

QM Lecture 5



EIGENSTATES OF ANG. MOM.

- RECALL $[J_i, J_j] = i\hbar \epsilon^{ijk} J_k$ (Einstein summation)

- SOME OPS. DON'T COMMUTE!

- WHAT ABOUT LENGTH OF $\hat{\vec{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$?

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

$$\text{IDENTITY: } [\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$$

note: will often drop hats!

$$\begin{aligned} [\hat{J}_z, \hat{J}^2] &= [\hat{J}_z, \hat{J}_x^2] + [\hat{J}_z, \hat{J}_y^2] + [\hat{J}_z, \hat{J}_z^2] \\ &= J_x [\hat{J}_z, J_x] + [\hat{J}_z, J_x] J_x + J_y [\hat{J}_z, J_y] + [\hat{J}_z, J_y] J_y \\ &= i\hbar [J_x J_y + \bar{J}_y J_x + J_y (-J_x) + (-J_x) J_y] = 0 \end{aligned}$$

$$\boxed{[J_i, J^2] = 0!} \quad (\text{so what?})$$

SIMULTANEOUS EIGENSTATES

SUPPOSE $\hat{A}|a\rangle = a|a\rangle$, $|a\rangle$ is ONLY A E-VEC w E-VAL a $\hat{A}(\lambda|a\rangle) = a(\lambda|a\rangle)$

$$\Rightarrow \hat{B}\hat{A}|a\rangle = \hat{B}(a|a\rangle) = a\hat{B}|a\rangle$$

$\hat{A}\hat{B}|a\rangle \stackrel{[A,B]=0}{=} \hat{B}\hat{A}|a\rangle \Rightarrow \hat{B}|a\rangle$ E-STATE OF \hat{A} w E-VAL a .

$$\Rightarrow \hat{B}|a\rangle = b|a\rangle \Rightarrow \boxed{|a\rangle \text{ E-VEC OF } \hat{B} \in \hat{A}, \text{ FROM } [\hat{A}, \hat{B}] = 0}$$

"ONLY" needed here say " $|a\rangle$ is a simult. e-state of $A \in B$ "

\hat{J}^2 : $[\hat{J}_i, \hat{J}^2] = 0 \Rightarrow$ TWO EIGENVALUES FOR STATE, CALL $|\lambda_m\rangle$

$$\hat{J}^2 |\lambda_m\rangle = \lambda_m^2 |\lambda_m\rangle$$

$$\hat{J}_z |\lambda_m\rangle = m_z |\lambda_m\rangle$$

HIGHER DIM. SPIN REPS

(key role will be played by raising & lowering ops)

(much more general than just "up" and "down")

"UP,DOWN" = 2-D REP.

\exists N-D REPS. $\forall N \in \mathbb{N} = \{1, 2, 3, \dots\}$

WANT: SYSTEMATIC THY FOR N-DIM SPIN REP

* BEING A SPIN REP REQ. MAT. REP'S OF J_i WITH CORRECT $[J_i, J_j]$

EX SPIN 1: $J_z = -\frac{\hbar}{2}, 0, \frac{\hbar}{2} \Rightarrow 3 \times 3$ MATRICES

$$\hat{J}_z \rightarrow \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \hat{J}_x \rightarrow \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{J}_y \rightarrow \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Q: IS THIS A REPRESENTATION? CHECK $[\hat{J}_i, \hat{J}_j] = i\hbar \epsilon^{ijk} \hat{J}_k$

$$\hat{J}_x \cdot \hat{J}_z = \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \hat{J}_x \cdot \hat{J}_z = \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\therefore [\hat{J}_x, \hat{J}_z] = \frac{\hbar^2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = i\hbar \hat{J}_y \quad \checkmark$$

OTHER $[\cdot, \cdot]$ 'S HOLD TOO!

FURTHERMORE CAN CHECK $[\hat{J}_i, \hat{J}^2] = 0 \quad \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = 2\hbar^2 \cdot \mathbb{I}_{3 \times 3}$

BASIS STATES $|s, m\rangle = |1, 1\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |s, m\rangle = |1, 0\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |s, m\rangle = |1, -1\rangle \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

RAISING & LOWERING OPS: $\hat{J}_{\pm} := \hat{J}_x \mp i\hat{J}_y$

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y = \sqrt{2} \frac{\hbar}{2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \hat{J}_- = \hat{J}_x - i\hat{J}_y = \sqrt{2} \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2} \frac{\hbar}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{J}_+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2} \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{J}_+ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0!$$

\hat{J}_+ RAISES m , ANNIHILATES "HIGHEST" m ST.

$$\underline{\text{Sum}}: \quad \hat{J}_- \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \hat{J}_- \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \hat{J}_- \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

J- LOWERS m, ANNI. LOWEST m STATE

PT: IF KNEW HIGHEST IN STATE, COULD "BUILD UP" REST WITH J_- !
 " " LOWEST " " " " " " " " " " J_+ !

