QM Lecture 11

POSITION SPACE & MOMENTUM SPACE.

Q: $[x, p_x] \neq 0 \Rightarrow |x\rangle$ NOT p_x- EIG STATE.

WHAT IS $\hat{p}_x$ IN X-BAS.?

HOW DOES $\hat{p}_x$ ACT ON $\psi(x)$?

CONSIDER: $\hat{T}(\delta x)|\psi\rangle \xrightarrow{|x\rangle\text{-BAS.}} \hat{T}(\delta x)\int dx\, |x\rangle\langle x|\psi\rangle$

$$= \int dx\, |x+\delta x\rangle\langle x|\psi\rangle$$

$x' = x+\delta x$

$$= \int dx'\, |x'\rangle \underbrace{\langle x'-\delta x|\psi\rangle}$$

$$\psi(x'-\delta x) = \underbrace{\psi(x') - \delta x \frac{\partial}{\partial x'}\psi(x')}$$

$$= \int dx' |x'\rangle\left(\langle x'|\psi\rangle - \delta x \frac{\partial}{\partial x'}\langle x'|\psi\rangle\right)$$

$$= \mathbb{1} - \int dx'\, |x'\rangle\, \delta x \frac{\partial}{\partial x'} \langle x'|\psi\rangle$$

$\hat{T}(\delta x) = \mathbb{1} - \frac{i}{\hbar}\delta x\, \hat{p}_x$

$$= \left(\mathbb{1} - \frac{i}{\hbar}\delta x\, \hat{p}_x\right)|\psi\rangle$$

$$\Rightarrow \hat{p}_x|\psi\rangle = \frac{\hbar}{i}\int dx'\, |x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle$$

MATRIX REP? NEED $\langle x|$

$$\langle x|\hat{P}_x|\psi\rangle = \frac{\hbar}{i}\int dx' \langle x|x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle = \frac{\hbar}{i}\int dx'\, \delta(x-x')\frac{\partial}{\partial x}\langle x'|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x) \Rightarrow \boxed{\hat{p}_x \xrightarrow{\hat{x}\text{-bas}} \frac{\hbar}{i}\frac{\partial}{\partial x}}$$

Q: what does this mean?

$|\varphi\rangle = \hat{p}_x|\psi\rangle$ $\quad \varphi(x) = \langle x|\varphi\rangle = \langle x|p_x|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x)$

COROLLARY 1 $\quad \langle\hat{p}_x\rangle = \langle\psi|\hat{p}_x|\psi\rangle = \int dx' \langle\psi|x'\rangle \frac{\hbar}{i}\frac{\partial}{\partial x'}\langle x'|\psi\rangle$

$$\Rightarrow \boxed{\langle\hat{p}_x\rangle = \int dx'\, \psi^*(x')\, \frac{\hbar}{i}\frac{\partial}{\partial x'}\psi(x')}$$

COROLLARY 2:

$$\boxed{\langle x|\hat{p}_x|x'\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|x'\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\delta(x-x')}$$

RHS INTERP!?

$$\int_{-\infty}^{\infty} dx\, f(x)\frac{\partial}{\partial x}\delta(x-x') \overset{IBP}{=} f(x)\,\delta(x-x')\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, \frac{df(x)}{dx}\delta(x-x')$$

if $f(\pm\infty)=0$ $\quad$ $RHS = -\left.\frac{df(x)}{dx}\right|_{x=x'}$

MOMENTUM SPACE

$\hat{p}_x$ AWK. IN X-BASIS

$\exists$ NATURAL BASIS. (Q: what is it?)

$$\hat{p}_x|p\rangle = p|p\rangle$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|\psi\rangle \qquad w/ \quad \langle p'|p\rangle = \delta(p'-p)$$

DEF'N: $\psi(p) = \langle p|\psi\rangle$ IS THE MOMENTUM SPACE WAVEFUNCTION

$\rightsquigarrow$ $dp\,|\psi(p)|^2$ = PROB $|\psi\rangle$ HAS MOM. BETWEEN p & $p+dp$

CHANGE OF BASIS (pos. space to momentum space)

ANALOG OF $\langle +z | +x \rangle$? $\Longleftrightarrow$ $\langle x | p \rangle$

(spin, diff x)

$$\langle x | \hat{p}_x | p \rangle = p \langle x | p \rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | p \rangle$$

DIFF. EQ. $\Longrightarrow$ $\langle x | p \rangle = N e^{ipx/\hbar}$

PIN. DOWN. N.

