Tonality structure in music

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Attempt to create a system of reference as a mnemonic rule.

Contents

Introduction

In an attempt, out of curiosity, to have a rudimentary understanding of musical harmony, without prior special knowledge, I read Open Music Theory, which looks similar to Integrated Musicianship: Theory, and two books by Arnold Schoenberg, THEORY OF HARMONY and Structural Functions Of Harmony. Schoenberg's books particularly influenced my perspective.

I know enough mathematics to discern the art behind it (it took me five years to discern the art behind general relativity) but I don't know enough music to discern the mathematics behind it. Logically, though, there can't be mathematical consistency in art.

With mathematics we consistently arrive at certain results, so the art lies in the freedom to choose how we arrive at them. In art, however, the freedom to choose an outcome leads to the fact that any rule or prohibition can be violated, as long as the violation does not occur by accident. Each violation must be justified by some structure that is the creation of the artist's arbitrary choice. However, in order to create any arbitrary structure that justifies any violation, a system of reference is needed to whose axes it refers. As with anything in the universe, mathematics will be inherent in this reference system, but finding a reference system with mathematical consistency and completeness is beyond my powers, my scant musical knowledge and my abilities. Therefore, I note my thoughts but scientific consistency and completeness cannot be expected in them.

My use of mathematical and physical concepts (such as fractals, category theory, group theory, dual spaces, energy, entropy, velocity, etc.) is intended as an attempt to create a system of reference, as a mnemonic rule, with as much consistency as possible to minimize exceptions to be memorized, and completeness to relate to as much material as possible from what I have read.

The present is not suitable for anyone who wants to learn harmony. There is a danger that a possibly distorted and incomplete view may be considered correct or complete. On the contrary, one already familiar with harmony can gain from a new perspective, even through demonstrating that this perspective is incorrect or incomplete.

Non-existence

1 Let there be time

Time allows for moments of creation. Without the existence of moments of time there can be no moment of creation. It imposes the end of these moments and the non-existence between these moments, otherwise non-change of existence would be equivalent to non-existence. Time is interwoven with the concept of change. Succession of birth - death, existence - non-existence, bit - zero, beat - silence. Without the aid of an extra-universal, supra-temporal observer, I cannot, only through the universe itself, distinguish whether time is actualized, or perceived, through changes, or whether changes are actualized, or perceived, through time. I believe that time is perhaps the most essential component of the universe and does not exist independently outside of it.[1]
Every birth presupposes the existence of time and refers to a moment in time, whenever it occurred. The birth of time is an oddity because of this very hidden self-reference of time to itself.

The time of the universe cannot define itself and claim "I was born then". It cannot define the Big Bang in a way that defines moments of time before it. What impressed me in general relativity is not that time is relative to the observer, nor that it coexists with space as spacetime, nor perhaps that within black holes time, despite the time arrow that differentiates it from spatial dimensions, is transmuted into space and space into time. I was struck by the fact that the end of infinite time appears "spatially" in the horizons of events. A mortal creature, heading towards a black hole, literally gains immortality relative to anyone observing it from afar and, being alive, it is forever approaching the black hole. It sounds strange but if you don't care if your loved one, for him, is killed in his next five minutes but you want him, for you, to descriptively live as long as trillions of years after your death and literally live for eternity, push him towards a black hole.

The measurement of time refers to discrete moments of existence. The structure of time, which allows for discrete moments of existence, allows for the mapping of moments of existence to integer numbers and thus for the comparison of time intervals by their number. The interval of nonexistence, if experienced, between moments of existence, is experienced subjectively by its observer and cannot be objectively measured without the error of self-reference in time itself[2]. Thus, we define that two time intervals are equal when during their duration equal changes of successive existence - non-existence of the same phenomenon occur. In 1967 it was defined that a second has a duration equal to that of $9,192,631,770$ of changes involving the chemical element Caesium-133.

[1]If you were to ask a quantum physicist about the nature of time in the microcosm, he would answer that time is absolute and exists independently of the universe. But I have not dabbled in quantum physics, nor do I aspire to resolve the contradictions between the two theories. I just find it quite amusing to weave relativity theory into notes on musical harmony.

[2]The same self-reference error occurs with the measurement of space. Hence, the theory of relativity, without the axiom of absolute space and time, considers events in space-time.

The succession of events at equal time intervals, on the structure of time, does not depend on the scale of these time intervals. Any structure of time, which makes it possible to create equal time intervals, allows it to define itself as fractal. It has the property of self-similarity. No matter how much we enlarge or shrink a time interval, it will have the same structure, allowing for the creation of equal time intervals within it[3]. Let us call the events at equal intervals *beats*, in the sense of the events of a metronome. The sequence of events, which we will refer to as the **reference sequence**, is measured by the inverse of the time interval between two consecutive beats. Its indication in musical scores is beats per minute (*BPM*), but where *beat* corresponds to a standard group of beats rather than the one beat of the reference sequence.

