# QM Lecture 16

## LAST TIME

1) 3D NOTATION

2) TWO BODY PROB.

   i) $V(|\vec{r}_1 - \vec{r}_2|)$

   ii) COM, REL VARS

   iii) $H_{rel}$

   iv) ANG MOM

$[\hat{H}, \hat{L}_z] = 0 \qquad [\hat{H}, \hat{\vec{L}}^2] = 0$

$\therefore$ ENERGY EIG. OF $\hat{H}_{rel}$ SIMULT EIG OF $E, l, m$

$$\boxed{\begin{aligned} \hat{H}_{rel}|E,l,m\rangle &= E|E,l,m\rangle \\ \hat{\vec{L}}^2|E,l,m\rangle &= l(l+1)\hbar^2|E,l,m\rangle \\ L_z|E,l,m\rangle &= m\hbar|E,l,m\rangle \end{aligned}}$$

# APP. OF $\vec{L}^2$ IN $H_{rel}$

## NEED IDENTS

LET'S PROVE: $\vec{L}^2 = \vec{r}^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2 + i\hbar\,\vec{r}\cdot\vec{p}$

$$\vec{L} = \vec{r}\times\vec{p} \xrightarrow{\text{comp}} L_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\, x_j p_k \overset{\text{Einstein}}{=:} \varepsilon_{ijk}\, x_j p_k$$

$$\boxed{L_i = \varepsilon_{ijk}\, x_j p_k}$$

$$\therefore\ \vec{L}^2 = \sum_{i=1}^{3} L_i L_i \overset{E}{=} L_i L_i = (\varepsilon_{ijk}\, x_j p_k)(\varepsilon_{imn}\, x_m p_n)$$

$$= x_j p_k x_m p_n \,(\varepsilon_{ijk}\,\varepsilon_{imn})$$

|| FACT

$$= x_j p_k x_m p_n \,(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})$$

$$= \underbrace{x_j p_k x_j p_k}_{\substack{= x_j (x_j p_k - i\hbar\,\delta_{jk}) p_k \\ = \vec{r}^2\vec{p}^2 - i\hbar\,\vec{r}\cdot\vec{p}}} - \underbrace{x_j p_k x_k p_j}$$

$$= x_j p_k (i\hbar\,\delta_{jk} + p_j x_k) = i\hbar\,\vec{r}\cdot\vec{p} + \overbrace{x_j p_k p_j x_k}^{= x_j p_j p_k x_k}$$

$$= i\hbar\,\vec{r}\cdot\vec{p} + \vec{r}\cdot\vec{p}\,(-i\hbar\,\delta_{kk} + x_k p_k)$$

$$= i\hbar\,\vec{r}\cdot\vec{p} + \vec{r}\cdot\vec{p}\,(-i\hbar\cdot 3 + x_k p_k) = -2i\hbar\,\vec{r}\cdot\vec{p} + (\vec{r}\cdot\vec{p})^2$$

$$\Longrightarrow\ \boxed{\vec{L}^2 = \vec{r}^2\vec{p}^2 + i\hbar\,\vec{r}\cdot\vec{p} + (\vec{r}\cdot\vec{p})^2}$$

SO WHAT? 3 PIECES TO $\langle\vec{r}|\vec{L}^2|\psi\rangle$

① $\langle\vec{r}|\vec{r}^2\vec{p}^2|\psi\rangle = r^2\langle\vec{r}|\vec{p}^2|\psi\rangle = r^2 \times$ kinetic

② $i\hbar\langle\vec{r}|\vec{r}\cdot\vec{p}|\psi\rangle = i\hbar\,\vec{r}\cdot\frac{\hbar}{i}\vec{\nabla}\langle\vec{r}|\psi\rangle = \hbar^2 r\frac{\partial}{\partial r}\langle\vec{r}|\psi\rangle$

③ $-\langle\vec{r}|(\vec{r}\cdot\vec{p})^2|\psi\rangle = \hbar^2\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right)\langle\vec{r}|\psi\rangle = \hbar^2\left[r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r}\right]\langle\vec{r}|\psi\rangle$

$$\therefore\ \langle\vec{r}|\vec{p}^2|\psi\rangle = \frac{1}{r^2}\left\{\langle\vec{r}|\vec{L}^2|\psi\rangle - \hbar^2\left[2r\frac{\partial}{\partial r}+r^2\frac{\partial^2}{\partial r^2}\right]\right\}\langle\vec{r}|\psi\rangle$$

$$\boxed{\therefore\ \langle\vec{r}|H_{rel}|\psi\rangle = E\langle\vec{r}|\psi\rangle = \frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\langle\vec{r}|\psi\rangle + \frac{\langle\vec{r}|\vec{L}^2|\psi\rangle}{2\mu r^2} + V(r)\langle\vec{r}|\psi\rangle}$$

