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Interaction of two unequal cracks in a prestressed fiber reinforced composite

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Abstract. We consider an elastic orthotropic material representing a fiber reinforced composite. The composite is prestressed and contains two collinear cracks having different lengths. The faces of the cracks are acted by symmetrically distributed constant normal incremental stresses. We determine the critical values of the applied incremental stresses for which the cracks tips start to propagate and we analyse the interaction of the cracks as function of their lengths and of the distance between the cracks.

Key words: Orthotropic composites, unequal collinear cracks, cracks interaction.

1. Introduction

We consider a prestressed, orthotropic, linear, elastic material representing a fiber reinforced composite. We take as coordinate planes the symmetry planes of the material, the x_1-axis being parallel to the fibers. We assume that the admissible incremental equilibrium states of the body are plane strain states relative to the x_1–x_2-plane. As was shown by Guz (1983, 1989, 1991), in the assumed circumstances the incremental equilibrium states of the material can be represented by two complex potentials defined in two complex planes. In Section 2 we give Guz's representation formulae.

We suppose that the material is unbounded and contains two collinear and unequal cracks situated in the same plane, parallel to the reinforcing fibers and to the initial applied stresses (see Figure 3.1). We assume that the upper and lower faces of the cracks are acted by symmetrically distributed normal incremental stresses. Our first aim is to determine the incremental elastic state produced in the body. To do this we use Guz's representation theorem. In Section 3 we give the boundary conditions which must be satisfied by Guz's complex potentials on the faces of the cracks and at large distances from the cracks. Using the theory of the Riemann–Hilbert problem we give the solution of our mathematical problem in Sections 4 and 5, assuming that the applied incremental stresses have a given constant value. Our second aim is to determine the critical values of the given incremental stress for which the tips of the cracks start to propagate. To do this we determine the singular parts of the elastic state near the cracks tips and we use Irwin's relation giving the energy release rate, as well as Griffith's energetical criterion. The obtained results are presented in Sections 6 and 7. Also, in Section 7 we give the critical value of the initial applied stress for which the resonance phenomenon can occur.

Our third aim is to study the interaction of the two cracks. This is done for cracks having the same length in Section 8. For an isotropic material, without initial applied stresses, we obtain for the critical stresses producing cracks propagation Willmore's classical results. In Section 9 we analyse the interaction of the cracks having different lengths. According to the obtained results this interaction is strong only if the distance between the cracks is much more smaller as their lengths. In such a situation, the inner tips start to propagate first and the cracks tend to unify. If the above distance is much grater as the lengths of the cracks, the interaction is weak and the tips of the longest crack start to propagate first. In our opinion, these results are plausible, in accordance with previously observed facts.

2. Guz's type representation of the incremental fields

We consider a prestressed, orthotropic, linear elastic material, representing a fiber reinforced composite. The coordinate planes are the symmetry planes of the body, the x_1-axis being parallel with the fibers. We assume that the incremental equilibrium states of the material are plane strain states relative to the x_1–x_2-plane. We suppose that the stress free reference configuration of the body is locally stable. As it was shown by Guz (see (1983) Chap. 2, (1989) Chap.1, (1991) Chap. 3), in the assumed circumstances the incremental elastic state of the material can be expressed by two analytic complex potentials $\Phi_j(z_j)$ defined in two complex planes $z_j,\ \ j = 1, 2$. We denote by u_1, u_2 the involved components of the incremental displacement field and by $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$ the involved components of the incremental nominal stress. We present Guz's representation formulae in a weakly modified form due to Soós (1996). We have

$$u_1 = 2\,\mathrm{Re}\{b_1\Phi_1(z_1) + b_2\Phi_2(z_2)\}, \qquad u_2 = 2\,\mathrm{Re}\{c_1\Phi_1(z_1) + c_2\Phi_2(z_2)\}, \tag{2.1}$$

$$\theta_{11} = 2\,\mathrm{Re}\{a_1\mu_1^2\Psi_1(z_1) + a_2\mu_2^2\Psi_2(z_2)\}, \qquad \theta_{12} = -2\,\mathrm{Re}\{\mu_1\Psi_1(z_1) + \mu_2\Psi_2(z_2)\}, \tag{2.2}$$

$$\theta_{21} = -2\,\mathrm{Re}\{a_1\mu_1\Psi_1(z_1) + a_2\mu_2\Psi_2(z_2)\}, \qquad \theta_{22} = 2\,\mathrm{Re}\{\Psi_1(z_1) + \Psi_2(z_2)\}. \tag{2.3}$$

In these relations

$$\Psi_j(z_j) = \Phi_j'(z_j) = \frac{\mathrm{d}\Phi_j(z_j)}{\mathrm{d}z_j}, \quad j = 1, 2 \tag{2.4}$$

and

$$z_j = x_1 + \mu_j x_2, \quad j = 1, 2. \tag{2.5}$$

The parameters μ_j are the roots of the algebraic equation

$$\mu^4 + 2A\mu^2 + B = 0 \tag{2.6}$$

with

$$A = \frac{\omega_{1111}\omega_{2222} + \omega_{1221}\omega_{2112} - (\omega_{1122} + \omega_{1212})^2}{2\omega_{2222}\omega_{2112}}, \qquad B = \frac{\omega_{1111}\omega_{1221}}{\omega_{2222}\omega_{2112}}. \tag{2.7}$$

The parameters a_j, b_j, c_j $(j = 1, 2)$ have the following expressions

$$a_j = (\omega_{2112}\omega_{1122}\mu_j^2 - \omega_{1111}\omega_{1212})/(B_j\mu_j^2), \tag{2.8}$$

$$b_j = -(\omega_{1122} + \omega_{1212})/B_j, \qquad c_j = (\omega_{2112}\mu_j^2 + \omega_{1111})/(B_j\mu_j), \tag{2.9}$$

with

$$B_j = \omega_{2222}\omega_{2112}\mu_j^2 + \omega_{1111}\omega_{2222} - \omega_{1122}(\omega_{1122} + \omega_{1212}). \tag{2.10}$$

The instantaneous elasticities ω_{klmn} $(k, l, m, n = 1, 2)$, involved in the above relations can be expressed through the elastic coefficients C_{11}, C_{12}, C_{22}, C_{66} of the material and through the initial applied stress σ^0 by the following relations

$$\begin{aligned} &\omega_{1111} = C_{11} + \sigma^0, \qquad \omega_{2222} = C_{22}, \qquad \omega_{1122} = C_{12}, \\ &\omega_{1212} = C_{66}, \qquad \omega_{1221} = C_{66} + \sigma^0, \qquad \omega_{2112} = C_{66}. \end{aligned} \tag{2.11}$$

In their turn the elastic coefficients can be expressed using the engineering constants of the composite; we have

$$\begin{aligned} C_{11} &= \frac{1 - \nu_{23}\nu_{32}}{E_2E_3H}, \qquad C_{22} = \frac{1 - \nu_{13}\nu_{31}}{E_1E_3H}, \\ C_{12} &= \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1E_3H}, \qquad C_{66} = G_{12}, \end{aligned} \tag{2.12}$$

with

$$H = \frac{(1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{21}\nu_{32}\nu_{13} - \nu_{12}\nu_{23}\nu_{31})}{(E_1E_2E_3)}. \tag{2.13}$$

In these relations E_1, E_2, E_3 are Young's moduli in the corresponding symmetry directions of the material, $\nu_{12}, \ldots, \nu_{32}$ are Poisson's ratios and G_{12} is the shear modulus in the symmetry plane x_1–x_2. By σ^0 we have designed the initial applied stress, acting in fibers direction.

It can be shown that if the initial deformed equilibrium configuration of the body is locally stable, the roots of the algebraic equation (2.6) are complex numbers, having nonvanishing immaginary parts. Also, for an orthotropic composite μ_1 and μ_2 are not equal, i.e.,

$$\mu_1 \neq \mu_2. \tag{2.14}$$

Ending this section we observe that for our fiber reinforced composite material Young's modulus E_1 is much greater as Young's modulus E_2 and as the shear modulus G_{12}; i.e.,

$$E_2 \ll E_1, \qquad G_{12} \ll E_1. \tag{2.15}$$

More details concerning the above presented elements of the three-dimensional linearized theory of elastic bodies can be find in Guz's fundamental monography (Guz, 1986).

We shall use Guz's complex representation of the incremental elastic state to study the interaction of two cracks having different lengths in a prestressed fiber reinforced composite material. Our approach is founded on that used by Soós (1996).