$$\langle p' | p \rangle = \int_{-\infty}^{\infty} dx \, \langle p' | x \rangle \langle x | p \rangle = N N^* \int_{-\infty}^{\infty} dx \, e^{i(p'-p)x/\hbar}$$

$$= N^* N \, (2\pi\hbar) \, \delta(p'-p) \Longrightarrow N = N^* = \sqrt{\frac{1}{2\pi\hbar}}$$

$$\boxed{\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}}$$

$$\boxed{\begin{aligned} \therefore \quad \psi(p) &= \langle p | \psi \rangle = \int dx \, \langle p | x \rangle \langle x | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \, e^{-ipx/\hbar} \psi(x) \\ \psi(x) &= \langle x | \psi \rangle = \int dp \, \langle x | p \rangle \langle p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \, e^{ipx/\hbar} \psi(p) \end{aligned}}$$

$$\psi(x) \xleftrightarrow{\text{Fourier series}} \psi(p)$$

WAVE PACKETS

Q: WHAT WAVE-FNS REP A PARTICLE?

NOT: $|\psi\rangle \xrightarrow{x} \psi(x) = e^{+ipx/\hbar}$

$$\underbrace{p_x|\psi\rangle \longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x) = p\,\psi(x)}$$

$p_x|\psi\rangle = p|\psi\rangle$, so $\psi(x) = e^{ipx/\hbar}$

p_x - eig!

$\Longrightarrow \Delta p_x = 0 \Rightarrow \Delta x = \infty$

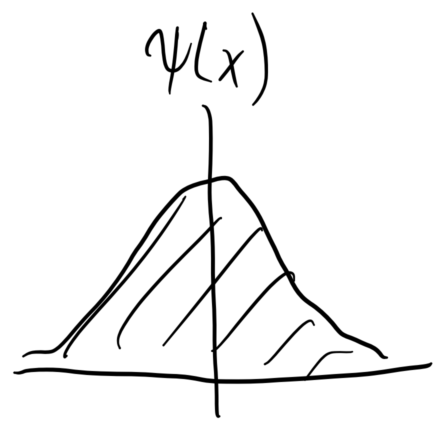
NOT A PARTICLE!

INSTEAD: GAUSSIAN WAVE-FN $\langle x|\psi\rangle = \psi(x) = N e^{-x^2/2a^2}$

NORM $1 = \int dx\, \psi^*(x)\psi(x) = |N|^2 \int_{-\infty}^{\infty} dx\, e^{-x^2/a^2}$

$\Longrightarrow N = \dfrac{1}{a^{1/2}\pi^{1/4}}$

$$\Longrightarrow \boxed{\psi(x) = \frac{1}{a^{1/2}\pi^{1/4}} e^{-x^2/2a^2}}$$



MORE LOCALIZED AS $a \longrightarrow 0$

* "SPREAD?" COMPUTE Δx

$$\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle = \int_{-\infty}^{\infty} dx \, \langle \psi | \hat{x} | x \rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} dx \quad x \, |\langle x | \psi \rangle|^2$$

$$= N^2 \int_{-\infty}^{\infty} dx \quad x \, e^{-x^2/a^2} = 0$$

$$\langle \hat{x}^2 \rangle = \langle \psi | \hat{x}^2 | \psi \rangle = \int_{-\infty}^{\infty} dx \; x^2 \, e^{-x^2/a^2} = N^2 \frac{1}{2} \sqrt{\pi} \, a^3 = \frac{a^2}{2}$$

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} = \frac{a}{\sqrt{2}}$$

* MOM. SPACE WAVE-FN?

$$\psi(p) = \langle p | \psi \rangle = \int_{-\infty}^{\infty} dx \; \langle p | x \rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} dx \; \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar} \frac{1}{a^{\frac{1}{2}} \pi^{\frac{1}{4}}} e^{-x^2/2a^2}$$

$$= \frac{1}{(2\pi a \hbar)^{\frac{1}{2}} \pi^{\frac{1}{4}}} \int_{-\infty}^{\infty} dx \; e^{-ipx/\hbar - x^2/2a^2}$$

ASIDE $$\int_{-\infty}^{\infty} dx \; e^{-ax^2 + bx} = e^{b^2/4a} \sqrt{\frac{\pi}{a}}$$

$$= \frac{1}{(2\pi a \hbar)^{\frac{1}{2}} \pi^{\frac{1}{4}}} \cdot \pi^{\frac{1}{2}} \cdot (2a^2)^{\frac{1}{2}} \, e^{-p^2 a^2/2\hbar^2}$$

$$= \frac{a^{\frac{1}{2}}}{\hbar^{\frac{1}{2}} \pi^{\frac{1}{4}}} e^{-p^2 a^2/2\hbar^2} \Longleftarrow \boxed{\psi(p) \text{ ALSO GAUSSIAN!}}$$