Q: J_f MORE GEN?

$$[J_z, J_{\pm}] = [J_z, J_x \pm iJ_y] = i\hbar J_y \mp i(-i\hbar J_x) = \hbar(J_y \mp J_x)$$

$$\boxed{[J_z, J_{\pm}] = \pm \hbar J_{\pm}} \quad (\text{This is universally true})$$

$$J_z J_{\pm} |x, m\rangle = \left(J_{\pm} J_z \pm t J_{\pm} \right) |x, m\rangle = \left(\hat{J}_{\pm} m \pm t \hat{J}_{\pm} \right) |x, m\rangle = (m \pm 1) t J_{\pm} |x, m\rangle$$

$\Rightarrow J_{\pm} |\lambda, m\rangle$ E-STATE OF J_z w/ E-VAC ($m \pm 1$) t

$$|\langle J_{\pm} | \lambda, m \rangle = C_{\pm} |\lambda, m \pm 1 \rangle| \quad (\text{more on constants later})$$

$$\underline{\text{ALSO}} \quad \vec{J}^2 \ J_{\pm} |x_{1m}\rangle = \lambda \vec{t}^2 \ J_{\pm} |x_{1m}\rangle$$

SPECTRUM OF ANG. MOM. OPS

FACT ① (convince yourself from $\langle \lambda, m | \hat{J}_x^2 + \hat{J}_y^2 | \lambda, m \rangle = (\lambda - m^2) \hbar^2 \langle \lambda, m | \lambda, m \rangle$)
 $\lambda \geq m^2$

\Rightarrow FACT ② \exists SOME MAX m , $m_{\max} := j$

DEFINED BY $\hat{J}_+ |\lambda, j \rangle = 0$

Q: WHAT IS j ?

$$\begin{aligned} J_- J_+ &= (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y] \\ &= \vec{J}^2 - J_z^2 - \hbar J_z \end{aligned}$$

$\Rightarrow |\lambda, m \rangle$ ARE E-STATES OF $J_- J_+$

$$J_+ |\lambda, j \rangle = 0 \Rightarrow 0 = J_- J_+ |\lambda, j \rangle = (\lambda - j^2 - j) \hbar^2 |\lambda, j \rangle$$

$$\Rightarrow \boxed{\lambda = j(j+1)}$$

Q: WHAT IS m_{\min} ? (any guesses?)

A: $m_{\min} = -j$

PROOF: m_{\min} DEF BY $J_- |\lambda, j_{\min} \rangle = 0$

$$\text{BUT } J_+ J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y] = \vec{J}^2 - J_z^2 + \hbar \hat{J}_z$$

$$\therefore 0 = J_+ J_- |\lambda, m_{\min} \rangle = (\lambda - m_{\min}^2 + m_{\min}) \hbar^2 |\lambda, m_{\min} \rangle$$

$$\Rightarrow \lambda = m_{\min} (m_{\min} - 1) \stackrel{\text{above}}{=} j(j+1)$$

TWO SOLS: a) $\boxed{m_{\min} = -j}$

b) $m_{\min} = j+1$

good? NO (would $\Rightarrow j$ is not max state)

LEARNED 2 THINGS

1) * J_- LOWERS M FROM $m_{\max} = j$ TO $m_{\min} = -j$ BY INTEGERS

$$\Rightarrow j - (-j) \in \mathbb{Z} \Rightarrow 2j \in \mathbb{Z} \Rightarrow j \in \frac{\mathbb{Z}}{2}$$

$$\boxed{\boxed{j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \dots}}$$

2) PICK j . THEN $\boxed{\exists \boxed{2j+1 \text{ STATES}}}$

$$\text{EG. } j \in \mathbb{Z} \Rightarrow m \in \{j, j-1, j-2, \dots, 0, \dots, -j+1, -j\}$$

