

# QM      Lecture 3

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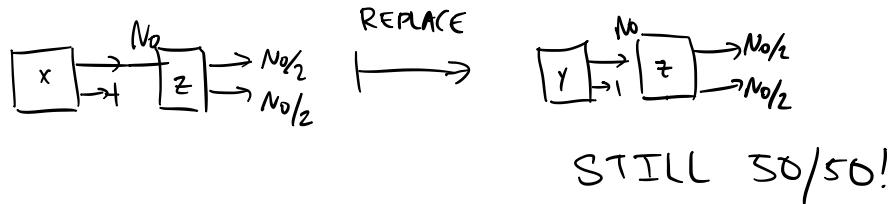
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PREVIOUSLY  $|x\rangle = \frac{1}{\sqrt{2}}(e^{i\delta_+}|+z\rangle + e^{i\delta_-}|-z\rangle)$   $\rightarrow \boxed{z} \rightarrow \boxed{x} \rightarrow \boxed{z} \rightarrow \begin{matrix} N_0 \\ N_0/2 \\ N_0/2 \end{matrix}$

\* PHYSICS DOESN'T CARE WHAT YOU CALL AXES!

REP.  $\boxed{x}$  BY  $\boxed{y}$ ?



$$|+y\rangle = \frac{1}{\sqrt{2}}(e^{i\gamma_+}|+z\rangle + e^{i\gamma_-}|-z\rangle) = \frac{e^{i\gamma_+}}{\sqrt{2}}(|+z\rangle + e^{i(\gamma_- - \gamma_+)}|-z\rangle)$$

SIM.

$\Rightarrow |<+y|x>|^2 = \frac{1}{2}$  EXP \*

\* COMPUTE  $\langle +y|+x\rangle = \left(\frac{e^{-i\gamma_+}}{\sqrt{2}}\right)\left(\frac{e^{i\delta_+}}{\sqrt{2}}\right)[\langle +z| + e^{-i\gamma_-}\langle -z|][(+z) + e^{i\delta_-}(-z)]$

comb terms 0  $\gamma = \delta_- - \delta_+$   $\gamma = \delta_- - \gamma_+$

$$\langle +y|+x\rangle = \frac{1}{2}e^{i(\delta_+ - \gamma_+)}(1 + e^{i(\delta - \gamma)})$$

$$\Rightarrow |\langle +y|+x\rangle|^2 = \frac{1}{4}[1 + e^{i(\delta - \gamma)} + e^{-i(\delta - \gamma)} + 1] = \frac{1}{2}[1 + \frac{1}{2}(e^{i(\delta - \gamma)} + e^{-i(\delta - \gamma)})]$$

matching exp @ ? comp  $= \frac{1}{2}(1 + \cos(\delta - \gamma))$

$$\therefore \frac{1}{2} = |\langle +y|+x\rangle|^2 = \frac{1}{2}(1 + \cos(\delta - \gamma)) \Rightarrow \delta - \gamma = \pm \frac{\pi}{2}$$

\* NB: ① CANNOT HAVE SAME PHASES

② MUST HAVE AT LEAST ONE COMPLEX COEFF.

EG: CAN TAKE  $\delta = 0$   $\Rightarrow |x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$

$$\Rightarrow \gamma = \pm \frac{\pi}{2} \quad |+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + e^{+i\pi/2}(-z)) = \frac{1}{\sqrt{2}}(|z\rangle + i|-z\rangle)$$

!!!

## QUANTUM STATES AS VECTORS

to make a little more obvious, write (for now)  $|q\rangle$  as col

$$\text{eg } V=\mathbb{R}^3 \quad \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \mapsto \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\text{eg: } |q\rangle = C_+ |+z\rangle + C_- |-z\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} \langle +z | q \rangle \\ \langle -z | q \rangle \end{pmatrix}$$

$$\left. \begin{array}{l} |+z\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |-z\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right\} \begin{array}{l} \text{follows from above general} \\ \text{result plus orthonormality} \end{array}$$

$$\text{OTHERS} \quad |x\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} \langle +z | +x \rangle \\ \langle -z | +x \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |y\rangle \xrightarrow{S_z\text{-bas}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

BRA STATES: TAKE CONJ. TRANSPOSE

$$\langle q | \mapsto (C_+, C_-)^* \quad \therefore \quad \langle +y | \mapsto \frac{1}{\sqrt{2}} (1, -i)$$

$$\langle +x | = \frac{1}{\sqrt{2}} (1, 1)$$

$$\text{CHECK: } \langle +y | +y \rangle = \frac{1}{\sqrt{2}} (1, -i) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} (1^2 - i^2) = 1 \checkmark \quad (\text{is what we expect!})$$

Q: WHAT ABOUT  $| -x \rangle$ ?

