

QM Lecture 15



# QM IN 3D (speculate with me: what happens?)

## NOTATION CHANGES

### POSITION EIG.

$$|\vec{r}\rangle = \underbrace{|x, y, z\rangle}_{\text{SIMULT. EIG.}}$$

$$\hat{x}|\vec{r}\rangle = x|\vec{r}\rangle$$

$$\hat{y}|\vec{r}\rangle = y|\vec{r}\rangle$$

$$\hat{z}|\vec{r}\rangle = z|\vec{r}\rangle$$

### ARB. STATES

$$|\psi\rangle = \int d^3r |\vec{r}\rangle \langle \vec{r} | \psi \rangle \rightarrow \iiint_{-\infty}^{\infty} dx dy dz |x, y, z\rangle \langle x, y, z|$$

$$\text{ORTHOGONALITY}, \quad \langle \vec{r} | \vec{r}' \rangle = \delta^3(\vec{r} - \vec{r}') \rightarrow \langle xyz | x'y'z' \rangle = \delta(x-x')\delta(y-y')\delta(z-z')$$

$$\text{NORM. } 1 = \langle \psi | \psi \rangle = \int d^3r |\langle \vec{r} | \psi \rangle|^2 \rightarrow \iiint dx dy dz |\langle xyz | \psi \rangle|^2$$

$$\text{PROB. DENS } P(\vec{r}) d^3r = |\langle \vec{r} | \psi \rangle|^2 d^3r$$

↪ prob of finding in some volume  $d^3\vec{r}$  near  $\vec{r}$

### SPATIAL TRANSLATIONS

$$\hat{T}(a)|\vec{r}\rangle = |\vec{r} + \vec{a}\rangle \rightarrow \begin{cases} \hat{T}(a_x)|xyz\rangle = |x+a_x, y, z\rangle \\ "a_y" " " " |x, y+a_y, z\rangle \\ "a_z" " " " |x, y, z+a_z\rangle \end{cases}$$

### LINEAR MOMENTA

$$\vec{T}(a_i) e^{-i\hat{p}_i a_i / \hbar} \quad i = x, y, z$$

Q: what should position/mom. comm rel be?

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar$$

$$\dot{\vec{r}} = [\vec{x}, \vec{y}] = [\vec{y}, \vec{z}] = [\vec{x}, \vec{z}] = 0 \quad (\text{SIMUL. EIG})$$

Q: what about momentum commutators?

$$\hat{T}(\vec{a}_y, \vec{j}) \hat{T}(\vec{a}_x, \vec{i}) - \hat{T}(\vec{a}_x, \vec{i}) \hat{T}(\vec{a}_y, \vec{j}) = 0$$

$$= \left[ e^{-i\vec{p}_y \cdot \vec{a}_y / \hbar}, e^{-i\vec{p}_x \cdot \vec{a}_x / \hbar} \right]$$

$$= \left[ 1 - i \frac{\hat{p}_y \cdot \vec{a}_y}{\hbar} - \frac{\hat{p}_y^2 \vec{a}_y^2}{2\hbar^2} + \dots, 1 - i \frac{\hat{p}_x \cdot \vec{a}_x}{\hbar} - \frac{1}{2} \frac{\hat{p}_x^2 \vec{a}_x^2}{\hbar^2} + \dots \right]$$

$\Rightarrow$  CONDS. ON EACH  $\vec{a}_x \vec{a}_y$  TERM

$$\vec{a}_x \vec{a}_y \text{ TERM} \Rightarrow -\frac{\vec{a}_x \cdot \vec{a}_y}{\hbar^2} [p_y, p_x] = 0$$

$$\Rightarrow \boxed{[p_x, p_y] = [p_x, p_z] = [p_y, p_z] = 0}$$

$$\text{(CORR: } \hat{T}(\vec{a}) = e^{-ip_x a_x / \hbar} e^{-ip_y a_y / \hbar} e^{-ip_z a_z / \hbar} = e^{-i\vec{p} \cdot \vec{a} / \hbar}$$

$$[\vec{x}, \hat{T}(\vec{a}_y, \vec{j})] = 0 \rightarrow [\hat{x}, p_y] = 0$$

SUMMARY

$$\boxed{[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}}$$

MOMENTUM OP. IN POS. SPACE

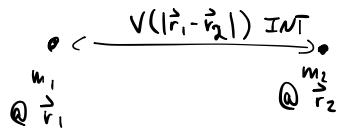
$$\langle \vec{r} | \psi \rangle = \Psi(\vec{r})$$

$$\langle \vec{r} | \vec{p} | \psi \rangle = \frac{i}{\hbar} \vec{\nabla} \langle \vec{r} | \psi \rangle$$

2016 Years

## TWO BODY PROBLEMS & CENTRAL POTENTIALS

PICTURE



HAMILTONIAN

any guesses?