(*)

# SEP. OF VAR. & CENTRAL POT.

- DEP. ON $r$, NOT $\theta, \phi$. (though $\psi$ has $\theta, \varphi$ dep)
- DEP ON $\vec{L}^2$, NOT $\hat{L}_z$. (not invariant, can't dep on "z-dir")

TAKE $|\psi\rangle = |E, \ell, m\rangle$

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2} + V(r)\right]\langle \vec{r}|E, \ell, m\rangle = E\langle \vec{r}|E, \ell, m\rangle$$

TRY SEPARABILITY

$$\Psi(\vec{r}) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$
$$=: R(r)\,Y_{\ell m}(\theta, \phi)$$

START w/ $\Phi(\phi)$

$$\hat{R}(d\phi\,\hat{k})|r, \theta, \phi\rangle = |r, \theta, \phi + d\phi\rangle$$

BRA: $\langle r, \theta, \phi|\,\hat{R}(d\phi\,\hat{k}) = \langle r, \theta, \phi - d\phi|$

$$\therefore \langle r, \theta, \phi|\hat{R}(d\phi\,\hat{k})|\psi\rangle = \langle r, \theta, \phi|\left(1 - \frac{i}{\hbar}\hat{L}_z d\phi\right)|\psi\rangle$$
$$= \langle r, \theta, \phi - d\phi|\psi\rangle = \langle r, \theta, \phi|\psi\rangle - d\phi\frac{\partial}{\partial\phi}\langle r, \theta, \phi|\psi\rangle$$

$$\Rightarrow \boxed{L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}}$$ ON SPHER. COORD. $\psi(\vec{r})$

BUT $\quad L_z |E, l, m\rangle = m\hbar\, |E, l, m\rangle$

$$\Rightarrow \langle r, \theta, \phi | \hat{L}_z | E, l, m\rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle r, \theta, \phi | E, l, m\rangle = m\hbar \langle r, \theta, \phi | E l m\rangle$$

$$\Rightarrow \Phi(\phi) = e^{im\phi}$$

Q: MAKE SENSE

A: NEED $\quad e^{im(\phi)} = e^{im(\phi + 2\pi)}$

$\Rightarrow$ ORB. ANG MOM. HAS $\quad m \in \mathbb{Z} \quad$ (not $\mathbb{Z}/2$)

## NOW $\Theta(\theta)$:

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} \xrightarrow{\text{POS}} \vec{r} \times \frac{\hbar}{i} \vec{\nabla} = r\hat{e}_r \times \frac{\hbar}{i}\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

$$= \frac{\hbar}{i}\left(\hat{e}_\phi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial \phi}\right)$$

CARTESIAN USEFUL FOR EIG

$$\hat{L}_x = \frac{\hbar}{i}\left(-\sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi}\right)$$

$$\hat{L}_y = \frac{\hbar}{i}\left(\cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi}\right)$$

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

TWO SOL'N METHODS

1. $\vec{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ & S. EQN!
2. RAISE & LOWER OPS (easier, first order PDE)

## METH. 2

$$L_\pm = L_x \pm i L_y$$

$$\exists \quad |l,l\rangle \quad \text{S.T.} \quad \hat{L}_+|l,l\rangle = 0$$

$$L_+|l,l\rangle = \frac{\hbar}{i}\left\{\left(-\sin\phi + i\cos\phi\right)\frac{\partial}{\partial\theta} - \cot\theta\left(\cos\phi + i\sin\phi\right)\frac{\partial}{\partial\phi}\right\}|l,l\rangle = 0$$

$$\Rightarrow \quad \frac{\hbar}{i} e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)|l,l\rangle = 0$$

BUT $|l,l\rangle = |E,l,l\rangle$ $\quad \phi$ DEP. IS $e^{il\phi}$

$$\Rightarrow \quad \frac{\hbar}{i} e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta \; il\right)\langle r,\theta,\phi|l,l\rangle = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial\theta} - l\cot\theta\right)\langle r,\theta,\phi|l,l\rangle = 0$$

$$\Rightarrow Y_{ll}(\theta,\phi) = \Phi_{ll}(\phi)\,\Theta_{ll}(\theta) = c_l e^{il\phi}\sin^l\theta$$

next ½ page about norm. only do if time permits

NORM.