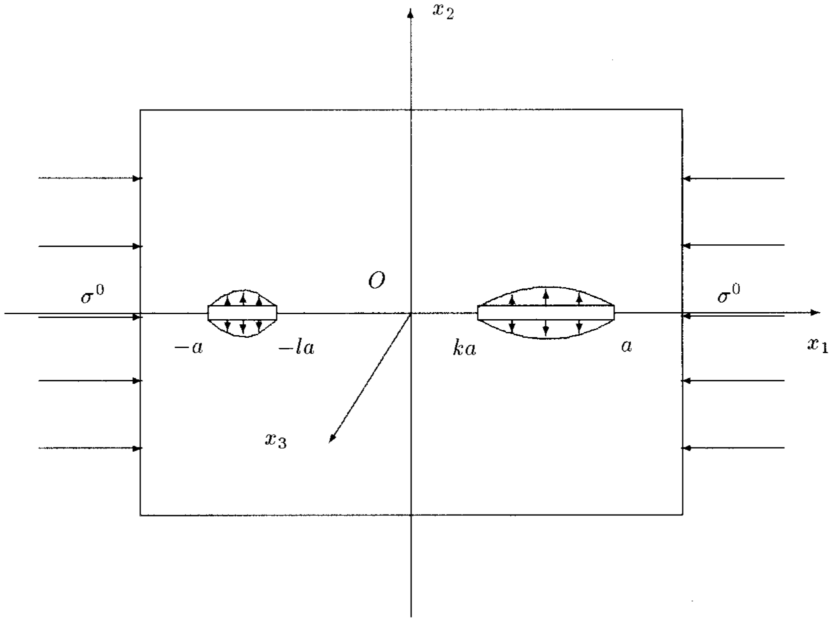

Figure 3.1. Two unequal cracks.

3. Composite containing unequal cracks. The mathematical problem

In all that follows we assume an unbounded material. We suppose that our composite contains two collinear and unequal cracks situated in the same plane, parallel with the reinforcing fibers and with the initial applied stress, as in Figure 3.1. The plane containing the cracks is taken as the x_1–x_3-plane, the x_2-axis being perpendicular to the cracks faces.

We suppose that on the superior and inferior faces of the cracks act given normal incremental stresses, symmetrically distributed relative to the plane containing the cracks (see Figure 3.1). The produced incremental elastic state is a plane deformation state, relative to the x_1–x_2-plane. Hence we can use Guz's complex representation to study our problem, corresponding to the first mode in classical fracture mechanics.

We denote by $\mathcal{L}$ the cut corresponding to the segments $(-a, -la)$ and (ka, a) as in Figure 3.1. Here $a > 0$ is a given positive constant, and l, k are given positive numbers satisfying the restriction $0 < l, k < 1$. Obviously, $-a$ and $-la$ are the coordinates of the tips of the left crack, and ka and a represent the coordinates of the tips of the right crack.

We design by $p = p(t)$, $t \in (-a, -la) \cup (ka, a)$ the given incremental normal stress.

According to the considered mechanical problem, the components θ_{21} and θ_{22} of the incremental nominal stress must satisfy the following boundary conditions

$$\begin{aligned} &\theta_{21}(t, 0^+) = \theta_{21}(t, 0^-) = 0, \\ &\theta_{22}(t, 0^+) = \theta_{22}(t, 0^-) = -p(t), \end{aligned} \tag{3.2}$$

for $t \in (-a, -la) \cup (ka, a)$.

Also, at large distances from the cracks the incremental displacements and nominal stresses must vanish; i.e.,

$$\lim_{r\to\infty} \{u_\alpha(x_1, x_2), \theta_{\alpha\beta}(x_1, x_2)\} = 0, \qquad r = \sqrt{x_1^2 + x_2^2}, \quad \alpha, \beta = 1, 2. \tag{3.3}$$

Consequently, from (2.1)–(2.3) we can conclude that Guz's complex potentials must satisfy the restrictions

$$\lim_{z_j \to \infty} \{\Phi_j(z_j), \Psi_j(z_j)\} = 0, \quad j = 1, 2. \tag{3.4}$$

To find the complex potentials satisfying the boundary conditions (3.1), (3.2) and the above restrictions we use Sih's and Leibwitz's method (1968) leading to a Riemann–Hilbert problem.

4. The Riemann–Hilbert problem

Let us consider a complex valued function $f = f(z)$ defined in the whole complex plane $z = x_1 + ix_2,\ i = \sqrt{-1}$. Let t be a real number. We denote by $f^+(t)$ and $f^-(t)$ the upper and the lower limits of the function $f(z)$; i.e.,

$$f^+(t) = \lim_{\substack{z \to t \\ x_2 > 0}} f(z) \quad \text{and} \quad f^-(t) = \lim_{\substack{z \to t \\ x_2 < 0}} f(z). \tag{4.1}$$

As usually, by a superposed bar we denote complex conjugation.

Starting with the function $f = f(z)$ we define a new function $\bar{f} = \bar{f}(z)$ by the following relation

$$\bar{f}(z) = \overline{f(\bar{z})}. \tag{4.2}$$

It is easy to see that

$$\bar{f}^+(t) = \overline{f^-(t)} \quad \text{and} \quad \bar{f}^-(t) = \overline{f^+(t)}. \tag{4.3}$$

From Guz's representation formula $(2.1)_2$ and from the boundary condition (3.1) it follows that the complex potentials $\Psi_j(z_j)$ must satisfy the following restrictions

$$\begin{aligned} &a_1\mu_1\Psi_1^+ + a_2\mu_2\Psi_2^+ + \overline{a_1\mu_1}\bar{\Psi}_1^- + \overline{a_2\mu_2}\bar{\Psi}_2^- = 0, \\ &a_1\mu_1\Psi_1^- + a_2\mu_2\Psi_2^- + \overline{a_1\mu_1}\bar{\Psi}_1^+ + \overline{a_2\mu_2}\bar{\Psi}_2^+ = 0, \end{aligned} \tag{4.4}$$

for $t \in (-a, -la) \cup (ka, a)$.

Adding and subtracting the above relations lead to

$$\begin{aligned} &(a_1\mu_1\Psi_1 + a_2\mu_2\Psi_2 + \overline{a_1\mu_1}\bar{\Psi}_1 + \overline{a_2\mu_2}\bar{\Psi}_2)^+ \\ &\quad +(a_1\mu_1\Psi_1 + a_2\mu_2\Psi_2 + \overline{a_1\mu_1}\bar{\Psi}_1 + \overline{a_2\mu_2}\bar{\Psi}_2)^- = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &(a_1\mu_1\Psi_1 + a_2\mu_2\Psi_2 - \overline{a_1\mu_1}\bar{\Psi}_1 - \overline{a_2\mu_2}\bar{\Psi}_2)^+ \\ &\quad -(a_1\mu_1\Psi_1 + a_2\mu_2\Psi_2 - \overline{a_1\mu_1}\bar{\Psi}_1 - \overline{a_2\mu_2}\bar{\Psi}_2)^- = 0. \end{aligned} \tag{4.6}$$

Using well known properties of Cauchy's integral (see (Muskhelishvili, 1953), Chap. 4) and taking into account the conditions (3.4) at large distances, from (4.6) we can conclude that the complex potentials must satisfy in the whole complex plane z the following relation

$$a_1\mu_1\Psi_1(z) + a_2\mu_2\Psi_2(z) - \overline{a_1\mu_1}\bar{\Psi}_1(z) - \overline{a_2\mu_2}\bar{\Psi}_2(z) = 0. \tag{4.7}$$

The restriction (4.5) is equivalent with a homogeneous Riemann–Hilbert problem (see (Muskhelishvili, 1953), Chap. 6) and its general solution is given by the relation

$$a_1\mu_1\Psi_1(z) + a_2\mu_2\Psi_2(z) + \overline{a_1\mu_1}\bar{\Psi}_1(z) + \overline{a_2\mu_2}\bar{\Psi}_2(z) = Q_0(z)X(z) \tag{4.8}$$

where

$$X(z) = \frac{1}{\sqrt{(z^2-a^2)(z+la)(z-ka)}} \tag{4.9}$$

is Plemelj function corresponding to the cut $\mathcal{L}$ and $Q_0(z)$ is an arbitrary polynomial having first degree and complex coefficients. In what follows we assume that

$$Q_0(z) \equiv 0. \tag{4.10}$$

As we shall see later, our choice is justified, since assuming (4.10) we shall be able to satisfy all boundary conditions. Hence the uniqueness theorem says us that other choice is not possible. Using (4.10) from (4.8) we get the second restriction which must be satisfied by the complex potentials in the whole complex plane z

$$a_1\mu_1\Psi_1(z) + a_2\mu_2\Psi_2(z) + \overline{a_1\mu_1}\bar{\Psi}_1(z) + \overline{a_2\mu_2}\bar{\Psi}_2(z) = 0. \tag{4.11}$$

Now, from (4.7) and (4.11) it results

$$a_1\mu_1\Psi_1(z) + a_2\mu_2\Psi_2(z) = 0. \tag{4.12}$$

Hence

$$\Psi_2(z) = -\frac{a_1\mu_1}{a_2\mu_2}\Psi_1(z). \tag{4.13}$$

Using the above result, and taking into account the boundary conditions (3.2) we can conclude that the complex potential $\Psi_1(z_1)$ must satisfy the following restrictions

$$\rho\Psi_1^+(t) + \bar{\rho}\bar{\Psi}_1^-(t) = -p(t), \qquad \rho\Psi_1^-(t) + \bar{\rho}\bar{\Psi}_1^+(t) = p(t) \tag{4.14}$$

for $t \in (-a, -la) \cup (ka, a)$.

