Stress concentration in an anisotropic body with three equal collinear cracks in Mode II of fracture. I. Analytical study

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Received 28 December 2012, revised 23 January 2013, accepted 9 February 2013
Published online 11 March 2013

Key words Three equal collinear cracks, Riemann-Hilbert problem, Lekhnitskii representation, asymptotic values.

One consider an anisotropic, unbounded elastic body containing three collinear, equal cracks subjected to asymmetrically tangential stresses, case corresponding to Mode II of classical Fracture. We determine the elastic state produced in the body using the formalism of Riemann-Hilbert problem and the representation of elastic stresses and displacements fields due to Lekhnitskii. Using the asymptotical analysis, we obtain the asymptotic values of the stress and the displacement fields in a vicinity of the cracks tips.

1 Introduction

The aim of this paper is to describe the elastic state in an anisotropic body with three equal, collinear cracks subjected to the Mode II of classical Fracture. The problem of a linear elastic solid with multiple cracks was investigated by many authors, using integral equations techniques, integral transform method, conformal mapping, Green's function method [1–12]. To the best of authors' knowledge there are no explicit analytical or asymptotical solution of the problem of three equal collinear cracks in an anisotropic body subjected to Mode II of classical fracture in complex potentials, using the formalism of the Riemann-Hilbert problem and the theory of Cauchy's integral. Present paper represents an extension of the paper [13] regarding the mathematical modeling of three equal and collinear cracks in an orthotropic solid, from Mode I to Mode II and from orthotropic material to an anisotropic one.

We consider an anisotropic elastic material containing three equal collinear cracks of a lengths $2a$ situated and having the distance between them equal with d in xz plane, as in Fig. 1. Let us suppose that the material is unbounded and the crack faces are subjected to the asymmetrically applied tangential stresses.

As usual in continuum mechanics, we suppose that the cracks are represented as cuts having two faces. We assume that the initial equilibrium configuration of the body is homogeneous and locally stable.

Our first aim is to determine the elastic state in the body. To do this, we use Lekhnitskii representation theorem of the stresses and displacements fields with complex potentials [14]. We give the boundary conditions which must be satisfied by the complex potentials on the faces and at large distances from the cracks. Using the theory of the Riemann-Hilbert homogeneous and non-homogeneous problems, Plemelj's function and Cauchy's integral [15–18] we obtain the general solution of our complex potentials.

Our second aim is to determine the asymptotical behavior of the stresses and displacements fields in a vicinity of the cracks tips. In Sect. 3, for an orthotropic material, assuming that our applied tangential forces have a given constant value we give an explicit solution of our mathematical problem. To do this we determine first the non-regular or asymptotical values of the complex potentials. The obtained results are presented in Sect. 3 and are important in a future research to determine the energy release rate and strain energy density for each tip of the cracks. In that moment we are able to study the cracks interaction, using numerical methods and later on, to validate these results with experimental tests which will be performed on particular types of anisotropic elastic composites.

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2 Field equations and boundary conditions

We consider an homogeneous anisotropic material in plane strain state relative to Oxy plane and we suppose that stress from reference configuration of our material is locally stable.

Following Lekhnitskii [14] the elastic state of the body can be expressed by two analytic complex potentials $\Phi_j(z_j)$ defined in two complex planes z_j, $j = 1, 2$ as following:

$$\sigma_x = 2\,\mathrm{Re}\{\mu_1^2\Psi_1(z_1) + \mu_2^2\Psi_2(z_2)\}, \quad \sigma_y = 2\,\mathrm{Re}\{\Psi_1(z_1) + \Psi_2(z_2)\},$$
$$\tau_{xy} = -2\,\mathrm{Re}\{\mu_1\Psi_1(z_1) + \mu_2\Psi_2(z_2)\}, \tag{2.1}$$

$$u_x = 2\,\mathrm{Re}\{p_1\Phi_1(z_1) + p_2\Phi_2(z_2)\}, \quad u_y = 2\,\mathrm{Re}\{q_1\Phi_1(z_1) + q_2\Phi_2(z_2)\}, \tag{2.2}$$

with

$$\Psi_j(z_j) = \frac{d}{dz_j}\Phi_j(z_j). \tag{2.3}$$

In these formulas

$$p_k = a_{11}\mu_k^2 + a_{12} - a_{16}\mu_k, \quad q_k = a_{12}\mu_k + \frac{a_{22}}{\mu_k} - a_{26}, \quad k = 1, 2. \tag{2.4}$$

The unequal complex parameters μ_j, $j = 1, 2$ and their complex conjugates $\overline{\mu}_j$, $j = 1, 2$ are the roots of the equation:

$$a_{11}\mu^4 - 2a_{16}\mu^3 + 2(a_{12} + a_{66})\mu^2 - 2a_{26}\mu + a_{22} = 0. \tag{2.5}$$

We denoted by a_{ij}, $i, j = 1, 2, 6$ the elastic coefficients of the anisotropic material.

