QM Lecture 15

QM IN 3D (speculate with me: what happens?)

NOTATION CHANGES

POSITION EIG. $\quad |\vec{r}\rangle = |\overbrace{x, y, z}^{\text{SIMULT. EIG.}}\rangle$

$$\hat{x}|\vec{r}\rangle = x|\vec{r}\rangle \qquad \hat{y}|\vec{r}\rangle = y|\vec{r}\rangle \qquad \hat{z}|\vec{r}\rangle = z|\vec{r}\rangle$$

ARB. STATES $\quad |\psi\rangle = \int d^3r\, |\vec{r}\rangle\langle\vec{r}|\psi\rangle \longrightarrow \iiint_{-\infty}^{\infty} dx\,dy\,dz\; |x,y,z\rangle\langle x,y,z|$

ORTHOGONALITY $\quad \langle\vec{r}|\vec{r}\,'\rangle = \delta^3(\vec{r}-\vec{r}\,') \longrightarrow \langle xyz|x'y'z'\rangle = \delta(x-x')\delta(y-y')\delta(z-z')$

NORM. $\quad 1 = \langle\psi|\psi\rangle = \int d^3r\, |\langle\vec{r}|\psi\rangle|^2 \longrightarrow \iiint dx\,dy\,dz\; |\langle xyz|\psi\rangle|^2$

PROB. DENS $\quad P(\vec{r})\, d^3r = |\langle\vec{r}|\psi\rangle|^2\, d^3r$

↳ prob of finding in some volume $d^3\vec{r}$ near $\vec{r}$

SPATIAL TRANSLATIONS

$$\hat{T}(a)|\vec{r}\rangle = |\vec{r}+\vec{a}\rangle \longrightarrow \begin{cases} \hat{T}(a_x\hat{i})|xyz\rangle = |x+a_x, y, z\rangle \\ \hat{T}(a_y\hat{j})|xyz\rangle = |x, y+a_y, z\rangle \\ \hat{T}(a_z\hat{k})|xyz\rangle = |x, y, z+a_z\rangle \end{cases}$$

LINEAR MOMENTA

$$\vec{T}(a_i)\, e^{-i\hat{p}_i a_i/\hbar} \qquad i = x, y, z$$

Q: what should position/mom. comm rel be?

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$$

$\therefore$ $[x,y]=[y,z]=[x,z]=0$ (SIMUL. EIG)

Q: what about momentum commutators?

$$\hat{T}(a_y \hat{j})\hat{T}(a_x\hat{i}) - \hat{T}(a_x\hat{i})\hat{T}(a_y\hat{j}) = 0$$

$$= \left[e^{-ip_y a_y/\hbar}, e^{-ip_x a_x/\hbar}\right]$$

$$= \left[1 - \frac{i\hat{p}_y a_y}{\hbar} - \frac{\hat{p}_y^2 a_y^2}{2\hbar^2} + \ldots,\ 1 - \frac{ip_x a_x}{\hbar} - \frac{1}{2}\frac{p_x^2 a_x^2}{\hbar^2} + \ldots\right]$$

$\Rightarrow$ CONDS. ON EACH $a_x^m a_y^n$ TERM

$a_x a_y$ TERM $\Rightarrow$ $-\frac{a_x a_y}{\hbar^2}[p_y, p_x] = 0$

$$\Rightarrow \boxed{[p_x, p_y] = [p_x, p_z] = [p_y, p_z] = 0}$$

CORR: $\hat{T}(\vec{a}) = e^{-ip_x a_x/\hbar} e^{-ip_y a_y/\hbar} e^{-ip_z a_z/\hbar} = e^{-i\vec{p}\cdot\vec{a}/\hbar}$

$$[x, \hat{T}(a_y\hat{j})] = 0 \Rightarrow [\hat{x}, p_y] = 0$$

SUMMARY $\boxed{[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}}$

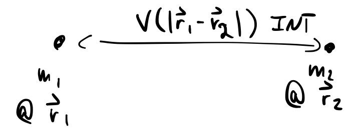
MOMENTUM OP. IN POS. SPACE $\langle \vec{r}|\psi\rangle = \psi(\vec{r})$

$$\langle \vec{r}|\vec{p}|\psi\rangle = \frac{\hbar}{i}\vec{\nabla}\langle \vec{r}|\psi\rangle$$

2016 vers

TWO BODY PROBLEMS & CENTRAL POTENTIALS

PICTURE



HAMILTONIAN any guesses?

$$\boxed{\hat{H} = \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + V(|\vec{r}_1-\vec{r}_2|)}$$

eg $\vec{p}_1^{\,2} = \hat{p}_{1x}^2 + \hat{p}_{1y}^2 + \hat{p}_{1z}^2$

$[\vec{p}_1, \vec{p}_2] = 0$

TAKE POS-BAS $|\vec{r}_1, \vec{r}_2\rangle$ (similar notation to two spin sys.)

