

QM Lernme 16



LAST TIME

1) 3D NOTATION

2) TWO BODY PROB.

i) $\sqrt{(\vec{r}_1 - \vec{r}_2)^2}$

ii) COM, REL VARS

iii) \hat{H}_{rel}

iv) ANG MOM

$$[\hat{H}, \hat{L}_z] = 0 \quad [\hat{H}, \hat{\vec{L}}^2] = 0$$

\therefore ENERGY EIG. OF \hat{H}_{rel} SIMULT EIG OF ϵ, ℓ, m

$$\hat{H}_{\text{rel}} |\epsilon, \ell, m\rangle = \epsilon |\epsilon, \ell, m\rangle$$

$$\hat{\vec{L}}^2 |\epsilon, \ell, m\rangle = \ell(\ell+1)\hbar^2 |\epsilon, \ell, m\rangle$$

$$\hat{L}_z |\epsilon, \ell, m\rangle = m\hbar |\epsilon, \ell, m\rangle$$

APP. OF \vec{L}^2 IN H_{rel}

NEED IDENTITIES

LET'S PROVE: $\vec{L}^2 = \vec{r}^2 \vec{p}^2 - (\vec{r} \cdot \vec{p})^2 + i\hbar \vec{r} \cdot \vec{p}$

$$\vec{L} = \vec{r} \times \vec{p} \xrightarrow{\text{comp}} L_i = \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} x_j p_k \stackrel{\text{Einstein}}{=} \epsilon_{ijk} x_j p_k$$

$$| L_i = \epsilon_{ijk} x_j p_k |$$

$$\therefore \vec{L}^2 = \sum_{i=1}^3 L_i L_i \stackrel{\mathbb{E}}{=} L_i L_i = (\epsilon_{ijk} x_j p_k) (\epsilon_{imn} x_m p_n)$$

$$= x_j p_k x_m p_n (\epsilon_{ijk} \epsilon_{imn})$$

|| FACT

$$= x_j p_k x_m p_n (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km})$$

$$= \underbrace{x_j p_k x_j p_k}_{= x_j (x_j p_k - i\hbar \delta_{jk}) p_k} - \underbrace{x_j p_k x_k p_j}_{= \vec{r}^2 \vec{p}^2 - i\hbar \vec{r} \cdot \vec{p}}$$

$$= x_j p_k x_k$$

$$\left. \begin{aligned} &= x_j p_k (i\hbar \delta_{jk} + p_j x_k) = i\hbar \vec{r} \cdot \vec{p} + \underbrace{x_j p_k p_j x_k}_{= x_j p_k x_k} \\ &= i\hbar \vec{r} \cdot \vec{p} + \vec{r} \cdot \vec{p} (-i\hbar \delta_{kk} + x_k p_k) \\ &= i\hbar \vec{r} \cdot \vec{p} + \vec{r} \cdot \vec{p} (-i\hbar \cdot 3 + x_k p_k) = -2i\hbar \vec{r} \cdot \vec{p} + (\vec{r} \cdot \vec{p})^2 \end{aligned} \right.$$

$$\Rightarrow \boxed{\vec{L}^2 = \vec{r}^2 \vec{p}^2 + i\hbar \vec{r} \cdot \vec{p} + (\vec{r} \cdot \vec{p})^2}$$

SO WHAT? 3 PIECES TO $\langle \vec{r} | \vec{L}^2 | \psi \rangle$

$$\textcircled{1} \quad \langle \vec{r} | \vec{r}^2 \vec{p}^2 | \psi \rangle = r^2 \langle \vec{r} | \vec{p}^2 | \psi \rangle = r^2 \times \text{kinetic}$$

$$\textcircled{2} \quad i\hbar \langle \vec{r} | \vec{r} \cdot \vec{p} | \psi \rangle = i\hbar \vec{r} \cdot \frac{\hbar}{i} \vec{\nabla} \langle \vec{r} | \psi \rangle = \hbar^2 r \frac{\partial}{\partial r} \langle \vec{r} | \psi \rangle$$

$$\textcircled{3} \quad -\langle \vec{r} | (\vec{r} \cdot \vec{p})^2 | \psi \rangle = \hbar^2 \left(r \frac{\partial}{\partial r} \right) \left(r \frac{\partial}{\partial r} \right) \langle \vec{r} | \psi \rangle = \hbar^2 \left[r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial^2}{\partial r^2} \right] \langle \vec{r} | \psi \rangle$$

$$\therefore \langle \vec{r} | \vec{p}^2 | \psi \rangle = \frac{1}{r^2} \left\{ \langle \vec{r} | \vec{L}^2 | \psi \rangle - \hbar^2 \left[2r \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial r^2} \right] \right\} \langle \vec{r} | \psi \rangle$$

$$\boxed{\therefore \langle \vec{r} | H_{\text{rel}} | \psi \rangle = E \langle \vec{r} | \psi \rangle = -\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \langle \vec{r} | \psi \rangle + \frac{\langle \vec{r} | \vec{L}^2 | \psi \rangle}{2\mu r^2} + V(r) \langle \vec{r} | \psi \rangle}$$



SEP. OF VAR. & CENTRAL POT.

