Report of HW9

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1 First Order Differential Equation

Code: `q1\q1.py`
We want to solve the first order differential equation below; Which governs the evolution of the charge of the capacitor in an RC circuit.

$$R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = V. \tag{1}$$

First, we try to make a dimensionless equation; It is easy to do by using the variable change below.

$$\begin{aligned} &x \equiv \frac{Q}{CV} - 1, \quad \tau \equiv \frac{t}{RC} \\ \Rightarrow &\frac{\mathrm{d}x}{\mathrm{d}\tau} + x = 0. \end{aligned} \tag{2}$$

The analytical solution to this equation is

$$x = x_0 \exp(-\tau) \Rightarrow Q = CV(1 - e^{-\frac{t}{RC}}). \tag{3}$$

This result is obtained simply by integrating the equation $\frac{\mathrm{d}x}{x} = -\mathrm{d}\tau$. I did the simulation for $t = 21ms$ or equally $\tau = 7$, $h = 10^{-4}$, $R = 3 \times 10^3\Omega$, $V = 10V$, $C = 10^{-6}F$ and zero initial charge. The result of simulation and comparison with analytical solution is given in the figure (1). The simulation result is in good matching with the analytical procedure. Finally, Euler's method relative error calculated and plotted for each varying h value in the figure (2). The slope of the best fitted line is approximately 0.993 and this result is in good matching with theoretical procedure.

2 Second Order Diffrential Equation and Comparison of Algorithms

Code: `q2\q2.py`
The simulation in this problem is done by using $h = 0.01$ as time step and $x_0 = 1, v_0 = 0$ as initial conditions. For some methods like Verlat's method I had to define extended initial conditions due to the instinct of the algorithms. The result of each method is plotted in the figure (3). It is obvious from the figure(3) that the Euler-Cromer, Verlat, velocity Verlat and Beeman methods are stable algorithms for this simulation; in other words, Euler method doesn't keep energy of the system constant.
Finally, I made plot of the phase space for each method. The plots are available in the figures 4 to 9. As it is obvious, the plots are circles for all methods except Euler method; which is a sign of changing the total mechanical energy of the system.

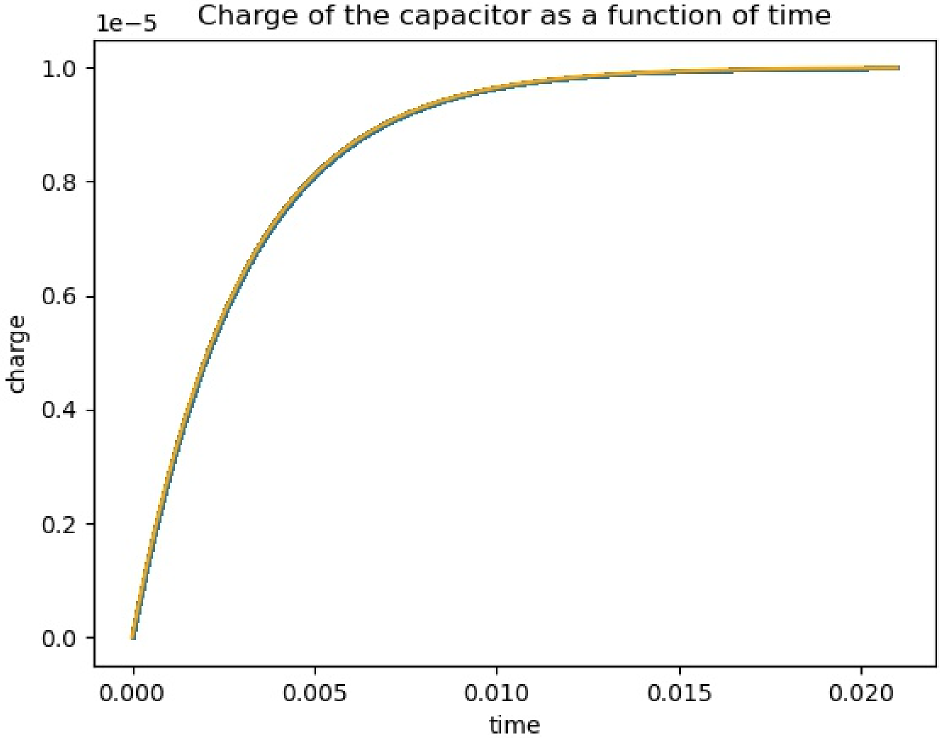

Figure 1: Analytical and numerical solution for the charging capacitor equation. The time step for the numerical solution here is $3 \times 10^{-8}s$

