

QM Lecture 14

LAST TIME T-INDEP SCH. EQN FOR HARM. OSC.

$\Psi_n(x)$ ODD FOR n ODD

$\Psi_n(x)$ EVEN FOR n EVEN

PARITY

DEFINE $\hat{\pi}$ S.T. $\hat{\pi}|x\rangle = |-x\rangle$

NOTE $\hat{\pi}^2 = \underline{1}$

$\hat{\pi}$ E-STATES

$$\hat{\pi}|\Psi_x\rangle = \lambda|\Psi_x\rangle \Rightarrow \hat{\pi}^2|\Psi_x\rangle = \lambda^2|\Psi_x\rangle = \underline{1}|\Psi_x\rangle = |\Psi_x\rangle$$

$$\Rightarrow \lambda = \pm 1$$

E-STATES $\lambda = 1$ $|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |-x\rangle)$

$\lambda = -1$ $|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|x\rangle - |-x\rangle)$

POSITION SPACE $\Psi_+(x) = \langle x|\Psi_+\rangle = \frac{1}{\sqrt{2}}(\Psi(x) + \Psi(-x))$

$$\Psi_-(x) = \langle x|\Psi_-\rangle = \frac{1}{\sqrt{2}}(\Psi(x) - \Psi(-x))$$

$$\Psi(x) \text{ EVEN} \Rightarrow \Psi(x) = \sqrt{2}\Psi_+(x) \Rightarrow \lambda = 1$$

$$\Psi(x) \text{ ODD} \Rightarrow \Psi(x) = \sqrt{2}\Psi_-(x) \Rightarrow \lambda = -1$$

\Rightarrow H.O. E-STATES ARE $\hat{\pi}$ (PARITY) E-STATES

HOW? HAPPENS IF $[\hat{\pi}, \hat{f}] = 0$

$$\langle x | \hat{\pi} | \hat{f} \rangle = \langle -x | \hat{H} | \hat{f} \rangle = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(-x) \right] \hat{f}(x) \quad (8)$$

$$\langle x | \hat{H} | \hat{f} \rangle = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \hat{f}(x)$$

$$(8) - (8) = 0 \text{ REQ } V(x) = V(-x)$$

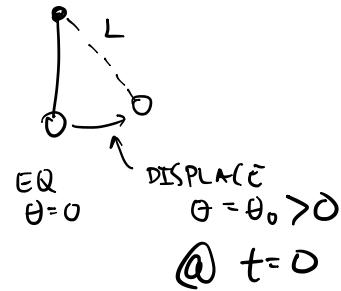
SATISFIED BY $V(x) = \frac{1}{2}m\omega^2 x^2$!
? SQUARE WELL!

SIGN N?

↓ norm
in case

$x=0$

TIME DEP. AND S.H.D.



Q: WHAT CORRESPONDS
TO MOTION OF
DISPLACED OSC?

- NOT $\langle n | \hat{x} | n \rangle$, WHICH $= 0$. NOR $\langle n | \hat{p}_x | n \rangle = 0$.

- EXPECTED SYMMETRY?

$t=0$ PROB $P(\theta, t=0) d\theta$ SHOULD FAVOR $\theta > 0$

$t = \frac{\pi}{2} = \pi \sqrt{\frac{L}{g}}$ $P(\theta, t=\frac{\pi}{2}) d\theta$ " " $\theta < 0$

NOTE $\int |\psi_n(x)|^2 dx$ SYMMETRIC \times
all $\neq 0$

A: A COHERENT STATE, $|\psi_{coh}\rangle = \sum c_n |n\rangle$

$$\underline{\text{WARM UP:}} \quad |\psi\rangle_{t=0} = |\psi(0)\rangle = C_n |n\rangle + C_{n+1} |n+1\rangle$$

(very special superposition, but a good start.)

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \quad \hat{H} = (\hat{N} + \frac{1}{2})\hbar\omega \\ &= C_n e^{-i(n+\frac{1}{2})\omega t} |n\rangle + C_{n+1} e^{-i(n+\frac{3}{2})\omega t} |n+1\rangle \\ &= e^{-i(n+\frac{1}{2})\omega t} (C_n |n\rangle + C_{n+1} e^{-i\omega t} |n+1\rangle) \end{aligned}$$

OSCILLATE?

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} [C_n^* \langle n| + C_{n+1}^* e^{i\omega t} \langle n+1|] (\hat{a} + \hat{a}^\dagger) [C_n |n\rangle + C_{n+1} e^{-i\omega t} |n+1\rangle]$$

use a, a^\dagger rels.

