QM Lecture 7

THM: $\hat{A}, \hat{B}, \hat{C}$ HERM. w $[\hat{A},\hat{B}] = i\hat{C}$. THEN $(\Delta A)(\Delta B) \geq \frac{|\langle C\rangle|}{2}$

PROOF

LEMMA: $|\alpha\rangle, |\beta\rangle$ QUANT. STATES. $\Rightarrow$ SCHWARZ INEQ.

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

PROOF: HMWK PROBLEM

DEFINE $|\alpha\rangle := (\hat{A} - \langle\hat{A}\rangle)|\psi\rangle$ $\qquad |\beta\rangle := \hat{B} - \langle\hat{B}\rangle|\psi\rangle$

$A^\dagger = A,\ B^\dagger = B \implies \langle\alpha|\alpha\rangle = \langle\psi|(\hat{A} - \langle\hat{A}\rangle)^2|\psi\rangle = (\Delta A)^2$

$\langle\beta|(\hat{B} - \langle B\rangle)^2|\beta\rangle = (\Delta B)^2$

$\implies$ LHS THM $= \Delta A\, \Delta B = \sqrt{\langle\alpha|\alpha\rangle}\sqrt{\langle\beta|\beta\rangle}$

SCHWARZ $\implies \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

$$\langle\alpha|\beta\rangle = \langle\psi|\underbrace{(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle B\rangle)}_{=:\hat{\Theta}}|\psi\rangle$$

$\hat{\Theta} = \hat{A}\hat{B} - \langle\hat{A}\rangle\hat{B} - \hat{A}\langle\hat{B}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle$

$\hat{\Theta}^\dagger = \hat{B}^\dagger\hat{A}^\dagger - \langle\hat{A}\rangle^*\hat{B}^\dagger - \hat{A}^\dagger\langle B\rangle^* + \langle A\rangle^*\langle B\rangle^*$

$= \hat{B}\hat{A} - \langle\hat{A}\rangle\hat{B} - \hat{A}\langle\hat{B}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle$

$\implies \hat{\Theta} - \hat{\Theta}^\dagger = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A},\hat{B}] = i\hat{C}$

WRITE $\hat{\Theta} = \frac{\hat{\Theta} + \hat{\Theta}^\dagger}{2} + \frac{\hat{\Theta} - \hat{\Theta}^\dagger}{2} = \frac{\hat{\Theta} + \hat{\Theta}^\dagger}{2} + \frac{i\hat{C}}{2}$

$\therefore |\langle\alpha|\beta\rangle|^2 = \left|\frac{1}{2}\langle\psi|\hat{\Theta} + \hat{\Theta}^\dagger|\psi\rangle + \frac{i}{2}\langle\psi|C|\psi\rangle\right|^2$

NOTE: $\underbrace{\langle\psi|\hat{O}+\hat{O}^{+}|\psi\rangle}_{a:}$ & $\underbrace{\langle\psi|C|\psi\rangle}_{=:b} \in \mathbb{R}$ (Q: WHY?)

$$|\langle\alpha|\beta\rangle|^2 = \left|\frac{a}{2}+i\frac{b}{2}\right|^2 = \left(\frac{a}{2}+i\frac{b}{2}\right)\left(\frac{a}{2}-i\frac{b}{2}\right) = \frac{a^2}{4}+\frac{b^2}{4}$$

$$= \frac{1}{4}|\langle\psi|\hat{O}+\hat{O}^{+}|\psi\rangle|^2 + \frac{1}{4}|\langle\psi|C|\psi\rangle|^2 \geq \frac{1}{4}|\langle\psi|C|\psi\rangle|^2 = \frac{|\langle C\rangle|^2}{4}$$

PUT IT TOGETHER

$$\boxed{\Delta A\, \Delta B \geq \frac{|\langle C\rangle|}{2}}$$

APPLICATION: $|\psi\rangle = |z\rangle \implies \langle J_z\rangle = \langle z|J_z|z\rangle = \frac{\hbar}{2}$

