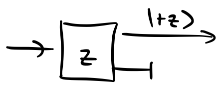
QM Lecture 4

QUANTUM INTERFERENCE

BACK TO T.E.'s



SG_z + WALL ACTS LIKE $\hat{P}_+$ IN Z-BASIS

SG_z PROJ ONTO ONE OR OTHER

- TE 1:

EXTRA SG_z? PROJECT AGAIN

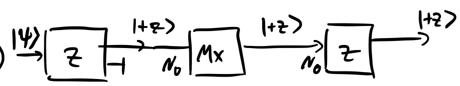
$$(\hat{P}_+)_z \cdot (\hat{P}_+)_z = |+z\rangle\langle +z|+z\rangle\langle +z| = |+z\rangle\langle +z| = (\hat{P}_+)_z$$

$$(\hat{P}_+)_z \cdot (\hat{P}_-)_z = |+z\rangle \underbrace{\langle +z|-z\rangle}_{0} \langle -z| = 0$$

$\Rightarrow$ OUTPUT OF SECOND MATCHES FIRST! *(and this matches the experiment!)*

- TE 4.

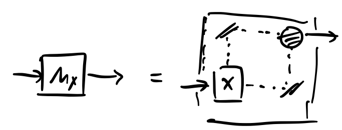
EXP (A)



$|M_x|$ = *modified* SG_x

S_x NOT MEASURED, JUST REDIRECT.

SO



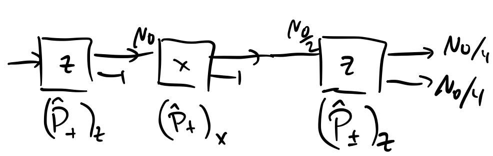
$$|\psi\rangle = C_+|+z\rangle + C_-|-z\rangle$$

FINAL STATE: $(\hat{P}_+)_z (\hat{\mathbb{I}})_z (\hat{P}_+)_z |\psi\rangle = (\hat{P}_+)_z |\psi\rangle = C_+|+z\rangle$

$$= C_+(|+x\rangle\langle +x|z\rangle + |-x\rangle\langle -x|z\rangle)$$
$$= C_+(C_+^x|+x\rangle + C_-^x|-x\rangle)$$

X-BAS.

EXP (B) Q: PUT IN WALL? $\hat{\mathbb{I}}$ REPL. BY $(\hat{P}_\pm)_x$ *(depend on wall loc.)*



EXP A $\Rightarrow |\langle -z|+z\rangle|^2 = 0$ (what happens if we exp. result in x-barrier? Same)

$$\begin{aligned} &\| \\ &|\langle -z|+x\rangle|^2 |\langle +x|+z\rangle|^2 + |\langle -z|-x\rangle|^2 |\langle -x|z\rangle|^2 \\ &+ \langle -z|+x\rangle\langle +x|-z\rangle \langle -z|-x\rangle^* \langle -x|+z\rangle^* \\ &+ \langle -z|+x\rangle^* \langle +x|+z\rangle^* \langle -z|-x\rangle\langle -x|+z\rangle \end{aligned} \Bigg\} \equiv 1 + 2 + 3 + 4$$

MTH: MEASURED TO HAVE

1 PROB $|+z\rangle$ IN MTH $S_x = \frac{\hbar}{2}$ THEN MTH $S_z = -\frac{\hbar}{2}$. EXP B $\Rightarrow$ $1 = \frac{1}{4}$

2 " " " " $S_x = -\frac{\hbar}{2}$ " " $S_z = -\frac{\hbar}{2}$. EXP B w REPL $\Rightarrow \frac{1}{4}$

$\Rightarrow 0 = \frac{1}{4} + \frac{1}{4} + 3 + 4 \Rightarrow$ 3, 4 CANCEL $\frac{1}{4} + \frac{1}{4}$.

TAKE AWAY:

i) 3 + 4 "INTERFERENCE TERMS"

ii) AMPLITUDES INTERFERE.

$\Rightarrow$ PROB. AMP'S ADD, SQUARE TO GET PROB, $\exists$ INTERFERING CROSS TERMS!