Let us assume that we have an unbounded anisotropic composite which contains three equal, collinear cracks situated in the some plane Oxz, the Oy axis being perpendicular to the crack faces which are acted by tangential stresses asymmetrically distributed relative to the plane containing the cracks (see Fig. 1).

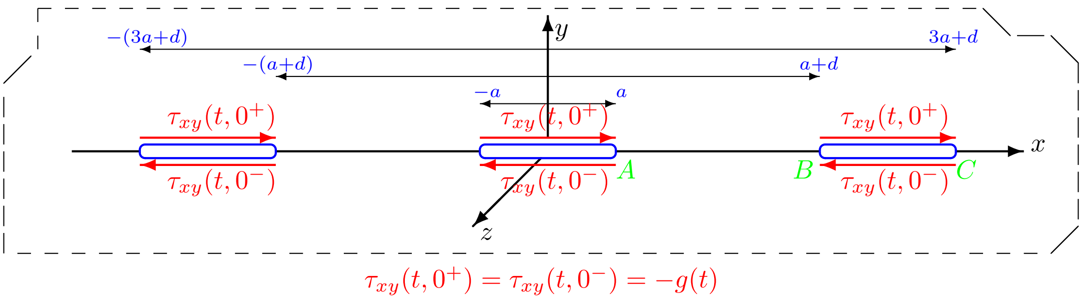

Fig. 1 Three equal and collinear cracks in acted by tangential loads.

We denote by $\mathscr{L}$ the cut corresponding to the segments $(-3a - d, -a - d)$, $(-a, a)$ and $(a + d, 3a + d)$, i.e.

$$\mathscr{L} = (-3a - d, -a - d) \cup (-a, a) \cup (a + d, 3a + d) \tag{2.6}$$

where $2a$ is the length of the cracks and d is the distance between the cracks, $a, d \in \mathbb{R}_+^*$. According to the assumptions made, the state of our body is a plane state relative to the plane Oxy. The stresses τ_{xy} and σ_y must satisfy the following boundary conditions on the two faces of the cracks:

$$\tau_{xy}(t, 0^+) = \tau_{xy}(t, 0^-) = -g(t), \quad \sigma_y(t, 0^+) = \sigma_y(t, 0^-) = 0, \quad t \in \mathscr{L}. \tag{2.7}$$

We assume that the tangential forces $g = g(t)$ satisfies the Hölder's condition [17].

At large distances from the cracks the displacements and stresses must vanish, i.e.

$$\lim_{r\to\infty}\{u_x(x, y), u_y(x, y), \sigma_x(x, y), \sigma_y(x, y), \tau_{xy}(x, y)\} = 0, \quad r = \sqrt{x^2 + y^2}. \tag{2.8}$$

From the above restriction (2.8) we conclude that the complex potentials $\Phi_j(z_j)$ and $\Psi_j(z_j)$ have to fulfill the following field restrictions:

$$\lim_{z_j\to\infty}\{\Phi_j(z_j), \Psi_j(z_j)\} = 0, \quad j = 1, 2. \tag{2.9}$$

We denote by $f^+(t)$ and $f^-(t)$ the upper and lower limits of a complex valued function $f(z)$ defined in the whole complex plane $z = x + iy, i = \sqrt{-1}$, i.e.

$$f^+(t) = \lim_{\substack{z\to t,\\ y>0}} f(z) \quad \text{and} \quad f^-(t) = \lim_{\substack{z\to t,\\ y<0}} f(z), \qquad t \in \mathscr{L}. \tag{2.10}$$

We denote by a superposed bar the complex conjugation. We define a new function $\overline{f} = \overline{f}(z)$ by the following relation:

$$\overline{f}(z) = \overline{f(\overline{z})}. \tag{2.11}$$

It is easy to see that

$$\overline{f}^+(t) = \overline{f^-(t)} \quad \text{and} \quad \overline{f}^-(t) = \overline{f^+(t)}, \qquad t \in \mathscr{L}. \tag{2.12}$$