Grouping of successive beats by one, two, three, four, etc, provides the infinite sequence of beats with considerable structure. Each position in a *typical* group[4] becomes a representative of *class (set) beats*. By grouping we identify which of the beats in the natural sequence are first, which are second, third, fourth, etc. within each group and thus we can creatively differentiate them, such as events with more intensity, less intensity, absence[5], with a different duration or pitch and others. As an example, suppose four beats (a, a, a, a, a). They represent the grouping of the sequence by one beat, i.e. they form a one-beat group repeated as $([a], [a], [a], [a])$. Grouping by two means a two-member group repeated sequentially as $([a, c], [a, c])$. Grouping by three beats means that a three-member group is repeated, and so on. *meters* in music scores is one such grouping. By the same logic, we can provide sequential groups with a similar structure by grouping them into supergroups. Grouping bimodal groups by two means that a bimodal group, but whose members are also bimodal groups, is repeated, e.g. $([[[a, c], [b, c]], [[a, c], [b, c]])$ which can be seen as equivalent to a four-member group $[a, c, b, c]$ that is repeated. The *hypermetre* in the scores are such groupings. The *metre* is just the basic, for the composer, grouping of the sequence.

The grouping distinction is made to clarify terminology used herein as follows:

In use groupings are those which, either because of differentiation between members of their typical group or as a one-beat group, are used in practice. They are divided into

[3]The same is true for rational numbers. Any interval between two rational numbers can be divided into equal intervals between rational numbers

[4]The isomorphism between the groupings and their typical groups is obvious, by the number of the latter, with the integers positive numbers. It does not matter whether a three-member typical group is represented as (A,a,a) or (1,2,3) or (a,A,a); only its length, which is equal to 3, matters.

[5]Absence, as an event opposite to presence, corresponds to *beat*. With it we achieve repetition of events at unequal intervals. Let $[a, a, x]$ be a group $[a, a, x]$ with x representing the absence of an event. During the iteration of the group, the event a occurs each time at a different time interval than it occurred the previous time.

Obvious which are used directly
π.χ. $(A, a, a)(A, a, a)(A, a, a)\ldots$

Hidden or missing appearing indirectly due to coexistence e.g. when (A, a, a) coexists with (B, b), hidden is (AB, ab, aB, Ab, aB, ab) of the six beats, as
$(AB, ab, aB, Ab, aB, ab)(AB, ab, aB, Ab, aB, ab)\ldots$.

Potential groupings are those that we refer to but, as far as we are concerned, without differentiating their typical group members, are in practice *superseded* by the groupings in use.
e.g. on the triplet $(a, a, a)(a, a, a)(a, a, a)\ldots$ the grouping by one beat dominates. In the scheme 1, even if there were no dotted half notes, they would potentially be candidates to exist.

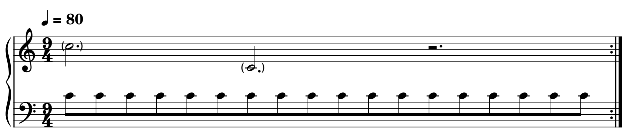

Figure 1: Potential grouping of 6 eighths

Rhythmicity is the *minimal* part of the event sequence that encloses integer all kinds of typical groups of the *in use* groupings. The rhythmicity is repeated at equal intervals. It is always in use, either as a hidden or missing grouping. It is measured by its number of beats and *is equal to the least common multiple of the number of beats of each typical group supporting it.* Without in-use groupings there can be no rhythmicity, except the trivial one with the typical one-beat group. The one-beat grouping supports any rhythmicity. And it is equivalent to repeating the same event over and over again at equal intervals.

Rhythm *of a grouping, relative to the rhythmicity it supports,* is the ratio of the number of members of the rhythmicity to the number of members of its typical group. In a different formulation, it is how many times the typical grouping group fits into the rhythmicity, or how much more often its typical group appears, in the sequence of events, than that of the rhythmicity. By definition of rhythmicity, it is always an integer. The rhythm of the largest possible group, that is, the rhythm of rhythmicity itself, is unity. The *time signature* convention is rather incomplete in describing every possible grouping that can be used. However, the upper signature number is clear as to how many beats are counted in each *meter*. The *meter* is simply the basic grouping for the composer, and there is nothing to prevent a signature, $\frac{3}{4}$ for example, from using six eighths, as many beats of rhythmicity, instead of three quarters, as would be, from the signature, expected. In any reference sequence there are infinitely more and more rhythmicities inherent in it. The largest of their rhythms, corresponding to the one-beat groups, have a limit of infinity, where an event occurs once, with zero frequency, and is not repeated.

The information economy is a necessity and a limitation imposed by the finite nature of the human brain. Our perception is limited to the correlation of information, not based on exact data but, rather, on simple patterns. If we look at a sea in a painting that occupies part of the work with a ratio of 2/3.236, i.e. the golden ratio, we will perceive it as occupying 2/3 of the frame. With respect to populations, the economy in information causes our perception to primarily seek the 1/2 ratio as the simplest comparison pattern[6]. Therefore, two groupings are identical in our perception when the typical group of one is twice as large as the other. The identification is transitional, so where [rhythmicities], including [rhythms], are mentioned, they mean *classes* of rhythmicities or rhythms, with the consequence that different rhythms are those whose ratio is not a power of 2. Our perception's need for simplicity endows the set of rhythms with *degree of proximity to rhythmicity* that they support. A rhythm is the closer to the rhythmicity it supports the smaller is the maximum odd number that divides it. For example, the rate 12 is closer to the rhythmicity 1 than the rate 5 because the maximum 3, which divides 12, is less than 5[7].