In the above relations we have used the following notations

$$\rho = \frac{\Delta}{a_2\mu_2} \tag{4.15}$$

and

$$\Delta = a_2\mu_2 - a_1\mu_1. \tag{4.16}$$

Using the same procedure as before, from (4.14) we get the following equivalent conditions

$$(\rho\Psi_1(t) + \bar{\rho}\bar{\Psi}_1(t))^+ + (\rho\Psi_1(t) + \bar{\rho}\bar{\Psi}_1(t))^- = -2p(t), \tag{4.17}$$

$$(\rho\Psi_1(t) - \bar{\rho}\bar{\Psi}_1(t))^+ - (\rho\bar{\Psi}_1(t) - \bar{\rho}\bar{\Psi}_1(t))^- = 0, \tag{4.18}$$

for $t \in (-a, -la) \cup (ka, a)$.

From the restriction (4.18), we can conclude, as before, that the potential Ψ_1 must satisfy in the whole complex plane z the following relation

$$\rho\Psi_1(z) - \bar{\rho}\bar{\Psi}_1(z) = 0. \tag{4.19}$$

The restriction (4.17) is equivalent with a nonhomogeneous Riemann–Hilbert problem. As it is known (see (Muskhelishvili, 1953), Chap. 6) its solution can be obtained using Plemelj function $X(z)$ introduced by the relation (4.9). In all what follows we use that branch of this multivalued function which satisfies the condition

$$\lim_{z\to\infty} (1/X(z)) = +\infty. \tag{4.20}$$

As it is easy to see, the upper and lower limits of this branch are given by the following relations

$$\begin{aligned} X^+(t) &= \frac{1}{iA(t)}, \qquad X^-(t) = -\frac{1}{iA(t)} \quad \text{for } ka < t < a, \\ X^+(t) &= -\frac{1}{iA(t)}, \qquad X^-(t) = \frac{1}{iA(t)} \quad \text{for } -a < t < -la, \end{aligned} \tag{4.21}$$

where

$$A(t) = \sqrt{(a^2 - t^2)(t + la)(t - ka)} > 0 \quad \text{for } t \in (-a, -la) \cup (ka, a). \tag{4.22}$$

In the above conditions the general solution of our nonhomogeneous Riemann–Hilbert problem is given by the relation

$$\rho\Psi_1(z) + \bar{\rho}\bar{\Psi}_1(z) = -\frac{X(z)}{\pi i}\int_{\mathcal{L}} \frac{p(t)\,\mathrm{d}t}{X^+(t)(t-z)} + P(z)X(z). \tag{4.23}$$

In this equation $\mathcal{L} = (-a, -la) \cup (ka, a)$ and $P(z)$ is an arbitrary polynomial having complex coefficients. Taking into account the restriction (3.4) at large distance we can conclude that $P(z)$ must be a first degree polynomial; i.e.,

$$P(z) = C_1 z + C_2, \tag{4.24}$$

where C_1 and C_2 are arbitrary complex numbers. Later we shall see that these constants can be uniquely determined imposing the uniformity of the incremental displacement field.

Now, from (4.13), (4.15), (4.19) and (4.23) we get the following expressions for the complex potentials

$$\Psi_1(z_1) = -\frac{a_2\mu_2}{\Delta}\frac{X(z_1)}{2\pi i}\int_{\mathcal{L}} \frac{p(t)\,\mathrm{d}t}{X^+(t)(t-z_1)} + \frac{a_2\mu_2}{2\Delta}P(z_1)X(z_1) \tag{4.25}$$

and

$$\Psi_2(z_2) = \frac{a_1\mu_1}{\Delta}\frac{X(z_2)}{2\pi i}\int_{\mathcal{L}} \frac{p(t)\,\mathrm{d}t}{X^+(t)(t-z_2)} - \frac{a_1\mu_1}{2\Delta}P(z_2)X(z_2). \tag{4.26}$$

We recall that the complex quantity Δ is defined by (4.16).

5. Normal incremental stresses having constant value

In all what follows we assume that the given incremental stresses acting on the cracks faces have a constant value; i.e.,

$$p(t) = p = \text{const.} > 0 \quad \text{for} \quad t \in (-a, -la) \cup (ka, a). \tag{5.1}$$

In this case the integrals involved in (4.25) and (4.26) can be calculated using known results of the theory of complex functions (see (Muskhelishvili, 1953), Chap. 6). Finally for the complex potentials we get the expressions

$$\begin{aligned}\Psi_1(z_1) \;=\; & \frac{pa_2\mu_2}{2\Delta}\left(\frac{z_1^2 - \alpha z_1 - \beta^2}{\sqrt{(z_1^2 - a^2)(z_1 + la)(z_1 - ka)}} - 1\right) \\ & + \frac{a_2\mu_2}{2\Delta}\frac{P(z_1)}{\sqrt{(z_1^2 - a^2)(z_1 + la)(z_1 - ka)}}\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\Psi_2(z_2) \;=\; & -\frac{pa_1\mu_1}{2\Delta}\left(\frac{z_2^2 - \alpha z_2 - \beta^2}{\sqrt{(z_2^2 - a^2)(z_2 + la)(z_2 - ka)}} - 1\right) \\ & - \frac{a_1\mu_1}{2\Delta}\frac{P(z_2)}{\sqrt{(z_2^2 - a^2)(z_2 + la)(z_2 - ka)}}.\end{aligned} \tag{5.3}$$

In these relations

$$\alpha = \frac{k - l}{2}a, \qquad \beta = \frac{(1 - l)^2 + 4(1 + l)(1 + k) + (1 - k)^2}{8}a^2. \tag{5.4}$$

From (5.2) and (5.3) we can conclude that the boundary condition (3.1) is satisfied if and only if $P(t) = \overline{P(t)}$ for $t \in (-a, -la) \cup (ka, a)$. This requirement shows that C_1 and C_2 from (4.24) must be real numbers.

According to (2.4) we have

$$\Phi_j(z_j) = \int \Psi_j(z_j)\,\mathrm{d}z_j, \qquad j = 1, 2. \tag{5.5}$$

Hence $\Phi_j(z_j)$ can be multivalued functions even if $\Psi_j(z_j)$ are univalued. Consequently to assure the uniformity of the incremental displacement field we must guarantee the uniformity of the potentials $\Phi_j(z_j)$ on closed path around the two cracks.This requirement uniquelly determines the polinomials $P(z_j)$ and leads to the following final expressions of the complex potentials $\Psi_j(z_j)$

$$\Psi_1(z_1) = \frac{pa_2\mu_2}{2\Delta}\left(\frac{z_1^2 - Maz_1 - a^2N}{\sqrt{(z_2^2 - a^2)(z_1 + la)(z_1 - ka)}} - 1\right),$$

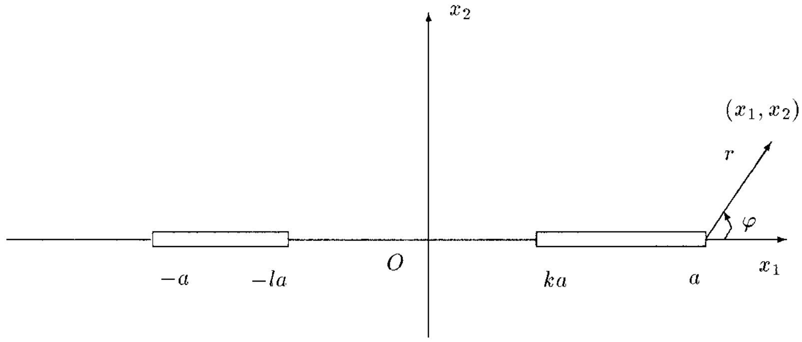

Figure 6.1. Asymptotic values near the crack tip a.

$$\Psi_2(z_2) = -\frac{pa_1\mu_1}{2\Delta}\left(\frac{z_2^2 - Maz_2 - a^2N}{\sqrt{(z_2^2 - a^2)(z_2 + la)(z_1 - ka)}} - 1\right) \tag{5.6}$$

where

$$M = M(k, l) = \frac{R_2S_0 - S_2R_0}{R_0S_1 + S_0R_1}, \qquad N = N(k, l) = \frac{R_2S_1 + S_2R_1}{R_0S_1 + S_0R_1}. \tag{5.7}$$

and

$$R_n = \int_k^1 \frac{t^n\,\mathrm{d}t}{R(t)}, \qquad S_n = \int_l^1 \frac{t^n\,\mathrm{d}t}{S(t)}, \quad n = 1, 2, 3 \tag{5.8}$$

with

$$R(t) = \sqrt{(1 - t^2)(t + l)(t - k)}, \qquad S(t) = \sqrt{(1 - t^2)(t - l)(t + k)}. \tag{5.9}$$

All details leading to the above relations are given in the Appendix.