Using (2.7) and $(2.1)_{2,3}$ we get:

$$\begin{aligned}
&\Psi_1^+(t) + \Psi_2^+(t) + \overline{\Psi_1^+(t)} + \overline{\Psi_2^+(t)} = 0,\\
&\Psi_1^-(t) + \Psi_2^-(t) + \overline{\Psi_1^-(t)} + \overline{\Psi_2^-(t)} = 0\\
\text{and}\\
&\mu_1\Psi_1^+(t) + \mu_2\Psi_2^+(t) + \overline{\mu_1\Psi_1^+(t)} + \mu_2\overline{\Psi_2^+(t)} = g,\\
&\mu_1\Psi_1^-(t) + \mu_2\Psi_2^-(t) + \overline{\mu_1\Psi_1^-(t)} + \mu_2\overline{\Psi_2^-(t)} = g
\end{aligned} \tag{2.13}$$

for $t \in \mathscr{L}$.

Taking into account Eqs. (2.12) and adding and subtracting $(2.13)_{1,2}$ we get

$$\begin{aligned}
&(\Psi_1 + \Psi_2 + \overline{\Psi}_1 + \overline{\Psi}_2)^+(t) + (\Psi_1 + \Psi_2 + \overline{\Psi}_1 + \overline{\Psi}_2)^-(t) = 0,\\
&(\Psi_1 + \Psi_2 - \overline{\Psi}_1 - \overline{\Psi}_2)^+(t) - (\Psi_1 + \Psi_2 - \overline{\Psi}_1 - \overline{\Psi}_2)^-(t) = 0
\end{aligned} \tag{2.14}$$

for $t \in \mathscr{L}$.

The second equation from (2.14) shows that the analytic function

$$\Psi_1(z) + \Psi_2(z) - \overline{\Psi}_1(z) - \overline{\Psi}_2(z), \tag{2.15}$$

depending on the complex variable $z = x + iy$ has a null jump across $\mathscr{L}$. Hence, for this function we have a homogeneous boundary value problem. The general solution [14,17] is given by

$$\Psi_1(z) + \Psi_2(z) - \overline{\Psi}_1(z) - \overline{\Psi}_2(z) = 0, \quad \text{for any } z = x + iy. \tag{2.16}$$

The first equation from (2.14) represents a homogeneous Riemann-Hilbert problem for the analytic function

$$\Psi_1(z) + \Psi_2(z) - \overline{\Psi}_1(z) - \overline{\Psi}_2(z), \tag{2.17}$$

depending on the complex variable $z = x + iy$.

The general solution of our Riemann-Hilbert problem is given by

$$\Psi_1(z) + \Psi_2(z) + \overline{\Psi}_1(z) + \overline{\Psi}_2(z) = 0, \text{ for any } z = x + iy. \tag{2.18}$$

From the Eqs. (2.16) and (2.18) we get

$$\Psi_1(z) + \Psi_2(z) = 0 \quad \text{and} \quad \overline{\Psi}_1(z) + \overline{\Psi}_2(z) = 0, \quad \text{for any} \quad z = x + iy. \tag{2.19}$$

Particularly, we shall have:

$$\Psi_1(x,0) + \Psi_2(x,0) = 0 \quad \text{and} \quad \overline{\Psi}_1(x,0) + \overline{\Psi}_2(x,0) = 0 \quad \text{for any} \quad -\infty < x < \infty. \tag{2.20}$$

Equations (2.20) can be used to eliminate $\Psi_2 = \Psi_2(z_2)$ from the boundary conditions $(2.13)_{3,4}$. In this way we get:

$$\alpha\Psi_1^+(t) + \bar{\alpha}\bar{\Psi}_1^-(t) = g(t),$$
$$\alpha\Psi_1^-(t) + \bar{\alpha}\bar{\Psi}_1^+(t) = g(t) \quad \text{for} \quad t \in \mathscr{L} \tag{2.21}$$

with

$$\alpha = \mu_1 - \mu_2. \tag{2.22}$$

Adding and subtracting the Eqs. (2.21) we obtain:

$$\left(\alpha\Psi_1 + \bar{\alpha}\bar{\Psi}_1\right)^+(t) + \left(\alpha\Psi_1 + \bar{\alpha}\bar{\Psi}_1\right)^-(t) = 2g(t),$$
$$\left(\alpha\Psi_1 + \bar{\alpha}\bar{\Psi}_1\right)^+(t) - \left(\alpha\Psi_1 + \bar{\alpha}\bar{\Psi}_1\right)^-(t) = 0 \quad \text{for} \quad t \in \mathscr{L}. \tag{2.23}$$