3 Instability in Algorithms

Code: `q3\q3.py`
In this problem I did a simulation for the equaiton (1) using the given algorithm below.

$$y_{n+1} = y_{n-1} + 2h\dot{y}_n \tag{4}$$

The result is plotted in the figure (10). Doing this simulation I used the variable change in the problem 1. According to the figure (10), the algorithm in equation (4) is more instable for bigger value of h; where h is time step in changed variable units (eq.2).

4 Chaos

Code: `q4\q4.py`
For calculating the periodic values of x for each r, I considered some random initial values for x in the interval $[0, 1)$. Then I did run the recursive equation (5) on each initial x for some given n_{max} as the number of iterations. The resulted plot is given in the figure (11).

$$x_{n+1} = 4rx_n(1 - x_n) \tag{5}$$

Then I did more exact simulation for calculating r_n values. The results are given in the figures (12) to (18). According to the figures (12) to (18), I concluded the results given in (6).

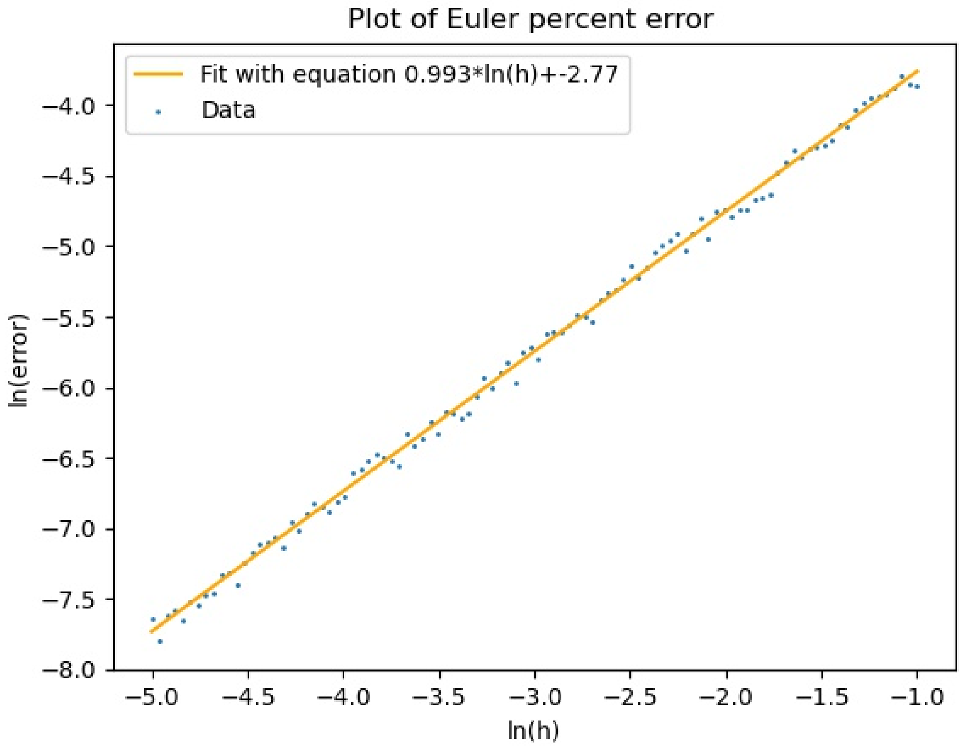

Figure 2: Analytical and numerical solution for the charging capacitor equation. The time step for the numerical solution here is $3 \times 10^{-8} s$

$$\begin{cases} r_0 = 0.250000 \pm 10^{-6} \\ r_1 = 0.749999 \pm 10^{-6} \\ r_2 = 0.862372 \pm 10^{-6} \\ r_3 = 0.886020 \pm 10^{-6} \\ r_4 = 0.891101 \pm 10^{-6} \\ r_5 = 0.892189 \pm 10^{-6} \\ r_\infty = 0.892487 \pm 10^{-6} \end{cases} \tag{6}$$