$[J_x, J_y] = i\hbar J_z$

$\implies$ FOR $|\psi\rangle = |z\rangle$ $\boxed{\Delta J_x\, \Delta J_y \geq \frac{|\langle\hbar J_z\rangle|}{2} = \frac{\hbar^2}{4}}$

APPLICATION WILL SEE $[\hat{x}, \hat{p}_x] = i\hbar$

$$\implies \boxed{\Delta x\, \Delta p_x \geq \frac{|\langle\hbar\rangle|}{2} = \frac{\hbar}{2}}$$

$\Downarrow$

HEISENBERG UNCERTAINTY PRINCIPLE

TIME EVOLUTION

* SAW SPATIAL ROTATION OP $\hat{R}(\theta \hat{n})|\psi\rangle =: |\psi'\rangle$

* NOW DYNAMICS, TRANSLATE IN TIME

$$|\psi(t)\rangle = \hat{U}(t)\,|\psi(t=0)\rangle$$

WRITE $|\psi(0)\rangle$ OR $|\psi_0\rangle$ FOR $|\psi(t=0)\rangle$

* NORM 1:

$$\langle\psi(t)|\psi(t)\rangle = 1 = \langle\psi(0)|\,\hat{U}(t)^\dagger\,\hat{U}(t)|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1 \checkmark$$

$\boxed{\hat{U}(t) \text{ UNITARY}}$

* SMALL TIME $t \to dt$

* WRITE W/ GENERATOR $\hat{U}(dt) = \mathbb{1} - \frac{i}{\hbar}\hat{H}\,dt$

GENERATOR

$\boxed{H^\dagger = H} \Rightarrow \hat{U}$ UNITARY

* LANG: - $\vec{J}\cdot\hat{n}$ GENERATES ROTATIONS
  - $\hat{H}$ GENERATES TIME TRANS.

* $t + dt$?

$$\hat{U}(t+dt) = \hat{U}(dt)\,\hat{U}(t) = \left(\mathbb{1} - \frac{i\hat{H}}{\hbar}dt\right)\hat{U}(t)$$

$$\Rightarrow \frac{\hat{U}(t+dt) - \hat{U}(t)}{dt} = -\frac{i\hat{H}}{\hbar}U(t)$$

$$\lim_{dt\to 0} \Rightarrow \boxed{i\hbar\frac{d}{dt}U(t) = \hat{H}\,\hat{U}(t)} \Longrightarrow \text{OP. EQ'N}$$

$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \implies \boxed{i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle}$

SCHRÖDINGER EQUATION

SCH. EQ'N

① WILL SEE $\hat{H}$ = HAMILTONIAN

∴ $\langle \hat{H} \rangle = E$ = ENERGY OF STATE / SYSTEM

in an eigenstate

② IF $\hat{H}$ TIME INDEP

$$\hat{U}(t) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{\hbar}\hat{H}\left(\frac{t}{N}\right)\right)^N = e^{-i\hat{H}t/\hbar}$$

$$\Rightarrow |\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$$

③ E-STATES OF $\hat{H}$ ARE ENERGY EIGENSTATES

$$\hat{H}|E\rangle = E|E\rangle$$

FACT: IF $|\psi(0)\rangle = |E\rangle$, $|\psi(t)\rangle = e^{-iEt/\hbar}|E\rangle$

$\Rightarrow$ TIME EVOLUTION DOESN'T CHANGE $|E\rangle$ EXCEPT BY A PHASE.

$\boxed{|E\rangle \text{ IS A "STATIONARY STATE"}}$

EVOLUTION OF EXPECTATION VALUES (how do observables change?)

$$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\left\{\langle\psi(t)|\hat{A}|\psi(t)\rangle\right\}$$

$$= \left\{\frac{d}{dt}\langle\psi(t)|\right\}\hat{A}|\psi(t)\rangle + \langle\psi(t)|\left\{\frac{d\hat{A}}{dt}\right\}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left\{\frac{d}{dt}|\psi(t)\rangle\right\}$$

$$= \left\{\frac{1}{-i\hbar}\langle\psi(t)|\hat{H}\right\}\hat{A}|\psi(t)\rangle + \langle\psi(t)|\frac{d\hat{A}}{dt}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left\{\frac{1}{i\hbar}\hat{H}|\psi(t)\rangle\right\}$$

$$\Longrightarrow \frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle + \langle\psi(t)|\frac{\partial\hat{A}}{\partial t}|\psi(t)\rangle$$