The first relation represent a nonhomogeneous Riemann-Hilbert problem. The general solution, satisfying the conditions imposed at large distances from the cracks is by

$$\alpha\Psi_1(z_1) + \bar{\alpha}\bar{\Psi}_1(z_1) = \frac{X(z_1)}{\pi i}\int_{\mathscr{L}} \frac{g(t)dt}{X^+(t)(t - z_1)} + P(z_1)X(z_1), \qquad t \in \mathscr{L} \tag{2.24}$$

where

$$X^+(t) = \frac{1}{i\sqrt{(a^2 - t^2)[t^2 - (a+d)^2][t^2 - (3a+d)^2]}}, \qquad t \in \mathscr{L} \tag{2.25}$$

represents the Plemelj's type function defined on the upper faces of the crack. We chose the branch of the Plemelj's function which is holomorfic in complex plane $\mathbb{C}$ with the cut $\mathscr{L}$ and satisfies the relations

$$\lim_{x\to\infty} \sqrt{(x^2 - a^2)[x^2 - (a+d)^2][x^2 - (3a+d)^2]} = +\infty, \quad x \in \mathbb{R}. \tag{2.26}$$

The second relation of (2.23) represents a homogeneous nul jump problem. The general solution satisfying the conditions imposed at large distances from the crack is given by

$$\alpha\Psi_1(z_1) - \bar{\alpha}\bar{\Psi}_1(\bar{z}_1) = 0. \tag{2.27}$$

Now from the system (2.24), (2.27) and using (2.19) we obtain

$$\Psi_1(z_1) = \frac{X(z_1)}{2\alpha\pi i}\int_{\mathscr{L}} \frac{g(t)}{X^+(t)(t - z_1)}dt + \frac{1}{2\alpha}P(z_1)X(z_1),$$
$$\Psi_1(z_2) = -\frac{X(z_2)}{2\alpha\pi i}\int_{\mathscr{L}} \frac{g(t)}{X^+(t)(t - z_2)}dt - \frac{1}{2\alpha}P(z_2)X(z_2), \quad t \in \mathscr{L}. \tag{2.28}$$

Taking into account the far field conditions we get that the polynomial $P(z)$ is a maximum second degree polynomial.

3 Tangential stresses having constant value

In what will follows, we consider an anisotropic material with three planes of elastic symmetry, *i.e.* an orthotropic material, and we assume that the given stress acting in the Mode II of fracture has a constant value,

$$g(t) = g = \text{const} > 0, \quad \text{for} \quad t \in \mathscr{L}. \tag{3.1}$$

Using (3.1) in (2.28) and Cauchy's integral proprieties [15], we have the following representation of the complex potentials $\Psi_j(z_j), j = 1, 2$:

$$\Psi_j(z_j) = \frac{(-1)^{j+1}}{\alpha}\left[g\left(\frac{\alpha_3 z_j^3 + \alpha_2 z_j^2 + \alpha_1 + \alpha_0}{\sqrt{(z_j^2 - a^2)[z_j^2 - (a+d)^2][z_j^2 - (3a+d)^2]}}\right) + \frac{P(z_j)}{\sqrt{(z_j^2 - a^2)[z_j^2 - (a+d)^2][z_j^2 - (3a+d)^2]}}\right]. \tag{3.2}$$

Using the same procedure as in [13,15] we get

$$\alpha_0 = -\frac{9a^8 + 8ad + 2d^2}{2}, \quad \alpha_1 = \alpha_2 = 0, \alpha_3 = 1. \tag{3.3}$$

From (3.2) we conclude that boundary condition $(2.1)_3$ is satisfied if and only if $P(t) = \overline{P(t)}$, $t \in \mathscr{L}$. The requirement above means that the coefficients of the polynomial $P(z)$ must be real numbers.