Using the values in (6), I calculated the δ parameter using the formula $\delta \approx \frac{r_4 - r_3}{r_5 - r_4}$. I found $\delta \approx 4.668$. Finally for calculating the value of α, I had to calculate the values of rs, say r_3 and r_4, more exactly. So I did the simulation more exactly about points r_3 and r_4. Resulted plots are given in the figures (19) and (20). Resulted more exact values for r_3 and r_4 are given in (7).

$$\begin{cases} r_3 = 0.888660 \pm 10^{-6} \\ r_4 = 0.891667 \pm 10^{-6} \end{cases} \tag{7}$$

Finally I calculated the length of the mouth of the plot, near the r_3 and r_4 and found the parameter α as given in the equation (8).

$$\alpha \approx \frac{0.04597434}{-0.01833120} \Longrightarrow \alpha \approx -2.507 \tag{8}$$

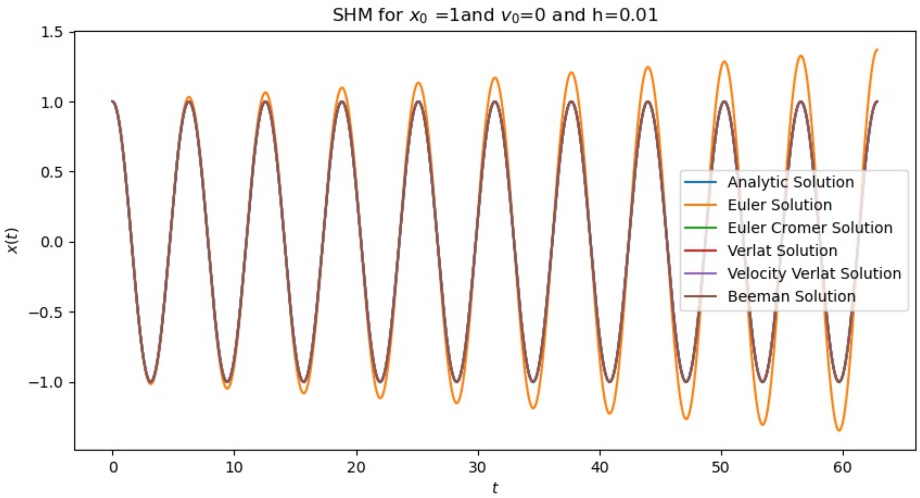

Figure 3: Plot of the simulation results using different methods from 0 to 20π with time step equal to 0.01.

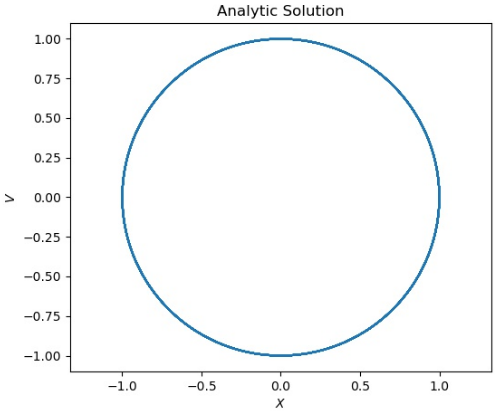

Figure 4: Phase space of the system due to the anaytical method. Time step is 0.01, initial velocity is 0 and initial position is 1.

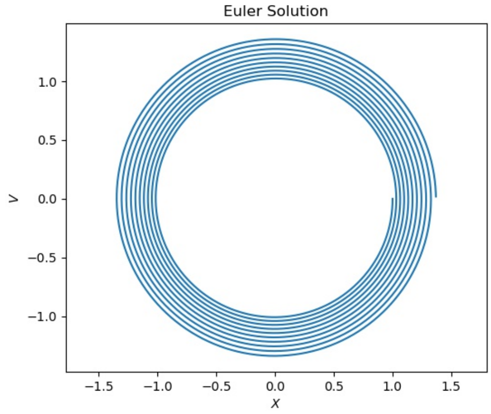

Figure 5: Phase space of the system due to the Euler method. Time step is 0.01, initial velocity is 0 and initial position is 1.