(if you just add probs, get 1 + 2. instead, must add amplitudes, then square to get probability! That process gives rise to interference terms.)

MATRIX REPS OF BASIS ROTS

(so far: matrix rep of $\mathbb{1}$ and $\hat{P}_i$, z-basis rep of $\hat{J}_z$.
now, matrix rep of rot op in quantum state space)

RECALL $|\psi\rangle \xrightarrow[S_z\text{-bas}]{} \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$

BRA EV* KET EV

FACT: OP. $\hat{A}: V \to V$ (Q: what maps $V^* \to V^*$)

ADJ. OP $\hat{A}^\dagger: V^* \to V^*$

IE $\hat{A}|\psi\rangle = |\psi'\rangle$, $\langle\psi|\hat{A}^\dagger = \langle\psi'|$

EXAMPLE $\hat{A} = \hat{R}(\theta\hat{n})$.

(Q: what is $(\hat{R}(\theta\hat{n}))^{-1}$?) $\hat{R}(\theta\hat{n})^{-1} = \hat{R}(-\theta\hat{n})$

ROT $|\psi\rangle$ BY $-\theta$ & MEAS $S_z = \frac{\hbar}{2}$ $= \langle +z|\hat{R}(-\theta\hat{n})|\psi\rangle = \langle +z|\hat{R}(\theta\hat{n})^{-1}|\psi\rangle$

BUT UNITARY $\Rightarrow$ $\hat{R}^\dagger(\theta\hat{n})R(\theta\hat{n}) = \mathbb{1}$

so $= \langle +z|\hat{R}(\theta\hat{n})^\dagger|\psi\rangle = \langle z'|\psi\rangle =$ MEAS. $S_{z'} = \frac{\hbar}{2}$ of $|\psi\rangle$ IN ROT. z-BAS. $|+z'\rangle = \hat{R}(\theta\hat{n})|z\rangle$

$R^\dagger(\theta\hat{n}) = ?$ GEN: $\overbrace{\langle i|\hat{A}^\dagger|j\rangle = \langle j|\hat{A}|i\rangle^*}^{(*)} \Rightarrow A^\dagger_{ij} = A^*_{ji}$

PROOF: (from above) $\langle\chi|\hat{A}|\psi\rangle = \langle\chi|\psi'\rangle$ & $\langle\psi|\hat{A}^\dagger|\chi\rangle = \langle\psi'|\chi\rangle$

THEN $\langle\psi'|\chi\rangle = \langle\chi|\psi'\rangle^* \Rightarrow (*)$

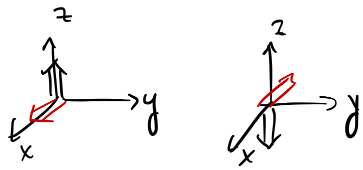
$\Longrightarrow$ FOR MATRIX REPS "DAGGER" MEANS CONJ. TRANSPOSE

CHECK $\hat{R}(\theta\hat{n}) = e^{-i\theta J_n/\hbar}$ w/ $J_n = J_n^\dagger$ (ie, generator is Hermitian)

$$\Rightarrow R^\dagger(\theta\hat{n}) = e^{i\theta J_n^\dagger/\hbar} = e^{i\theta J_n/\hbar} = \hat{R}(-\theta\hat{n})$$

(matches previous direct result, but here we just took the conjugate transpose!)

ROTATE $|\psi\rangle$



$$|\pm x\rangle = \hat{R}\left(\tfrac{\pi}{2}\hat{j}\right)|\pm z\rangle \Rightarrow \langle \pm x| = \langle \pm z|\hat{R}^\dagger\left(\tfrac{\pi}{2}\hat{j}\right)$$