According to (2.3), hence $\Phi_j(z_j)$ can be multivalued functions even if $\Psi_j(z_j), j = 1, 2$ are univalued. To assure the uniformity of the displacement field we have to satisfy the uniformity of the potentials $\Phi_j(z_j)$ an closed path around the three cracks.

We denote by $\Lambda_j, j = 1, 2$ the corresponding closed curves corresponding around the crack $(T_r, T_s) \in \{(-3a-d, -a-d), (-a, a), (a+d, 3a+d)\}$.

Squeezing the curve Λ_j around the crack (T_r, T_s) and taking into account that the integrals involved in the expression of the displacement field by the $analytical functions$ $\Phi_j(z_j), j = 1, 2$ are path independent we conclude that the uniformity of u_x and u_y will be satisfied if we have fulfilled the following relations:

$$\begin{aligned} &\mathrm{Re}\left\{\int_{T_r}^{T_s} \left(p_1\Psi_1^+(t) + p_2\Psi_2^+(t)\right) dt + \int_{T_r}^{T_s} \left(q_1\Psi_1^-(t) + q_2\Psi_2^-(t)\right) dt\right\} = 0, \\ &\mathrm{Re}\left\{\int_{T_r}^{T_s} \left(q_1\Psi_1^+(t) + q_2\Psi_2^+(t)\right) dt + \int_{T_r}^{T_s} \left(p_1\Psi_1^-(t) + p_2\Psi_2^-(t)\right) dt\right\} = 0. \end{aligned} \tag{3.4}$$

Using (3.2) we get the following equations

$$\begin{aligned} &\int_{T_r}^{T_s} \left(p_1\Psi_1^+(t) + p_2\Psi_2^+(t)\right) dt = \frac{\Gamma_0}{2}\left\{p\int_{T_r}^{T_s}\left(\frac{Q(t)}{iA(t)} - 1\right) dt + \int_{T_r}^{T_s}\frac{P(t)}{iA(t)} dt\right\}, \\ &\int_{T_r}^{T_s} \left(q_1\Psi_1^+(t) + q_2\Psi_2^+(t)\right) dt = -\frac{\Gamma_1}{2}\int_{T_r}^{T_s}\frac{pQ(t) + P(t)}{A(t)} dt + i\frac{p\Gamma}{2}(T_r - T_s), \end{aligned} \tag{3.5}$$

where

$$\Gamma_0 = \frac{p_1\mu_2 - p_2\mu_1}{\mu_2 - \mu_1}, \tag{3.6}$$

$$\Gamma_1 = \frac{q_1\mu_2 - q_2\mu_1}{\mu_2 - \mu_1}, \tag{3.7}$$

$$Q(t) = t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0, \qquad t \in \mathscr{L} \tag{3.8}$$

$$A(t) = \sqrt{(a^2 - t^2)((a+d)^2 - t^2)((3a+d)^2 - t^2)}, \qquad t \in \mathscr{L}. \tag{3.9}$$

After long manipulations [17-18] we obtain that Γ_0 and $i\Gamma_1$ are real numbers. Taking into account that the values $\Psi_1^-(t)$ and $\Psi_2^-(t)$ can be obtain from the limit value of $\Psi_1^+(t)$ and $\Psi_2^+(t)$ replacing i by $-i$ we get that the relations $(3.4)_1$, is fulfilled, i.e. the uniformity of u_x is fulfilled for all three cracks.

Using the representations (3.5) we get that (3.6) is fulfilled i.e. the uniformity of u_y is fulfilled, if and only if, we have:

$$\int_{T_r}^{T_s}\frac{P(t)}{A(t)} dt = -p\int_{T_r}^{T_s}\frac{Q(t)}{A(t)} dt. \tag{3.10}$$

The three equations algebraic system (2.7), for $(T_r, T_s) \in \{(-3a-d, -a-d), (-a, a), (a+d, 3a+d)\}$ has the following solution:

$$C_0 = C_2 = 0 \quad \text{and} \quad C_1 = -p\left(\frac{\mathscr{H}_3}{\mathscr{H}_1} + \alpha_1\right), \tag{3.11}$$

where

$$\mathscr{H}_k = \int_{a+d}^{3a+d}\frac{t^k}{A(t)} dt. \tag{3.12}$$

Finally, using (3.1), (3.8), and (3.2) we obtain the final form of the complex potentials $\Psi_j(z_j), j = 1, 2$

$$\Psi_j(z_j) = \frac{(-1)^{j+1}g}{\alpha}\left[\frac{z_j^3 - \frac{\mathscr{H}_3}{\mathscr{H}_1} z_j}{Y(z_j)} - 1\right], \quad j = 1, 2. \tag{3.13}$$

4 Asymptotic behavior of the stress and displacement field

In this section we shall analyze the asymptotical behavior of the stress and displacement field in the vicinity of the cracks tips. This analysis is important since, in this way, the energy release rate, and the strain energy density corresponding to a crack tip and predicting the crack propagation or crack stability can be examined and evaluated. To obtain these quantities we have to know the non regular or the asymptotic values of the complex potentials in a small neighborhood of the crack tip.