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; |c_l|^2 \sin\theta \sin^{2l}\theta$$

FACT $\quad \displaystyle\int_0^{\pi} d\theta \sin^p\theta = \sqrt{\pi}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}$

$\Gamma(z)$ = Euler Gamma Function

- FOR $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$
- IDENTITIES $\quad \Gamma(n+1) = n\Gamma(n)$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{2n!}{4^n n!}\sqrt{\pi}$$

$$\Gamma(n)\Gamma\left(n+\tfrac{1}{2}\right) = 2^{1-2n}\sqrt{\pi}\;\Gamma(2n)$$

ABOVE $\Longrightarrow$ $$|c_\ell|^2 = \frac{1}{2\pi}\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{2\ell+1}{2}+1\right)}{\Gamma\left(\frac{2\ell+1+1}{2}\right)} = \frac{1}{2\pi}\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\ell+\frac{3}{2}\right)}{\Gamma(\ell+1)}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{\pi}}\frac{2\cdot 2^{-2(\ell+1)}\sqrt{\pi}\,\Gamma(2\ell+2)}{\Gamma(\ell+1)\,\Gamma(\ell+1)}$$

$$= \frac{1}{4\pi}\frac{(2\ell+1)!}{(\ell!)^2}\frac{1}{(2^\ell)^2}$$

$$\Longrightarrow \quad Y_{\ell,\ell}(\theta,\phi) = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{2\ell+1}{4\pi}}\, e^{i\ell\phi}\sin^\ell\theta$$

phase inserted for reasons that will become clear shortly

## $m<\ell$ CASES? USE $L_-$

$$\hat{L}_-|\ell,m\rangle = \sqrt{\ell(\ell+1) - m(m-1)}\;\hbar\,|\ell,m-1\rangle$$

$$\hat{L}_- = \frac{\hbar}{i}e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\Longrightarrow \quad \hat{L}_-|\ell,\ell\rangle = \frac{\hbar}{i}e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)c_\ell\, e^{i\ell\phi}\sin^\ell\theta$$

$$= -\hbar e^{-i\phi}c_\ell e^{i\ell\phi}\sin^{\ell-1}\theta\cos\theta - \frac{\hbar}{i}e^{-i\phi}\cot\theta\sin^\ell\theta\, c_\ell(i\ell)e^{i\ell\phi}$$

$$= -\hbar\ell e^{-i\phi}c_\ell e^{i\ell\phi}\left(\sin^{\ell-1}\theta\cos\theta + \frac{\cos\theta}{\sin\theta}\sin^\ell\theta\right)$$

$$= -2\hbar(\ell c_\ell)\sin^{\ell-1}\theta\cos\theta\, e^{i(\ell-1)\phi}$$

$$= \sqrt{\ell(\ell+1)-\ell(\ell-1)}\;\hbar\,|\ell,\ell-1\rangle = \sqrt{2\ell}\,\hbar\,|\ell,-\ell\rangle$$

$$\Longrightarrow \quad Y_{\ell,\ell-1} = -\frac{2\ell}{\sqrt{2\ell}}c_\ell\sin^{\ell-1}\theta\cos\theta\, e^{i(\ell-1)\phi}$$

LOWERING:

1. ONE LESS $\sin\theta$
2. ONE MORE $\cos\theta$
3. ONE MORE $-1$

prob 4.18

IN GENERAL

$$\boxed{Y_{\ell m}(\theta,\phi) = \frac{(-1)^{\ell}}{2^{\ell}\ell!}\sqrt{\frac{(2\ell+1)(\ell+m)!}{4\pi(\ell-m)!}}\, e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{\ell-m}}{d(\cos)^{\ell-m}}\sin^{2\ell}\theta}$$

AGREES WITH $|\ell,\ell\rangle$ & $|\ell,\ell-1\rangle$ CASES.

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{20} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\, e^{\pm i\phi}\sin\theta \qquad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\, e^{\pm i\phi}\sin\theta\cos\theta$$

$$Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\, e^{\pm 2i\phi}\sin^2\theta \qquad Y_{\ell,0} = \pm\sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

WHERE $P_\ell$ a LEGENDRE POLY.