ROTATE:

$$R^\dagger\left(\tfrac{\pi}{2}\hat{j}\right)|\psi\rangle =: |\psi'\rangle \xrightarrow[S_z\text{-bas}]{} \begin{pmatrix}\langle +z|\psi'\rangle \\ \langle -z|\psi'\rangle\end{pmatrix} = \begin{pmatrix}\langle +z|\hat{R}^\dagger(\frac{\pi}{2}\hat{j})|\psi\rangle \\ \langle -z|\hat{R}^\dagger(\frac{\pi}{2}\hat{j})|\psi\rangle\end{pmatrix} = \begin{pmatrix}\langle x|\psi\rangle \\ \langle x|\psi\rangle\end{pmatrix} \xleftarrow[S_x\text{-bas}]{} |\psi\rangle$$

ACTIVE: ROTATE STATE

PASSIVE: ROTATE BASIS

(point: this rotated state in unrotated basis and unrotated state in this rotated basis are same thing!)

RELATE TWO COL. VECTORS? (does the above suggest how to do it?)

$$\underbrace{\begin{pmatrix}\langle +x|\psi\rangle \\ \langle -x|\psi\rangle\end{pmatrix}_x}_{(|\psi\rangle)_{S_x}} = \begin{pmatrix}\langle +x|z\rangle\langle +z|\psi\rangle + \langle +x|-z\rangle\langle -z|\psi\rangle \\ \langle -x|z\rangle\langle +z|\psi\rangle + \langle -x|-z\rangle\langle -z|\psi\rangle\end{pmatrix} = \underbrace{\begin{pmatrix}\langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|-z\rangle & \langle -x|-z\rangle\end{pmatrix}}_{\underline{S}^\dagger(\frac{\pi}{2}\hat{j})} \underbrace{\begin{pmatrix}\langle +z|\psi\rangle \\ \langle -z|\psi\rangle\end{pmatrix}_z}_{(|\psi\rangle)_{S_z}}$$

$$\underline{\text{SO}} \quad (|\psi\rangle_{S_x}) = \underline{S}^\dagger (|\psi\rangle)_{S_z}$$

$\underline{S}^\dagger$ reps $R^\dagger$ IN S_z-BAS. IE (with rot ∡ $\frac{\pi}{2}\hat{j}$)

$$\begin{pmatrix}\langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle\end{pmatrix} = \begin{pmatrix}\langle +z|R^\dagger(\frac{\pi}{2}\hat{j})|+z\rangle & \langle +z|R^\dagger(\frac{\pi}{2}\hat{j})|-z\rangle \\ \langle -z|R^\dagger(\frac{\pi}{2}\hat{j})|-z\rangle & \langle -z|R^\dagger(\frac{\pi}{2}\hat{j})|-z\rangle\end{pmatrix}$$

INVERSE? ① SAME DER. W $\left(|\psi\rangle\right)_{S_z}$ ON LHS

$$\Rightarrow \left(\underline{S}\right)_{S_z} = \begin{pmatrix} \langle +z|\hat{R}(\frac{\pi}{2}\hat{j})|+z\rangle & \langle +z|\hat{R}(\frac{\pi}{2}\hat{j})|-z\rangle \\ \langle -z|\hat{R}(\frac{\pi}{2}\hat{j})|+z\rangle & \langle -z|\hat{R}(\frac{\pi}{2}\hat{j})|-z\rangle \end{pmatrix}$$

$S^\dagger$ REPS $R^\dagger$, S REPS R!

ROTATE OPERATORS? (instead of states?)

$$\hat{A} \xrightarrow{S_x\text{-BAS}} \begin{pmatrix} \langle x|\hat{A}|x\rangle & \langle x|\hat{A}|-x\rangle \\ \underbrace{\langle -x|\hat{A}|x\rangle}_{=\langle -z|\hat{R}^\dagger(\frac{\pi}{2}\hat{j})\,\hat{A}\,\hat{R}(\frac{\pi}{2}\hat{j})|z\rangle} & \langle -x|\hat{A}|-x\rangle \end{pmatrix} = \left(\underline{A}\right)_{S_x}$$

* MAT. REP OF PRODUCT OF $\hat{O}$'s?

$$\underbrace{\langle i|\hat{A}\hat{B}|j\rangle}_{=(AB)_{ij}} = \langle i|\hat{A}\Big(\sum_k |k\rangle\langle k|\Big)\hat{B}|j\rangle = \sum_k \langle i|\hat{A}|k\rangle\langle k|\hat{B}|j\rangle = \sum_k A_{ik}B_{kj} = \underline{A}\cdot\underline{B}$$

MATRIX MULT!