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(C_n^* C_{n+1} \sqrt{n+1} e^{-i\omega t} + C_{n+1}^* C_n \sqrt{n+1} e^{i\omega t} \right)$$

$$\frac{A_n e^{i\delta}}{2} := \sqrt{\frac{\hbar}{2m\omega}} C_{n+1}^* C_n \sqrt{n+1}$$

$$\Rightarrow \boxed{\langle x \rangle = \frac{A_n e^{-i(\delta + \omega t)}}{2} + \frac{A_n}{2} e^{i(\delta + \omega t)} = A_n \cos(\omega t + \delta)}$$

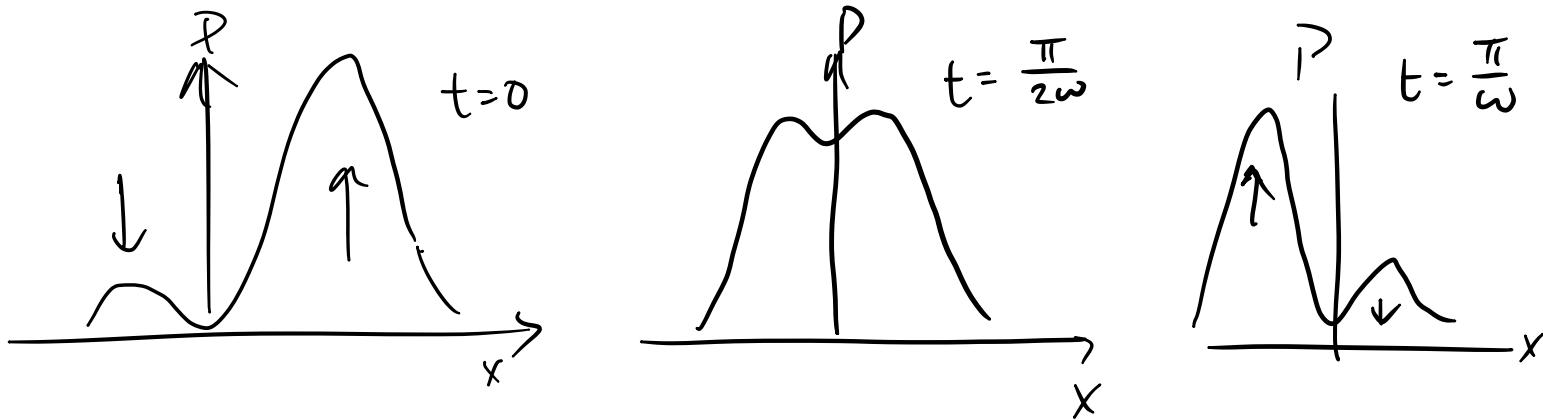
NB: $f = \frac{\omega}{2\pi}$, AMP A, PHASE δ . MATCHES CLASSICAL EXPECT!

$$\text{EX: } |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\psi(t)\rangle = e^{-i\omega t/2} \left(|0\rangle + e^{-i\omega t} |1\rangle \right)$$

$$\langle x \rangle = A \cos \omega t \quad (\delta=0)$$

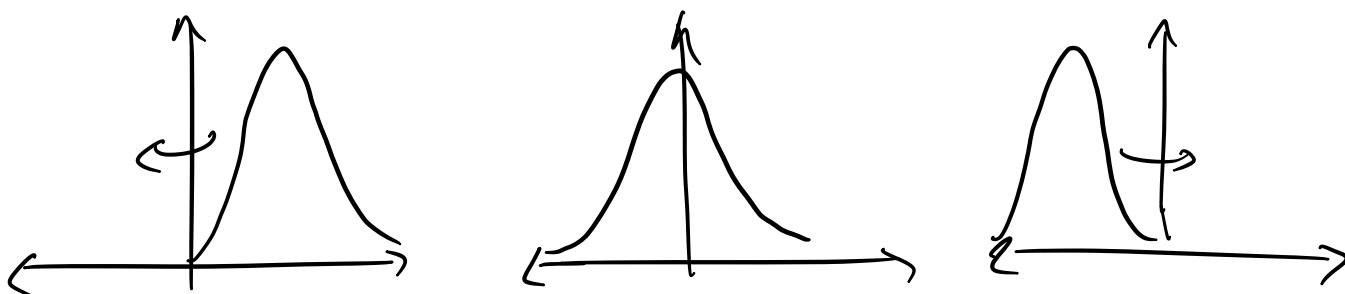
PLOT $|\langle \psi(t) | x \rangle \langle x | \psi(t) \rangle|^2 = P(x, t)$



GET SLOSHING

COHERENT STATES

WHAT MOVES LIKE THIS?