Q: WHAT IF WE TRANSLATE m_1 & m_2 BY $\vec{a}$? $|\vec{r}_1,\vec{r}_2\rangle \to |\vec{r}_1+\vec{a}, \vec{r}_2+\vec{a}\rangle$

ie, both particles

(how does H act on it?)

A: $\hat{H}\,\vec{T}(\vec{a})\,|\vec{r}_1,\vec{r}_2\rangle = \vec{T}(\vec{a})\,\hat{H}\,|\vec{r}_1,\vec{r}_2\rangle$

$\parallel$

$\hat{H}\,|\vec{r}_1+\vec{a}, \vec{r}_2+\vec{a}\rangle$ **NB** IF BOTH ARE $\hat{H}$-ESTATES

HAVE SAME E!

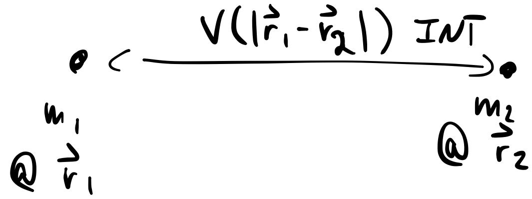
WHAT IS $\vec{T}(a)$? $T(\vec{a}) = e^{-i\vec{P}\cdot\vec{a}/\hbar}$

w/ $\vec{P} = \vec{P}_1 + \vec{P}_2 \Rightarrow \vec{T}(a) = e^{-i\vec{p}_1\cdot\vec{a}/\hbar}\, e^{-i\vec{p}_2\cdot\vec{a}/\hbar} = T_1(\vec{a})\,T_2(\vec{a})$

NOTE ALSO $[\hat{H}, \hat{\vec{P}}] = 0 \Rightarrow \frac{d\langle\hat{\vec{P}}\rangle}{dt} = 0$ SINCE $\frac{d\hat{\vec{P}}}{dt} = 0$

TWO BODY PROBLEMS & CENTRAL POTENTIALS

PICTURE



HAMILTONIAN any guesses?

$$\boxed{\hat{H} = \frac{\vec{P}_1^{\,2}}{2m} + \frac{\vec{P}_2^{\,2}}{2m} + V(|\vec{r}_1 - \vec{r}_2|)}$$

eg $\vec{P}_1^{\,2} = \hat{P}_{1x}^2 + \hat{P}_{1y}^2 + \hat{P}_{1z}^2$

$[\vec{P}_1, \vec{P}_2] = 0$

TAKE POS-BAS $|\vec{r}_1, \vec{r}_2\rangle$ (similar notation to two spin sys.)

ie, both particles

HOW DO WE TRANSLATE m_1 & m_2 BY $\vec{a}$? $|\vec{r}_1, \vec{r}_2\rangle \rightarrow |\vec{r}_1 + \vec{a}, \vec{r}_2 + \vec{a}\rangle$

(how does H act on it?)

$$T_1(\vec{a})\, T_2(\vec{a})\, |\vec{r}_1, \vec{r}_2\rangle = e^{i\vec{p}_1 \cdot \vec{a}/\hbar}\, e^{i\vec{p}_2 \cdot \vec{a}/\hbar}\, |\vec{r}_1, \vec{r}_2\rangle = |\vec{r}_1 + \vec{a}, \vec{r}_2 + \vec{a}\rangle$$

$$\parallel$$

$$T(\vec{a})\, |\vec{r}_1, \vec{r}_2\rangle = e^{i\vec{P}\cdot\vec{a}/\hbar}\, |\vec{r}_1, \vec{r}_2\rangle \implies \vec{P} = \vec{P}_1 + \vec{P}_2$$

PHYSICS: $[H, T] = 0$ $\therefore$ $[\hat{H}, \hat{P}] = 0$

NOTE ALSO $\implies \dfrac{d\langle\hat{P}\rangle}{dt} = 0$ SINCE $\dfrac{d\hat{\vec{P}}}{dt} = 0$

(translational invariance of the Hamiltonian guarantees total momentum conservation)