* MOM. SPACE "SPREAD"? Δp, $\langle \hat{p} \rangle = 0$

$$\langle \hat{p}^2 \rangle = \frac{a}{\sqrt{\pi}\,\hbar} \int_{-\infty}^{\infty} dp \; p^2 e^{-p^2 a^2/2\hbar} = \frac{a}{\sqrt{\pi}\,\hbar} \frac{\sqrt{\pi}}{2} \left(\frac{\hbar}{a}\right)^3 = \frac{\hbar^2}{2a^2}$$

$$\Rightarrow \quad \Delta p_x = \sqrt{\langle \hat{p}_x^2 \rangle - \langle p_x \rangle^2} = \frac{\hbar}{\sqrt{2}\, a}$$

NOTE: ① $a \to 0$ $\quad \psi(x)$ MORE LOCALIZED

$\quad\quad\quad\quad \& \ \psi(p)$ LESS LOCALIZED

② $a \to \infty$: OPPOSITE

③ $\boxed{\Delta x \Delta p_x = \frac{\hbar}{2}}$ $\Rightarrow$ GAUSS. WVFN HAS MINIMAL UNCERTAINTY!

TIME EVOL. OF FREE PARTICLE

FREE PARTICLE: $V(x) = 0 \quad \Rightarrow \quad \hat{H} = \frac{\hat{p}_x^2}{2m}$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle = e^{-i p_x^2 t/2m\hbar} \int dp\, |p\rangle \langle p|\psi(0)\rangle$$

$$\boxed{|\psi(t)\rangle = \int dp\, e^{-ip^2 t/2m\hbar} |p\rangle \langle p|\psi(0)\rangle}$$

NOTE: GENERAL EXPRESSION. STUDY EX'S

EX: GAUSSIAN WAVEPACKET

HAS

$$\langle x|\psi\rangle = \psi(x) = \frac{1}{\pi^{1/4}a^{1/2}} e^{-x^2/2a^2}$$

$$\langle p|\psi\rangle = \psi(p) = \frac{\sqrt{a}}{\sqrt{\hbar}\,\pi^{1/4}} e^{-p^2a^2/2\hbar^2}$$

$t=0$

PROP. FORWARD

$$\psi(x,t) = \langle x|\psi(t)\rangle = \int dp\; e^{-ip^2t/2m\hbar} \langle x|p\rangle\langle p|\psi(0)\rangle$$

$$= \int dp\; e^{-ip^2t/2m\hbar} \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \sqrt{\frac{a}{\hbar}} \frac{1}{\pi^{1/4}} e^{-p^2a^2/2\hbar^2}$$

$$= \sqrt{\frac{a}{2\pi\hbar^2}} \frac{1}{\pi^{1/4}} \int dp\; e^{-ap^2+bp} \quad w/\; a = \left(\frac{a^2}{2\hbar^2} + \frac{it}{2m\hbar}\right)$$

$$b = \frac{ix}{\hbar}$$

$$= \sqrt{\frac{a}{2\pi\hbar^2}} \frac{1}{\pi^{1/4}} \pi^{1/2} \left(\frac{a^2}{2\hbar^2} + \frac{it}{2m\hbar}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{\hbar^2}\left(\frac{2a^2}{\hbar^2} + \frac{2it}{m\hbar}\right)^{-1}}$$

$$\Longrightarrow \boxed{\psi(x,t) = \left[\sqrt{\pi}\left(a + \frac{i\hbar t}{ma}\right)\right]^{-\frac{1}{2}} e^{-x^2/2a^2[1+i\hbar t/ma^2]}}$$

DELOCALIZATION / DECOHERENCE

Q: HOW DOES Δx TIME EVOLVE?

$$c := \frac{\hbar t}{ma^2} \qquad A := \left[\sqrt{\pi}\,(a + ic)\right]^{-\frac{1}{2}}$$

$$\psi^*(x,t)\psi(x,t) = |A|^2 e^{-x^2/2a^2(1+ic)} \, e^{-x^2/2a^2(1-ic)}$$

$$= |A|^2 e^{\left\{-\frac{x^2}{2a^2} \frac{(1-ic)+(1+ic)}{1+c^2}\right\}}$$

$$= \left[\pi\left(a^2 + \frac{\hbar^2 t^2}{m^2 a^2}\right)\right]^{-\frac{1}{2}} e^{-\frac{x^2}{a^2\left(1+\frac{\hbar^2 t^2}{m^2 a^4}\right)}}$$