$$\hat{H} = \frac{\hat{\vec{p}}_1^2}{2m} + \frac{\hat{\vec{p}}_2^2}{2m} + V(|\vec{r}_1 - \vec{r}_2|)$$

$$\text{eg } \hat{\vec{P}}_1^2 = \hat{P}_{1x}^2 + \hat{P}_{1y}^2 + \hat{P}_{1z}^2$$

$$[\hat{\vec{p}}_1, \hat{\vec{p}}_2] = 0$$

TAKE POS-KETS

$|\vec{r}_1, \vec{r}_2\rangle$  (similar notation to two spin sys.)  
(ie, both particles)

Q: WHAT IF WE TRANSLATE  $m_1 \leftrightarrow m_2$  BY  $\vec{a}$ ?  $|\vec{r}_1, \vec{r}_2\rangle \rightarrow |\vec{r}_1 + \vec{a}, \vec{r}_2 - \vec{a}\rangle$   
(how does  $H$  act on it?)

$$\text{A: } \hat{H} \hat{T}(\vec{a}) |\vec{r}_1, \vec{r}_2\rangle = \hat{T}(\vec{a}) \hat{H} |\vec{r}_1, \vec{r}_2\rangle$$

$$\hat{H} |\vec{r}_1 + \vec{a}, \vec{r}_2 - \vec{a}\rangle \quad \text{NB IF BOTH ARE } \hat{H}\text{-ESTATES}$$

HAVE SAME  $E$ !

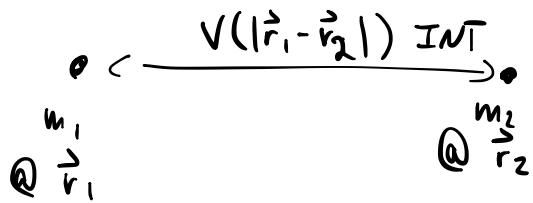
WHAT IS  $\hat{T}(\vec{a})$ ?  $T(\vec{a}) = e^{-i\vec{p} \cdot \vec{a}/\hbar}$

$$\omega/ \hat{\vec{P}} = \hat{\vec{P}}_1 + \hat{\vec{P}}_2 \Rightarrow \hat{T}(\vec{a}) = e^{-i\hat{\vec{p}}_1 \cdot \vec{a}/\hbar} e^{-i\hat{\vec{p}}_2 \cdot \vec{a}/\hbar} = T_1(\vec{a}) T_2(\vec{a})$$

$$\text{NOTE ALSO } [\hat{H}, \hat{\vec{P}}] = 0 \Rightarrow \frac{d\langle \hat{\vec{P}} \rangle}{dt} = 0 \quad \text{SINCE } \frac{d\hat{\vec{P}}}{dt} = 0$$

# TWO BODY PROBLEMS & CENTRAL POTENTIALS

## PICTURE



## HAMILTONIAN

Any guesses?

$$\hat{H} = \frac{\vec{P}_1^2}{2m} + \frac{\vec{P}_2^2}{2m} + V(|\vec{r}_1 - \vec{r}_2|)$$

$$\text{eg } \vec{P}_1^2 = \hat{P}_{1x}^2 + \hat{P}_{1y}^2 + \hat{P}_{1z}^2$$

$$[\vec{P}_1, \vec{P}_2] = 0$$

## TAKE POS-BS

$|\vec{r}_1, \vec{r}_2\rangle$  (similar notation to two spin sys.)  
*(ie, both particles)*

HOW DO WE TRANSLATE  $m_1 \leftrightarrow m_2$  BY  $\vec{a}$ ?  $|\vec{r}_1, \vec{r}_2\rangle \rightarrow |\vec{r}_1 + \vec{a}, \vec{r}_2 - \vec{a}\rangle$   
 (how does  $H$  act on it?)

$$T_1(\vec{a}) T_2(\vec{a}) |\vec{r}_1, \vec{r}_2\rangle = e^{i\vec{P}_1 \cdot \vec{a}/\hbar} e^{i\vec{P}_2 \cdot \vec{a}/\hbar} |\vec{r}_1, \vec{r}_2\rangle = |\vec{r}_1 + \vec{a}, \vec{r}_2 - \vec{a}\rangle$$

"

$$T(\vec{a}) |\vec{r}_1, \vec{r}_2\rangle = e^{i\vec{P} \cdot \vec{a}/\hbar} |\vec{r}_1, \vec{r}_2\rangle \Rightarrow \vec{P} = \vec{P}_1 + \vec{P}_2$$

$$\text{PHYSICS: } [H, T] = 0 \quad \therefore [\hat{H}, \hat{P}] = 0$$

$$\text{NOTE ALSO} \quad \Rightarrow \frac{d\langle \hat{P} \rangle}{dt} = 0 \quad \text{SINCE } \frac{d\vec{P}}{dt} = 0$$

