QM Lecture 9

TIME EVOLVE:

$$|\Psi(t)\rangle = |1(t)\rangle = (\text{phase})\left(\frac{1}{\sqrt{2}}|I\rangle + \frac{e^{-2i\Delta E t/\hbar}}{\sqrt{2}}|II\rangle\right)$$

Q: HOW MUCH TIME PASSES BEFORE ΔE CHANGES "A LOT"

$\Rightarrow$ DET'D BY RELATIVE PHASE

$$e^{-2i\Delta E t/\hbar} \longrightarrow e^{-2i\Delta E \Delta t/\hbar} =: e^{-i\phi}$$

"A LOT" $:= \{\phi \gtrsim 1\}$ $\Rightarrow$ $\boxed{\Delta E \Delta t \gtrsim \frac{\hbar}{2}}$

NOTE: $\Delta E = 0 \Rightarrow \Delta t \to \infty$, STATIONARY STATE ✓

LOOKS LIKE UNCERTAINTY REL, BUT VAGUE.

PRECISE - LET A, B, C HERM, $[\hat{A},\hat{B}] = iC$ THEN $\Delta A\, \Delta B \geq \frac{|\langle C\rangle|}{2}$

LET $\hat{B} = \hat{H}$ $\Rightarrow C = -i[\hat{A},\hat{H}]$

$$\therefore\ \Delta A\, \Delta E \geq \frac{1}{2}\left|\langle\Psi|[\hat{A},\hat{H}]|\Psi\rangle\right|$$

RECALL $\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\Psi(t)|[H,A]|\Psi(t)\rangle$ if $\frac{dA}{dt} = 0$

$\boxed{\Delta A \Delta E \geq \frac{\hbar}{2}\left|\frac{d\langle A\rangle}{dt}\right|}$ $\Rightarrow$ ΔE REL TO $\frac{d\langle A\rangle}{dt}$

DEFINE $\Delta t := \dfrac{\Delta A}{\left|\frac{d\langle A\rangle}{dt}\right|}$ $\begin{array}{l}\longrightarrow \text{uncertainty} \\ \longrightarrow \text{rate of change}\end{array}$

$\frac{1}{\Delta t}$ IS CHANGE RATE REL TO UNC.

$\Rightarrow$ $\boxed{\Delta E \Delta t \geq \frac{\hbar}{2}}$

UNIT FIVE: ADDITION OF ANGULAR MOMENTUM

Q: WHAT IF YOU HAVE PROTON AND ELECTRON? (both are spin $\frac{1}{2}$)

A: EACH HAVE EITHER SPIN $\uparrow$ OR $\downarrow$ IN Z-DIR

$$|\psi\rangle = |\psi\rangle_{proton} \otimes |\psi\rangle_{electron} =: |\psi_p, \psi_e\rangle$$

$|\pm z\rangle$ BAS. FOR $|\psi\rangle_p$ & $|\psi_e\rangle$

$\Rightarrow$ $|\psi_p, \psi_e\rangle$ BASIS

$$|+z, +z\rangle =: |1\rangle$$
$$|+z, -z\rangle =: |2\rangle$$
$$|-z, +z\rangle =: |3\rangle$$
$$|-z, -z\rangle =: |4\rangle$$

& ORTHONORMAL $\langle m|n\rangle = \delta_{mn}$

"FOUR STATE SYSTEM"

* $\vec{S}_1$ & $\vec{S}_2$, ONE FOR EACH
* S_{1z}, S_{2z} ACT AS YOU'D THINK

$$S_{1z}|z, -z\rangle = \frac{\hbar}{2}|z, -z\rangle \qquad S_2|z, -z\rangle = -\frac{\hbar}{2}|z, -z\rangle$$

SPIN-SPIN INTERACTIONS

CANONICAL EXAMPLE: $\boxed{\hat{H} = \frac{2A}{\hbar^2} \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2}$

EG: HYPERFINE STRUCT. OF ATOM. $\vec{S}_e \cdot \vec{S}_p$

* simplest possible way that spins can interact

$\Rightarrow$ ENERGY DEPENDS ON BOTH SPINS

Q: HOW TO TACKLE THIS?