$([x_i, \hat{P}_j]) = 0$

CENTER OF MASS

Define

RELATIVE OPS

$$\hat{\vec{r}} = \hat{\vec{r}}_1 - \hat{\vec{r}}_2$$

$$\hat{\vec{p}} = \frac{m_2 \hat{\vec{p}}_1 - m_1 \hat{\vec{p}}_2}{m_1 + m_2} = \mu \frac{d\hat{\vec{r}}}{dt}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

COM OPS

$$\hat{\vec{R}} = \frac{m_1 \hat{\vec{r}}_1 + m_2 \hat{\vec{r}}_2}{m_1 + m_2}$$

$$\hat{\vec{P}} = \hat{\vec{p}}_1 + \hat{\vec{p}}_2$$

COMPONENTS

$\hat{p}_i \quad \hat{p}_{1i}$

$\hat{P}_i \quad \hat{p}_{2i}$

$\hat{x}_i \quad X_i$

$\hat{x}_{1i}$

$\hat{x}_{2i}$

FACTS

$$[\hat{X}_i, \hat{P}_j] = i\hbar \delta_{ij}$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

OTHERS COMMUTE

$$" \frac{1}{m_1+m_2} [\hat{x}_{1i} - \hat{x}_{2i}, m_2 \hat{p}_{1j} - m_1 \hat{p}_{2j}] = \frac{1}{m_1+m_2} i\hbar \{m_1 \delta_{ij} + m_2 \delta_{ij}\}$$

$$= i\hbar \delta_{ij} \checkmark$$

* POINT: USE $|\vec{r}, \hat{\vec{R}}\rangle$ BASIS INSTEAD OF $|\vec{r}_1, \vec{r}_2\rangle$

NOTE: 1) $V(|\vec{r}_1 - \vec{r}_2|) = V(|\vec{r}|)$

2) CLAIM $\hat{H}_{kin} = \dfrac{\hat{\vec{P}}^2}{2M} + \dfrac{\hat{\vec{p}}^2}{2\mu}$ $\quad M = m_1 + m_2$ ("total mass") $\quad \mu = \dfrac{m_1 m_2}{m_1 + m_2}$ ("reduced mass")

$$\frac{\vec{P}^2}{2M} = \frac{\hat{\vec{p}}_1^2 + \hat{\vec{p}}^2 + 2\,\hat{\vec{p}}_1 \cdot \hat{\vec{p}}_2}{2(m_1 + m_2)}$$

$$\frac{\vec{p}^2}{2\mu} = \frac{(m_1 + m_2)}{m_1 m_2} \quad \frac{m_2^2\,\hat{\vec{p}}_1^2 + m_1^2\,\hat{\vec{p}}_2^2 - 2 m_1 m_2\,\hat{\vec{p}}_1 \cdot \hat{\vec{p}}_2}{2(m_1 + m_2)^2}$$

$$\therefore \hat{H}_{kin} = \frac{1}{2(m_1 + m_2)} \cdot \left\{ \hat{\vec{p}}_1^2 \left(1 + \frac{m_2}{m_1}\right) + \hat{\vec{p}}_2^2 \left(1 + \frac{m_1}{m_2}\right) \right\}$$

$$= \frac{\hat{\vec{p}}_1^2}{2m_1} + \frac{\hat{\vec{p}}_2^2}{2m_2} \checkmark$$

$$\therefore \hat{H} = \underbrace{\hat{H}_{cm}}_{\frac{\hat{\vec{P}}^2}{2M}} + \underbrace{\hat{H}_{rel}}_{\frac{\hat{\vec{p}}^2}{2\mu} + V(|\hat{\vec{r}}|)}$$

(Q: Can you say anything about $\vec{H}$ e-states already?)
(A: CM & REL ops commute, $\therefore$ sol E-states of both parts separately.)

$$[CM, REL] = 0 \implies \hat{H}|E_{cm}, E_{rel}\rangle = (\hat{H}_{cm} + \hat{H}_{rel})|E_{cm}, E_{rel}\rangle$$
$$= (E_{cm} + E_{rel})|E_{cm}, E_{rel}\rangle = E_{TOT}|\ \rangle$$

ALREADY SOLVED $\hat{H}_{CM}$: FREE "PARTICLE" (system)

$$\langle \vec{R} | \vec{P} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{-i\vec{P}\cdot\vec{R}/\hbar}$$

<u>POINT</u> INTERESTING PART ALL IN $\hat{H}_{rel}$

(in CM frame $\vec{P}=0$, just seek sol to H_{rel})

(could solve many two body probs: spec. $V(|\vec{r}|)$!)

<u>ROT. INV.</u> <u>OF $\hat{H}_{rel}$</u>

Q: WHY CAN SIMP. USE ROT. INV?