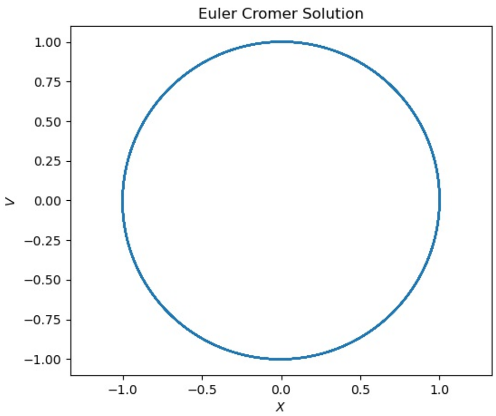

Figure 6: Phase space of the system due to the Euler-Cromer method. Time step is 0.01, initial velocity is 0 and initial position is 1.

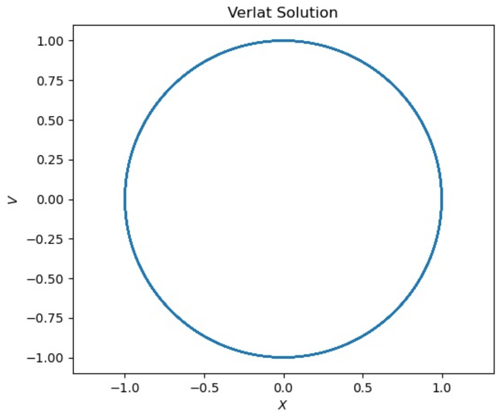

Figure 7: Phase space of the system due to the Verlat method. Time step is 0.01, initial velocity is 0 and initial position is 1.

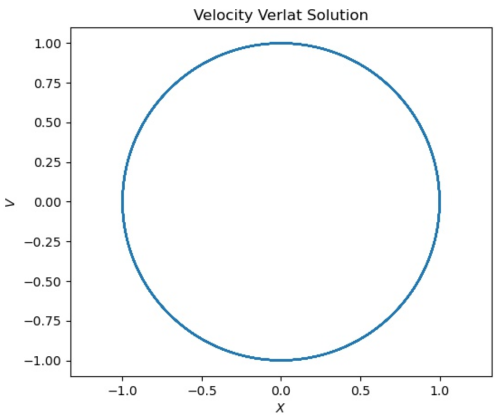

Figure 8: Phase space of the system due to the velocity Verlat method. Time step is 0.01, initial velocity is 0 and initial position is 1.

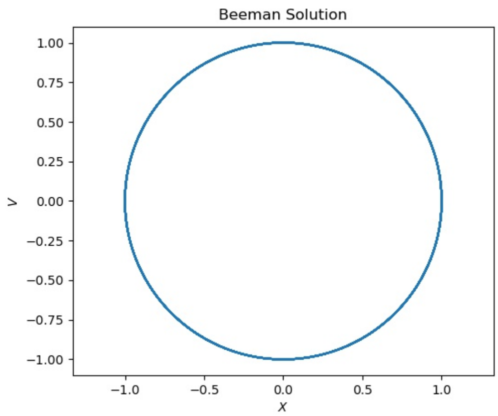

Figure 9: Phase space of the system due to the Beeman method. Time step is 0.01, initial velocity is 0 and initial position is 1.

