

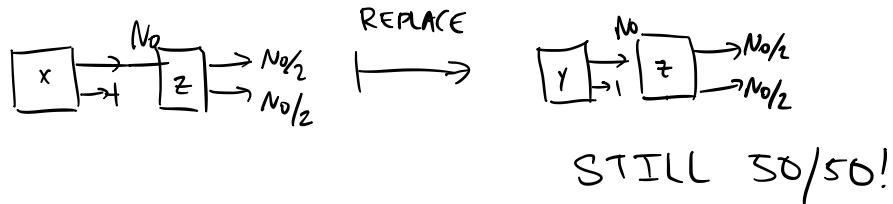
QM Lecture 3



PREVIOUSLY $|x\rangle = \frac{1}{\sqrt{2}}(e^{i\delta_+}|+z\rangle + e^{i\delta_-}|-z\rangle)$ $\rightarrow \boxed{z} \rightarrow \boxed{x} \rightarrow \boxed{z} \rightarrow \begin{matrix} N_0 \\ N_0/2 \\ N_0/2 \end{matrix}$

* PHYSICS DOESN'T CARE WHAT YOU CALL AXES!

REP. \boxed{x} BY \boxed{y} ?



$$|+y\rangle = \frac{1}{\sqrt{2}}(e^{i\gamma_+}|+z\rangle + e^{i\gamma_-}|-z\rangle) = \frac{e^{i\gamma_+}}{\sqrt{2}}(|+z\rangle + e^{i(\gamma_- - \gamma_+)}|-z\rangle)$$

SIM.

$$\Rightarrow |\langle +y | x \rangle|^2 = \frac{1}{2} \quad \text{EXP } \otimes$$

* COMPUTE $\langle +y | +x \rangle = \left(\frac{e^{-i\gamma_+}}{\sqrt{2}}\right)\left(\frac{e^{i\delta_+}}{\sqrt{2}}\right)[\langle +z | +e^{-i\delta} \langle -z |] [(+z) + e^{i\delta} (-z)]$

comb terms 0 $\delta = \delta_- - \delta_+ \quad \gamma = \gamma_- - \gamma_+$

$$\langle +y | +x \rangle = \frac{1}{2} e^{i(\delta_+ - \gamma_+)} (1 + e^{i(\delta - \gamma)})$$

$$\Rightarrow |\langle +y | +x \rangle|^2 = \frac{1}{4} [1 + e^{i(\delta - \gamma)} + e^{-i(\delta - \gamma)} + 1] = \frac{1}{2} [1 + \frac{1}{2} (e^{i(\delta - \gamma)} + e^{-i(\delta - \gamma)})]$$

matching exp \otimes & comp $= \frac{1}{2} (1 + \cos(\delta - \gamma))$

$$\therefore \frac{1}{2} = |\langle +y | x \rangle|^2 = \frac{1}{2} (1 + \cos(\delta - \gamma)) \Rightarrow \delta - \gamma = \pm \frac{\pi}{2}$$

* NB: ① CANNOT HAVE SAME PHASES

② MUST HAVE AT LEAST ONE COMPLEX COEFF.

EG: CAN TAKE $\delta = 0$ $\Rightarrow |x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$

$$\Rightarrow \gamma = \pm \frac{\pi}{2} \quad |+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + e^{+i\frac{\pi}{2}}(-z)) = \frac{1}{\sqrt{2}}(|z\rangle + i|-z\rangle)$$

!!!

QUANTUM STATES AS VECTORS

to make a little more obvious, write (for now) $|q\rangle$ as col

$$\text{eg } V=\mathbb{R}^3 \quad \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \mapsto \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\text{eg: } |q\rangle = C_+ |+z\rangle + C_- |-z\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \begin{pmatrix} \langle +z | q \rangle \\ \langle -z | q \rangle \end{pmatrix}$$

$$\left. \begin{array}{l} |+z\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |-z\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right\} \begin{array}{l} \text{follows from above general} \\ \text{result plus orthonormality} \end{array}$$

$$\text{OTHERS} \quad |x\rangle \xrightarrow{S_z\text{-bas}} \begin{pmatrix} \langle +z | +x \rangle \\ \langle -z | +x \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |y\rangle \xrightarrow{S_z\text{-bas}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

BRA STATES: TAKE CONJ. TRANSPOSE

$$\langle q | \mapsto (C_+, C_-)^* \quad \therefore \quad \langle +y | \mapsto \frac{1}{\sqrt{2}} (1, -i)$$

$$\langle +x | = \frac{1}{\sqrt{2}} (1, 1)$$

$$\text{CHECK: } \langle +y | +y \rangle = \frac{1}{\sqrt{2}} (1, -i) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} (1^2 - i^2) = 1 \checkmark \quad (\text{is what we expect!})$$

Q: WHAT ABOUT $| -x \rangle$?