THEN: IF $\frac{d\hat{A}}{dt}=0$ & $[\hat{H},\hat{A}]=0$

1. $\langle A\rangle$ IS CONSTANT
2. $\hat{H}|E\rangle = E|E\rangle$, $\hat{A}|E\rangle = a|E\rangle$ SIMUL. E-STATES

EXAMPLE: SPIN-$\frac{1}{2}$ IN CONSTANT B-FIELD

* $\vec{B} = B_0\hat{k}$
* $\hat{H} = -\hat{\vec{\mu}}\cdot\vec{B} = -\frac{gq}{2mc}\hat{\vec{S}}\cdot\vec{B} = \frac{ge}{2mc}\hat{S}_z B_0 =: \omega_0\hat{S}_z$ (q: CHARGE; e: ELECTRON CHG.)

* $|E\rangle = ?$

$$\hat{H}|\pm z\rangle = \omega_0\hat{S}_z|\pm z\rangle = \pm\frac{\hbar\omega_0}{2}|\pm z\rangle \Longrightarrow |E\rangle \text{ ARE } |\pm z\rangle,\ \hat{H}|\pm z\rangle = E_\pm|\pm z\rangle$$

$$E_\pm := \pm\frac{\hbar\omega_0}{2}$$

* $\hat{U}$? $\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega_0\hat{S}_z t/\hbar} =: e^{-i\phi\hat{S}_z/\hbar} = R(\phi\hat{k})$

$\hat{U}(t)$ = TIME DEP. ROT GEN, $\phi = \omega_0 t$

$$\hat{U}(t)|z\rangle = e^{-i\omega_0 \hat{S}_z t/\hbar}|z\rangle = e^{-i\omega_0 t/2}|z\rangle \implies |z\rangle \text{ STATIONARY STATE}$$

$\underline{|x\rangle ?}$

$$|\psi(0)\rangle = |x\rangle = \underbrace{\tfrac{1}{\sqrt{2}}|z\rangle}_{\text{ENERGY } E_+} + \underbrace{\tfrac{1}{\sqrt{2}}|-z\rangle}_{\text{ENERGY } E_-} \implies \text{CAN'T ASSIGN ENERGY EIGENVALUE}$$

<u>EVOLVE:</u> $$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\frac{1}{\sqrt{2}}\left(|z\rangle + |-z\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|z\rangle + e^{-iE_-t/\hbar}|-z\rangle\right)$$

$$= \frac{e^{-i\omega_0 t/2}}{\sqrt{2}}|z\rangle + \frac{e^{i\omega_0 t/2}}{\sqrt{2}}|-z\rangle$$

$$\implies |\psi(t)\rangle = \frac{e^{-i\omega_0 t/2}}{\sqrt{2}}\left(|z\rangle + e^{i\omega_0 t}|-z\rangle\right)$$

<u>HOW DO PROBS CHANGE IN</u> TIME?

$$|\langle \pm z|\psi(t)\rangle|^2 = \left|\frac{e^{\mp i\omega_0 t/2}}{\sqrt{2}}\right|^2 = \frac{1}{2} \implies \text{DOESN'T CHANGE}$$

$$\langle x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\langle z| + \langle -z|\right)\frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|z\rangle + e^{i\omega_0 t/2}|-z\rangle\right) = \frac{1}{2}\left(e^{-i\omega_0 t/2} + e^{i\omega_0 t/2}\right)$$

$$= \cos\left(\frac{\omega_0 t}{2}\right) \implies \boxed{|\langle x|\psi(t)\rangle|^2 = \cos^2\left(\frac{\omega_0 t}{2}\right) \implies \text{CHANGES!}}$$

$\Longrightarrow$ PROB CHANGES, SINCE $[\hat{H}, S_x] \propto [\hat{S}_z, \hat{S}_x] \neq 0$