6. Asymptotic values

The complex potentials determined in Section 5 will be used to evaluate the energy release rate corresponding to a crack tip. To obtain this quantity we must know the nonregular parts or the asymptotic values of the complex potentials in a small neighbourhood of the crack tip. This problem will be solved in this section.

We can get the asymptotic values of the potentials near the crack tip a of the crack (ka, a) taking (see Figure 6.1)

$$x_1 = a + r\cos\varphi, \qquad x_2 = r\sin\varphi, \tag{6.1}$$

where the polar coordinates r and φ designate, respectively, the radial distance from the crack tip and the angle between the radial line and the line ahead of the crack.

Using (2.5) and (6.1) we get

$$z_j - a = r(\cos\varphi + \mu_j \sin\varphi), \quad j = 1, 2. \tag{6.2}$$

For simplicity we introduce the functions

$$\chi_j(\varphi) = \cos\,\varphi + \mu_j\,\sin\,\varphi, \quad j = 1, 2 \tag{6.3}$$

and from (6.2) we obtain

$$z_j - a = r\chi_j(\varphi), \quad j = 1, 2. \tag{6.4}$$

We note that in a small neighbourhood of the tip a

$$z_j \approx a, \quad j = 1, 2. \tag{6.5}$$

Using the above observation and the relation (6.4) near the tip we get the following approximate equation

$$\sqrt{(z_j^2 - a^2)(z_j + la)(z_j - ka)} \approx a\sqrt{a}\sqrt{(1+l)(1-k)}\sqrt{2r\chi_j(\varphi)}, \quad j = 1, 2. \tag{6.6}$$

Now, from (5.6) we obtain the asymptotic values or nonregular parts of the complex potentials in a small neighbourhood of the tip a

$$\Psi_1(z_1) \approx \frac{p\sqrt{a}\,a_2\mu_2 m_1}{2\Delta\sqrt{2r\chi_1(\varphi)}}, \qquad \Psi_2(z_2) \approx -\frac{p\sqrt{a}\,a_1\mu_1 m_1}{2\Delta\sqrt{2r\chi_2(\varphi)}}, \tag{6.7}$$

where

$$m_1 = m_1(k, l) = \frac{1 - M - N}{\sqrt{(1+l)(1-k)}}. \tag{6.8}$$

To obtain the asymptotic values of the complex potentials $\Phi_j(z_j)$ we use the relations (5.5). Since the involved integrals are path independent we can take

$$\mathrm{d}z_j = \mathrm{d}r\chi_j(\varphi), \quad j = 1, 2. \tag{6.9}$$

In this way from (6.7) we get

$$\Phi_1(z_1) \approx \frac{p\sqrt{a}\,a_2\mu_2 m_1}{2\Delta}\sqrt{2r\chi_1(\varphi)}, \qquad \Phi_2(z_2) \approx -\frac{p\sqrt{a}\,a_1\mu_1 m_1}{2\Delta}\sqrt{2r\chi_2(\varphi)}. \tag{6.10}$$

To obtain the critical incremental stress producing the propagation of the tip a we must know the asymptotic values of the normal incremental displacement u_2 and of the normal incremental stress θ_{22} near the tip a. These asymptotic values can be obtained using Guz's representation formulae $(2.1)_2$ and $(2.3)_2$, and the relations (6.7), (6.10). Elementary calculus lead to the following result

$$u_2(r, \varphi) \approx p\sqrt{a}\,m_1\sqrt{2r}\,\mathrm{Re}\left\{\frac{1}{\Delta}(c_1 a_2\mu_2\sqrt{\chi_1(\varphi)} - c_2 a_1\mu_1\sqrt{\chi_2(\varphi)})\right\}, \tag{6.11}$$

$$\theta_{22}(r, \varphi) \approx \frac{p\sqrt{a}\,m_1}{\sqrt{2r}}\,\mathrm{Re}\left\{\frac{1}{\Delta}\left(\frac{a_2\mu_2}{\sqrt{\chi_1(\varphi)}} - \frac{a_1\mu_1}{\sqrt{\chi_2(\varphi)}}\right)\right\}. \tag{6.12}$$

We consider now the tip ka of the crack (ka, a). The same procedure as before leads to the following asymptotic values

$$u_2(r,\varphi) \approx p\sqrt{a}\, m_2 2\sqrt{r}\,\mathrm{Re}\left\{\frac{1}{i\Delta}\left(c_1 a_2 \mu_2 \sqrt{\chi_1(\varphi)} - c_2 a_1 \mu_1 \sqrt{\chi_2(\varphi)}\right)\right\}, \tag{6.13}$$

$$\theta_{22}(r,\varphi) \approx \frac{p\sqrt{a}\, m_2}{\sqrt{r}}\,\mathrm{Re}\left\{\frac{1}{i\Delta}\left(\frac{a_2\mu_2}{\sqrt{\chi_1(\varphi)}} - \frac{a_1\mu_1}{\sqrt{\chi_2(\varphi)}}\right)\right\}, \tag{6.14}$$

where

$$m_2 = m_2(k,l) = \frac{k^2 - kM - N}{\sqrt{(1-k^2)(k+l)}}. \tag{6.15}$$

In this relations r in the radial distance from the tip ka of the crack (ka, a).

We determine now the involved asymptotic values near the tip $-la$ of the crack $(-a, -la)$. The same method gives us the following result

$$u_2(r,\varphi) \approx p\sqrt{a}\, m_3 2\sqrt{r}\,\mathrm{Re}\left\{\frac{1}{\Delta}\left(c_1 a_2 \mu_2 \sqrt{\chi_1(\varphi)} - c_2 a_1 \mu_1 \sqrt{\chi_2(\varphi)}\right)\right\}, \tag{6.16}$$

$$\theta_{22}(r,\varphi) \approx \frac{p\sqrt{a}\, m_3}{\sqrt{r}}\,\mathrm{Re}\left\{\frac{1}{\Delta}\left(\frac{a_2\mu_2}{\sqrt{\chi_1(\varphi)}} - \frac{a_1\mu_1}{\sqrt{\chi_2(\varphi)}}\right)\right\}, \tag{6.17}$$

where

$$m_3 = m_3(k,l) = \frac{l^2 + lM - N}{\sqrt{(1-l^2)(k+l)}}. \tag{6.18}$$

To get the asymptotic values near the tip $-a$ of the crack $(a, -la)$ we use the same procedure and in this way we obtain

$$u_2(r,\varphi) \approx p\sqrt{a}\, m_4 \sqrt{2r}\,\mathrm{Re}\left\{\frac{1}{i\Delta}(c_1 a_2 \mu_2 \sqrt{\chi_1(\varphi)} - c_2 a_1 \mu_1 \sqrt{\chi_2(\varphi)})\right\}, \tag{6.19}$$

$$\theta_{22}(r,\varphi) \approx \frac{p\sqrt{a}\, m_4}{\sqrt{2r}}\,\mathrm{Re}\left\{\frac{1}{i\Delta}\left(\frac{a_2\mu_2}{\sqrt{\chi_1(\varphi)}} - \frac{a_1\mu_1}{\sqrt{\chi_2(\varphi)}}\right)\right\}, \tag{6.20}$$

where

$$m_4 = m_4(k,l) = \frac{1 + M - N}{\sqrt{(1-l)(1+k)}}. \tag{6.21}$$

In the relations (6.16), (6.11) r is the radial distance from the tip $-la$ of the crack $(-a, -la)$, and in (6.19), (6.20), r represents the radial distance from the tip $-a$ of the same crack.