MODERN PHYS NOTATION $|j, m\rangle$ NOT $|J, m\rangle$

$$\vec{J}^2 |j, m\rangle = j(j+1) \hbar^2 |j, m\rangle$$

$$J_z |j, m\rangle = m \hbar |j, m\rangle$$

EXAMPLE REPS

$j=0$

$$|0, 0\rangle$$

$j=\frac{1}{2}$

$$|\frac{1}{2}, \frac{1}{2}\rangle = "1+z"$$

$$|\frac{1}{2}, -\frac{1}{2}\rangle = "1-z"$$

$j=1$

$$|1, 1\rangle$$

$$|1, 0\rangle$$

$$|1, -1\rangle$$

$j=\frac{3}{2}$

$$|\frac{3}{2}, \frac{3}{2}\rangle$$

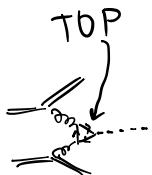
$$|\frac{3}{2}, \frac{1}{2}\rangle$$

$$|\frac{3}{2}, -\frac{1}{2}\rangle$$

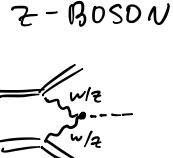
$$|\frac{3}{2}, -\frac{3}{2}\rangle$$

EG: HIGGS!

ELECTRON



W-BOSON



GRAVITINO?

FIX CONSTANTS $C_{\pm} \quad J_{\pm}|j,m\rangle = \pm C_{\pm}|j,m \pm 1\rangle$

$$\langle j,m | J_+^{\dagger} J_+ | j,m \rangle = C_+^* C_+ \hbar^2 \langle j,m+1 | j,m+1 \rangle$$

NOTE: $J_+^{\dagger} = J_x^{\dagger} + (iJ_y)^{\dagger} = J_x - iJ_y = J_-$

$$\boxed{J_{\pm}^{\dagger} = J_{\mp}}$$

$$\therefore J_+^{\dagger} J_+ = J_- J_+ = \overrightarrow{J^2} - J_z^2 - \hbar J_z$$

$$\rightarrow \cancel{|C_+|^2 \hbar^2 \langle j,m+1 | j,m+1 \rangle} = \langle j,m | \overrightarrow{J^2} - J_z^2 - \hbar J_z | j,m \rangle \\ = \hbar^2 (j(j+1) - m^2 - m) \cancel{\langle j,m | j,m \rangle}$$

RHS REAL $\Rightarrow \boxed{\begin{array}{l} C_+ = [j(j+1) - m(m+1)]^{1/2} \\ C_- = [j(j+1) - m(m-1)]^{1/2} \end{array}} \quad (\text{explain this})$

just say depending on time:

CHECK RECALL $J_+|j,m=j\rangle = 0$

$$J_-|j,m=-j\rangle = 0$$

BUT $C_+|_{m=j} = 0 ? \quad C_-|_{m=-j} = 0 \quad \checkmark$

$|j,m\rangle$ MATRIX ELEMENTS

$$\langle j,m' | j,m \rangle = \delta_{m,m'}$$

$$\boxed{\langle j,m' | J_+ | j,m \rangle = \sqrt{j(j+1) - m(m+1)} \hbar \langle j,m' | j,m+1 \rangle = \sqrt{j(j+1) - m(m+1)} \hbar \delta_{m',m+1}}$$

$$\boxed{\langle j,m' | \hat{J}_- | j,m \rangle = \sqrt{j(j+1) - m(m-1)} \hbar \delta_{m',m-1}}$$

$$\underline{\text{EX: }} j=1 \quad |1,1\rangle = |+\rangle \quad |1,0\rangle = |0\rangle \quad |1,-1\rangle = |- \rangle$$

$|+\rangle, |0\rangle, |- \rangle$ O-NORM BASIS.

$$\text{IN THIS BASIS: } |+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |- \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$J_+ |+\rangle = 0 \quad J_+ |0\rangle = \sqrt{2+0} \neq |+\rangle = \sqrt{2} \neq |+\rangle \quad J_+ |- \rangle = \sqrt{2-(-1)\cdot 0} \neq |0\rangle = \sqrt{2} \neq |0\rangle$$

$$\Rightarrow J_+ \xrightarrow[+0-\text{BASIS}]{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\text{SIM.}} \quad J_- \xrightarrow{+0-} \sqrt{2} \neq \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

NOTE: RAISES / LOWERS!