NB:  $\langle x | -x \rangle = 0$

$$|x\rangle \xrightarrow{z-b} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad | -x \rangle = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \langle x | -x \rangle = \frac{1}{\sqrt{2}} (a^* + b^*) = 0$$

$$\begin{array}{l} \nearrow \\ \text{follows from result} \\ \text{on last page} \end{array} \quad a = -b \Rightarrow | -x \rangle = a \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{BUT } \langle -x | -x \rangle = 1 = 2 |a|^2 \Rightarrow |a| = \frac{1}{\sqrt{2}}$$

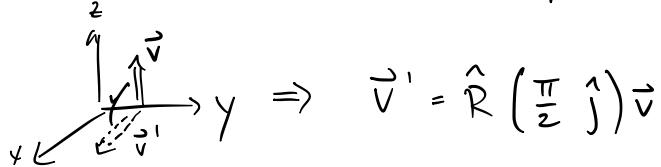
(overall phase isn't fixed,  
but unphysical, set to 1)

# ROTATIONS & CHANGE OF BASIS

\* PHYSICS MUST NOT DEP. ON BASIS!

(different ways to express the same state)

\* ROTATIONS  $\hat{R}(\theta \hat{n})$  = ROT BY  $\theta$  ABOUT  $\hat{n}$



\* QM:  $|+x\rangle = \hat{R}\left(\frac{\pi}{2} \hat{j}\right) |+z\rangle$  (see above picture)

\*  $\hat{R}$  LINEAR OPERATORS IE.

$$\hat{R}(\theta \hat{n})(a|+\rangle + b|-\rangle) = a(\hat{R}(\theta \hat{n})|+\rangle) + b(\hat{R}(\theta \hat{n})|-\rangle)$$

\* Q: Z-BAS. TO X-BAS?  $|+\rangle = c_+|+z\rangle + c_-|-z\rangle$

$$\hat{R}\left(\frac{\pi}{2} \hat{j}\right) |+z\rangle = |+x\rangle \Rightarrow \hat{R}\left(\frac{\pi}{2} \hat{j}\right) |-\rangle = c_+|+x\rangle + c_-|-x\rangle$$

((ADJOINT OPERATOR)) TO EACH LIN. OP.  $\Theta: V \rightarrow V$   
ASSOC. ADJ L.OP.  $\Theta^*: V^* \rightarrow V^*$

KEY:

$$|\psi'\rangle = \Theta|\psi\rangle \Rightarrow \langle\psi'| = \langle\psi|\Theta^*$$

Pronounced "O-dagger"

IF:  $\Theta$  a MATRIX,  $\Theta^* = (\Theta^*)^T$

UNITARY: DEF'N ANY OPERATOR  $\Theta$  SATISFYING  $\Theta^*\Theta = \mathbb{1}$  IS SAID TO BE UNITARY.

PRESERVES PROB:  $|\psi'\rangle = \Theta|\psi\rangle, |\psi'\rangle = \Theta|\psi\rangle$

$$\Rightarrow |\langle\psi'|\psi'\rangle|^2 = |\langle\psi|\Theta^*\Theta|\psi\rangle|^2 = |\langle\psi|\psi\rangle|^2$$

E.G. CHANGE OF BASIS  $|v'_i\rangle = \Theta|v_i\rangle$

(now is prob. preserved, as you would hope!)

## GENERATOR OF ROTATIONS

$SO(3)$  Special  
 $\det = 1$  Orthog.  $\downarrow_{\text{dim}=3}$   
 $O^T O = \mathbb{1}_L$   $3 \times 3$

IDEA: FOR  $\theta \ll 1$ ,  $R \approx \mathbb{1} - \delta\phi A$

EX  $R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$  ROT ABOUT z-AX

BUT FOR  $\theta \ll 1$  WRITE  $\theta = \delta\phi$  (some small change)

$$\cos \delta\phi \approx 1 + \dots \quad \sin \delta\phi \approx \delta\phi + \dots$$

$$\Rightarrow R = \mathbb{1} - \delta\phi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \mathbb{1} - \delta\phi A$$

SAY: A "GENERATES ROTATION ABOUT z-AX"

QM: (want to rotate in complex QM vs.)