NB: $\langle x | -x \rangle = 0$

$$|x\rangle \xrightarrow{z-b} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad | -x \rangle = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \langle x | -x \rangle = \frac{1}{\sqrt{2}} (a^* + b^*) = 0$$

↑
follows from result
on last page

$$\text{BUT } \langle -x | -x \rangle = 1 = 2 |a|^2 \Rightarrow |a| = \frac{1}{\sqrt{2}}$$

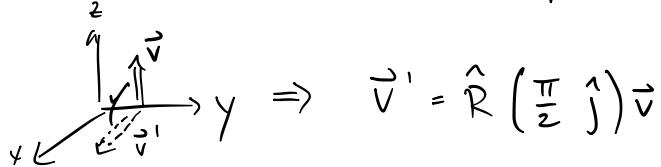
(overall phase isn't fixed,
but unphysical, set to 1)

ROTATIONS & CHANGE OF BASIS

* PHYSICS MUST NOT DEP. ON BASIS!

(different ways to express the same state)

* ROTATIONS $\hat{R}(\theta \hat{n})$ = ROT BY θ ABOUT \hat{n}



* QM: $|_{+x}\rangle = \hat{R}\left(\frac{\pi}{2} \hat{j}\right) |_{+z}\rangle$ (see above picture)

* \hat{R} LINEAR OPERATORS IE.

$$\hat{R}(\theta \hat{n})(a|\psi\rangle + b|\phi\rangle) = a(\hat{R}(\theta \hat{n})|\psi\rangle) + b(\hat{R}(\theta \hat{n})|\phi\rangle)$$

* Q: Z-BAS. TO X-BAS? $|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle$

$$\hat{R}\left(\frac{\pi}{2} \hat{j}\right) |_{\pm z}\rangle = |_{\pm x}\rangle \Rightarrow \hat{R}\left(\frac{\pi}{2} \hat{j}\right) |\psi\rangle = c_+|+x\rangle + c_-|-x\rangle$$

((ADJOINT OPERATOR)) TO EACH LIN. OP. $\Theta: V \rightarrow V$
ASSOC. ADJ L.OP. $\Theta^*: V^* \rightarrow V^*$

KEY:

$$|\psi'\rangle = \Theta|\psi\rangle \Rightarrow \langle\psi'| = \langle\psi|\Theta^*$$

Pronounced "O-dagger"

IF: Θ a MATRIX, $\Theta^* = (\Theta^*)^T$

UNITARY: DEF'N ANY OPERATOR Θ SATISFYING $\Theta^*\Theta = \mathbb{1}$ IS SAID TO BE UNITARY.

PRESERVES PROB: $|\psi'\rangle = \Theta|\psi\rangle, |\psi'\rangle = \Theta|\psi\rangle$

$$\Rightarrow |\langle\psi'|\psi'\rangle|^2 = |\langle\psi|\Theta^*\Theta|\psi\rangle|^2 = |\langle\psi|\psi\rangle|^2$$

E.G. CHANGE OF BASIS $|v'_i\rangle = \Theta|v_i\rangle$

(now ? prob. preserved, as you would hope!)

GENERATOR OF ROTATIONS

$SO(3)$ Special
 $\det = 1$ Orthog. $\downarrow_{\text{dim}=3}$
 $O^T O = \mathbb{1}_L$ 3×3

IDEA: FOR $\theta \ll 1$, $R \approx \mathbb{1} - \delta\phi A$

EX $R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ROT ABOUT z-AX

BUT FOR $\theta \ll 1$ WRITE $\theta = \delta\phi$ (some small change)

$$\cos \delta\phi \approx 1 + \dots \quad \sin \delta\phi \approx \delta\phi + \dots$$

$$\Rightarrow R = \mathbb{1} - \delta\phi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \mathbb{1} - \delta\phi A$$

SAY: A "GENERATES ROTATION ABOUT z-AX"

