QM Lecture 5

EIGENSTATES OF ANG. MOM.

- RECALL $[J_i, J_j] = i\hbar \varepsilon^{ijk} J_k$ (Einstein summation)
- SOME OPS. DON'T COMMUTE!
- WHAT ABOUT LENGTH OF $\hat{\vec{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$?

$$\hat{\vec{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

note: will often drop hats!

IDENTITY: $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$

$$[\hat{J}_z, \hat{J}^2] = [\hat{J}_z, \hat{J}_x^2] + [J_z, J_y^2] + [J_z, J_z^2]$$

$$= J_x[J_z, J_x] + [J_z, J_x]J_x + J_y[J_z, J_y] + [J_z, J_y]J_y$$

$$= i\hbar[J_x J_y + J_y J_x + J_y(-J_x) + (-J_x)J_y] = 0$$

$$\boxed{[J_i, J^2] = 0!}$$

(so what?)

SIMULTANEOUS EIGENSTATES

SUPPOSE $\hat{A}|a\rangle = a|a\rangle$, $|a\rangle$ is ONLY A E-VEC W E-VAL a (up to constant, $\hat{A}(\lambda|a\rangle) = a(\lambda|a\rangle)$) & $[\hat{A}, \hat{B}] = 0$

$$\Rightarrow \hat{B}\hat{A}|a\rangle = \hat{B}(a|a\rangle) = a\hat{B}|a\rangle$$

$[A,B]=0$: $\hat{A}\hat{B}|a\rangle \Rightarrow \hat{B}|a\rangle$ E-STATE OF $\hat{A}$ W E-VAL a.

$\Rightarrow \hat{B}|a\rangle = b|a\rangle \Rightarrow$ $\boxed{|a\rangle \text{ E-VEC OF } \hat{B} \text{ \& } \hat{A}, \text{ FROM } [\hat{A}, \hat{B}] = 0}$

"ONLY" needed here

say "$|a\rangle$ is a simult. e-state of A & B"

$\underline{\hat{J}}$: $[J_i, J^2] = 0 \Rightarrow$ TWO EIGENVALUES FOR STATE, CALL $|\lambda, m\rangle$

$$\hat{\vec{J}}^2 |\lambda, m\rangle = \lambda\hbar^2 |\lambda, m\rangle$$

$$\hat{J}_z |\lambda, m\rangle = m\hbar |\lambda, m\rangle$$

HIGHER DIM. SPIN REPS

(much more general than just "up" and "down")

"UP, DOWN" = 2-D REP.

$\exists$ N-D REPS. $\forall N \in \mathbb{N} = \{1, 2, 3, \dots\}$

WANT: SYSTEMATIC THY FOR N-DIM SPIN REP
(will have raising, lowering ops)

$*$ BEING A SPIN REP REQ. MAT. REP'S OF J_i WITH CORRECT $[J_i, J_j]$

EX SPIN 1: $J_z = -\hbar, 0, \hbar \Rightarrow 3\times 3$ MATRICES

$$\hat{J}_z \to \hbar\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \qquad \hat{J}_x \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_y \to \frac{\hbar}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}$$

Q: IS THIS A REPRESENTATION? CHECK $[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon^{ijk}\hat{J}_k$

$$\hat{J}_x \cdot \hat{J}_z = \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} & & \\ 1 & & -1 \\ & & \end{pmatrix} \qquad J_z \cdot J_x = \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} & 1 & \\ & & \\ & -1 & \end{pmatrix}$$

$$\therefore [\hat{J}_x, \hat{J}_z] = \frac{\hbar^2}{\sqrt{2}}\begin{pmatrix} & -1 & \\ 1 & & -1 \\ & 1 & \end{pmatrix} = i\hbar J_y \checkmark$$

OTHER $[\cdot,\cdot]$'s HOLD TOO!

FURTHERMORE CAN CHECK $[\hat{J}_i, \hat{J}^2] = 0$ $\quad \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = 2\hbar^2 \cdot \mathbb{I}_{3\times 3}$

BASIS STATES $|\lambda, m\rangle = |1, 1\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad |\lambda, m\rangle = |1, 0\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad |\lambda, m\rangle = |1, -1\rangle \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

RAISING & LOWERING OPS: $\hat{J}_\pm := \hat{J}_x \pm i\hat{J}_y$

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y = \sqrt{2}\,\hbar\begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix} \qquad \hat{J}_- = \hat{J}_x - i\hat{J}_y = \sqrt{2}\,\hbar\begin{pmatrix} & & \\ 1 & & \\ & 1 & \end{pmatrix}$$

$\hat{J}_+\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $J_+\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{J}_+\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0!$ $\quad J_+$ RAISES m, ANNIHILATES "HIGHEST" m ST.