Simple rhythmicity is the rhythmicity whose class is the class of trivial grouping by one beat, i.e. it is of class [2]. This has the consequence that only rhythms that are power of 2 also get used. Such as, for example, when in a time signature $\frac{4}{4}$ four quarter notes, two half notes and one whole note are heard in one measure. The whole note represents the grouping of beats by (2^2) four beats, coinciding with that of measure and rhythmicity, the half notes the grouping by (2^1) two, and each quarter note the grouping by (2^0) one beat. The number of beats in each group is a power of 2, so all rhythms, including rhythmicity, are identified since they belong to the same class $[1] = [2]$. **Compound rhythmicity** is any other rhythmicity.

Co-hearing of rhythms exists in any case where the rhythmicities are supported by more than one rhythm class. Suppose that a measure uses one dotted half note and two dotted quarter notes. Obviously, the rhythmicity is expressed as two dotted quarters and the two rhythms belong to the $[\frac{2}{2}] = [\frac{2}{1}] = [2]$ class of simple rhythm. But if we add the development of the dotted half note, using three fourths, then the rhythmicity is now expressed in six eighths, and this addition, which belongs to the rhythm class $[\frac{6}{1}] = [6] = [3]$, is co-heared with the rhythm class $[\frac{6}{6}] = [\frac{6}{3}] = [2]$ of the rest. *The grouping of length equal*

[6]Moreover, by denoting by a the presence of an event and x its absence, because in each one-beat grouping the presence of the event de facto alternates with its absence, this grouping can be understood as a double grouping $[a, x]$ per two beats, but in which the potential one-beat grouping is hidden.

[7]I speculate that if our perception detects reason $\frac{1}{3}$ it might expect the same pattern, so there might be a match for this reason as well. By this reasoning there will be as many rhythm classes as there are prime numbers greater than unity.

to the rhythmicity, which is a class rate [2]*, is, either as hidden or as missing, always in use.*

The classes of rhythmicities and their rhythms will be denoted by the maximum odd[8] number that divides them. The next class, in order of proximity to the simple [2], is [3]. The next class [5] is rarer. Except for [9] and [15], the following theoretical classes [7], [11], [13], [17], *etc* do not seem to have any practical use. Those classes which do not correspond to prime numbers may correspond to a co-hearing of one or more groupings. To clarify, the class rhythmicity [15] may result from the co-hearing of the rhythms [2] (as a hidden or missing class), [3] and [5]. In both measures of 2 the rhythmicity is [15]. In the first part, C is rhythm $[\frac{15}{15}] = [1] = [2]$, G is rhythm $[\frac{15}{5}] = [3]$, E is rhythm $[\frac{15}{3}] = [5]$ and B is rhythm $[\frac{15}{1}] = [15]$. In the second measure rhythm [15] is hidden.

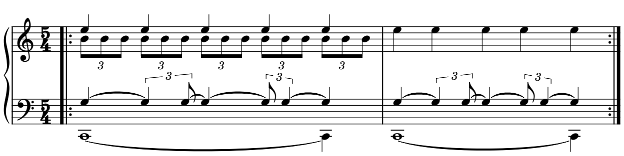

Figure 2: Co-hearing of rhythms

The algebraic structure *of monoid* seems to be distinguished in the above mentioned. Let S be the set of groupings. Each element of it is identified with the typical group of the grouping and the trivial equal-length rhythmicity. We endow this set with the binary operation $S \times S \rightarrow S$ of the co-hearing, denoted by $+$. The result is the grouping by typical group that has its number of members equal to the least common multiple of the number of members of the two typical groups. The pair $(S, +)$ is *monoid* because it satisfies the next two axioms:

- Associativity: For each a, b and c belonging to S, the relation $(a+b)+c = a+(b+c)$ holds. Rhythmicity with more than two rhythms is a result of this property.
- Identity element: There is an element e in S, that of the grouping of the one-beat typical group, such that for every element a in S, the equations $e+a=a$ and $a+e=a$ hold.

Because of the commutative property, where $a+b=b+a$ for every $a, b \in S$, $(S, +)$ is *commutative* or else *abelian monoid*. Like any commutative monoid, $(S, +)$, i.e. the set of typical groups with their co-hearings, is equipped with

[8]They will be denoted by the product of the odd prime numbers, if we identify each prime power with itself.

the weak order $\leq$, by which it is defined that $a \leq b$ if there exists c such that $a + c = b$, i.e. the length of a divides the length of b[9].