$$\hat{R}(S\phi \hat{A}) = \mathbb{1} - iS\phi \hat{A} \quad ? \quad \hat{R}^+(S\phi \hat{A}) = \mathbb{1} + iS\phi \hat{A}^+$$

(free to choose  $i$  factor since def  $\hat{A}$ )

$$\begin{aligned} \hat{R} \text{ UNITARY} \Rightarrow \mathbb{1} &= (\mathbb{1} + iS\phi \hat{A}^+) (\mathbb{1} - iS\phi \hat{A}) = \mathbb{1} + iS\phi (\hat{A}^+ - \hat{A}) \\ &\Rightarrow \hat{A}^+ = \hat{A} \quad R^+ \quad R \end{aligned}$$

DEF'N: AN OPERATOR  $\Theta$  SATIS.  $\Theta^+ = \Theta$  IS HERMITIAN

FINITE ROTATIONS:  $\hat{R}(\phi \hat{A}) = \lim_{N \rightarrow \infty} \left(1 - i\left(\frac{\phi}{N}\right) \hat{A}\right)^N = e^{-i\phi \hat{A}}$

(repeated application of infinitesimal)

# EIGENSTATES (recall from lin. alg.)

DEF'N IF  $\vec{v} \neq 0 \in V$  AND  $A: V \rightarrow V$  LIN. OP.

THEN  $\vec{v}$  IS AN EIGENVECTOR OF A IF

$$A\vec{v} = \lambda(A, \vec{v})$$

WITH  $\lambda(A, \vec{v})$  A #, THE ASSOC. EIGENVALUE

NB: IF  $\Theta = e^{-iA}$  &  $A|4\rangle = \lambda(A, |4\rangle) |4\rangle$

THEN  $\Theta|4\rangle = \lambda(\Theta, |4\rangle) |4\rangle$ .

PHYSICS EXAMPLE: GEN. OF ROTS  $\hat{A}$  CORRESP TO SPIN ANG. MOM

$$* \hat{A} = \frac{\hat{J}_z}{\hbar} \quad J_z^+ = J_z \quad \hat{R}(\phi \hbar) = e^{-i\hat{J}_z \phi / \hbar}$$

$$* |\pm z\rangle E\text{-STATES}: \quad \hat{J}_z |\pm z\rangle = \pm \frac{\hbar}{2} |\pm z\rangle$$

$$\therefore \hat{R}(\phi \hbar) |+z\rangle = \left(1 - \frac{i\phi}{2} + \frac{1}{2!} \left(-\frac{i\phi}{2}\right)^2 + \dots\right) |+z\rangle = e^{-i\phi/2} |+z\rangle$$

$$\text{SIM } \hat{R}(\phi \hbar) |-z\rangle = e^{i\phi/2} |-z\rangle$$

$$* \underline{\text{CRAZY FACT}}: \quad \hat{R}(2\pi \hbar) |+z\rangle = e^{-i\pi} |z\rangle = -|z\rangle$$

$$" \quad |-z\rangle = -|-z\rangle$$

verified  
in  
exp' tally  
ch 4!

$\Rightarrow$  ROT. BY  $\frac{\pi}{2}$  FOR  $|z\rangle \mapsto (-z)$

\* RELATED TO TOPOLOGY:  $\Pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$

phone turn:  
hold forward,  
turn right.

$\mathrm{SU}(2)$  is a double cover of  $\mathrm{SO}(3)$

elbow, pos  
= spin!

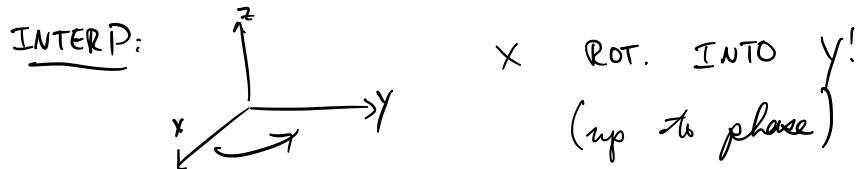
ROTATE ARB STATES (  $\langle \text{NOW} | \psi_i \rangle, |\psi_i\rangle \}$  A BASIS? KNOW  $|\psi\rangle$

$$|\psi\rangle = \sum c_i |\psi_i\rangle \rightarrow |\psi\rangle = \sum c_i |\psi_i\rangle$$

HERE:  $|\psi\rangle = c_+ |+\rangle + c_- |-\rangle \Rightarrow \hat{R}(\phi_k) |\psi\rangle = c_+ e^{-i\phi_k} |+\rangle + c_- e^{i\phi_k} |-\rangle$