WANT: GAUSSIAN THAT MOVES BACK AND FORTH

BETWEEN CLASSICAL T.P.'S w/ PER. $T = \frac{2\pi}{\omega}$

→ COHERENT STATE

- COH. ST. ARE \hat{a} E-ST.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad (*)$$
$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad (**)$$

$c_n = ?$ $\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} \sqrt{n} c_n |n-1\rangle \stackrel{!}{=} \alpha \sum_{n=0}^{\infty} c_n |n\rangle$

RELABEL $n' = n-1$, THEN n' BACK TO n

$$\sum_{n=0}^{\infty} \sqrt{n+1} c_{n+1} |n\rangle = \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

$$\Rightarrow c_{n+1} = \frac{\alpha c_n}{\sqrt{n+1}} = \frac{\alpha^n}{\sqrt{n+1} \sqrt{n}} c_1$$

GEN: $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0 \quad \Rightarrow \quad |\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

$$1 = \langle \alpha | \alpha \rangle = |c_0|^2 \left[\sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \langle n' | \right] \left[\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right]$$
$$= |c_0|^2 \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{(\alpha^*)^{n'}}{\sqrt{n'!}} \frac{\alpha^n}{\sqrt{n!}} \delta_{nn'} = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2}$$

$$\Rightarrow c_0 = e^{-|\alpha|^2/2}$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

PONDER ① SUP. OF ALL EN. E-ST.

② LARGER $|\alpha| \Rightarrow$ LARGE n STATES MATTER more

STILL NEED TO SEE:

- 1) $|\alpha(\epsilon)\rangle$ GAUSSIAN At
- 2) "BEHAVES CLASSICALLY"

TIME EV. OF COH. ST.

$$|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar} |\alpha\rangle = e^{-| \alpha |^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i(\alpha + \frac{i}{2})\omega t} |n\rangle$$

$$= e^{-i\omega t/2} e^{-| \alpha |^2/2} \sum_{n=0}^{\infty} \underbrace{\left(\frac{e^{-i\omega t}}{\sqrt{n!}} \right)^n}_{\text{SAME AS } |\alpha\rangle \text{ EXP. IF REPL. } \alpha \rightarrow \alpha e^{-i\omega t}} |n\rangle$$

(CALL $|\alpha e^{-i\omega t}\rangle$)

$$\Rightarrow \boxed{|\alpha(t)\rangle = e^{-i\omega t/2} |\alpha e^{-i\omega t}\rangle}$$

(Q: why might this bother you? why should it not?)

CLASSICAL BEHAV. WANT TO VERIFY EHRENFEST'S THM.

$$\frac{d\langle p_x \rangle}{dt} = \left\langle -\frac{dV}{dx} \right\rangle \quad \text{w/} \quad \langle p_x \rangle = m \frac{d\langle x \rangle}{dt}$$

$$\Rightarrow \langle x \rangle = \langle \alpha(t) | \hat{x} | \alpha(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha(t) | (\hat{a} + \hat{a}^\dagger) | \alpha(t) \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} [\alpha(t) + \alpha^*(t)] \quad \text{LET } \alpha = |\alpha| e^{-i\delta}$$

$$\therefore \langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} 2|\alpha| \cos(\omega t + \delta)$$

$$\Rightarrow \langle p_x \rangle = \langle \alpha(t) | \hat{p}_x | \alpha(t) \rangle = -i \sqrt{\frac{m\omega\hbar}{2}} \langle \alpha(t) | (\hat{a} - \hat{a}^\dagger) | \alpha(t) \rangle$$

$$= -i \sqrt{\frac{m\omega\hbar}{2}} [\alpha(t) - \alpha^*(t)]$$

$$\therefore \langle p_x \rangle = -\sqrt{\frac{m\omega\hbar}{2}} 2|\alpha| \sin(\omega t + \delta)$$

$$\frac{d\langle x \rangle}{dt} = -\omega \sqrt{\frac{\hbar}{2m\omega}} 2|\alpha| \sin(\omega t + \delta) = \frac{\langle p_x \rangle}{m} \checkmark$$

$$\frac{d\langle p_x \rangle}{dt} = -\omega \sqrt{\frac{m\omega\hbar}{2}} 2|\alpha| \cos(\omega t + \delta)$$

$$\left\langle -\frac{dV}{dx} \right\rangle = -m\omega^2 \langle x \rangle$$

$$= -\omega \sqrt{\frac{m\omega\hbar}{2}} 2|\alpha| \cos(\omega t + \delta)$$

EHRENFEST'S THM. IS SATISFIED!