7. Energy release rates and crack propagation conditions. Resonance

As it is known the energy release rates can be obtained using Irwin's relation (see for instance (Sih and Leibowitz, 1968; Rice, 1968; Leblond, 1991)).

In our calculus we take into account the symmetry involved in our problem. Due to this symmetry, we have

$$u_2(t, 0^+) = -u_2(t, 0^-) \quad \text{for} \quad t \in (-a, -la) \cup (ka, a). \tag{7.1}$$

We denote by $G(a)$ the energy release rate corresponding to the propagation of the tip a of the crack (ka, a). According to Irwin's formula, $G(a)$ can be obtained using the relation (see also Figure 6.1)

$$G(a) = \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta \theta_{22}(\delta - t, 0) u_2(t, \pi)\, \mathrm{d}t, \quad \delta > 0. \tag{7.2}$$

As this formula shows, the energy release rate depends only on the singular part of the normal incremental stress θ_{22} in a small neighbourhood of the tip a.

Since, according to (6.3)

$$\chi_j(0) = 1 \quad \text{and} \quad \chi_j(\pi) = i, \quad j = 1, 2, \tag{7.3}$$

from (6.9) and (6.10) we obtain

$$u_2(t, \pi) = p\sqrt{a}\, m_1 \Gamma \sqrt{2t}, \qquad \theta_{22}(\delta - t, 0) = \frac{p\sqrt{a}\, m_1}{\sqrt{2(t - \delta)}}, \tag{7.4}$$

where the real number Γ is defined by the relation (A.9).

Introducing (7.4) in (7.2) we get

$$G(a) = ap^2 m_1^2 \Gamma \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta \sqrt{\frac{t}{\delta - t}}\, \mathrm{d}t. \tag{7.5}$$

Now we use the relation

$$\int_0^\delta \sqrt{\frac{t}{\delta - t}}\, \mathrm{d}t = \frac{\pi\delta}{2}. \tag{7.6}$$

In this way for $G(a)$ we obtain the following value

$$G(a) = \tfrac{1}{2}\pi a p^2 m_1^2 \Gamma. \tag{7.7}$$

We use again Irwin's relation to obtain the energy release rate $G(ka)$ corresponding to the propagation of the tip ka of the crack (ka, a). In this way we get

$$G(ka) = \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta \theta_{22}(\delta - t, \pi) u_2(t, 0)\, \mathrm{d}t. \tag{7.8}$$

From (6.11) and (6.12) it results

$$u_2(t, 0) = -2p\sqrt{a}\, m_2 \Gamma \sqrt{t}, \qquad \theta_{22}(\delta - t, \pi) = -\frac{p\sqrt{a}\, m_2}{\sqrt{\delta - t}}. \tag{7.9}$$

Introducing (7.9) in (7.8) we obtain

$$G(ka) = \pi a p^2 m_2^2 \Gamma. \tag{7.10}$$

We denote by $G(-la)$ and $G(-a)$ the energy release rates corresponding to the propagation of the crack tips $-la$ and $-a$, respectively. Using the involved asymptotic values and the same procedure as before, after elementary calculus we get

$$G(-la) = \pi a p^2 m_3 \Gamma \quad \text{and} \quad G(-a) = \tfrac{1}{2}\pi a p^2 m_4 \Gamma. \tag{7.11}$$

In order to obtain the critical values of the applied incremental stress, for which one of the tips starts to propagate we shall use Griffith's energetical criterion. We denote by $\gamma > 0$ the surface tension of the composite and we assume that γ is a material constant, having the same value for all tips and being independent on the initial applied stress.

According to Griffith's criterion (see for instance (Sih and Leibowitz, 1968)) the tip a starts to propagate if the condition

$$G(a) = 2\gamma \tag{7.12}$$

is fulfilled.

Denoting by $p(a)$ the critical value of the applied incremental stress, for which the tip a starts to propagate, from (7.7) and (7.12) we obtain

$$p(a) = \frac{2}{\sqrt{\Gamma}}\sqrt{\frac{\gamma}{\pi a}}\frac{\sqrt{(1+l)(1-k)}}{1-M-N}. \tag{7.13}$$

In similar manner we design by $p(ka)$, $p(-la)$ and $p(-a)$ the critical values of the applied incremental stress for which the tips ka, $-la$ and $-a$ start to propagate. Using the obtained results and the same criterion as before we get

$$p(ka) = \frac{\sqrt{2}}{\sqrt{\Gamma}}\sqrt{\frac{\gamma}{\pi a}}\frac{\sqrt{(1-k^2)(k+l)}}{-k^2+kM+N}, \tag{7.14}$$

$$p(-la) = \frac{\sqrt{2}}{\sqrt{\Gamma}}\sqrt{\frac{\gamma}{\pi a}}\frac{\sqrt{(1-l^2)(k+l)}}{-l^2-lM+N}, \tag{7.15}$$

$$p(-a) = \frac{2}{\sqrt{\Gamma}}\sqrt{\frac{\gamma}{\pi a}}\frac{\sqrt{(1-l)(1+k)}}{1+M+N}. \tag{7.16}$$

The numerical tests realised by us show that $1-M-N$, $-k^2+kM+N$, $-l^2-lM+N$ and $1+M-N$ are positive quantities for any k and l from the interval $(0, 1)$. This fact justifies the choice made by us to express the involved critical values by (7.13)–(7.16).

In the same time we observe that the above relations are meaningful only if the real number Γ is a positive quantity. The equation (A.9) shows that the values of Γ depend on the elastic constants of the material and on the initial applied stress σ^0. We assume for a moment $\sigma^0 = 0$, i.e., we suppose a stress free reference configuration. We assume also transverse isotropy in

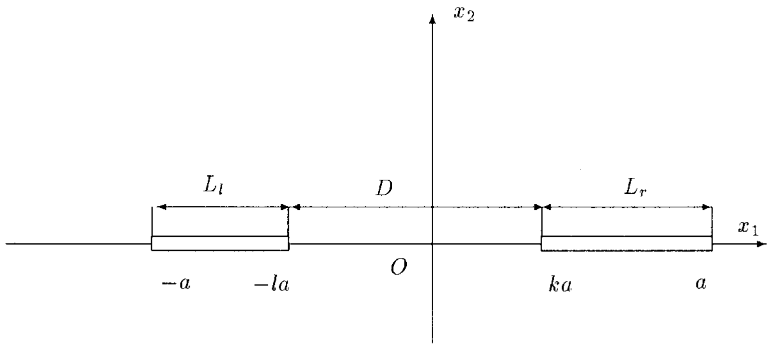

Figure 7.1. Geometrical characteristics.

the x_1–x_2-plane. In this case, using the expressions (2.11) of the instantaneous elasticities, after elementary calculus we get

$$\Gamma = \frac{C_{11} + C_{12}}{2C_{11}(C_{11} - C_{12})} > 0. \tag{7.17}$$

The above inequality takes place since the reference configuration of the material was assumed to be locally stable. From continuity consideration we can conclude that Γ will be positive also if σ^0 is not vanishing, its value being situated in a sufficiently small neighbourhood of zero. Consequently the relation (7.13)–(7.16) are meaningful even if the initial applied stress is not zero and fullfils the above mentioned condition. Similar results are valuable also for an orthotropic, fiber reinforced and prestressed composite material.

However, even in the above specified conditions can arise the following question: May exist a critical value σ_c^0 of the initial applied stress σ^0 such that when σ^0 starting from zero converges to σ_c^0 , the coefficient Γ converges to infinity? For a fiber reinforced composite the answer to this question was given by Guz and it is positive (see (Guz, 1983), Chap. 2). Taking into account the relations (2.15), valuable for a fiber reinforced composite, Guz was able to show that the critical value σ_c^0 is given by the following relation

$$\sigma_c^0 \approx -G_{12}\left\{1 - \frac{G_{12}^2}{E_1E_2}(1 - \nu_{13}\nu_{31})(1 - \nu_{23}\nu_{31})\right\} < 0. \tag{7.18}$$

Since $E_2 \ll E_1$ and $G_{12} \ll E_1$, the critical compression stress σ_c^0 produces only infinitesimal strains in the prestressed material. Hence the value of the coefficient Γ can become very large in the domain of infinitesimal strains. In such a situation the incremental critical stresses $p(a)$, $p(ka)$, $p(-la)$ and $p(-a)$ become very small, and the resonance phenomena can occur. To avoid such dangerous situations, leading to the total rupture of the composite, the initial applied compression stress must be drastically limited.