SO MATRIX REP. OF PRODUCT OF OPS

= PRODUCT OF MAT. REP OF OPS

$$\Longrightarrow \boxed{\left(\underline{A}\right)_{S_x} = S^\dagger\left(\underline{A}\right)_{S_z} S}$$

(linear alg.: just a change of basis!)

APPLICATION   COMPUTE $(\hat{J}_z)_{S_x}$ (note: z & x!)

ABOVE $\Rightarrow$ $(\hat{J}_z)_{S_x} = \underline{S}^\dagger (\hat{J}_z)_{S_z} \underline{S}$ NEED $\underline{S}$

$$|\pm x\rangle = \frac{1}{\sqrt{2}}\left(\pm|+z\rangle + |-z\rangle\right)$$ (WLOG choice to make purely real)

$$\underline{S} = \begin{pmatrix} \langle z|x\rangle & \langle z|-x\rangle \\ \langle -z|x\rangle & \langle -z|-x\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix}$$

$$\underline{S}^\dagger = \underline{S}^{T*} = \underline{S}^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad (J_z)_{S_z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore (\hat{J}_z)_{S_x} = \frac{1}{2}\frac{\hbar}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{\hbar}{4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

OFF DIAGONAL? $\Longrightarrow$ $|\pm x\rangle$ NOT EIGENVECTORS OF J_z
(why?)

EIG-STATES ARE BASIS-IND

EG:

$$(\hat{J}_z)_{S_x}\left(|+z\rangle\right)_{S_x} = \frac{\hbar}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2}\left(|+z\rangle\right)_{S_x} \checkmark$$

$$|z\rangle \xrightarrow{S_x\text{-BAS}} \begin{pmatrix} \langle x|z\rangle \\ \langle -x|z\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

ALT: (express in z-basis and rotate)

$$\left(|z\rangle\right)_{S_x} = \underline{S}^\dagger\left(|z\rangle\right)_{S_z} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \checkmark$$

ROTATION NON-COMMUTATION

($\underline{R}$ act on $v \in \mathbb{R}^3$)

$$\underline{R}(\phi\hat{\imath}) = \begin{pmatrix} 1 & & \\ & C_\phi & -S_\phi \\ & S_\phi & C_\phi \end{pmatrix} \qquad \underline{R}(\phi\hat{\jmath}) = \begin{pmatrix} C_\phi & & S_\phi \\ & 1 & \\ -S_\phi & & C_\phi \end{pmatrix} \qquad \underline{R}(\phi\hat{k}) = \begin{pmatrix} C_\phi & -S_\phi & \\ S_\phi & C_\phi & \\ & & 1 \end{pmatrix}$$

$$C_\phi := \cos\phi \qquad S_\phi := \sin\phi$$

NOTE: - $\det \underline{R} = \mathbb{1}$

- ϕ SMALL: $C_\phi = 1 - \frac{1}{2}\phi^2 + \dots \qquad \sin\phi = \phi + \dots$

- ROT. DON'T COMMUTE! (demonstrate)

- NEITHER DO MATRICES $AB \neq AB$ GEN.

DEFINITION THE EXTENT TO WHICH TWO OPERATORS A, B DO NOT COMMUTE IS MEASURED BY THE COMMUTATOR

$$[A, B] = AB - BA$$

(if matrix reps, just compute!)