GAUSSIAN? MUST HAVE MIN UNC. CHECK.

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \langle \alpha(t) | (\alpha + \alpha^\dagger)^2 | \alpha(t) \rangle = \frac{\hbar}{2m\omega} \langle \alpha(t) | (\alpha^2 + \alpha^{\dagger 2} + \alpha\alpha^\dagger + \alpha^\dagger\alpha) | \alpha(t) \rangle$$

$$= \frac{\hbar}{2m\omega} \left\{ \alpha(t)^2 + \alpha^*(t)^2 + 2|\alpha(t)|^2 + 1 \right\}$$

$$\text{BUT } \langle x \rangle^2 = \frac{\hbar}{2m\omega} \left\{ \alpha(t)^2 + \alpha^*(t)^2 + 2|\alpha(t)|^2 \right\}$$

$$\therefore (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{2m\omega} \quad (\text{why surprising? time indep!})$$

$$\langle p_x^2 \rangle = \frac{m\omega^2}{2} \left\{ -\alpha(t)^2 - \alpha^*(t)^2 + 2|\alpha(t)|^2 + 1 \right\}$$

$$\langle p_x \rangle^2 = \frac{m\omega^2}{2} \left\{ -\alpha(t)^2 - \alpha^*(t)^2 + 2|\alpha(t)|^2 \right\}$$

$$\Rightarrow (\Delta p_x)^2 = \frac{m\omega\hbar}{2}$$

$$\Rightarrow \boxed{\therefore \Delta x \Delta p_x = \frac{\hbar}{2} \quad \forall t, \text{ NO SPREADING!}}$$

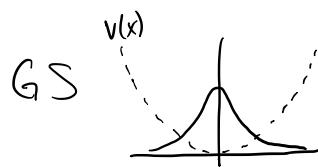
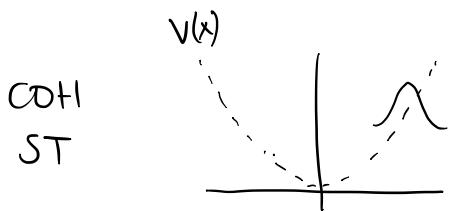
CONCLUSIONS

① MOVES BACK & FORTH

② SOME CLASSICAL BEHAVIOR

③ ALWAYS MIN UNC!

DISPLACEMENT & COHERENT STATES



$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \xrightarrow{\alpha \rightarrow 0} |0\rangle$$

$|\alpha\rangle \sim \text{DISPLACED } |0\rangle$

Q: DOES $|\alpha\rangle = \hat{T}(d)|0\rangle$ for some d ?

A: YES

$$\hat{T}(d) = e^{-ip_x d / \hbar} = e^{\frac{\sqrt{m\omega}}{2\hbar} d (\hat{a}^\dagger - \hat{a})} =: e^{\beta \hat{a}^\dagger + (-\beta^* \hat{a})} e^{-|\beta|^2/2}$$

$$=: e^{A+B} = e^A e^B e^{-[A,B]/2} = e^{\beta \hat{a}^\dagger} e^{-\beta^* \hat{a}} e^{+|\beta|^2 [\hat{a}^\dagger, \hat{a}] / 2}$$

$$\text{so } \hat{T}(d)|0\rangle = e^{-|\beta|^2/2} e^{\beta \hat{a}^\dagger} e^{-\beta^* \hat{a}} |0\rangle = " " \underbrace{|0\rangle}_{|0\rangle} \quad (\text{b/c only first moves})$$

$$= " " e^{\beta \hat{a}^\dagger} |0\rangle$$

$$= " " \sum_{n=0}^{\infty} \frac{\beta^n}{n!} (\hat{a}^\dagger)^n |0\rangle$$

$$= e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |0\rangle = |\beta\rangle$$

ie $\hat{T}(d)|0\rangle$ is coh. st. w/ $\alpha = \beta = \sqrt{\frac{m\omega}{2\hbar}} d$

$ \alpha\rangle = \hat{T}\left(\sqrt{\frac{m\omega}{2\hbar}} \alpha\right) 0\rangle$	(point: nice intenp)
---	----------------------