Deconstruction of rhythmicity I consider it to be the mutation into the rhythmicity of some strict subset of the groupings that support it[10]. Each rhythmicity is supported by whatever groupings the composer uses, just as a house is supported by its columns. A solid rhythmicity(building) cannot be built without using groupings(columns) that support it. Over time, entropy increases, hence the natural deterioration of the rhythmicity(building), leaving, as ruins, some of the groupings(columns) that supported it. *The supportive relationship is not reversed.* If G is a grouping(column) of the rhythmicity(building) C, then the latter, as a grouping(column) C, cannot support the rhythmicity(building) G. This holds in short because if $[G]$ is a grouping of $[C]$, then $[G] = \frac{[C]}{2*k+1} \Rightarrow [C] = (2*k+1)*[G],\ with\, k \neq 0,\ [C] = 2^n * C\, and\, [G] = 2^m * G$. So $[C]$ does not have the required form $[C] = \frac{[G]}{2*k+1}$, so it is not a grouping of $[G]$[11]. More simply, outside of classes, if G divides C then C does not divide G.

Rhythmicity progression is the change to another class of rhythmicity. The rhythmicity of a composition may be left unchanged. However, if there is a desire to change it, taking into account the aforementioned supporting relation and the simulation of the entropy effect, we can distinguish the progression into *passive* and *active*.

Passive progress (or *weak* or *decreasing* progress) of a rhythmicity is *its decomposition* into a different class. Hence, *there is no passive transition from the simple* [2] *class rhythmicity.* If we restrict ourselves only to the [2] and [3] classes of rhythmicity, then the only possible passive transition is from the $[3] = [6] = [2] + [3]$ class of rhythmicity to the [2] class of simple rhythmicity.

Active progress (or *strong* or *ascending* progress) of a rhythmicity is any *not passive* progress of it to a new rhythmicity of a different class. The original rhythmicity may or may not support, as a grouping, the new rhythmicity. If we restrict ourselves only to the [2] and [3] classes of rhythmicities, then the only possible active transition is that from the [2] class of rhythmicity to the [3] class of rhythmicity which, in addition to the particular [2] class grouping, now obvious or hidden, is also supported by a $[3] = [6] = [2] + [3]$ class grouping.
In the figure 3, despite the time signature, in the first measure the rhythms of one

[9]The degree of proximity of the page 6 is different layout. Given a rhythmicity b, the proximity degree orders the divisors a of her length.

[10]The rhythmicity of empty grouping can be assumed to be identical to the reference sequence, hence to the one-beat grouping, so it is of class [2]. But I believe that the empty set should not be considered as a grouping that supports rhythmicity. So deconstruction cannot be done on trivial rhythmicity supported only by its own one-beat rhythm grouping.

[11]If $k = 0$ then the grouping is identical to the rhythmicity it supports. But then it is not a supporting relation but a tautology, since C is a column of rhythmicity C then C is a column of rhythmicity C.

dotted whole note of length 4 and four dotted quarters of length 1 are co-heard, so the rhythmicity $[1]+[4]=[4]$, of the simple class rhythm [2] supporting it, is in use. From the first to the second measure, active rhythmicity progression occurs. In the second measure, with twelve 1-length eighths, a rhythm $[\frac{12}{1}]=[12]=[3]$ is energetically added, co-heard with the rhythm [4], supporting the rhythmicity $[4]+[3]=[12]$, class [3]. In the third measure both the rhythmicity and its class

Figure 3: Progress of rhythmicity

remain unchanged, since the conversion of the four dotted quarters, of class $[\frac{12}{3}]=[4]=[2]$, into two dotted halves does not affect their $[\frac{12}{6}]=[2]$ class. From the third measure to the fourth measure, passive rhythmicity progression occurs, because the class rhythmicity $[3]=[3]+[2]$ is decomposed into a class rhythmicity $[4]=[2]$.

The fractal self-similarity of the structure of time projects the aforementioned structure of rhythmicity into the concept of the structure of tonality. We refer to rhythmicity, rhythms, co-hearing and beat groupings when the frequency of beats is of the order of one beat per second ($1Hz$). We refer to tonality, tones, chords and frequencies respectively, if we "get away", about 20 to 20,000 times "farther", so that we perceive the time intervals to be shorter and $1Hz$ now corresponds to sound frequencies. If we even take into account the aforementioned structure of the *monoid*, then frequencies and chords can be considered elements of the same set and perhaps not as different as I thought.

Limits to self-similarity are set by the finiteness of the acoustic spectrum. Beats of 1 Hz will be heard as a sound frequency if accelerated 440 times, but a sound frequency of 440 Hz will not be heard if decelerated 440 times, because the air pressure that produces it is subject to continuous variation. It should have been a different nature of the event from the start, such as a shock rather than an acoustic wave. The tonality is often quite a large sub-multiple of the frequencies that support it. I cannot easily accept that our perception will impart entity to tonality that is below the threshold of hearing. That is why, in a co-hearing, a bass frequency near the $20Hz$ threshold, when it does not itself identify with the co-hearing's tonality class, since the latter will be outside the auditory spectrum, sounds independent, outside the umbrella, of the tonality produced by the other frequencies, creating a tension that calls for resolution. For the same reason, in a resonance, a high frequency, near the $20000Hz$ limit, cannot have the role of tonality supported by other higher frequencies, but only the role of supporting a tonality sub-multiple of it.