WHAT COULD YOU LEARN?

A: EIGENSPECTRUM $\Rightarrow$ DYNAMICS VIA SCH. EQN

MATRIX REP. MIGHT HELP

$$\vec{S}_1 \cdot \vec{S}_2 = S_{1x} S_{2x} + S_{1y} S_{2y} + S_{1z} S_{2z}$$

RECALL: $S_x = \frac{S_+ + S_-}{2}$ $\quad S_y = \frac{S_+ - S_-}{2i}$

$$= \left(\frac{S_{1+} + S_{1-}}{2}\right)\left(\frac{S_{2+} + S_{2-}}{2}\right) + \left(\frac{S_{1+} - S_{1-}}{2}\right)\left(\frac{S_{2+} - S_{2-}}{2}\right) + S_{1z} S_{2z}$$

$$= S_{1+} S_{2+}\left(\tfrac{1}{4} - \tfrac{1}{4}\right) + S_{1-} S_{2+}\left(\tfrac{1}{4} + \tfrac{1}{4}\right) + S_{1+} S_{2-}\left(\tfrac{1}{4} + \tfrac{1}{4}\right) + S_{1-} S_{2-}\left(\tfrac{1}{4} - \tfrac{1}{4}\right)$$

$$+ S_{1z} S_{2z}$$

$$\therefore \; 2 \vec{S}_1 \cdot \vec{S}_2 = S_{1+} S_{2-} + S_{1-} S_{2+} + 2 S_{1z} S_{2z}$$

CONVENIENT FORM (why?)

COMPUTE MATRIX REP

$\| \langle I | \hat{H} | J \rangle \|$ IN STEPS

$$\hat{H}|1\rangle = \hat{H}|z,z\rangle = \frac{A}{\hbar^2}\left(S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}\right)|z,z\rangle$$

$$= \frac{A}{\hbar^2}\left(2\frac{\hbar^2}{4}\right)|z,z\rangle = \frac{A}{2}|1\rangle$$

WHY?

$$\Rightarrow \langle J|\hat{H}|1\rangle = \frac{A}{2}\delta_{J1}$$

SIM. $\hat{H}|4\rangle = \frac{A}{\hbar^2}\left(2\left(-\frac{\hbar}{2}\right)^2\right) = \frac{A}{2}$

$$\Rightarrow \langle J|\hat{H}|4\rangle = \frac{A}{2}\delta_{J4}$$

SO FAR $H \xrightarrow{1,2,3,4 \text{ BAS}} \begin{pmatrix} A/2 & 0 & 0 & 0 \\ 0 & & & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & A/2 \end{pmatrix}$

STILL NEED: $\langle 2|\hat{H}|2\rangle$ $\langle 3|\hat{H}|3\rangle$ $\langle 2|\hat{H}|3\rangle$

(why not $\langle 3|\hat{H}|2\rangle$)

$$\langle 2|\hat{H}|2\rangle = \frac{A}{\hbar^2}\langle +z,-z|\cancel{S_{1+}S_{2-}}^{\to 0} + S_{1-}S_{2+} + 2S_{1z}S_{2z}|z,-z\rangle$$

$$= \frac{A}{\hbar^2}\cancel{\langle +z,-z|\text{stuff}|-z,z\rangle}^{\to 0} + 2\frac{A}{\hbar^2}\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right) = -\frac{A}{2}$$

SIM $\langle 3|\hat{H}|3\rangle = -\frac{A}{2}$

$$\langle 3|\hat{H}|2\rangle = \frac{A}{\hbar^2}\langle -z,z|\cancelto{0}{S_{1+}S_{2-}} + S_{1-}S_{2+} + \cancelto{0}{2S_{1z}S_{2z}}|z,-z\rangle$$

$$= \frac{A}{\hbar^2}\langle -z,z|\left[\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\right]^{\frac{1}{2}}\left[\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}+1\right)\right]^{\frac{1}{2}}|-z,z\rangle$$

$$= \frac{A}{\hbar^2}\langle -z,z|(1)(1)\hbar^2|-z,z\rangle = A$$

$$\hat{H} \xrightarrow{|1\rangle-|4\rangle \text{ BAS}} \begin{pmatrix} A/2 & & & \\ & -A/2 & A & \\ & A & -A/2 & \\ & & & A/2 \end{pmatrix}$$