SIM: $\hat{J}_-\begin{pmatrix}1\\0\\0\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\hat{J}_-\begin{pmatrix}0\\1\\0\end{pmatrix} = \sqrt{2}\hbar\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\hat{J}_-\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$

J_- LOWERS m, ANNI. LOWEST m STATE

PT: IF KNEW HIGHEST m STATE, COULD "BUILD UP" REST WITH J_-!
" " LOWEST " " " " " " J_+!

Q: $J_\pm$ MORE GEN?

$$[J_z, J_\pm] = [J_z, J_x \pm iJ_y] = i\hbar J_y \pm i(-i\hbar J_x) = \hbar(iJ_y \pm J_x)$$

$$\boxed{[J_z, J_\pm] = \pm\hbar J_\pm}$$ (this is universally true)

$$J_z J_\pm |\lambda, m\rangle = (J_\pm J_z \pm \hbar J_\pm)|\lambda, m\rangle = (\hat{J}_\pm m\hbar \pm \hbar \hat{J}_\pm)|\lambda, m\rangle$$
$$= (m \pm 1)\hbar J_\pm |\lambda, m\rangle$$

$\Rightarrow$ $J_\pm|\lambda, m\rangle$ E-STATE OF J_z W/ E-VAL $(m\pm 1)\hbar$

$$\boxed{J_\pm|\lambda, m\rangle = c_\pm|\lambda, m\pm 1\rangle}$$ (more on constants later)

ALSO $\vec{J}^2 J_\pm|\lambda, m\rangle = \lambda\hbar^2 J_\pm|\lambda, m\rangle$

SPECTRUM OF ANG. MOM. OPS

FACT ① (convince yourself from $\langle\lambda,m|\hat{J}_x^2+\hat{J}_y^2|\lambda,m\rangle = (\lambda-m^2)\hbar^2\langle\lambda,m|\lambda,m\rangle$)

$\lambda \geq m^2$

$\Rightarrow$ FACT ② $\exists$ SOME MAX m, CALL j

DEFINED BY $\hat{J}_+|\lambda,j\rangle = 0$

Q: WHAT IS j?

$J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y]$

$= \vec{J}^2 - J_z^2 - \hbar J_z$

$\Rightarrow |\lambda,m\rangle$ ARE E-STATES OF $J_\pm$!

$J_+|\lambda,j\rangle = 0 \Rightarrow 0 = J_-J_+|\lambda,j\rangle = (\lambda - j^2 - j)\hbar^2|\lambda,j\rangle$

$$\Rightarrow \boxed{\lambda = j(j+1)}$$

Q: WHAT IS m_{min}? (any guesses?)

A: $m_{min} = -j$

PROOF: m_{min} DEF BY $J_-|\lambda, j_{min}\rangle = 0$

BUT $J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y] = \vec{J}^2 - J_z^2 + \hbar\hat{J}_z$

$\therefore\ 0 = J_+J_-|\lambda, m_{min}\rangle = (\lambda - m_{min}^2 + m_{min})\hbar^2|\lambda, m_{min}\rangle$

$\Rightarrow \lambda = m_{min}(m_{min}-1) \overset{\text{above}}{=} j(j+1)$

TWO SOLS: a) $\boxed{m_{min} = -j}$ b) $m_{min} = j+1$

good? NO (would $\Rightarrow$ j is not max state)