A: $\hat{H}_{rel} = f(|\hat{\vec{p}}|^2, |\vec{r}|)$ LENGTHS

→ <u>NOT</u> SPIN. ANG MOM AS YOU KNOW IT!

<u>ROT</u> <u>OP.</u> $\hat{R}(\phi\hat{k}) = e^{-i\hat{L}_z\phi/\hbar}$

<u>RECALL</u> SPATIAL ROT

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

ϕ small $d\phi$ $\Longrightarrow$
$$A_x' = A_x - d\phi\, A_y$$
$$A_y' = A_y + d\phi\, A_x$$

$$\Longrightarrow \quad \hat R(d\phi\,\hat k)\,|x,y,z\rangle = |x - y\,d\phi,\, y + x\,d\phi,\, z\rangle$$

$$= \left[1 - \tfrac{i}{\hbar}\hat p_x(-\hat y\,d\phi)\right]\left[1 - \tfrac{i}{\hbar}\hat p_y(\hat x\,d\phi)\right]|x\,y\,z\rangle$$

$$= \left[1 - \tfrac{i}{\hbar}\left(\hat p_y \hat x - \hat p_x \hat y\right)d\phi + O(d\phi^2)\right]|x\,y\,z\rangle$$

$$\Longrightarrow \quad \hat R(d\phi\,\hat k) = 1 - \tfrac{i}{\hbar}\hat L_z\,d\phi = 1 - \tfrac{i}{\hbar}\left(\hat x \hat p_y - \hat y \hat p_x\right)$$

$$\Longrightarrow \quad \boxed{\hat L_z = \hat x \hat p_y - \hat y \hat p_x}$$

GEN. $\quad \boxed{\hat{\vec L} = \hat{\vec r} \times \hat{\vec p}}$

L IS ORBITAL ANG. MOMENTUM!

PROVE ROT. INV. $\quad$ ie $\quad [H, L_z] = 0$

by $\quad$ a) $[L_z, \vec r^{\,2}] = 0 \quad$ b) $[L_z, \vec p^{\,2}] = 0$

$$[L_z, \vec p^{\,2}] = [L_z, p_x^2 + p_y^2 + p_z^2] = p_x[L_z, p_x] + [L_z, p_x]p_x + p_y[L_z, p_y] + [L_z, p_y]p_y + p_z\underbrace{\cancel{[L_z, p_z]}}_{0} + \underbrace{\cancel{[L_z, p_z]}}_{0}p_z$$

$L_z = xp_y - yp_z$
COMM. p_z

$$[L_z, p_x] = [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x]$$

$$= -x\cancel{[p_y, p_x]} - [p_x, x]p_y + y\cancel{[p_x, p_x]} + \cancel{[p_x, y]}p_x$$

$$= i\hbar\, p_y$$

SIM $[L_z, p_y] = -i\hbar p_x$

$\therefore [L_z, \vec{p}^2] = i\hbar p_x p_y + i\hbar p_y p_x - i\hbar p_y p_x - i\hbar p_x p_y = 0$

ALSO $[L_z, \vec{r}^2] = 0$

$\Rightarrow [\hat{H}, \hat{L}_z] = 0 \qquad [\hat{H}, \hat{\vec{L}}^2] = 0$

$\therefore$ ENERGY EIG. OF $\hat{H}_{rel}$ SIMULT EIG OF E, l, m

$$\boxed{\begin{aligned} \hat{H}_{rel}|E,l,m\rangle &= E|E,l,m\rangle \\ \hat{\vec{L}}^2|E,l,m\rangle &= l(l+1)\hbar^2|E,l,m\rangle \\ L_z|E,l,m\rangle &= m\hbar|E,l,m\rangle \end{aligned}}$$

APP. OF $\vec{L}^2$ IN H_{rel}

NEED IDENTS

LET'S PROVE: $\vec{L}^2 = \vec{r}^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2 + i\hbar\,\vec{r}\cdot\vec{p}$

$\vec{L} = \vec{r}\times\vec{p} \xrightarrow{\text{comp}} L_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\, x_j p_k \overset{\text{Einstein}}{=:} \varepsilon_{ijk}\, x_j p_k$

$$\boxed{L_i = \varepsilon_{ijk}\, x_j p_k}$$

$\therefore \vec{L}^2 = \sum_{i=1}^{3} L_i L_i \overset{E}{=} L_i L_i = (\varepsilon_{ijk}\, x_j p_k)(\varepsilon_{imn}\, x_m p_n)$