EIGENVALUE PROB.

$$\det\begin{pmatrix} A/2-E & & & \\ & -A/2-E & A & \\ & A & -A/2-E & \\ & & & A/2-E \end{pmatrix} = 0$$

$$= \underbrace{\left(\frac{A}{2}-E\right)^2}\underbrace{\left[\left(\frac{A}{2}+E\right)^2-A^2\right]} = 0$$

$\Rightarrow \boxed{E = \frac{A}{2}}$ $\quad = 0 \Rightarrow E^2 + EA - \frac{3}{4}A^2 = \left(E-\frac{A}{2}\right)\left(E+\frac{3A}{2}\right)$

$\downarrow$

2 TIMES $\quad \Rightarrow \boxed{E = \frac{A}{2} \text{ AGAIN}} \quad \boxed{E = -\frac{3A}{2}}$

$\Rightarrow E = \frac{A}{2}$ 3 TIMES

EIGENVECTORS

$\boxed{E = \frac{A}{2}}$ $\Rightarrow$ $$\underbrace{\begin{pmatrix} -A & A & & \\ A & -A & & \\ & & & \\ & & & \end{pmatrix}}_{(H - E\mathbb{I})\big|_{E=\frac{A}{2}}} \begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \\ \langle 3|\psi\rangle \\ \langle 4|\psi\rangle \end{pmatrix} = 0$$

THREE SOL: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |z, z\rangle$ $\qquad$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |-z, -z\rangle$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|z, -z\rangle + |-z, z\rangle \right)$$

↳ why $\frac{1}{\sqrt{2}}$ here? NORMALIZE

$\boxed{E = \frac{-3A}{2}}$

$$\underbrace{\begin{pmatrix} -A & & & \\ & A & A & \\ & A & A & \\ & & & -A \end{pmatrix}}_{(H - E\mathbb{I})\big|_{E=-\frac{3A}{2}}} \begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \\ \langle 3|\psi\rangle \\ \langle 4|\psi\rangle \end{pmatrix} = 0 \Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|+z, -z\rangle - |-z, z\rangle \right)$$

PHYSICS: $\hat{H}$ INT. SPLITS ENERGY LEVELS (EG. e-p HYP. SPLIT)

$$\Delta E = \frac{A}{2} - \left(\frac{-3A}{2}\right) = 2A = h\nu$$

↳ e-p sys. $\Delta E = 5.9 \times 10^{-6}\,\text{eV}$ $\Rightarrow$ $\boxed{\nu = 1420\text{ MHz} \quad \lambda = 21\text{ cm}}$

NOTE: 21 CM LINE + DOPPLER $\Rightarrow$ GALAXY DYN. $\Rightarrow$ D.M.

IN SUMMARY

3 "TRIPLET STATES"

$$2\vec{S}_1\cdot\vec{S}_2\,|k\rangle = \frac{\hbar^2}{2}|k\rangle \qquad |k\rangle \in \left\{ |z,z\rangle, \frac{1}{\sqrt{2}}\left(|z,-z\rangle + |-z,z\rangle\right), |-z,-z\rangle \right\}$$

$$2\vec{S}_1\cdot\vec{S}_2\left[\frac{1}{\sqrt{2}}\left(|z,-z\rangle - |-z,z\rangle\right)\right] = -\frac{3\hbar^2}{2}\left[\text{same}\right]$$

"SINGLET" STATE

TOTAL ANGULAR MOMENTUM

$\vec{S} := \vec{S}_1 + \vec{S}_2$ "TOTAL ANG. MOM. OP"

WANT $\vec{S}^2|s,m\rangle = s(s+1)\hbar^2|s,m\rangle$
$S_z|s,m\rangle = m\hbar|s,m\rangle$

$$\vec{S}^2 = (\vec{S}_1+\vec{S}_2)^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$S_z = S_{1z} + S_{2z}$$