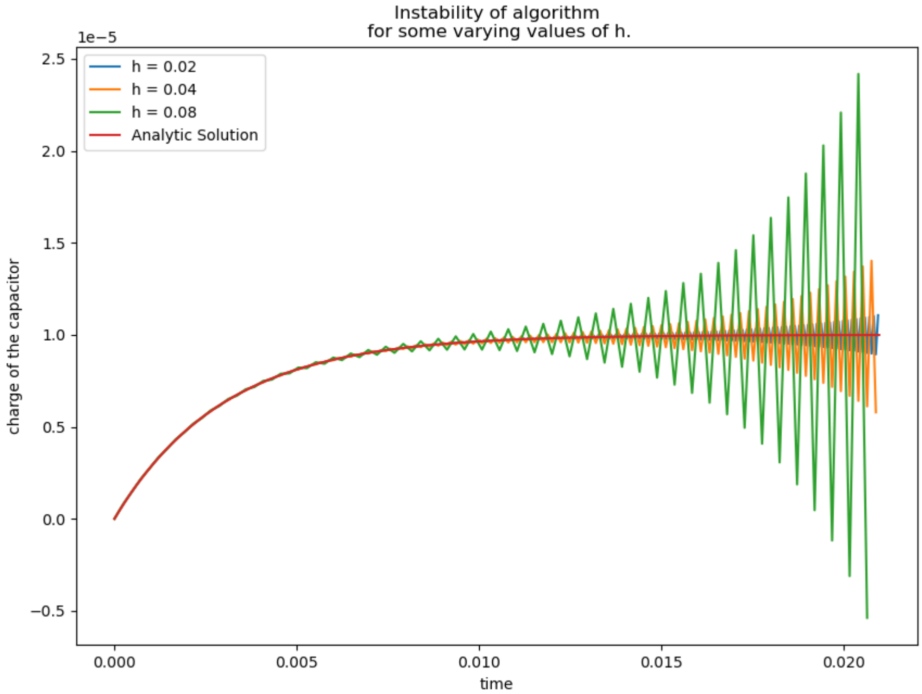

Figure 10: Simulation result using the given algorithm in equaiton (4) and comparison with analytical solution.

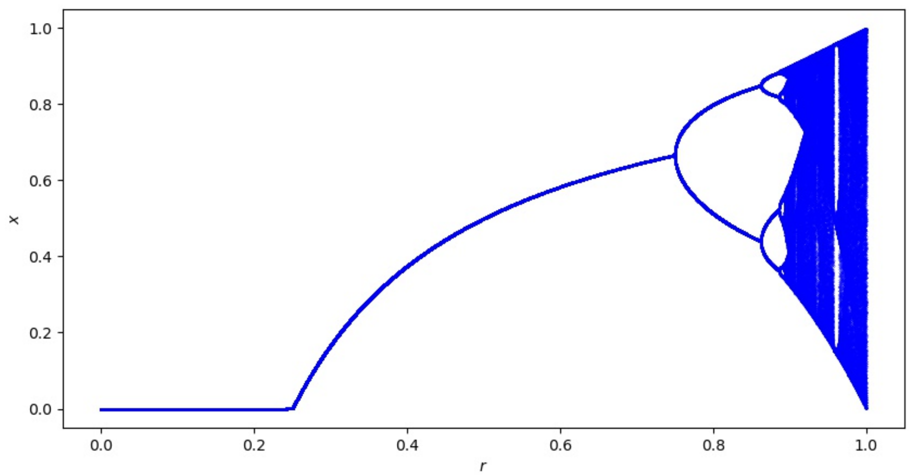

Figure 11: Bifurcation plot for $\Delta r = 10^{-4}$ and 100 x for each r and number of iterations is 10^3.

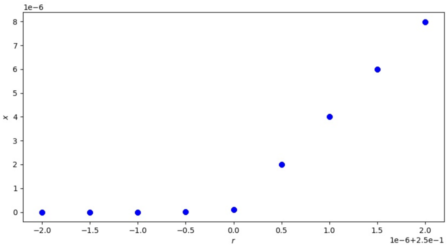

Figure 12: Bifurcation plot about r_0 for $\Delta r = 10^{-6}$ and 100 x for each r and number of iterations is 10^7.

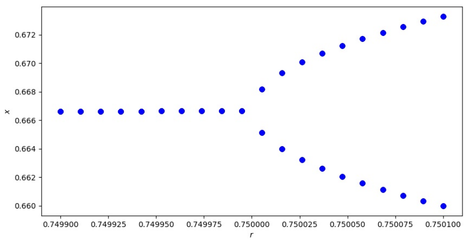

Figure 13: Bifurcation plot about r_1 for $\Delta r = 10^{-5}$ and 100 x for each r and number of iterations is 10^6.