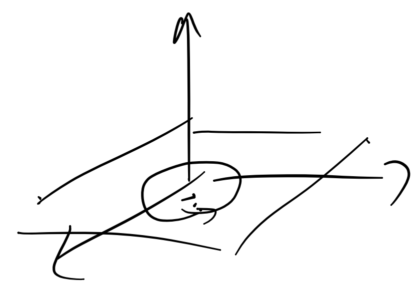
$$\langle -x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\langle z| - \langle -z|\right)\frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t/2}|z\rangle + e^{i\omega_0 t/2}|-z\rangle\right)$$

$$= \frac{1}{2}\left(e^{-i\omega_0 t/2} - e^{+i\omega_0 t/2}\right) = -i\sin\left(\frac{\omega_0 t}{2}\right)$$

$$\Longrightarrow \boxed{|\langle -x|\psi(t)\rangle|^2 = \sin^2\left(\frac{\omega_0 t}{2}\right)}$$

$$\langle S_x\rangle = \frac{\hbar}{2}|\langle x|\psi(t)\rangle|^2 - \frac{\hbar}{2}|\langle -x|\psi(t)\rangle|^2 = \frac{\hbar}{2}\left(\cos^2\frac{\omega_0 t}{2} - \sin^2\frac{\omega_0 t}{2}\right)$$

$$\Longrightarrow \boxed{\langle S_x\rangle = \frac{\hbar}{2}\cos\omega_0 t}$$



<u>OR</u> $\langle S_x\rangle = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle$

$$\Big\Downarrow z$$

$$\frac{1}{\sqrt{2}}\left(e^{i\omega_0 t/2}, e^{-i\omega_0 t/2}\right)\frac{\hbar}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t/2} \\ e^{i\omega_0 t/2}\end{pmatrix}$$

$$= \frac{\hbar}{4}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) = \frac{\hbar}{2}\cos\omega_0 t \checkmark$$

EXAMPLE: MAGNETIC RESONANCE

* ADD OSCILLATING FIELD $\vec{B}$

$$\vec{H} = -\frac{gq}{2mc}\hat{\vec{S}}\cdot\vec{B} = \frac{-gq}{2mc}\hat{\vec{S}}\cdot(B_1\cos\omega t\,\hat{i} + B_0\hat{k})$$

$$\omega_0 := \frac{eg B_0}{2mc} \qquad \omega_1 := \frac{eg B_1}{2mC}$$

$$\Longrightarrow \quad \hat{H} = \omega_0\hat{S}_z + \omega_1\cos(\omega t)\hat{S}_x$$

NB: $\hat{H} = \hat{H}(t) \Rightarrow$ CAN'T USE EXP FORM OF $\hat{U}$

- Q: WHY NOT?
- NB: SCH. EQ REQ ONLY $U(dt) \simeq \mathbb{1} - \frac{i}{\hbar}\hat{H}dt$

SETUP: $|\psi(0)\rangle = |z\rangle \xrightarrow[z]{} \begin{pmatrix}1\\0\end{pmatrix}$

* $|\psi(t)\rangle$? SCHRODINGER EQN

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle \qquad |\psi(t)\rangle \xrightarrow[z]{} \begin{pmatrix}a(t)\\b(t)\end{pmatrix}$$

$$\Big\downarrow z$$

$\cdot :=$ t-DERIV

$$\left\{\frac{\hbar}{2}\begin{pmatrix}\omega_0 & \\ & -\omega_0\end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} & \omega_1\cos\omega t\\ \omega_1\cos\omega t & \end{pmatrix}\right\}\begin{pmatrix}a(t)\\b(t)\end{pmatrix} = i\hbar\begin{pmatrix}\dot{a}(t)\\\dot{b}(t)\end{pmatrix}$$

* MIXING $a, b \Rightarrow$ HARD TO SOLVE IN GEN.