NOTE $\vec{S}_1^2|I\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2|I\rangle$ – FOR $I=1,2,3,4$ ✓
$\vec{S}_2^2|I\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2|I\rangle$

$\Rightarrow$ $\boxed{\text{E-STATES OF } \vec{S}_1\cdot\vec{S}_2 \text{ ARE } \vec{S}_1^2\ S_z \text{ E-STATES!}}$

$$H = \frac{2A}{\hbar^2}\vec{S}_1\cdot\vec{S}_2$$

$$\vec{S}^2|\text{trip}\rangle = \left(\vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2\right)|\text{trip}\rangle = \left(\frac{1}{2}\left(\frac{1}{2}+1\right) + \frac{1}{2}\left(\frac{1}{2}+1\right) + \frac{1}{2}\right)\hbar^2|\text{trip}\rangle = 2\hbar^2|\text{trip}\rangle$$

$$\vec{S}^2|\text{sing}\rangle = \left(\frac{1}{2}\left(\frac{1}{2}+1\right) + \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{3}{2}\right)\hbar^2|\text{sing}\rangle = 0$$

$\Rightarrow$ $\boxed{|\text{trip}\rangle \text{ is } s=1}$ $\boxed{|\text{sing}\rangle \text{ is } s=0}$

EIGENSTATES OF $\vec{S}^2, S_z$ IN $|s,m\rangle$ NOT.:

$$|1,1\rangle = |z,z\rangle = |1\rangle$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}\Big(|z,-z\rangle + |-z,z\rangle\Big) = \frac{1}{\sqrt{2}}\big(|2\rangle + |3\rangle\big)$$
$$|1,-1\rangle = |-z,-z\rangle = |4\rangle$$

TRIPLET, HIGHER EN.

$$|0,0\rangle = \frac{1}{\sqrt{2}}\Big(|z,-z\rangle - |-z,z\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|2\rangle - |3\rangle\Big)$$

SINGLET, LOWER EN.

Q: $|1,1\rangle$, $|1,-1\rangle$, SPINS ALIGNED.
WHAT ABOUT $|1,0\rangle$, $|0,0\rangle$?

$$|\pm z\rangle = \frac{1}{\sqrt{2}}\Big(|+x\rangle \pm |-x\rangle\Big)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\Big(|z,-z\rangle + |-z,z\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|z\rangle_1|-z\rangle_2 + |-z\rangle_1|z\rangle_2\Big)$$

$$= \frac{1}{\sqrt{2}}\Big[\frac{1}{\sqrt{2}}\big(|x\rangle_1 + |-x\rangle_1\big)\Big]\Big[\frac{1}{\sqrt{2}}\big(|x\rangle_2 - |-x\rangle_2\big)\Big]$$
$$+ \frac{1}{\sqrt{2}}\Big[\frac{1}{\sqrt{2}}\big(|x\rangle_1 - |-x\rangle_1\big)\Big]\Big[\frac{1}{\sqrt{2}}\big(|x\rangle_2 + |-x\rangle_2\big)\Big]$$
$$= \frac{1}{2}\frac{1}{\sqrt{2}}\Big[|x\rangle_1|x\rangle_2 + |-x\rangle_1|x\rangle_2 - |x\rangle_1|-x\rangle_2 - |-x\rangle_1|-x\rangle_2$$
$$+ |x\rangle_1|x\rangle_2 + |x\rangle_1|-x\rangle_2 - |-x\rangle_1|x\rangle_2 - |-x\rangle_1|-x\rangle_2\Big]$$
$$= \frac{1}{\sqrt{2}}\Big(|x\rangle_1|x\rangle_2 - |-x\rangle_1|-x\rangle_2\Big) = \frac{1}{\sqrt{2}}\Big(|x,x\rangle - |-x,-x\rangle\Big)$$

All 5.38

$$|0,0\rangle = \frac{1}{\sqrt{2}}\Big(|-x,x\rangle - |x,-x\rangle\Big)$$

<u>upshot:</u> triplet aligned, w/ middle state
$(m=0)$ in x-dir.

singlet truly anti aligned!