2 Tonality

2.1 Homomorphism

Rhythms and tones are connected by a relation of homomorphism, which is the mapping of the time intervals of the phenomena of one to the time intervals of the phenomena of the other. The subscript r will be used for the rhythm phenomena and the subscript t for the tone phenomena. Thus we have:

Sound wave reference. The beat reference sequence of frequency σ_r and period $T_r = 1/\sigma_r$ *corresponds a reference sound wave* of frequency σ_t and period $T_t = 1/\sigma_t$.

Frequency σ_t^1. The grouping σ_r^1 per one beat *corresponds* to a sound wave of frequency $\sigma_t^1 = \sigma_t$ and period $T_t^1 = 1/\sigma_t^1 = 1/\sigma_t$.

Συχνότητα σ_t^n. Η ομαδοποίηση σ_r^n ανά n κτύπους *αντιστοιχεί* σε ηχητικό κύμα περιόδου $T_t^n = n * T_t = n/\sigma_t$ και συχνότητας $\sigma_t^n = 1/\mathrm{T}_t^n = \sigma_t/n$.

Tonality. The rhythmicity $\sigma_r^{n,m,\cdots}$ of the σ_r^n grouping co-hearing *corresponds* to *sound wave* (**tonality**) $\sigma_t^{n,m,\cdots}$ of period $T_t^{n,m,\cdots} = lcm(T_t^n, T_t^m, \ldots) = \frac{lcm(n,m,\ldots)}{\sigma_t}$ and frequency $\sigma_t^{n,m,\cdots} = \frac{\sigma_t}{lcm(n,m,\ldots)}$, where $lcm()$ is the least common multiple. Equivalently, it is also expressed in terms of greatest common divisor gcd[12].

$$\sigma_t^{n,m,\cdots} = \frac{1}{lcm(1/\sigma_t^n, 1/\sigma_t^m, \ldots)} = gcd(\sigma^n, \sigma^m, \ldots)$$

Tone. The rhythm λ_r^n of a grouping σ_r^n, relative to a rhythmicity $\sigma_r^{n,m,\cdots}$ that it supports, *corresponds* to the **tone**

$$\lambda_t^n = \frac{T_t^{n,m,\cdots}}{T_t^n} = \frac{\sigma_t^n}{\sigma_t^{n,m,\cdots}} = \frac{lcm(n,m,\ldots)/\sigma_t}{n/\sigma_t} = \frac{lcm(n,m,\ldots)}{n}$$

and is a dimensionless integer. Equivalently expressed as

$$\lambda_{n,m,\ldots}^n = \frac{\sigma_t^n}{gcd(\sigma_t^n, \sigma_t^m, \ldots)}$$

Substituted, in λ, the index t to clarify that the tone refers to the tonality it supports. The pitch of the tonality $\lambda_t^{n,m,\cdots} = \frac{T_t^{n,m}}{T_t^{n,m}} = 1$ is always equal to unity. Therefore, *tonality is a frequency and tones are ratios, integer multiples, of the in use frequencies to it. The degree of proximity* is defined the same as in page 6.

[12] If lcd is expressed in prime numbers, its exponents will be of the form $max(m_i, n_i, \ldots) = -min(-m_i, -n_i, \ldots)$ which proves the relation, because min corresponds to gcd.

> The frequency of tonality, because it is supported, potentially, by the infinite frequencies of its integer multiples, *can not* define well a particular set of supporting tones in use.
> *The frequencies in use are those that determine the frequency of tonality.*

The frequency chord. The co-hearing operation $+ \colon S \times S \to S$, with S the set of groupings, corresponds to *frequency chord* and is naturally projected as $\sigma_t^n + \sigma_t^m = \sigma_t^{n,m} = gcd(\sigma_t^n, \sigma_t^m)$, where gcd is the greatest common divisor. With σ_t^1 as the neutral element $e = \sigma_t^1$, **the structure of the commutative monoid**(page:7), with its weak order $\leq$, is obvious. Note however that, because we are now referring to frequencies rather than periods, the order $a \leq b$ means that the frequency a is divided by b.

2.2 Classes

The frequency classes $[\sigma]$ defined by the *equivalence relation* $\sim$, where two frequencies σ_t^n and σ_t^m are considered equivalent if their ratio is a power of 2, i.e. $[\sigma_t^n] = [\sigma_t^m] \iff \sigma_t^n \sim \sigma_t^m \iff \frac{\sigma_t^n}{\sigma_t^m} = 2^{\pm k}$, $k \in$ Z. The set of all classes is the $S/\sim := \{[\sigma] : \sigma \in S\}$.