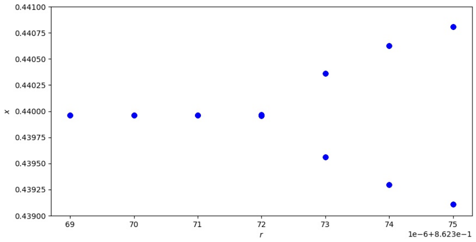

Figure 14: Bifurcation plot about r_2 for $\Delta r = 10^{-6}$ and 100 x for each r and number of iterations is 10^6.

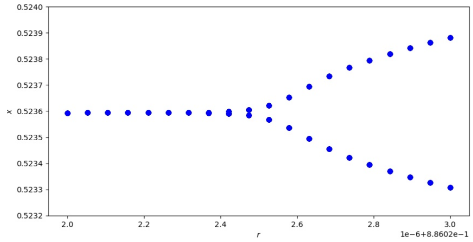

Figure 15: Bifurcation plot about r_3 for $\Delta r = 5 \times 10^{-8}$ and 100 x for each r and number of iterations is 10^6.

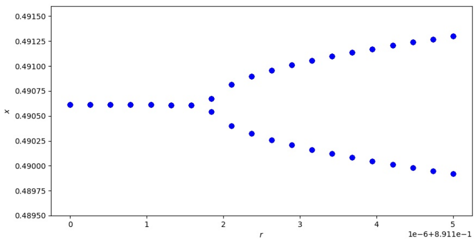

Figure 16: Bifurcation plot about r_4 for $\Delta r = 2.5 \times 10^{-7}$ and 100 x for each r and number of iterations is 10^6.

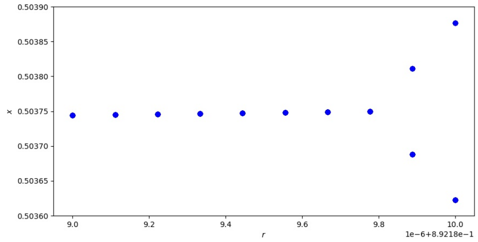

Figure 17: Bifurcation plot about r_5 for $\Delta r = 10^{-6}$ and 100 x for each r and number of iterations is 10^6.

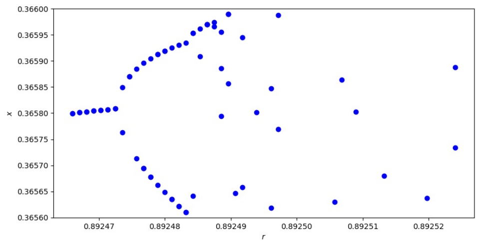

Figure 18: Bifurcation plot about r_∞ for $\Delta r = 10^{-6}$ and 100 x for each r and number of iterations is 10^6.

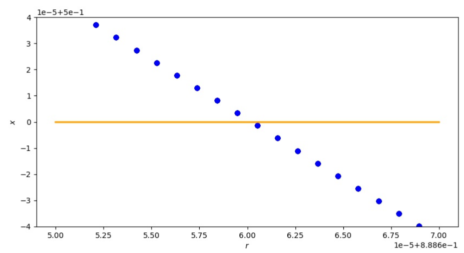

Figure 19: Bifurcation plot about r_3 for $\Delta r = 10^{-6}$ and 100 x for each r and number of iterations is 10^5.

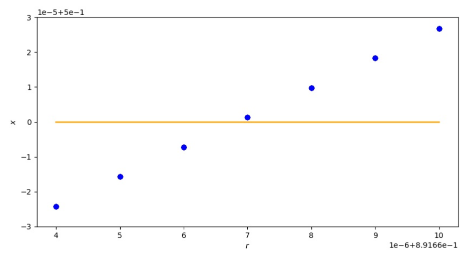

Figure 20: Bifurcation plot about r_4 for $\Delta r = 10^{-6}$ and 100 x for each r and number of iterations is 10^5.