$\langle x \rangle = 0$ STILL

$$\langle \hat{x}^2 \rangle = \langle \psi_t | \hat{x}^2 | \psi_t \rangle = \int_{-\infty}^{\infty} dx\, \langle \psi_t | \hat{x}^2 | x \rangle \langle x | \psi_t \rangle = \int_{-\infty}^{\infty} dx\, x^2\, \psi^*(x,t)\psi(x,t)$$

$$= \frac{a^2}{2}\left[1 + \left(\frac{\hbar t}{ma^2}\right)^2\right]$$

$\Delta x = \sqrt{\langle \hat{x} \rangle^2} \implies$ WAVE PACKET SPREADS OUT IN TIME

NOTE

(1) HOW FAST DOES FREE e^- SPREAD

$$T_{loc} \simeq \frac{ma^2}{\hbar^2} = 10^{-16}\,\text{s} \quad \text{FOR } a = 10^{-10}\,\text{m} \quad m = m_e$$

(2) MACROSCOPIC $\quad a = 10^{-3}\,\text{m} \quad m = 1g$ (what?)

$\implies T = 3 \times 10^{17}$ years $\sim 1000\ T_{universe}$

(3) PACKET SPREADS, BUT V_{CENTER} DOES NOT CHANGE!

$\left(\frac{d}{dt}\langle p_x \rangle \sim \langle \frac{\partial V}{\partial x} \rangle = 0 \quad \text{EHRENFEST}\right)$

④ $\psi(p,t) = \psi(p)\, e^{-ip^2t/2m\hbar} \Rightarrow \psi^*(p,t)\psi(p,t)$ SAME!

$\Rightarrow \Delta p_x = \frac{\hbar}{\sqrt{2}a}$ STILL CONSTANT

$\Rightarrow \Delta x \Delta p_x = \frac{\hbar}{2}\left[1+\left(\frac{\hbar t}{ma}\right)^2\right]^{1/2}$

TIME EVOL. INCREASES UNCERT!

1D POTENTIALS

* $\hat{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$

POS. SPACE: $\langle x|\hat{H}|\psi(t)\rangle = \langle x|\left[\frac{\hat{p}_x^2}{2m} + V(\hat{x})\right]|\psi(t)\rangle$

$= \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\langle x|\psi(t)\rangle$

$$\therefore \boxed{\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)}$$

↳ TIME-DEP. SCH. EQN

* IF $|\psi(t)\rangle = |E\rangle e^{-iEt/\hbar} \Rightarrow \psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\,\psi(x)}$$

↳ T-INDEP. S. EQN

(RECALL FROM EARLIER:)
$\hat{p}_x|\psi\rangle = \frac{\hbar}{i}\int dx'\, |x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle$

EX: INFINITE SQUARE WELL

$$V(x) = \begin{cases} 0 & |x| < \frac{a}{2} \\ \infty & |x| > \frac{a}{2} \end{cases} \qquad \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi =: -k^2\psi$$

outside wall $\psi(x) = 0$ BY PHYSICS

* INSIDE: $\psi(x) = A\sin kx + B\cos kx$

* BOUND. CONDS:

$$\left.\begin{aligned} \psi\left(\tfrac{a}{2}\right) &= 0 = A\sin\tfrac{ka}{2} + B\cos\tfrac{ka}{2} \\ \psi\left(-\tfrac{a}{2}\right) &= 0 = -A\sin\tfrac{ka}{2} + B\cos\tfrac{ka}{2} \end{aligned}\right\} \quad \underbrace{\begin{pmatrix} \sin\frac{ka}{2} & \cos\frac{ka}{2} \\ -\sin\frac{ka}{2} & \cos\frac{ka}{2} \end{pmatrix}}_{K} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

- $K\begin{pmatrix} A \\ B \end{pmatrix} = 0$ HAS NON-TRIV SOLN

WHEN $\det(K) = 0$

- $2\sin\left(\frac{ka}{2}\right)\cos\left(\frac{ka}{2}\right) = \sin(ka) = 0 \Rightarrow \boxed{ka = n\pi \quad n \in \mathbb{Z}}$

n ODD $\Rightarrow$ SIN

n EVEN $\Rightarrow$ COS

* SOLUTIONS: $\psi_n(x) = \begin{cases} B_n \cos\frac{n\pi x}{a} & n = 1, 3, 5, \dots \\ A_n \sin\frac{n\pi x}{a} & n = 2, 4, 6, \dots \end{cases}$

* NORMALIZE: $1 = |B_n|^2 \int_{-a/2}^{a/2} dx \cos^2\left(\frac{n\pi x}{a}\right)$

$$\Rightarrow \boxed{B_n = \sqrt{\frac{2}{a}}, \quad \text{SIM } A_n = \sqrt{\frac{2}{a}} \quad \forall n}$$

* ENERGY: $k_n^2 = \frac{2mE_n}{\hbar^2} \Rightarrow E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2\pi^2 n^2}{2ma^2}$

n large, E_n large!