Frequency Class representative $[\sigma] \in S/\sim$, of a frequency σ, is selected the number of the form $\frac{\sigma}{\sigma^0} * 2^k$, $k \in$ Z which belongs to the interval $[1, 2)$ of irrational numbers, so it corresponds the reference frequency σ^0 to unity, $[\sigma^0] = 1$[13]. Because tonalities are frequencies, *the* $[\sigma^{n,m,\cdots}]$ *classes of tonalities are represented by points of the same interval* $[1, 2)$ *of irrational numbers.*

The tone class $[\lambda^n]$ defined by their projection

$$[\lambda^n_{n,m,\ldots}] = \frac{[\sigma_t^n]}{[gcd(\sigma_t^n, \sigma_t^m, \ldots)]}$$

thus

$$[\lambda_t^n] = [\lambda_t^m] \iff \lambda_t^n \sim \lambda_t^m \iff \frac{\lambda_t^n}{\lambda_t^m} = 2^{\pm k},\ k \in \mathrm{Z}$$

Representative of the tone class $[\lambda^n_{n,m,\ldots}] \in$ Z$/\sim$, of the tone $\lambda^n_{n.m.\ldots}$, is chosen the rational number of the form $\lambda^n_{n,m,\ldots} * 2^{-k}$, $k \in$ Z belonging to the interval $[1, 2)$ of rational numbers.

[13]The seven common notes or more fully the twelve common tones are, seven or twelve respectively, classes of frequencies corresponding to points in the interval $[1, 2]$ over the irrational numbers. Notes are used as frequency classes in musical analyses and are notated as frequencies in musical scores.

The monoid of tone classes is defined by their projection. Obviously the projection π, of the chord act $+$, onto $S/\sim$, as $\pi : S \to S/\sim$, $\pi(\sigma) = [\sigma]$, is self-evident and well defined. So

$$[\sigma^n] + [\sigma^m] = [gcd(\sigma^n, \sigma^m)]$$

The weak order, however, changes slightly in meaning. $[a] \leq [b]$ means that there exists k such that $a * 2^k$ is divisible by b.
Extending the above relation and dividing by $\sigma^{n.m.\cdots}$ we have that, with respect to the tones λ^n of a chord, *applies to the chord operation* (note, not the usual addition)

$$[\lambda^n] + [\lambda^m] + \ldots = [1]$$

If we assume some tonality and the relationship does not hold then obviously our assumption was wrong and the frequency chord supports another tonality. For example, if we assume the frequency of C as tonality and class [1] and we hear in a chord C [1], G [3/2] = [3] and D [9/8] = [9] then $[1] + [3] + [9] = [gcd(1, 3, 9)] = [1]$ correctly the tonality of the chord is C. But if we subtract C then the chord of G and D $[3] + [9] = [gcd(3, 9)] = [3]$, so obviously C is no longer the correct tonality but G [3].

Simple tonality is the tonality whose class is the class of the reference frequency. This has the consequence that only frequencies belonging to $[\sigma^0]$ are in chord.

Deconstruction of tonality is its mutation into the tonality of some strict subset of the frequencies that support it.

Tonality progression is the change of tonality to another tonality class.

Passive progression (or *weak* or *decreasing* progression) of a tonality is *its deconstruction* to a different class.

Active progression (or *strong* or *ascending* progression) of a tonality is any *not passive* progression of that tonality to a new tonality of a different class.

2.3 The space of tones and tonality

The set of frequencies is a one-dimensional space, i.e. it is isomorphic to the line R of the irrational numbers. We have to define some frequency of this space as *reference frequency*, to which the representatives of the frequency classes presented in the previous sections refer. The frequency $\sigma^0 = 440Hz$ is usually referred to as the reference frequency[14].

[14] It is actually referred to as standard pitch.

The tone and tonality torus is the space where they are placed. The identification of the edges of the interval of frequency classes $[1,2)$ needs one more dimension to be visualized as a circle. That is, it needs two-dimensional space in order to bend the frequency line within it. However, despite the visualization of the world of tonality classes in *tone circle*, euclidean proximity remains an illusion, probably quite contrary to the ordering relations mentioned above. By the same logic, all tone classes $[\lambda^n]$ of a tonality $\sigma^{n,m,\cdots}$, correspond to *tone cycle* $[1,2)$. From each point $[\sigma^{n,m,\cdots}]$ of the tonality circle passes a tone circle $[1,2)$, where the unit $[\lambda^{n,m,\cdots}] = [1]$ coincides with the tonality class $[\sigma^{n,m,\cdots}]$. Thus, we still need one more dimension to visualize the tone cycle in a different dimension from that of the tonality cycle. So, in three-dimensional space, the circles of all possible tone classes, on each point of the tonality circle, are surface of a torus. The torus is the universe of the classes of tonalities and their tones.

Every tonality $\sigma^{n,m,\cdots}$ *always exists*, as hidden or obvious, at the sound of any frequency $\sigma^n = \sigma^0/n$ that supports it[15]. However, the tone λ^n, of frequency σ^n, is almost never alone in the tone cycle of $\sigma^{n,m,\cdots}$. It is the set of notes $\{[\lambda^n], [\lambda^m], \ldots\}$, i.e., the set of chord tones, that belongs to the tone cycle passing through the tonality class $[\sigma^{n,m,\cdots}]$. A strict subset or superset of this may well, as a chord, belong to a tone cycle of different tonality. Although the tone circle of a tonality does not contain isolated tones, but only chords of tones, nevertheless the tone circle $[1,2)$ contains infinite points, and the question arises which of them may be candidates to be *elements of the sets* $\{[\lambda^n], [\lambda^m], \ldots\}$, i.e. the tones that create the chords. Since the tonality $\sigma^{n,m,\cdots}$ is a common divisor of all the σ^n that support it, it is obvious that the only suitable points of the circle of its tones, for elements of $\{[\lambda^n], [\lambda^m], \ldots\}$, are those which, by definition, correspond to its harmonics, i.e. to the frequencies $n * \sigma^{n,m,\cdots}$, $n \in \mathrm{Z}^+$. So $[\lambda^n] = [n]$, i.e. the tone classes coincide with the integer classes. Thus, the representatives of the tone classes are terms of the sequence $\{1, 3/2, 5/4, 7/4, 9/8, 11/8, \ldots\}$[16].