- DEP. ON r , NOT θ, ϕ . (though Ψ has θ, ϕ dep)
- DEP ON \vec{L}^2 , NOT L_z . (not invariant, can't dep on "z-dir")

TAKE $|4\rangle = |E, l, m\rangle$

$$\left[-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) \right] \langle \vec{r} | E, l, m \rangle = E \langle \vec{r} | E, l, m \rangle$$

TRY SEPARABILITY

$$\begin{aligned} \Psi(\vec{r}) &= R(r) \Theta(\theta) \Phi(\phi) \\ &=: R(r) Y_{lm}(\theta, \phi) \end{aligned}$$

START w/ $\Phi(\phi)$

$$\hat{R}(d\phi \hat{k}) |r, \theta, \phi\rangle = |r, \theta, \phi + d\phi\rangle$$

BRA: $\langle r, \theta, \phi | \hat{R}(d\phi \hat{k}) = \langle r, \theta, \phi - d\phi |$

$$\begin{aligned} \langle r, \theta, \phi | \hat{R}(d\phi \hat{k}) |4\rangle &= \langle r, \theta, \phi | \left(1 - \frac{i}{\hbar} \hat{L}_z d\phi \right) |4\rangle \\ &= \langle r, \theta, \phi - d\phi | 4 \rangle = \langle r, \theta, \phi | 4 \rangle - d\phi \frac{\partial}{\partial \phi} \langle r, \theta, \phi | 4 \rangle \end{aligned}$$

$$\Rightarrow \boxed{L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}} \quad \text{ON SPHER. COORD. } \Psi(\vec{r})$$

$$\text{BUT } L_z |E, \ell, m\rangle = m \hbar |E, \ell, m\rangle$$

$$\Rightarrow \langle r, \theta, \phi | \hat{L}_z |E, \ell, m\rangle = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \langle r, \theta, \phi | E, \ell, m\rangle = m \hbar \langle r, \theta, \phi | E, \ell, m\rangle$$

$$\Rightarrow \Phi(\phi) = e^{im\phi}$$

Q: MAKE SENSE

$$A: NEED \quad e^{im(\phi)} = e^{im(\phi+2\pi)}$$

\Rightarrow ORB. ANG MOM. HAS $m \in \mathbb{Z}$ (not $\mathbb{Z}/2$)

NOW $\hat{\Theta}^+(0)$:

$$\begin{aligned} \hat{L} &= \hat{r} \times \hat{p} \xrightarrow{\text{pos}} \hat{r} \times \frac{\hbar}{i} \hat{v} = \hat{r} \hat{e}_r \times \frac{\hbar}{i} \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\ &= \frac{\hbar}{i} \left(\hat{e}_\phi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \end{aligned}$$

CARTESIAN USEFUL FOR EIG

$$\hat{L}_x = \frac{\hbar}{i} \left(-\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_y = \frac{\hbar}{i} \left(\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right)$$

$$\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}$$

TWO SOLN METHODS

$$\textcircled{1} \quad \vec{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \quad \dot{\vdash} \text{ S. EQN!}$$

\textcircled{2} RAISE $\dot{\vdash}$ LOWER OPS (easier, first order PDE)

METH. 2

$$L_z = L_x \pm iL_y$$

$$\exists |l, l\rangle \text{ s.t. } \hat{L}_z |l, l\rangle = 0$$

$$L_z |l, l\rangle = \frac{\hbar}{i} \left\{ (-\sin\phi + i\cos\phi) \frac{\partial}{\partial\theta} - \cot\theta (\cos\phi + i\sin\phi) \frac{\partial}{\partial\phi} \right\} |l, l\rangle = 0$$

$$\Rightarrow \frac{\hbar}{i} e^{i\phi} \left(i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi} \right) |l, l\rangle = 0$$

BUT $|l, l\rangle = |E, l, l\rangle$ ϕ DEP. IS $e^{il\phi}$

$$\Rightarrow \frac{\hbar}{i} e^{i\phi} \left(i \frac{\partial}{\partial\theta} - \cot\theta il \right) \langle r, \theta, \phi | l, l \rangle = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial\theta} - l \cot\theta \right) \langle r, \theta, \phi | l, l \rangle = 0$$

$$\Rightarrow Y_{ll}(r, \theta, \phi) = \overline{\Theta}_{ll}(r) \Theta_{ll}(\theta) = C_l e^{il\phi} \sin^l \theta$$

NORM.