$$\underline{R}(\phi\hat{\imath})\,\underline{R}(\phi\hat{\jmath}) = \begin{pmatrix} C_\phi & 0 & S_\phi \\ S_\phi^2 & C_\phi & -S_\phi C_\phi \\ -S_\phi C_\phi & S_\phi & C_\phi^2 \end{pmatrix} \simeq \begin{pmatrix} 1-\frac{1}{2}\phi^2 & 0 & \phi \\ \phi^2 & 1-\frac{1}{2}\phi^2 & -\phi \\ -\phi & \phi & 1-\frac{1}{2}\phi^2 \end{pmatrix} + \mathcal{O}(\phi^3)$$

$$\underline{R}(\phi\hat{\jmath})\,\underline{R}(\phi\hat{\imath}) = \begin{pmatrix} C_\phi & S_\phi^2 & S_\phi C_\phi \\ & C_\phi & -S_\phi \\ -S_\phi & C_\phi S_\phi & C_\phi^2 \end{pmatrix} \simeq \begin{pmatrix} 1-\frac{1}{2}\phi^2 & \phi^2 & \phi \\ 0 & 1-\frac{1}{2}\phi^2 & -\phi \\ -\phi & \phi & 1-\frac{1}{2}\phi^2 \end{pmatrix}$$

$$[\underline{R}(\phi\hat{\imath}), \underline{R}(\phi\hat{\jmath})] \simeq \begin{pmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underline{R}[\phi^2\hat{k}] - \mathbb{1}$$

(gen rot around third axis!)

$*_2$

* NOTE: DON'T COMM. @ $\mathcal{O}(\phi^2)$

IMPLICATIONS FOR GENERATORS $\hat{J}_x$ & $\hat{J}_y$

RECALL $\underline{R}$ IS MATRIX REP OF $\hat{R}$

$$\hat{R}(\phi\hat{k}) = e^{-i\hat{J}_z\phi/\hbar} \quad \& \quad \hat{R}(\phi\hat{i}) = e^{-i\hat{J}_x\phi/\hbar} \quad \& \quad \hat{R}(\phi\hat{j}) = e^{-i\hat{J}_y\phi/\hbar}$$

ϕ SMALL $\quad \hat{R}(\phi\hat{n}) \simeq 1 - i\,\vec{J}\cdot\hat{n}\,\frac{\phi}{\hbar} + \frac{1}{2!}\left(\frac{i\vec{J}\cdot\hat{n}\,\phi}{\hbar}\right)^2 + \dots$

$$\text{COMM.} \Rightarrow \left(\mathbb{1} - \frac{i\hat{J}_x\phi}{\hbar} - \frac{1}{2}\left(\frac{\hat{J}_x\phi}{\hbar}\right)^2\right)\left(\mathbb{1} - \frac{iJ_y\phi}{\hbar} - \frac{1}{2}\left(\frac{J_y\phi}{\hbar}\right)^2\right)$$

$- (\text{second})(\text{first}) + \dots$

$$= \left\{ -\frac{i\phi}{\hbar}\left[J_x + J_y - J_y - J_x\right] + \left(\frac{\phi}{\hbar}\right)^2\left(J_xJ_y + \frac{J_x^2}{2} + \frac{J_y^2}{2} - J_yJ_x - \frac{J_y^2}{2} - \frac{J_x^2}{2}\right)\right\} + \dots$$

$$= \left\{-\frac{1}{2}\left(\frac{\phi}{\hbar}\right)^2\left[J_xJ_y - J_yJ_x\right]\right\} = -\left(\frac{\phi}{\hbar}\right)^2\left[\hat{J}_x, \hat{J}_y\right]$$

$= (*_2)$ LHS

$(*_2)$ RHS $= \left\{\mathbb{1} - \frac{iJ_z\phi^2}{\hbar}\right\} - \mathbb{1} \quad \Rightarrow \quad \| \boxed{[J_x, J_y] = i\hbar J_z} \|$

GEN $\quad \| \boxed{[J_i, J_j] = i\hbar\,\epsilon^{ijk} J_k} \| \qquad \epsilon^{ijk} = \begin{cases} 0 & \text{ind rep} \\ 1 & \text{even perm } 1,2,3 \\ -1 & \text{odd perm } 1,2,3 \end{cases}$

$\Rightarrow [J_z, J_z] = 0 \qquad [J_z, J_x] = i\hbar J_x \qquad [J_y J_x] = -[J_x, J_y] = -i\hbar J_z$