The elements of the tone set that can support a tonality, apart from belonging to the positive integers, can be drastically limited due to the limit set by the acoustic spectrum mentioned on page 9. By definition, it is true that

$$\lambda^n = \frac{\sigma^n}{\sigma^{n,m,\ldots}} \Rightarrow 20Hz \lesssim \sigma^{n,m,\cdots} = \frac{\sigma^n}{\lambda^n} \Rightarrow \lambda^n \lesssim \frac{\sigma^n}{20Hz}$$

It's completely subjective but, if we want the $440Hz$ frequency to support tonality within the acoustic spectrum we need $\lambda^n \lesssim \frac{440}{20} = 22$. The closer to unity some tones are, the more it is perceived that they are indeed elements of the tonality they support. The closest elements other than the unit are $[3/2]$ and

[15]Its smaller subfrequencies, $\sigma_n^k = \sigma^n/k$, always exist in potential. However, in our perception, they are overshadowed by the sound of σ^n. In other words, the opposite happens to what happens to the harmonics $k * \sigma^n$, which are sometimes generated by musical instruments and sometimes not, depending on the instrument producing the fundamental frequency σ.

[16]It is a matter of musical culture how many and which of them are used and make up a set of tones in use in practice.

[5/4]. We will not add another element so that the tonality does not easily stray out of the audible range[17].

The dualism of tones and tonalities is a necessary additional structure, on top of the torus on which they are placed, in order to limit the classes of tonality which, so far, are represented by the infinite points of the interval $[1, 2)$ of the irrational numbers. Each tone λ^n, multiplied by the tonality $\sigma^{n,m,\cdots}$ it supports, is uniquely assigned to the frequency $\sigma^n = \lambda^n * \sigma^{n,m,\cdots}$. With the additional structure, on the tonality circle, we allow, as tonality classes, only the points corresponding to tone frequencies. It is of course sufficient to specify a reference frequency σ^0, so that the tonality $\sigma^{n,m,\cdots}$ corresponds to a specific frequency $\sigma^{n,m,\cdots} = \lambda * \sigma^0$. We thus create an ***tonality tree***, where each tonality is connected to two other tonalities through the two branches corresponding to the two basic tones [3/2] and [5/4]. The figure 4 shows part of the tree, where, centered on the reference tonality [1], we see the two-level tonality classes, both in the direction from the two fundamental tones to the tonality they support, where they are marked with 3 the [3/2] and 5 the [5/4], and two-level classes in the opposite direction, since, according to our desired additional structure, the central tonality [1] also has to support some other tonalities. In the figure, the beginning of the arrow supports its apex.

All classes, with respect to [1], are considered simple ratios using numbers less than 22, except [32/25] and [25/16]. Both of these approximate either the simple classes [9/7] and [11/7][18] or the simple classes [5/4] and [8/5] respectively. Let us accept that we ignore them or the habit and economy of our perception, of the page 6, makes us perceive the sound of [32/25] as [5/4] and [25/16] as [8/5]. By denoting the tonality class [1] by C and the others according to just intonation, the relationships of the tonality classes are shown in Fig. 5. In each direction, vertical, horizontal, diagonal left and diagonal right, there is ± 1[19], ± 3, ± 4 and ± 5 constant half steps variation respectively. The torus topology is strong and, if we mapped the frequency triads, there would be a match with the diagram Tonnetz of Neo-Riemannian theory. Note that the above relationships are transitional. If the tonality class $[X]$ supports $[Y]$ and $[Y]$ supports $[Z]$ then $[X]$ supports $[Z]$, just the support is more distant. In the figure above, the eleven known tones are shown, and the tone $F\#$ corresponding to the interval of the tritone $CF\#$ is missing[20]. Unfortunately, no matter how clear the tonalities around the central class [1] sound, the further we move away from it, the more dense the classes in the $[1, 2]$ interval become, as we will never again end up with a [1] class tonality. Thus, the distant tonalities become indistinguishable

[17] If we had added the element [7/4], for example, the product with [5] would require a limit of $[5 * 7] = [35]$, so the usual frequencies would have to be one octave higher at $880Hz$, so $880/20 = 44 \gtrsim 35$.

[18] But we are not used to hearing or using tones based on 7.

[19] The major scales of all $C\#, A\#, G\#, D\#$ in the figure usually have key signatures with flats, so perhaps they should be denoted as Db, Bb, Ab, Eb. But the notation with sharps demonstrates the ± 1 relationship in the vertical direction.

[20] Consistency with the relationships of the other classes places $[F\#]$ in the middle of the figure to the left of $[D\#]$ or below $[G]$.