Returning to (7.13)–(7.16) we note that the values of the critical incremental stresses are given as functions of a and of the dimensionless parameters k and l. This variant is useful for a cantitative analysis of cracks interactions. However, the primary geometrical characteristics of the problem concerns the cracks lengths and their distance. Thus sometimes it is useful to express the obtained results, using the above quantities. To do this we denote by L_l and L_r the length of the cracks and by D the distance between them, as in Figure 7.1.

From Figure 7.1. it is easy to see that

$$2a = L_l + L_r + D, \qquad a(1-l) = L_l, \qquad a(1-k) = L_r \tag{7.19}$$

and

$$a = \tfrac{1}{2}(L_l + L_r + D), \qquad k = \frac{L_r - L_l + D}{L_r + L_l + D}, \qquad l = \frac{L_l - L_r + D}{L_r + L_l + D}. \tag{7.20}$$

We shall use these relations in the next section, devoted to the case in which the two cracks have the same length.

8. Equal cracks

To test our results we consider the case of two cracks having the same length L. In this situation the values of the critical stresses are already known for an isotropic material without initial stresses (see (Sneddon and Lowengrub, 1969), Chap. 2).

According to the assumption made

$$L_l = L_r = L \tag{8.1}$$

and from (7.20) we get

$$a = \tfrac{1}{2}(2L + D), \qquad k = l = \frac{D}{2L + D}. \tag{8.2}$$

Since $k = l$, from (5.8) and (5.9) it results

$$R_n = S_n = \int_k^1 \frac{t^n \, \mathrm{d}t}{\sqrt{(1-t^2)(t^2-k^2)}}, \quad n = 0, 1, 2. \tag{8.3}$$

In this way from (5.7) we obtain

$$M = M(k) = 0, \qquad N = N(k) = \frac{R_2(k)}{R_0(k)}. \tag{8.4}$$

Moreover, elementary calculus gives

$$R_0(k) = F(k_1) = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - k_1^2 \sin^2 \varphi}} \geqslant 1 \tag{8.5}$$

and

$$R_2(k) = E(k_1) = \int_0^{\pi/2} \sqrt{1 - k_1^2 \sin^2 \varphi} \leqslant 1. \tag{8.6}$$

Here $F(k_1)$ and $E(k_1)$ are the complete elliptical integrals of the first and second kind, respectively, (see (Sneddon and Lowengrub), Chap. 9), of modulus

$$k_1 = \sqrt{1 - k^2}. \tag{8.7}$$

As the second equation (8.4) shows, N is expressed by the rate of the above elliptical integrals, i.e.,

$$N = N(k) = \frac{E(k_1)}{F(k_1)} \leqslant 1. \tag{8.8}$$

Now it is easy to see, using (8.13)–(8.16) and the relations (8.4), that the values of the critical incremental stresses are given by the following formulae

$$p(a) = p(-a) = \frac{2}{\sqrt{\Gamma}}\sqrt{\frac{\gamma}{\pi a}}\frac{\sqrt{1-k^2}}{1-N(k)}, \tag{8.9}$$

$$p(ka) = p(-ka) = \frac{2}{\sqrt{\Gamma}}\sqrt{\frac{\gamma}{\pi a}}\frac{\sqrt{k(1-k^2)}}{N(k)-k^2}. \tag{8.10}$$

For an isotropic material and without initial stresses the above formulae were obtained by Willmore (1949) in 1949, and independently by Tranter (1961).

Our formulae generalise these classical results for a prestressed orthotropic material.

We assume now an isotropic material, without initial stresses, considering thus the problem first analysed by Willmore. Using (2.11), (2.12) and (A.9), after long, but elementary calculus we get for the coefficient Γ the following value

$$\Gamma = \frac{1-\nu}{\mu} > 0. \tag{8.11}$$

In the above relation μ is the shear modulus and ν is Poisson's ratio.

Introducing (8.11) in (8.9) and (8.10) we obtain Willmore's classical result (see (Sneddon and Lowengrub, 1969), Chap. 2)

$$p(a) = 2\sqrt{\frac{\gamma\mu}{\pi a(1-\nu)}}\frac{\sqrt{1-k^2}}{1-N(k)}, \tag{8.12}$$

$$p(ka) = 2\sqrt{\frac{\gamma\mu}{\pi a(1-\nu)}}\frac{\sqrt{k(1-k^2)}}{N(k)-k^2}. \tag{8.13}$$

The last conclusion can increase our trust in the correctness of the obtained results in a more complex case of an orthotropic prestessed material.

Obviously Willmore's formulae itself can be tested.

To do this we assume first that the distance D converges to zero, that is the parameter k tends to zero. In this case k_1 tends to 1, and $N(k)$ converges to zero. Hence from (8.12) we obtain

$$p(a) = 2\sqrt{\frac{\gamma\mu}{\pi a(1-\nu)}} \quad \text{if } D \to 0. \tag{8.14}$$

This is a well-known result, and gives the critical stress corresponding to a single crack having length $2a$ (see for instance (Sih and Leibowitz)).

Let us suppose now the length L of the crack fixed and let us assume that the distance D becomes greater and greater. To obtain the critical stresses corresponding to the limiting case in which D converges to infinity we express the parameters a and D through L and k. From (8.2) it results

$$a = \frac{L}{1-k}, \qquad D = \frac{2kL}{1-k}. \tag{8.15}$$

Introducing the first relation in (8.12) and (8.13) we get the expressions of the critical stresses as function of the length L and of the parameter k

$$p(a) = 2\sqrt{1+k}\sqrt{\frac{\gamma\mu}{\pi(1-\nu)L}}\frac{1-k}{N(k)-k^2}, \tag{8.16}$$

$$p(ka) = 2\sqrt{k(1+k)}\sqrt{\frac{\gamma\mu}{\pi(1-\nu)L}}\frac{1-k}{1-N(k)}.$$

From the second relation (8.15) we can see that the distance D converges to infinity if and only if the parameter k tends to 1. Hence to obtain the critical stresses in the analysed limiting case we must find the limit values of $p(a)$ and $p(ka)$ when k converges to 1. This problem can be solved using the developments ((Janke et al., 1960), Chap. 9):

$$\frac{2}{\pi}E(k_1) \approx 1 - \frac{k_1^2}{4} + \cdots, \qquad \frac{2}{\pi}F(k_1) = 1 + \frac{k_1^2}{4} + \cdots \quad \text{for } k_1 \ll 1. \tag{8.17}$$

Elementary calculus shows that $p(a)$ and $p(k)$ have the same limit value. Denoting this common value by $p(\infty)$ we get

$$p(\infty) = 2\sqrt{2}\sqrt{\frac{\gamma\mu}{\pi(1-\nu)L}}. \tag{8.18}$$

Again, this is a well-known result and gives the critical stress corresponding to a single crack having length L. Thus we can see that if the distance D between the cracks becomes greater and greater, their interaction becomes weaker and weaker.

The above obtained results are plausible and increase our trust in the correctness of Willmore's formulae.

Taking into account this fact we return now to our orthotropic prestressed composite containing to equal cracks. In this situation we shall analyse the interaction of the cracks. To do this we return to (8.9) and (8.10), assuming a fixed and supposing that the parameter k takes values in the interval (0,1). Our aim is to establish the relation existing between the critical incremental stresses $p(a)$ and $p(ka)$. For values of k for which $p(a) > p(ka)$ the tips $\pm ka$ start to propagate first. For values of k for which $p(a) < p(ka)$ the tips $\pm a$ start to propagate first. To establish the above order relations we introduce the dimensionless quantities $q(a)$ and $q(ka)$ defined by the equations

$$q(a) = \frac{\pi a p^2(a)}{4\gamma\Gamma} = \frac{\sqrt{1-k^2}}{1-N(k)}, \qquad q(ka) = \frac{\pi a p^2(ka)}{4\gamma\Gamma} = \frac{\sqrt{k(1-k^2)}}{N(k)-k^2}. \tag{8.19}$$

According to the assumptions made $q(a)$ and $q(ka)$ depend only on $k \in (0,1)$. The order relation existing between the critical incremental stressed $p(a)$ and $p(ka)$ is entirely determined by the ratio

$$r(k) = \frac{q(ka)}{q(a)} = \sqrt{k}\frac{1-N(k)}{N(k)-k^2}, \quad k \in (0,1). \tag{8.20}$$

Taking into account the developments (8.17) it is easy to see that

$$\lim_{k \to 1} r(k) = 1. \tag{8.21}$$

To calculate the limit value of the ratio $r(k)$ when k converges to zero, that is k_1 tends to 1, we use the asymptotic developments (see (Janke et al., 1960), Chap. 9)

$$E(k_1) \approx 1 + \frac{1}{2}\left(\ln\frac{4}{k_1} - \frac{1}{2}\right)k_1^2 + \cdots,$$

$$F(k_1) \approx \ln\frac{4}{k_1} - \frac{\ln(4/k_1) - 1}{4}k_1^2 + \cdots, \tag{8.22}$$

valuable for $k_1 \approx 1$. In this way, elementary calculus gives

$$\lim_{k \to 0} r(k) = 0. \tag{8.23}$$

The values of the ratio $r(k)$ for values of the parameter k in the interval $(0, 1)$ were determined by numerical methods. In Table 8.1 we give the obtained results for 10 values of k.