* LIM: $\omega_0 \gg \omega_1$ APPROX DIAG $\quad \hbar\frac{\omega_0}{2}a(t) = i\hbar\frac{da(t)}{dt} \Longrightarrow a(t) = e^{-i\omega_0 t/2}$

SIM. $\quad b(t) = e^{i\omega_0 t/2}$

THESE APPROX. CORRECT

TRY $\begin{pmatrix}a(t)\\b(t)\end{pmatrix} = \begin{pmatrix}c(t)\,e^{-i\omega_0 t/2}\\d(t)\,e^{i\omega_0 t/2}\end{pmatrix}$

$$\Rightarrow \dot{a} = \dot{c}e^{-i\omega_0 t/2} - \frac{i\omega_0}{2} c e^{-i\omega_0 t/2} = \dot{c}e^{-i\omega_0 t/2} - \frac{i\omega}{2} a$$

$\Rightarrow$ SCHROD.

$$\frac{\omega_1}{2}\cos\omega t \begin{pmatrix} d\, e^{i\omega_0 t/2} \\ c\, e^{-i\omega_0 t/2} \end{pmatrix} = i \begin{pmatrix} \dot{c} & e^{-i\omega_0 t/2} \\ \dot{d} & e^{i\omega_0 t/2} \end{pmatrix}$$

$$\Rightarrow i \begin{pmatrix} \dot{c} \\ \dot{d} \end{pmatrix} = \frac{\omega_1}{2}\cos\omega t \begin{pmatrix} d\, e^{i\omega_0 t} \\ c\, e^{-i\omega_0 t} \end{pmatrix} = \frac{\omega_1}{4} \begin{pmatrix} d\left[e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t}\right] \\ c\left[e^{i(\omega-\omega_0)t} + e^{-i(\omega_0+\omega)t}\right] \end{pmatrix} \quad (*)$$

NB. THREE FREQUENCIES

$\omega_0 \longrightarrow$ ASSOC. W/ B_0

$\omega_1 \longrightarrow$ " " B_1

$\omega \longrightarrow$ OSC. FREQ OF B_1

* FOR $\underbrace{\omega \simeq \omega_0}_{\text{RES. FREQ.}}$, $(*)$ HAS $\underbrace{\text{NEARLY CONST TERM}}_{e^{0t}}$ & $\underbrace{\text{RAP. OSC. TERM}}_{e^{2i\omega_0 t}}$.

$$\boxed{i\begin{pmatrix} \dot{c}(t) \\ \dot{d}(t) \end{pmatrix} = \frac{\omega_1}{4}\begin{pmatrix} d(t) \\ c(t) \end{pmatrix}} \Longrightarrow i\begin{pmatrix} \ddot{c} \\ \ddot{d} \end{pmatrix} \simeq \frac{\omega_1}{4}\begin{pmatrix} \dot{d} \\ \dot{c} \end{pmatrix} = \frac{\omega_1}{4i}\,\frac{\omega_1}{4}\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} \ddot{c}(t) \\ \ddot{d}(t) \end{pmatrix} = -\left(\frac{\omega_1}{4}\right)^2 \begin{pmatrix} c \\ d \end{pmatrix}}$$

CHOSEN SO $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \text{FOR } \omega \simeq \omega_0 \gg \omega_1 \quad |\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{\omega_1 t}{4}\right) e^{-i\omega_0 t/2} \\ -i\sin\left(\frac{\omega_1 t}{4}\right) e^{i\omega_0 t/2} \end{pmatrix}$$

$$\Rightarrow |\langle -z|\psi(t)\rangle|^2 = \sin^2\left(\frac{\omega_1 t}{4}\right) \qquad |\langle z|\psi(t)\rangle|^2 = \cos^2\frac{\omega_1 t}{4}$$

$$\Longrightarrow \quad |\langle -z|\psi(t)\rangle|^2 = \sin^2\left(\frac{\omega_1 t}{4}\right) \qquad |\langle z|\psi(t)\rangle|^2 = \cos^2 \frac{\omega_1 t}{4}$$

STATE OSCILLATES BETWEEN $|\pm z\rangle$!

- EACH FLIP REQ. $\Delta E = |E_+ - E_-| = \hbar\omega_0$
  (energy supplied by the B-field)