We start to get the asymptotic values of complex potentials and the stress and displacement fields for the first tip (point A, Fig. 1) letting

$$x = a + r\cos\varphi, y = r\sin\varphi \tag{4.1}$$

and by assuming that r is small in comparison with the half crack length a. The polar coordinates r and φ designate (see Fig. 2), respectively, the radial distance from the considered crack tip and the angle between the radial line extending the crack.

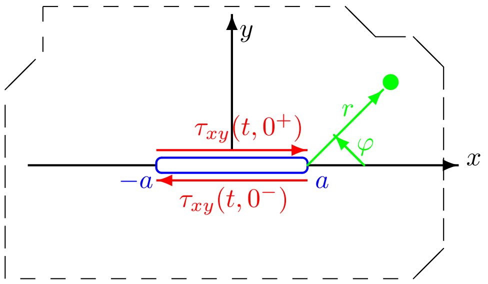

Fig. 2 Polar coordinates.

In a small neighborhood of the crack tip a, we have

$$x \approx a, \quad y \approx 0 \tag{4.2}$$

thus

$$z_1 = z_2 \approx a. \tag{4.3}$$

Denoting by

$$\chi_j(\varphi) = \cos\varphi + \mu_j \sin\varphi \quad \text{and} \quad k = \frac{d}{a}, \tag{4.4}$$

we have

$$z_j - a = r\chi_j(\varphi) \tag{4.5}$$

and

$$Y(z_j) = a^5\sqrt{2r\chi_j(\varphi)}\sqrt{k(k+2)^2(k+4)}. \tag{4.6}$$

Using above Eqs. (4.6) in (3.13) we get the following asymptotic values of the complex potentials $\Psi_j(z_j)$:

$$\Psi_j(z_j) - (-1)^{j+1}\frac{g}{\alpha}\sqrt{\frac{a}{2r\chi_j(\varphi)}}m_1 \tag{4.7}$$

with

$$m_1 = \frac{1 - \frac{\mathscr{H}_3}{\mathscr{H}_1 a^2}}{(k+2)\sqrt{k(k+2)}}. \tag{4.8}$$

Taking into account that

$$\Phi_j(z_j) = \int_{\mathscr{L}} \Psi_j(z_j)dz_j \tag{4.9}$$

and due to the fact that the above integrals are path independent, we have

$$dz_j = dr\chi_j(\varphi). \tag{4.10}$$

So, we get the following values of the complex potentials $\Phi_j(z_j)$:

$$\Phi_j(z_j) = (-1)^{j+1}\frac{g\sqrt{a}}{\alpha}\sqrt{2r\chi_j(\varphi)}m_1, \quad j = 1, 2. \tag{4.11}$$

Using Lekhnitskii representation of the stress and displacement fields (2.1), (2.2) and the asymptotic values of the complex potentials (4.7), (4.11) elementary calculus lead to the following asymptotic values of the stress and displacement fields:

$$\begin{aligned}
\sigma_x &= 2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{\mu_1^2}{\chi_1(\varphi)} - \frac{\mu_2^2}{\chi_2(\varphi)}\right]m_1\right\},\\
\sigma_y &= 2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{1}{\chi_1(\varphi)} - \frac{1}{\chi_2(\varphi)}\right]m_1\right\},\\
\tau_{xy} &= -2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{\mu_1}{\chi_1(\varphi)} - \frac{\mu_2}{\chi_2(\varphi)}\right]m_1\right\},\\
u_x &= 2Re\left\{\frac{g\sqrt{2ar}}{\alpha}\left[p_1\sqrt{\chi_1(\varphi)} - p_2\sqrt{\chi_2(\varphi)}\right]m_1\right\},\\
u_y &= 2Re\left\{\frac{g\sqrt{2ar}}{\alpha}\left[q_1\sqrt{\chi_1(\varphi)} - q_2\sqrt{\chi_2(\varphi)}\right]m_1\right\}.
\end{aligned} \tag{4.12}$$

