C_n^* C_{n+1}\sqrt{n+1}\,e^{-i\omega t} + C_{n+1}^* C_n\sqrt{n+1}\,e^{i\omega t}\right)$$

$$\frac{A}{2}e^{i\delta} := \sqrt{\frac{\hbar}{2m\omega}}\,C_{n+1}^* C_n\sqrt{n+1}$$

$$\Rightarrow \boxed{\langle x\rangle = \frac{A}{2}e^{-i(\delta+\omega t)} + \frac{A}{2}e^{i(\delta+\omega t)} = A\cos(\omega t+\delta)}$$

NB: $f = \frac{\omega}{2\pi}$, AMP A, PHASE δ. MATCHES CLASSICAL EXPECT!

EX: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

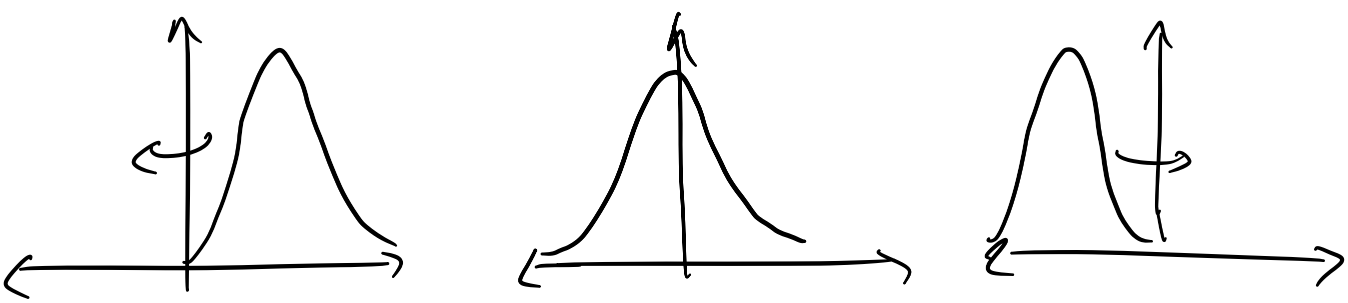
$$|\psi(t)\rangle = \frac{e^{-i\omega t/2}}{\sqrt{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right) \quad *$$

$$\langle x\rangle = A\cos\omega t \quad (\delta = 0)$$

PLOT $|\langle\psi(t)|\psi(t)\rangle|^2 = P(x,t)$ GET SLOSHING

COHERENT STATES

WHAT MOVES LIKE THIS?



WANT: GAUSSIAN THAT MOVES BACK AND FORTH BETWEEN CLASSICAL T.P.'s w/ PER. $T = \frac{2\pi}{\omega}$

$\Longrightarrow$ COHERENT STATE

- COH. ST. ARE $\hat{a}$ E-ST.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad (**)$$

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad (*)$$

$c_n = ?$

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} \sqrt{n}\, c_n |n-1\rangle \overset{!}{=} \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

RELABEL $n' = n-1$, THEN n' BACK TO n

$$\sum_{n=0}^{\infty} \sqrt{n+1}\, c_{n+1} |n\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

$$\Rightarrow \quad c_{n+1} = \frac{\alpha c_n}{\sqrt{n+1}}$$

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \quad \Longrightarrow \quad |\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2 \left[\sum_{n'=0}^{\infty} \frac{(\alpha^*)^{n'}}{\sqrt{n'!}} \langle n'|\right]\left[\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\right]$$

$$= |c_0|^2 \sum_{n'=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha^*)^{n'}}{\sqrt{n'!}} \frac{\alpha^n}{\sqrt{n!}} \delta_{nn'} = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2}$$

$$\Rightarrow \quad c_0 = e^{-|\alpha|^2/2}$$

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

PONDER
1. SUP. OF ALL EN. E-ST.
2. LARGER $|\alpha| \Rightarrow$ LARGE n STATES MATTER MORE

STILL NEED TO SEE:
1) $|\alpha(t)\rangle$ GAUSSIAN $\forall t$
2) "BEHAVES CLASSICALLY"

TIME EV. OF COH. ST.

$$|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i(n+\frac{1}{2})\omega t}|n\rangle$$

$$= e^{-i\omega t/2}\underbrace{e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}|n\rangle}$$

SAME AS $|\alpha\rangle$ EXP. IF REPL. $\underbrace{\alpha \longrightarrow \alpha e^{-i\omega t}}$

CALL $|\alpha e^{-i\omega t}\rangle$

(Q: why might this bother you? why should it not?)

$$\Rightarrow \boxed{|\alpha(t)\rangle = e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle}$$

CLASSICAL BEHAV. WANT TO VERIFY EHRENFEST'S THM.