(translational invariance of the Hamiltonian guarantees total momentum conservation)

## CENTER OF MASS

$$([x_i; \hat{P}_j]) = 0$$

Define

### RELATIVE OPS

$$\hat{r} = \hat{r}_1 - \hat{r}_2$$

$$\hat{\vec{p}} = \frac{m_2 \hat{\vec{p}}_1 - m_1 \hat{\vec{p}}_2}{m_1 + m_2} = \mu \frac{d\hat{r}}{dt}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

### COM OPS

$$\hat{\vec{R}} = \frac{\hat{m}_1 \hat{r}_1 + \hat{m}_2 \hat{r}_2}{\hat{m}_1 + \hat{m}_2}$$

$$\hat{\vec{P}} = \hat{\vec{p}}_1 + \hat{\vec{p}}_2$$

### COMPONENTS

$$\hat{p}_i$$

$$\hat{p}_{1i}$$

$$\hat{p}_{2i}$$

$$\begin{matrix} \hat{x}_i \\ \hat{x}_{1i} \\ \hat{x}_{2i} \end{matrix}$$

$$\hat{x}_i$$

## FACTS

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

OTHERS COMMUTE

$$\frac{1}{m_1 + m_2} [\hat{x}_{1i} - \hat{x}_{2i}, m_2 \hat{p}_{1j} - m_1 \hat{p}_{2j}] = \frac{1}{m_1 + m_2} i\hbar \{m_1 \delta_{ij} + m_2 \delta_{ij}\}$$

$$= i\hbar \delta_{ij} \checkmark$$

\* POINT USE  $|\vec{r}, \vec{R}\rangle$  BASIS INSTEAD OF  $(\vec{r}_1, \vec{r}_2)\rangle$

NOTE: 1)  $V(|\vec{r}_1 - \vec{r}_2|) = V(|\vec{r}|)$

$$2) \text{ CLAIM } \hat{H}_{\text{kin}} = \frac{\hat{\vec{P}}^2}{2M} + \frac{\hat{\vec{P}}^2}{2\mu} \quad M = m_1 + m_2 \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\frac{\hat{\vec{P}}^2}{2M} = \frac{\hat{\vec{P}}_1^2 + \hat{\vec{P}}_2^2 + 2 \hat{\vec{P}}_1 \cdot \hat{\vec{P}}_2}{2(m_1 + m_2)}$$

$$\frac{\hat{\vec{P}}^2}{2\mu} = \frac{(m_1 + m_2)}{m_1 m_2} \frac{m_2^2 \hat{\vec{P}}_1^2 + m_1^2 \hat{\vec{P}}_2^2 - 2 m_1 m_2 \hat{\vec{P}}_1 \cdot \hat{\vec{P}}_2}{2(m_1 + m_2)^2}$$

$$\therefore \hat{H}_{\text{kin}} = \frac{1}{2(m_1 + m_2)} \cdot \left\{ \hat{\vec{P}}_1^2 \left( 1 + \frac{m_2}{m_1} \right) + \hat{\vec{P}}_2^2 \left( 1 + \frac{m_1}{m_2} \right) \right\}$$

$$= \frac{\hat{\vec{P}}_1^2}{2m_1} + \frac{\hat{\vec{P}}_2^2}{2m_2}$$

$$\therefore \hat{H} = \underbrace{\frac{\hat{\vec{P}}^2}{2M}}_{\hat{H}_{\text{com}}} + \underbrace{\frac{\hat{\vec{P}}^2}{2\mu} + V(|\vec{r}|)}_{\hat{H}_{\text{rel}}}$$

(Q: Can you say anything about  $\hat{H}$  e-states already?)

(A: cm & rel ops commute,  $\therefore$  sol E-states of both parts separately.)

$$[\text{CM}, \text{REL}] = 0 \Rightarrow \hat{H} |\bar{E}_{\text{cm}}, \bar{E}_{\text{rel}}\rangle = (\hat{H}_{\text{cm}} + \hat{H}_{\text{rel}}) |\bar{E}_{\text{cm}}, \bar{E}_{\text{rel}}\rangle$$

$$= (\bar{E}_{\text{cm}} + \bar{E}_{\text{rel}}) |\bar{E}_{\text{cm}}, \bar{E}_{\text{rel}}\rangle = E_{\text{tot}} |\rangle$$

ALREADY SOLVED  $\hat{H}_{\text{CM}}$ : FREE "PARTICLE" (system)

$$\langle \vec{R} | \vec{P} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} e^{-i\vec{P} \cdot \vec{R}/\hbar}$$

POINT INTERESTING PART ALL IN  $\hat{H}_{\text{rel}}$

(in CM frame  $\vec{P}=0$ , just seek sol to  $H_{\text{rel}}$ )

(could solve many two body probs: spec.  $V(|\vec{r}|)$ .)