We consider now the second tip (point B, Fig. 1).
In this case we have

$$z_j \approx a + d, z_j - a + d = r\chi_j(\varphi), \tag{4.13}$$

and

$$Y_j = 2a^{\frac{5}{2}}\sqrt{2r\chi_j(\varphi)}im_2 \tag{4.14}$$

with

$$m_2 = \frac{1 - \frac{\mathscr{H}_3}{\mathscr{H}_1 a^2}}{2(k+2)\sqrt{k(k+1)}}. \tag{4.15}$$

In above relations r represents the radial distance from the crack tip $a + d$ to any arbitrary point.
A similar manner as for the case of the tip a we obtain the following asymptotic values of the complex potentials $\Psi_j(z_j)$ and $\Phi_j(z_j)$:

$$\Psi_j(z_j) = (-1)^j\frac{g}{2\alpha}\sqrt{\frac{a}{2r\chi_j(\varphi)}}im_2, \qquad \Phi_j(z_j) = (-1)^j\frac{g}{2\alpha}\sqrt{2r\chi_j(\varphi)}im_2. \tag{4.16}$$

For the asymptotic values of the stress and displacement fields we get:

$$\begin{aligned}
\sigma_x &= 2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{\mu_1^2}{\chi_1(\varphi)} - \frac{\mu_2^2}{\chi_2(\varphi)}\right]im_2\right\},\\
\sigma_y &= 2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{1}{\chi_1(\varphi)} - \frac{1}{\chi_2(\varphi)}\right]im_2\right\},\\
\tau_{xy} &= -2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{\mu_1}{\chi_1(\varphi)} - \frac{\mu_2}{\chi_2(\varphi)}\right]im_2\right\},
\end{aligned} \tag{4.17}$$

$$u_x = 2Re\left\{\frac{g\sqrt{2ar}}{\alpha}\left[p_1\sqrt{\chi_1(\varphi)} - p_2\sqrt{\chi_2(\varphi)}\right]im_2\right\},$$

$$u_y = 2Re\left\{\frac{g\sqrt{2ar}}{\alpha}\left[q_1\sqrt{\chi_1(\varphi)} - q_2\sqrt{\chi_2(\varphi)}\right]im_2\right\}.$$

Using the same procedure as in the previous two crack tips a and $a+d$, we get the following asymptotic values corresponding to the third tip (point C, Fig. 1):

$$\Psi_j(z_j) = (-1)^{j+1}\frac{g}{2\alpha}\sqrt{\frac{a}{2r\chi_j(\varphi)}}m_3, \quad \Phi_j(z_j) = (-1)^{j+1}\frac{g}{2\alpha}\sqrt{2r\chi_j(\varphi)}m_3, \tag{4.18}$$

$$\begin{aligned}
\sigma_x &= 2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{\mu_1^2}{\chi_1(\varphi)} - \frac{\mu_2^2}{\chi_2(\varphi)}\right]m_3\right\},\\
\sigma_y &= 2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{1}{\chi_1(\varphi)} - \frac{1}{\chi_2(\varphi)}\right]m_3\right\},\\
\tau_{xy} &= -2Re\left\{\frac{g}{\alpha}\sqrt{\frac{a}{r}}\left[\frac{\mu_1}{\chi_1(\varphi)} - \frac{\mu_2}{\chi_2(\varphi)}\right]m_3\right\},\\
u_x &= 2Re\left\{\frac{g\sqrt{2ar}}{\alpha}\left[p_1\sqrt{\chi_1(\varphi)} - p_2\sqrt{\chi_2(\varphi)}\right]m_3\right\},\\
u_y &= 2Re\left\{\frac{g\sqrt{2ar}}{\alpha}\left[q_1\sqrt{\chi_1(\varphi)} - q_2\sqrt{\chi_2(\varphi)}\right]m_3\right\}
\end{aligned} \tag{4.19}$$

with

$$m_3 = \frac{1 - \frac{\mathscr{H}_3}{\mathscr{H}_1 a^2}}{2(k+2)\sqrt{(k+3)(k+4)}}. \tag{4.20}$$

Due to the symmetry of the considered problem we have the following relations that are giving the complex potentials corresponding to the tips $-3a-d$, $-a+d$, and $-a$:

$$\Psi_j^{(-3a-d)}(z_j) = i\Psi_j^{(3a+d)}(z_j), \Psi_j^{(-a-d)}(z_j) = i\Psi_j^{(a+d)}(z_j), \Psi_j^{(-a)}(z_j) = i\Psi_j^{(a)}(z_j). \tag{4.21}$$

5 Conclusions

We considered an homogeneous elastic anisotropic solid containing three equal and collinear cracks acted by asymmetrically tangential stresses.