QM: (want to rotate in complex QM vs.)

$$\hat{R}(S\phi \hat{A}) = \mathbb{1} - iS\phi \hat{A} \quad ? \quad \hat{R}^+(S\phi \hat{A}) = \mathbb{1} + iS\phi \hat{A}^+$$

(free to choose i factor since def \hat{A})

$$\begin{aligned} \hat{R} \text{ UNITARY} \Rightarrow \mathbb{1} &= (\mathbb{1} + iS\phi \hat{A}^+) (\mathbb{1} - iS\phi \hat{A}) = \mathbb{1} + iS\phi (\hat{A}^+ - \hat{A}) \\ &\Rightarrow \hat{A}^+ = \hat{A} \quad R^+ \quad R \end{aligned}$$

DEF'N: AN OPERATOR Θ SATIS. $\Theta^+ = \Theta$ IS HERMITIAN

FINITE ROTATIONS: $\hat{R}(\phi \hat{A}) = \lim_{N \rightarrow \infty} \left(1 - i\left(\frac{\phi}{N}\right) \hat{A}\right)^N = e^{-i\phi \hat{A}}$

(repeated application of infinitesimal)

EIGENSTATES (recall from lin. alg.)

DEF'N IF $\vec{v} \neq 0 \in V$ AND $A: V \rightarrow V$ LIN. OP.

THEN \vec{v} IS AN EIGENVECTOR OF A IF

$$A\vec{v} = \lambda(A, \vec{v}) \vec{v}$$

WITH $\lambda(A, \vec{v})$ A #, THE ASSOC. EIGENVALUE

NB: IF $\Theta = e^{-iA}$ & $A|4\rangle = \lambda(A, |4\rangle) |4\rangle$

THEN $\Theta|4\rangle = \lambda(\Theta, |4\rangle) |4\rangle$.

PHYSICS EXAMPLE: GEN. OF ROTS \hat{A} CORRESP TO SPIN ANG. MOM

$$* \hat{A} = \frac{\hat{J}_z}{\hbar} \quad J_z^+ = J_z \quad \hat{R}(\phi \hbar) = e^{-i\hat{J}_z \phi / \hbar}$$

$$* |\pm z\rangle E\text{-STATES}: \quad \hat{J}_z |\pm z\rangle = \pm \frac{\hbar}{2} |\pm z\rangle$$

$$\therefore \hat{R}(\phi \hbar) |+z\rangle = \left(1 - \frac{i\phi}{2} + \frac{1}{2!} \left(-\frac{i\phi}{2}\right)^2 + \dots\right) |+z\rangle = e^{-i\phi/2} |+z\rangle$$

$$\text{SIM } \hat{R}(\phi \hbar) |-z\rangle = e^{i\phi/2} |-z\rangle$$

$$* \underline{\text{CRAZY FACT}}: \quad \hat{R}(2\pi \hbar) |+z\rangle = e^{-i\pi} |z\rangle = -|z\rangle$$

$$" \quad |-z\rangle = -|-z\rangle$$

verified
in
exp' tally
ch 4!

\Rightarrow ROT. BY $\frac{\pi}{2}$ FOR $|z\rangle \mapsto (-z)$

* RELATED TO TOPOLOGY: $\Pi_1(\text{SO}(3)) = \mathbb{Z}_2$

phone turn:
hold forward,
turn right.

$SU(2)$ is a double cover of $\text{SO}(3)$

elbow, rot
= spin!

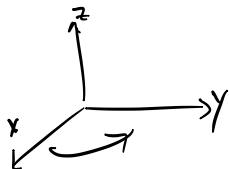
ROTATE ARB STATES ($\langle \text{NOW} | \psi_i \rangle, |\psi_i\rangle \}$ A BASIS? KNOW $|\psi\rangle$

$$|\psi\rangle = \sum c_i |\psi_i\rangle \rightarrow |\psi\rangle = \sum c_i |\psi_i\rangle$$

HERE: $|\psi\rangle = c_+ |+\rangle + c_- |-\rangle \Rightarrow \hat{R}(\phi_k) |\psi\rangle = c_+ e^{-i\phi_k} |+\rangle + c_- e^{i\phi_k} |-\rangle$