SPECIFIC  $|x\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$

$$\begin{aligned} \rightarrow \hat{R}\left(\frac{\pi}{4} \hat{h}\right) |x\rangle &= \frac{1}{\sqrt{2}} \left( e^{-i\frac{\pi}{4}} |+\rangle + e^{i\frac{\pi}{4}} |-\rangle \right) \\ &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} (|+\rangle + i|-\rangle) \\ &= e^{-i\frac{\pi}{4}} |+\rangle \end{aligned}$$



## MATRIX REPRESENTATIONS

IDEA: REP. OP. BY MATRIX

concrete notions for "adjoint, Hermitian"

CONSIDER  $\hat{A} |\psi\rangle = |\psi\rangle \quad (*)$

$|\pm z\rangle$  A BAS.  $\Rightarrow |\psi\rangle = |+\rangle \langle +z|\psi\rangle + |-\rangle \langle -z|\psi\rangle$

$$|\psi\rangle = " " |\psi\rangle + " " |\psi\rangle$$

THEN  $(*) \Rightarrow \hat{A} [|+\rangle \langle +z|\psi\rangle + |-\rangle \langle -z|\psi\rangle] = |+\rangle \langle +z|\psi\rangle + |-\rangle \langle -z|\psi\rangle$

\* ACT w  $\langle \pm z | \Rightarrow \langle +z | \hat{A} |z\rangle \langle z | \psi\rangle + \langle z | \hat{A} | -z\rangle \langle -z | \psi\rangle = \langle z | \psi\rangle$

$$\langle -z | \hat{A} | z\rangle \langle z | \psi\rangle + \langle -z | \hat{A} | -z\rangle \langle -z | \psi\rangle = \langle -z | \psi\rangle$$

\* EQUIV  $\begin{pmatrix} \langle z | \hat{A} | z \rangle & \langle z | \hat{A} | -z \rangle \\ \langle -z | \hat{A} | z \rangle & \langle -z | \hat{A} | -z \rangle \end{pmatrix} \begin{pmatrix} \langle z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix} = \begin{pmatrix} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix}$

(note arb.  $A$  encoded in a matrix!)

$$\underline{\text{GEN}} \quad \{ |v_i\rangle \} \text{ A BASIS} \quad A_{ij} = \langle v_i | \hat{A} | v_j \rangle$$

DEFN THE MATRIX ELEMENTS  
OF  $\hat{A}$ .

### PROJ. & ID OPS

1: RECALL  $|y\rangle = \sum c_i |v_i\rangle$  BUT  $c_i = \langle v_i | y \rangle$

so  $|y\rangle = \left( \sum_i |v_i\rangle \langle v_i| \right) |y\rangle \Rightarrow \underline{I} = \sum_i |v_i\rangle \langle v_i|$

PROJ.: WANT  $P_i$  ST  $P_i |y\rangle = c_i |v_i\rangle$  (*ie project onto one component*)

GOOD GUESS:  $P_i = |v_i\rangle \langle v_i|$

$$P_i |y\rangle = |v_i\rangle \langle v_i | y \rangle = c_i |v_i\rangle \checkmark$$

NOTE:  $\underline{I} = \sum_i P_i$

ANALOGY

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

$$P_x \vec{v} = v_x \hat{i}$$

$$P_x = \hat{i} (\cdot \cdot) \quad P_x \vec{v} = \hat{i} (\cdot \cdot (v_x \hat{i} + v_y \hat{j})) = v_x \hat{i}$$

### MATRIX REPS OF $J_z$

$$\hat{J}_z \xrightarrow{\text{S}_z\text{-bas}} \begin{pmatrix} \langle z | \hat{J}_z | z \rangle & \langle +z | \hat{J}_z | -z \rangle \\ \langle -z | \hat{J}_z | z \rangle & \langle -z | \hat{J}_z | -z \rangle \end{pmatrix} = \begin{pmatrix} \hbar/2 \langle z | z \rangle & -\frac{\hbar}{2} \langle +z | -z \rangle \\ \frac{\hbar}{2} \langle -z | z \rangle & -\frac{\hbar}{2} \langle -z | -z \rangle \end{pmatrix}$$

$$\Rightarrow \hat{J}_z \xrightarrow{\text{S}_z\text{-bas}} \frac{\hbar}{2} \hat{P}_+ - \frac{\hbar}{2} \hat{P}_-$$

=  $\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$

(ops. do something to state.  
Can account for SG?)