The elastic state produced in the body is determined using the theory of Riemann-Hilbert problem by complex potentials. For the problem of three equal and collinear cracks in an anisotropic material acted by applied tangential stresses in Mode II of fracture an explicit new solution in terms of complex potentials were given in this paper.

For an orthotropic material with three equal and collinear cracks acted by constant tangential stress, we obtained that in a neighborhood of the cracks tips the asymptotical values of the components of the stresses behave like $r^{-1/2}$ and of the displacements behave like $r^{1/2}$ and so, the total elastic energy is bounded in any finite domain containing the cracks. This fact is essential in cracks stability analysis based on Griffith's type criterion of cracks propagation. With these obtained asymptotical values we are able to compute, the energy release rate and the strain energy density corresponding to each tip of the three cracks. Later on, we intend to study the crack interaction versus the distance between them, i.e. which one of the tips start to propagate first, using numerical methods. These and the experimental test which will be performed on particular types of anisotropic elastic composites have to be in good agreement and will represent our future research studies.

Acknowledgements The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007–2013), FP7-REGPOT-2009-1, under grant agreement No. 245479; CEMCAST.

The support by Polish Ministry of Science and Higher Education – Grant No. 1471-1/7PR UE/2010/7 and within statutory research number S/20/2012.

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Book Review

Adrian Bejan, Convection Heat Transfer, John Wiley & Sons, Hoboken, New Jersey, 4th ed., 2013, 690 pp., 55 Tab., 260 Fig., Hardcover, EUR 142.00, USD 160.00, ISBN 978-0-470-90037-6

After 1984, 1995, and 2004 the revised fourth edition of this book is presented now. The book contains twelve chapters and five appendixes.

Chapter 1 deals with the balance equations for mass, momentum, energy, and entropy. Some remarks on scale analysis and visualising results are done. The idea of boundary layers is in the focus of Chapter 2 because of their great importance for convective heat transfer models. First only laminar conditions are considered. Three methodologies are described for solving the boundary layer equation for the case of an isothermal plate in a parallel flow: scale analysis, integral solution, and similarity solution. Then the discussion follows on the influence of other wall heating conditions, of longitudinal pressure gradients, of blowing or suction, and of coating. Flow and heat transfer in laminar ducts are subject of Chapter 3. Typical special cases are presented and remarks on optimal geometry are given. The presentation of the external and internal natural convection follows in Chapter 4 resp. 5. Different solution possibilities and results for many constellations are shown after a scale analysis. Chapter 6 deals with the transition to turbulence. Empirical results and theoretical possibilities for prediction of the beginning of transition are given for different situations. Chapter 7 continues with turbulent boundary layers. Time-averaged balance equations, their simplifications for boundary layers, and the mixing length model are introduced. Velocity and temperature profiles are described. The resulting heat transfer equations are summarised for different situations. Velocity profiles and heat transfer in turbulent duct flow are topics of Chapter 8. Chapter 9 shows result for free turbulent flows. Considered examples are jets, plumes, and wakes. Problems with phase changing are the content of Chapter 10. This is a broad field. Thus, this part can only show some selected problems of condensation, boiling, and melting. Chapter 11 discusses the analogy between heat and mass transfer and presents results for some mass transfer situations, most without reaction. Convection in porous media is in the focus of Chapter 12. The describing equations are derived and discussed. An overview of some forced and natural convection problems trough porous media coupled with heat transfer follows. The appendixes contain constants, some material properties, and mathematical formulas.

The book gives a very good overview and is a compendium of useful equations for describing a great number of convection heat transfer processes. It contains some interesting historical details at the beginning and a great number of problems at the end of each chapter. Many figures give a clearer picture of the subject and show dependencies. The book is very useful for students, practicing engineers, and for researchers. It is highly recommended.

Chemnitz **Bernd Platzer**