SPECIFIC $|x\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$

$$\begin{aligned} \rightarrow \hat{R}\left(\frac{\pi}{4} \hat{h}\right) |x\rangle &= \frac{1}{\sqrt{2}} \left(e^{-i\frac{\pi}{4}} |+\rangle + e^{i\frac{\pi}{4}} |-\rangle \right) \\ &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{2}} (|+\rangle + i|-\rangle) \\ &= e^{-i\frac{\pi}{4}} |+\rangle \end{aligned}$$

INTERP:  X ROT. INTO Y!
(up to phase)

MATRIX REPRESENTATIONS

IDEA: REP. OP. BY MATRIX

concrete notions for "adjoint, Hermitian"

CONSIDER $\hat{A}|\psi\rangle = |\psi\rangle$ (*)

inserting the identity $|\pm z\rangle$ A BAS. $\Rightarrow |\psi\rangle = |+\rangle \langle +z|\psi\rangle + |-\rangle \langle -z|\psi\rangle$

$$|\psi\rangle = " " |\psi\rangle + " " |\psi\rangle$$

THEN (*) $\Rightarrow \hat{A}[|+\rangle \langle +z|\psi\rangle + |-\rangle \langle -z|\psi\rangle] = |+\rangle \langle +z|\psi\rangle + |-\rangle \langle -z|\psi\rangle$

* ACT w $\langle \pm z | \Rightarrow \langle +z | \hat{A} | z \rangle \langle z | \psi \rangle + \langle z | \hat{A} | -z \rangle \langle -z | \psi \rangle = \langle z | \psi \rangle$

$$\langle -z | \hat{A} | z \rangle \langle z | \psi \rangle + \langle -z | \hat{A} | -z \rangle \langle -z | \psi \rangle = \langle -z | \psi \rangle$$

* EQUIV $\begin{pmatrix} \langle z | \hat{A} | z \rangle & \langle z | \hat{A} | -z \rangle \\ \langle -z | \hat{A} | z \rangle & \langle -z | \hat{A} | -z \rangle \end{pmatrix} \begin{pmatrix} \langle z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix} = \begin{pmatrix} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix}$

(note: we see arbitrary A can be rep'd as matrix!)

GEN $\{|\psi_i\rangle\}$ A BASIS $A_{ij} = \langle v_i | \hat{A} | v_j \rangle$

DEFN THE MATRIX ELEMENTS OF \hat{A} .

PROJ. $\{|\psi_i\rangle\}$ ID OPS

$$1: \text{RECALL } |\psi\rangle = \sum c_i |v_i\rangle \quad \text{BUT} \quad c_i = \langle v_i | \psi \rangle$$

$$\text{so } |\psi\rangle = \left(\sum_i |v_i\rangle \langle v_i| \right) |\psi\rangle \Rightarrow \mathbb{I} = \sum_i |v_i\rangle \langle v_i|$$

PROJ.: WANT P_i ST $P_i |\psi\rangle = c_i |v_i\rangle$ (*ie project onto one component*)

GOOD GUESS: $P_i = |v_i\rangle \langle v_i|$

$$P_i |\psi\rangle = |v_i\rangle \langle v_i | \psi \rangle = c_i |v_i\rangle \checkmark$$

NOTE: $\mathbb{I} = \sum_i P_i$

ANALOGY

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

$$P_x \vec{v} = v_x \hat{i}$$

$$P_x = \hat{i} (\cdot \cdot) \quad P_x \vec{v} = \hat{i} (\cdot \cdot (v_x \hat{i} + v_y \hat{j})) = v_x \hat{i}$$

MATRIX REPS OF J_z

$$\hat{J}_z \xrightarrow{\text{S}_z \text{ bas}} \begin{pmatrix} \langle z | \hat{J}_z | z \rangle & \langle +z | \hat{J}_z | -z \rangle \\ \langle -z | \hat{J}_z | z \rangle & \langle -z | \hat{J}_z | -z \rangle \end{pmatrix} = \begin{pmatrix} \frac{\hbar}{2} \langle z | z \rangle & -\frac{\hbar}{2} \langle +z | -z \rangle \\ \frac{\hbar}{2} \langle -z | z \rangle & -\frac{\hbar}{2} \langle -z | -z \rangle \end{pmatrix}$$

$$\Rightarrow \hat{J}_z \xrightarrow{\text{S}_z \text{ bas}} \frac{\hbar}{2} \hat{P}_+ - \frac{\hbar}{2} \hat{P}_-$$

$$= \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

(ops. do something to state.
Can account for SG?)