ROT. INV. OF  $\hat{H}_{\text{rel}}$

Q: WHY CAN SIMPL. USE ROT. INV?

A:  $\hat{H}_{\text{rel}} = f(|\vec{p}|, |\vec{r}|)$  LENGTHS

→ NOT SPIN. ANG MOM AS  
YOU KNOW IT!

ROT OP.  $\hat{R}(\phi \hat{k}) = e^{-i\hat{L}_k \phi / \hbar}$

RECALL SPATIAL ROT

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\phi \text{ small } d\phi \Rightarrow A'_x = A_x - d\phi A_y$$

$$A'_y = A_y + d\phi A_x$$

$$\Rightarrow \hat{R}(d\phi \hat{k}) |_{x,y,z} = |_{x-yd\phi, y+xd\phi, z}$$

$$= \left[ 1 - \frac{i}{\hbar} \hat{P}_x (-\hat{y} d\phi) \right] \left[ 1 - \frac{i}{\hbar} \hat{P}_y (\hat{x} d\phi) \right] |_{x,y,z}$$

$$= \left[ 1 - \frac{i}{\hbar} (\hat{P}_y \hat{x} - \hat{P}_x \hat{y}) d\phi + O(d\phi^2) \right] |_{x,y,z}$$

$$\Rightarrow \hat{R}(d\phi \hat{k}) = 1 - \frac{i}{\hbar} \hat{L}_z d\phi = 1 - \frac{i}{\hbar} (\hat{x}\hat{P}_y - \hat{y}\hat{P}_x)$$

$$\Rightarrow \boxed{\hat{L}_z = \hat{x}\hat{P}_y - \hat{y}\hat{P}_x}$$

GEN.  $\boxed{\hat{L} = \hat{r} \times \hat{p}}$

L IS ORBITAL ANG. MOMENTUM!

PROVE ROT. INV. i.e.  $[H, L_z] = 0$

by a)  $[L_z, \vec{r}^2] = 0$  b)  $[L_z, \vec{p}^2] = 0$

$$[L_z, \vec{p}^2] = [L_z, P_x^2 + P_y^2 + P_z^2] = P_x [L_z, P_x] + [L_z, P_x] P_x$$

$$+ P_y [L_z, P_y] + [L_z, P_y] P_y + P_z [L_z, P_z] + [L_z, P_z] P_z$$

$$L_z = \cancel{x P_y - y P_x} \quad \cancel{0} \quad \cancel{0}$$

Comm.  $P_z$

$$[L_z, P_x] = [x P_y - y P_x] = [x P_y, P_x] - [y P_x, P_x]$$

$$= -x [P_y, P_x] - [P_x, x] P_y + y [P_x, P_x] + [P_x, \cancel{y}] P_x$$

$$= i\hbar P_y$$

$$\text{SIM} \quad [L_z, p_y] = -i\hbar p_x$$

$$\therefore [L_z, \vec{p}^2] = i\hbar p_x p_y + i\hbar p_y p_x - i\hbar p_y p_x - i\hbar p_x p_y = 0$$

$$\text{ALSO} \quad [L_z, \vec{r}^2] = 0$$

$$\Rightarrow [\hat{H}, \hat{L}_z] = 0 \quad [\hat{H}, \hat{\vec{L}}^2] = 0$$

$\therefore$  ENERGY EIG. OF  $\hat{H}_{\text{rel}}$  SIMULT EIG OF  $E, l, m$

$$\hat{H}_{\text{rel}} |E, l, m\rangle = E |E, l, m\rangle$$

$$\hat{\vec{L}}^2 |E, l, m\rangle = l(l+1)\hbar^2 |E, l, m\rangle$$

$$L_z |E, l, m\rangle = m\hbar |E, l, m\rangle$$

APP. OF  $\vec{L}^2$  IN  $H_{\text{rel}}$

NEED IDENTITIES

$$\text{LET'S PROVE: } \vec{L}^2 = \vec{r}^2 \vec{p}^2 - (\vec{r} \cdot \vec{p})^2 + i\hbar \vec{r} \cdot \vec{p}$$

$$\vec{L} = \vec{r} \times \vec{p} \xrightarrow{\text{comp}} L_i = \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} x_j p_k \stackrel{\text{Einstein}}{=} \epsilon_{ijk} x_j p_k$$

$$L_i = \epsilon_{ijk} x_j p_k$$

$$\therefore \vec{L}^2 = \sum_{i=1}^3 L_i L_i \stackrel{E}{=} L_i L_i = (\epsilon_{ijk} x_j p_k)(\epsilon_{imn} x_m p_n)$$