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta |C_l|^2 \sin^l \theta \sin^{2l} \theta$$

FACT $\int_0^\pi d\theta \sin^p \theta = \sqrt{\pi} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)}$

$\Gamma(z)$ = Euler Gamma Function

next
in part
about
now.
only do
time
if
permits

- FOR $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$

- IDENTITIES $\Gamma(n+1) = n \Gamma(n)$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{4^n n!} \sqrt{\pi}$$

$$\Gamma(n) \Gamma\left(n + \frac{1}{2}\right) = 2^{1-2n} \sqrt{\pi} \Gamma(2n)$$

$$\begin{aligned}
 \text{ABOVE} \rightarrow |c_\ell|^2 &= \frac{1}{2\pi} \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{2\ell+1}{2} + 1\right)}{\Gamma\left(\frac{2\ell+1}{2}\right)} = \frac{1}{2\pi} \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\ell + \frac{3}{2}\right)}{\Gamma(\ell+1)} \\
 &= \frac{1}{2\pi} \frac{1}{\sqrt{\pi}} \frac{2 \cdot 2^{-2(\ell+1)} \sqrt{\pi} \Gamma(2\ell+2)}{\Gamma(\ell+1) \Gamma(\ell+1)} \\
 &= \frac{1}{4\pi} \frac{(2\ell+1)!}{(\ell!)^2} \frac{1}{(2^\ell)^2}
 \end{aligned}$$

$$\Rightarrow Y_{\ell,m}(\theta, \phi) = \frac{(-1)^\ell}{2^\ell \ell!} \sqrt{\frac{2\ell+1}{4\pi}} e^{im\phi} \sin^\ell \theta$$

phase inserted for reasons that
 will become clear shortly

$m < \ell$ CASES? USE \hat{L}_-

$$\hat{L}_- |l, m\rangle = \sqrt{l(l+1) - m(m-1)} \hat{t} |l, m-1\rangle$$

$$\hat{L}_- = \frac{\hbar}{i} e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right)$$

$$\begin{aligned}
 \Rightarrow \hat{L}_- |l, \ell\rangle &= \frac{\hbar}{i} e^{-i\phi} \left(-i \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \phi} \right) c_\ell e^{il\phi} \sin^\ell \theta \\
 &= -\hbar e^{-i\phi} c_\ell e^{il\phi} \sin^{\ell-1} \theta \cos \theta - \frac{\hbar}{i} e^{-i\phi} \cot \theta \sin^\ell \theta c_\ell (i\ell) e^{il\phi} \\
 &= -\hbar l e^{-i\phi} c_\ell e^{il\phi} (\sin^{\ell-1} \theta \cos \theta + \frac{\cot \theta}{\sin \theta} \sin^\ell \theta) \\
 &= -2\hbar (l c_\ell) \sin^{\ell-1} \theta \cos \theta e^{i(l-1)\phi} \\
 &= \sqrt{l(l+1) - l(l-1)} \hat{t} |l, l-1\rangle = \sqrt{2\ell} \hat{t} |l, l-1\rangle \\
 \Rightarrow Y_{l,l-1} &= -\frac{\hbar l}{\sqrt{2\ell}} c_\ell \sin^{\ell-1} \theta \cos \theta e^{i(l-1)\phi}
 \end{aligned}$$

LOWERING:

- ① ONE LESS $\sin \theta$
- ② ONE MORE $\cos \theta$
- ③ ONE MORE -1

IN GENERAL

prob 9.18

$$Y_{lm}(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(l+m)!}{4\pi (l-m)!}} e^{im\phi} \frac{1}{\sin^m \theta} \frac{d^{\ell-m}}{d(\cos \theta)^{\ell-m}} \sin^{2l} \theta$$

AGREES WITH $|l, l\rangle \neq |l, l-1\rangle$ CASES.

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \quad Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2 \theta - 1)$$

$$Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} e^{\pm i\phi} \sin \theta \quad Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} e^{\pm i\phi} \sin \theta \cos \theta$$

$$Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\phi} \sin^2 \theta \quad Y_{l,0} = \pm \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$$

WHERE P_l a LEGENDRE POLY.