Table 8.1. Numerical values of the ratio $r(k)$

k	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.98
$r(k)$	0.4	0.8	0.94	0.97	0.98	0.994	0.997	0.9992	0.9997	0.99999

In this way we can conclude that

$$r(k) < 1 \quad \text{and} \quad p(ka) < p(a) \quad \text{for } 0 < k < 1. \tag{8.24}$$

Thus we can conclude that always the internal tips $\pm ka$ start to propagate first. Moreover, since $D/L = 2k/(1-k)$ we can see that $p(ka) \ll p(a)$ if $D \ll L$, and $p(ka) \approx p(a)$ if $D \gg L$. Hence the interaction between the cracks is strong if the distance between the cracks is much smaller as their length, and is weak if the distance D is much greater as the common length L of the cracks. These results are plausible, being in accordance with observed facts and with the theoretical results given by Kachanov ((1974), Chap. 8) and Slepian ((1981), Chap. 1).

We stress the fact that the above presented behaviour of the cracks does not depend on the initial applied stress.

In the next section we shall analyse the behaviour of the cracks assuming that they can have different lengths.

9. Interaction of two cracks with different lengths

We return now to the general relations (7.13)–(7.16) giving the critical incremental stresses for two cracks having different length. As in Section 8 we assume a fixed and suppose that k

and l have values in the interval $(0, 1)$. To study the interaction of the cracks we introduce the following dimensionless quantities

$$
\begin{aligned}
\Pi_1(k,l) &= \frac{p(a)}{\Pi} = \frac{\sqrt{2(1+l)(1-k)}}{1-M_0-N_0},\\
\Pi_2(k,l) &= \frac{p(ka)}{\Pi} = \frac{\sqrt{(1-k^2)(k+l)}}{-k^2+kM_0+N_0},\\
\Pi_3(k,l) &= \frac{p(-la)}{\Pi} = \frac{\sqrt{(1-l^2)(k+l)}}{-l^2-lM_0+N_0},\\
\Pi_4(k,l) &= \frac{p(-a)}{\Pi} = \frac{\sqrt{2(1-l)(1+k)}}{1+M_0-N_0},
\end{aligned}
\tag{9.1}
$$

with

$$
\Pi = \sqrt{\frac{2\gamma}{\pi a\Gamma}}. \tag{9.2}
$$

Again, to analyse the interaction of cracks we must know the order relation existing between the four critical incremental stresses for different values of the dimensionless parameters k and l. Obviously, the necessary order relation will be known if the values of the dimensionless quantities Π_1, Π_2, Π_3 and Π_4 are known. These values can be obtained by numerical methods and in Table 9.1 we give the results obtained by us for 36 values of the pair (k, l).

To interpret the obtained results we use (7.19) to express the geometrical characteristics L_l, L_r and D of the cracks as functions of a, k and l. We get (see Figure 7.1)

$$
L_l = a(1-l), \qquad L_r = a(1-k), \qquad D = a(k+l). \tag{9.3}
$$

From Table 9.1 we get that if $k = l$, than $\Pi_1 \approx \Pi_4$, $\Pi_2 \approx \Pi_3$ and $\Pi_2 < \Pi_1$. Hence if the cracks have nearly the same length, the internal tips start to propagate first, as we already have seen in Section 8. Let us take $k = 0.001$ and $l = 0.0002$. In this case $L_l \approx L_r \approx a$, $D = 0.003a$ hence $D \ll L_l$. From Table 9.1 we get $\Pi_2 \approx \Pi_3 \approx 0.013$, $\Pi_1 \approx \Pi_4 = 1.6$, hence $\Pi_2 \ll \Pi_1$.

Consequently the critical incremental stresses for which the inner tips start to propagate are much smaller as those which can produce the propagation of the external tips. Thus we can see that if the distance between the cracks is much smaller as their length there exist a strong interaction between the crack and they tend to unify.

Let us consider now $k = 0.901$ and $l = 0.902$. In this case $L_l \approx L_r \approx 0.099a$, $D \approx 1.8a$, hence $L_l \ll D$. From Table 9.1 we obtain $\Pi_1 \approx \Pi_2 \approx \Pi_3 \approx \Pi_4 \approx 6.3$. Consequently all tips start to propagate simultaneously. Consequently if the distance between the cracks is much greater as their length, the interaction between the cracks is weak. These results are in good agreement with those obtained in Section 8.

Let us consider now two examples corresponding to cracks having different length.

First we take $k = 0.001$ and $l = 0.902$. In this case $L_l = 0.098a$, $L_r = 0.999a$, $D \approx 0.903a$, hence $L_l < L_r$ and $D \approx L_r$. From Table 9.1 we get $\Pi_1 = 1.983$, $\Pi_2 = 2.017$, $\Pi_3 = 6.048$ and $\Pi_4 = 5.828$. Thus we can conclude that the tips of the longer crack start

Table 9.1. Numerical values of the parameters Π_1, Π_2, Π_3, Π_4

k/l		0.002	0.202	0.402	0.602	0.802	0.902
	Π_1	1.577	1.864	1.932	1.957	1.980	1.983
0.001	Π_2	0.012	1.649	1.881	1.974	2.011	2.017
	Π_3	0.015	1.741	2.256	2.901	4.230	6.048
	Π_4	1.673	2.018	2.371	2.972	4.144	5.828
	Π_1	2.007	2.112	2.168	2.198	2.212	2.214
0.201	Π_2	1.745	2.032	2.163	2.226	2.253	2.258
	Π_3	1.648	2.028	2.433	3.045	4.379	6.270
	Π_4	1.872	2.112	2.461	3.021	4.262	5.988
	Π_1	2.361	2.453	2.506	2.168	2.550	2.553
0.401	Π_2	2.260	2.438	2.534	2.585	2.608	2.612
	Π_3	1.880	2.160	2.529	3.128	4.470	2.180
	Π_4	1.938	2.174	2.514	3.078	4.336	6.087
	Π_1	2.914	3.009	3.067	3.100	3.116	3.118
0.601	Π_2	2.909	3.052	3.136	3.183	3.204	3.208
	Π_3	1.972	2.222	2.580	3.174	4.498	6.421
	Π_4	1.970	2.204	2.544	3.110	4.378	6.142
	Π_1	4.123	4.241	4.315	4.358	4.379	4.382
0.801	Π_2	2.248	4.396	4.486	4.538	4.563	4.568
	Π_3	2.008	2.245	2.603	3.195	4.544	6.480
	Π_4	1.985	2.217	2.558	3.126	4.399	6.172
	Π_1	5.789	5.948	6.048	6.106	6.134	6.139
0.901	Π_2	6.120	6.306	6.421	6.487	6.520	6.525
	Π_3	2.014	2.254	2.606	3.199	4.548	6.485
	Π_4	1.980	2.219	2.560	3.128	4.402	6.177

to propagate nearly simultaneously, and the interaction between the cracks is relatively weak when the distance between them is relatively large, compared with the length of the shorter crack.

As the last example we consider $k = 0.801$ and $l = 0.402$. Now $L_l = 0.599a$, $L_r = 0.198a$, $D = 1.203a$, hence $L_r < L_l < D$. Table 9.1 gives $\Pi_1 = 4.315$, $\Pi_2 = 4.486$, $\Pi_3 = 2.603$ and $\Pi_4 = 2.588$. Thus we can see that the tips of the larger crack start to propagate nearly simultaneously. The interaction between the cracks is very weak, since the distance between them is relatively large, compared with their lengths.

10. Final remarks

The main conclusions of our paper can be summarised as follows:

(i) If the cracks have the same length and this is greater as the distance between the cracks, the inner tips start to propagate first. The cracks tend to unify and the interaction between them is strong.