LEARNED 2 THINGS

1) * J_- LOWERS M FROM $m_{max} = j$ TO $m_{min} = -j$ BY INTEGERS

$$\Rightarrow j - (-j) \in \mathbb{Z} \Rightarrow 2j \in \mathbb{Z} \Rightarrow j \in \mathbb{Z}/2$$

$$\boxed{j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2 \ldots}$$

2) PICK j. THEN $\boxed{\exists \ 2j+1 \ \text{STATES}}$

E.G. j EVEN $\Rightarrow m \in \{j, j-1, j-2, \ldots, 0, \ldots, -j+1, -j\}$

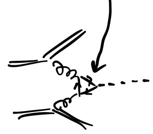
MODERN PHYS NOTATION $|j,m\rangle$ NOT $|\lambda, m\rangle$

$$\vec{J}^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle$$

$$J_z |j,m\rangle = m\hbar |j,m\rangle$$

EXAMPLE REPS

$j=0$	$j=\frac{1}{2}$	$j=1$	$j=\frac{3}{2}$
$\lvert 0,0\rangle$	$\lvert \frac{1}{2},\frac{1}{2}\rangle$ = " $\lvert +z\rangle$ "	$\lvert 1,1\rangle$	$\lvert \frac{3}{2},\frac{3}{2}\rangle$
	$\lvert \frac{1}{2},-\frac{1}{2}\rangle$ = " $\lvert -z\rangle$ "	$\lvert 1,0\rangle$	$\lvert \frac{3}{2},\frac{1}{2}\rangle$
		$\lvert 1,-1\rangle$	$\lvert \frac{3}{2},-\frac{1}{2}\rangle$
			$\lvert \frac{3}{2},-\frac{3}{2}\rangle$
EG: HIGGS!	ELECTRON	W-BOSON	GRAVITINO?
	TOP	Z-BOSON	



FIX CONSTANTS $C_\pm$ $\quad J_\pm|j,m\rangle = \hbar\, C_\pm |j,m\pm 1\rangle$

$$\langle j,m|J_+^\dagger J_+|j,m\rangle = C_+^* C_+ \hbar^2 \langle j,m+1|j,m+1\rangle$$

NOTE: $J_+^\dagger = J_x^\dagger + (iJ_y)^\dagger = J_x - iJ_y = J_-$

$$\boxed{J_\pm^\dagger = J_\mp}$$

$\therefore \quad J_+^\dagger J_+ = J_- J_+ \overset{\text{earlier}}{=} \vec{J}^2 - J_z^2 - \hbar J_z$

$$|C_+|^2 \hbar^2 \langle j,m+1|\cancel{j,m+1}\rangle^{1} = \langle j,m|\vec{J}^2 - J_z^2 - \hbar J_z|j,m\rangle$$
$$= \hbar^2\left(j(j+1) - m^2 - m\right)\langle j,m|\cancel{j,m}\rangle^{1}$$

RHS REAL $\Rightarrow$
$$\boxed{\begin{aligned} C_+ &= [j(j+1) - m(m+1)]^{\frac{1}{2}} \\ C_- &= [j(j+1) - m(m-1)]^{\frac{1}{2}} \end{aligned}}$$
(explain this)

just say depending on time:

CHECK RECALL $J_+|j,m=j\rangle = 0$

$J_-|j,m=-j\rangle = 0$

BUT $C_+\big|_{m=j} = 0$ & $C_-\big|_{m=-j} = 0$ ✓

$|j,m\rangle$ MATRIX ELEMENTS $\quad \langle j,m'|j,m\rangle = \delta_{m,m'}$

$$\boxed{\begin{aligned} \langle j,m'|J_+|j,m\rangle &= \sqrt{j(j+1) - m(m+1)}\,\hbar \langle j,m'|j,m+1\rangle = \sqrt{j(j+1) - m(m+1)}\,\hbar\, \delta_{m',m+1} \\ \langle j,m'|\hat{J}_-|j,m\rangle &= \sqrt{j(j+1) - m(m-1)}\,\hbar\, \delta_{m',m-1} \end{aligned}}$$

EX: $j=1$ $\quad |1,1\rangle =: |+\rangle \quad |1,0\rangle =: |0\rangle \quad |1,-1\rangle =: |-\rangle$

$|+\rangle, |0\rangle, |-\rangle$ O-NORM BASIS.

IN THIS BASIS: $|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$J_+|+\rangle = 0 \quad J_+|0\rangle = \sqrt{2+0}\,\hbar\,|+\rangle = \sqrt{2}\,\hbar\,|+\rangle \quad J_+|-\rangle = \sqrt{2-(-1)\cdot 0}\,\hbar\,|0\rangle = \sqrt{2}\,\hbar\,|0\rangle$$

$$\Rightarrow \quad J_+ \xrightarrow{+0-\text{ BAS}} \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

SIM. $\quad J_- \xrightarrow{+0-} \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

NOTE: RAISES / LOWERS!