$$\frac{d\langle p_x\rangle}{dt} = \left\langle -\frac{dV}{dx}\right\rangle \quad w/ \quad \langle p_x\rangle = m\frac{d\langle x\rangle}{dt}$$

$$\Rightarrow \langle x\rangle = \langle\alpha(t)|\hat{x}|\alpha(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha(t)|(\hat{a}+\hat{a}^\dagger)|\alpha(t)\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}[\alpha(t)+\alpha^*(t)] \qquad \text{LET } \alpha = |\alpha|e^{-i\delta}$$

$$\therefore \langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,2|\alpha|\cos(\omega t+\delta)$$

$$\Rightarrow \langle p_x\rangle = \langle\alpha(t)|\hat{p}_x|\alpha(t)\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\langle\alpha(t)|(\hat{a}-\hat{a}^\dagger)|\alpha(t)\rangle$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}[\alpha(t)-\alpha^*(t)]$$

$$\therefore \langle p_x\rangle = -\sqrt{\frac{m\omega\hbar}{2}}\,2|\alpha|\sin(\omega t+\delta)$$

$$\frac{d\langle x\rangle}{dt} = -\omega\sqrt{\frac{\hbar}{2m\omega}}\,2|\alpha|\sin(\omega t+\delta) = \frac{\langle p_x\rangle}{m} \checkmark$$

$$\frac{d\langle p_x\rangle}{dt} = -\omega\sqrt{\frac{m\omega\hbar}{2}}\,2|\alpha|\cos(\omega t+\delta) \qquad \left\langle -\frac{dV}{dx}\right\rangle = -m\omega^2\langle x\rangle$$

$$= -\omega\sqrt{\frac{m\omega\hbar}{2}}\,2|\alpha| \times \cos(\omega t+\delta)$$

=

EHRENFEST'S THM. IS SATISFIED!

GAUSSIAN? MUST HAVE MIN UNC. CHECK.

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle\alpha(t)|(a+a^\dagger)^2|\alpha(t)\rangle = \frac{\hbar}{2m\omega}\langle\alpha(t)|(a^2+a^{\dagger 2}+aa^\dagger+a^\dagger a)|\alpha(t)\rangle$$

$$= \frac{\hbar}{2m\omega}\left\{\alpha(t)^2+\alpha^*(t)^2+2|\alpha(t)|^2+1\right\}$$

BUT $\langle x\rangle^2 = \frac{\hbar}{2m\omega}\left\{\alpha(t)^2+\alpha^*(t)^2+2|\alpha(t)|^2\right\}$

$$\therefore (\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}$$ (why surprising? time indep!)

$$\langle p_x^2\rangle = \frac{m\omega\hbar}{2}\left\{-\alpha(t)^2-\alpha^*(t)^2+2|\alpha(t)|^2+1\right\}$$

$$\langle p_x\rangle^2 = \frac{m\omega\hbar}{2}\left\{-\alpha(t)^2-\alpha^*(t)^2+2|\alpha(t)|^2\right\}$$

$$\Rightarrow (\Delta p_x)^2 = \frac{m\omega\hbar}{2}$$

$\Rightarrow$ $\boxed{\therefore \Delta x\,\Delta p_x = \frac{\hbar}{2}\quad \forall t,\ \text{NO SPREADING!}}$

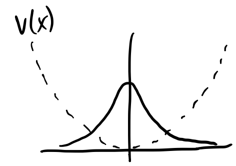
CONCLUSIONS

1. MOVES BACK & FORTH
2. SOME CLASSICAL BEHAVIOR
3. ALWAYS MIN UNC!

DISPLACEMENT & COHERENT STATES

COH ST

GS



$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \xmapsto{\alpha \to 0} \quad |0\rangle$$

$|\alpha\rangle \sim$ DISPLACED $|0\rangle$

Q: DOES $|\alpha\rangle = \hat{T}(d)|0\rangle$ for some d?

A: YES

$$\hat{T}(d) = e^{-ip_x d/\hbar} = e^{\sqrt{\frac{m\omega}{2\hbar}}\, d\,(\hat{a}^+ - \hat{a})} =: e^{\beta \hat{a}^+ + (-\beta^* a)}$$

$$=: e^{A+B} = e^{A} e^{B} e^{-[A,B]/2} = e^{\beta \hat{a}^+} e^{-\beta^* a} \overbrace{e^{+|\beta|^2[a^+,a]/2}}^{e^{-|\beta|^2/2}}$$

so $\hat{T}(d)|0\rangle = e^{-|\beta|^2/2} e^{\beta \hat{a}^+} e^{-\beta^* a} |0\rangle$

$= \quad '' \quad '' \quad \underbrace{\phantom{e^{-\beta^* a}}}_{|0\rangle}$ (b/c only first survives)

$= \quad '' \quad e^{\beta \hat{a}^+} |0\rangle$

$= \quad '' \quad \sum_{n=0}^{\infty} \frac{\beta^n}{n!} (\hat{a}^+)^n |0\rangle$

$= e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |0\rangle = |\beta\rangle$

ie $\hat{T}(d)|0\rangle$ is coh. st. w/ $\alpha = \beta = \sqrt{\frac{m\omega}{2\hbar}}\, d$

$$\boxed{|\alpha\rangle = \hat{T}\left(\sqrt{\frac{2\hbar}{m\omega}}\, \alpha\right)|0\rangle}$$

(point: nice interp)