(ii) If the common length of the cracks is much smaller as their distance, all tips start to propagate simultaneously. The interaction between the cracks is weak.

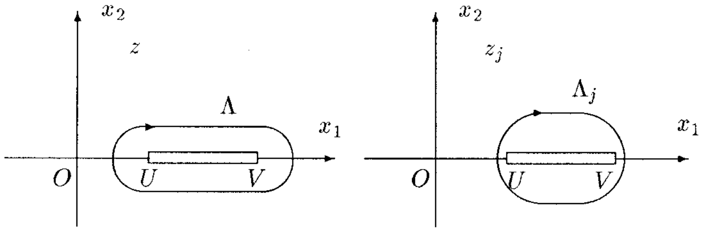

Figure A.1. Uniformity of the incremental displacement field.

(iii) If the cracks have different length, and one of the length is much smaller as the distance between the crack, the tips of the longer crack start to propagate nearly in the same time. The interaction between the cracks is weak again.

(iv) There exists strong interaction between two unequal cracks only if the distance between them is much smaller as their lengths. In such a situation the cracks tend to unify for relatively small values of the critical incremental stresses.

In our opinion the obtained results are plausible and in good agreement with observed cracks behaviour. (See Kachanov, 1974; Slepian, 1981).

We note that the obtained properties do not depend on the initial applied stress, if its magnitude is limited, to avoid the occurrence of the resonance phenomena.

A. Appendix

To assure the uniformity of the potentials $\Phi_j(z_j)$ let us design by (U, V) one of the two cracks. Also we design by Λ a simple closed curve around the crack (U, V) in the complex plane $z = x_1 + ix_2$, as in Figure A.1. Let Λ_j be the corresponding simple closed curves around the crack (U, V) in the complex planes $z_j = x_1 + \mu_j x_2$, $j = 1, 2$, as in Figure A.1.

According to the relations $(2.1)_2$ and (5.5) the uniformity of u_1 is assured if the potentials $\Psi_j(z_j)$ satisfy the restriction

$$\oint_{\Lambda_1} \{b_1\Psi_1(z_1)dz_1 + \overline{b_1\Psi_1(z_1)\,\mathrm{d}z_1}\} + \oint_{\Lambda_2} \{b_2\Psi_2(z_2)\,\mathrm{d}z_2 + \overline{b_2\Psi_2(z_2)dz_2}\} = 0. \tag{A.1}$$

Since $\Psi_j(z_j)$ are analytical functions in the complex planes z_j with the cut $\mathcal{L}$ the integrals involved in (A.1) rest unchanged if Λ is changed. Taking into account this fact and squeezing the curve Λ around the crack we can conclude that the uniformity of u_1 will be guaranteed if the following restriction will be satisfied

$$\mathrm{Re}\left\{\int_U^V (b_1\Psi_1^+(t) + b_2\Psi_2^+(t))\,\mathrm{d}t + \int_V^U (b_1\Psi_1^-(t) + b_2\Psi_2^-(t))\,\mathrm{d}t\right\} = 0. \tag{A.2}$$

To obtain the limit values involved in (A.2) we use the relations (5.2), (5.3) and (4.21), (4.22). First we observe that the restriction is homogeneous and the replacement of the crack (ka, a) with the crack $(-a, -la)$ implies only the replacement of i with $-i$ in the involved limit values. Hence we can consider only the crack (ka, a). In this way from (4.21) and (4.25), (4.26) we get

$$\Psi_1^+(t) = \frac{pa_2\mu_2}{2\Delta}\left(\frac{Q(t)}{iA(t)} - 1\right) + \frac{a_2\mu_2}{2\Delta}\frac{P(t)}{iA(t)},$$

$$\Psi_2^+(t) = -\frac{pa_1\mu_1}{2\Delta}\left(\frac{Q(t)}{iA(t)} - 1\right) - \frac{a_1\mu_1}{2\Delta}\frac{P(t)}{iA(t)}, \tag{A.3}$$

where

$$Q(t) = t^2 - \alpha t - \beta^2. \tag{A.4}$$

The limit values $\Psi_1^-(t)$ and $\Psi_2^-(t)$ can be obtained from (A.3), replacing i by $-i$. Using (A.3) and returning to the crack (U, V) we get

$$\int_U^V (b_1\Psi_1^+(t) + b_2\Psi_2^+(t))\,\mathrm{d}t = \frac{\Gamma_0}{2}\left\{p\int_U^V \left(\frac{Q(t)}{iA(t)} - 1\right)\mathrm{d}t + \int_U^V \frac{P(t)}{iA(t)}\,\mathrm{d}t\right\}, \tag{A.5}$$

where

$$\Gamma_0 = \frac{b_1a_2\mu_2 - b_2a_1\mu_1}{\Delta}. \tag{A.6}$$

As has been shown by Guz (1989), and independently by Soós (1996), Γ_0 is a real number if the initial deformed equilibrium configuration is locally stable. Taking into account this fact, from (A.5) and the observation made concerning the limit values $\Psi_1^-(t)$, $\Psi_2^-(t)$, we can conclude that the uniformity condition (A.2) is satisfied for both cracks.

We consider now the incremental displacement field u_2. According to the representation formula $(2.1)_2$ the uniformity if u_2 will be assured if the condition

$$\mathrm{Re}\left\{\int_U^V (c_1\Psi_1^+(t) + c_2\Psi_2^+(t))\,\mathrm{d}t + \int_V^U (c_1\Psi_1^-(t) + c_2\Psi_2^-(t))\,\mathrm{d}t\right\} = 0 \tag{A.7}$$

is fulfilled. Using the same procedures as before we get

$$\int_U^V (c_1\Psi_1^+(t) + c_2\Psi_2^+(t))\,\mathrm{d}t = -\frac{\Gamma i}{2}\int_U^V \frac{pQ(t) + P(t)}{A(t)}\,\mathrm{d}t + i\frac{p\Gamma}{2}(V - U), \tag{A.8}$$

where

$$\Gamma = \frac{c_1a_2\mu_2 - c_2a_1\mu_1}{\Delta}i. \tag{A.9}$$

Guz (1989), and independently Soós (1996), have shown that Γ is a real number if the initial deformed equilibrium configuration is locally stable. Using this fact, the relation (A.8) and the observation concerning the limit values $\Psi_1^-(t)$, $\Psi_2^-(t)$, we can see that the uniformity condition (A.7) will be satisfied if the condition

$$\int_U^V \frac{pQ(t) + P(t)}{A(t)}\,\mathrm{d}t = 0 \tag{A.10}$$

will be fulfilled. Taking in (A.10) $U = ka$, $V = a$ and $U = -a$, $V = -la$, respectively, we are leaded to the following two conditions of uniformity

$$\int_{ka}^a \frac{P(t)\,\mathrm{d}t}{A(t)} = -p\int_{ka}^a \frac{Q(t)\,\mathrm{d}t}{A(t)}, \qquad \int_{-a}^{-la} \frac{P(t)\,\mathrm{d}t}{A(t)} = -p\int_{-a}^{-la} \frac{Q(t)\,\mathrm{d}t}{A(t)}. \tag{A.11}$$

The above relations will be used to determine the real coefficient C_1 and C_2, appearing in the expression (4.24) of the polynomial $P(z)$.

To express the system (A.11) in a simpler form we introduce the following integrals

$$I_n = \int_{ka}^{a} \frac{t^n\,\mathrm{d}t}{A(t)}, \qquad J_n = \int_{la}^{a} \frac{t^n\,\mathrm{d}t}{B(t)}, \quad n = 0, 1, 2 \tag{A.12}$$

with

$$A(t) = \sqrt{(a^2 - t^2)(t + la)(t - ka)}, \qquad B(t) = \sqrt{(a^2 - t^2)(t - la)(t + ka)}. \tag{A.13}$$

Now, using the expression (A.4) of the polynomial $Q(t)$, we can express (A.11) in the following equivalent form

$$\begin{aligned} &I_1 C_1 + I_0 C_2 = -p(I_2 - \alpha I_1 - \beta^2 I_0), \\ &-J_1 C_1 + J_0 C_2 = -p(J_2 + \alpha J_1 - \beta^2 J_0). \end{aligned} \tag{A.14}$$

Now it is easy to see that

$$C_1 = -p(M - \alpha), \qquad C_2 = -p(N - \beta^2), \tag{A.15}$$

where $M = M(k,l)$ and $N = N(k,l)$ are given by (5.7)–(5.9).

Using (A.15) in (5.2) and (5.3) we are leaded to the relations (5.6)–(5.9) given in Section 5.

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