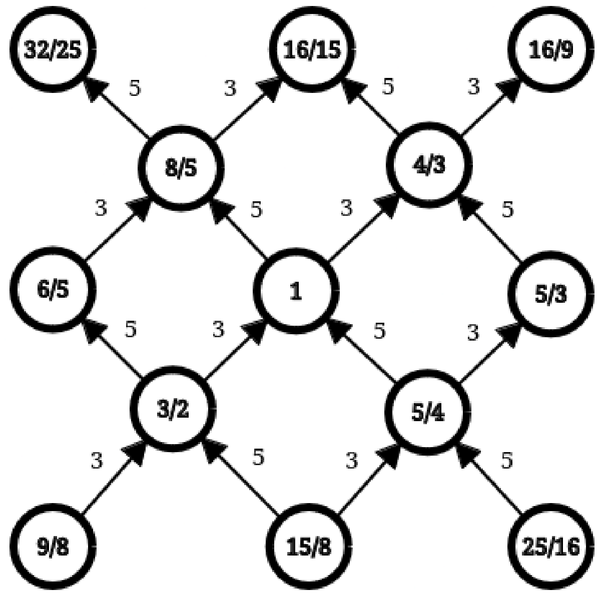

Figure 4: Tonality relationships as a tone of each other

from the classes of simple analogies, and their sound becomes muddled. The above difficulty is overcome to some extent by sharing the error of the distant classes over all classes, by adopting tonality classes that divide 2, the octave, into k equal proportional intervals, with such a ratio a that $a^k = 2$. This creates an **equal temperament system** of k frequencies in use. Although there are no longer tones as integers, since there is no greatest common divisor of the irrational frequencies in a chord, nevertheless if a^k are very close to simple ratios, the need for economy of information causes them to be taken as simple rational ratios. The number in use for k depends on musical culture, familiarity with the frequency relationship it evokes, and how well the ratios approximate simple ones. In modern times, $k = 12$ has almost universally prevailed with the twelve-tone equal temperament system. In this system $a = 2^{1/12}$. And the ratios of the eleven tonality classes, of the portion of the tree we have considered, are very well approximated by the in-use $a^k = 2^{k/12}$. Since we have arrived at twelve points on the tonality circle then, in each tone circle, we will allow twelve class points to select the λ^n tones of the chords. Corresponding unity to the class $[C]$, the best approximations[21] of $2^{k/12}$ with simple ratios are shown in the table 1.

[21] The approximation of $2^{1/12} = 1.059463 \sim 17/16 = 1.0625$ is better than $16/15 = 1.0666$. That of $2^{6/12} = 1.41421 \sim 17/12 = 1.41666$ is better than $45/32 = 1.40625$. That of $2^{10/12} = 1.78179 \sim 16/9 = 1.77777$ is better than $9/5 = 1.8$. All three are better than those listed in the just intonation system.

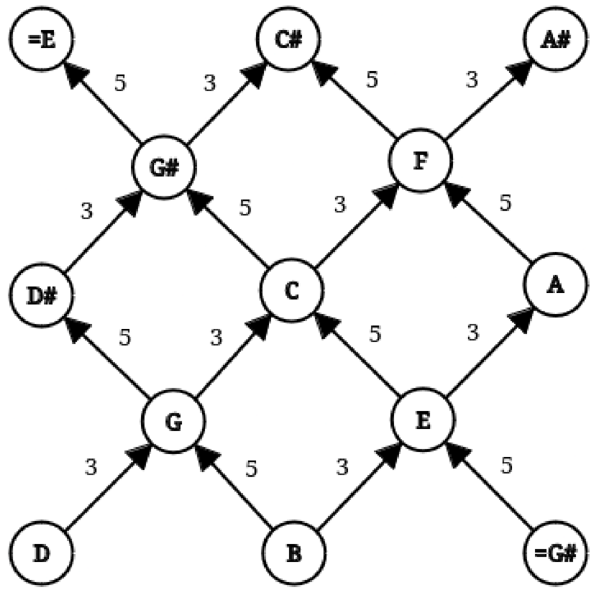

Figure 5: Local map related to $[C]$ tonality.

k	0:C	1:C#,Db	2:D	3:D#,Eb	4:E	5:F
$[\lambda]$	1	17/16	9/8	6/5	5/4	4/3
k	6:F#,Gb	7:G	8:G#,Ab	9:A	10:A#,Bb	11:B
$[\lambda]$	17/12	3/2	8/5	5/3	16/9	15/8

Table 1: Approximations of $2^{k/12}$ with simple ratios

Simple tonality is extended in the equal temperament system as *the trivial tonality based only on its unit tone*. This is because all tonalities acquire equal importance[22], since each octave of each of them is divided into equal proportional parts by the others. All unaccompanied melodies, where single frequencies are heard, not being an element of a chord, are a sequence of simple, trivial tonalities. Hence each step of the melodic sequence is an energetic tonality progression, since it is not identified with a strict subset of frequencies of the preceding chord.

About manifolds. Imagine a creature living on the surface of the torus of tones and tonalities on page 13. The creature has the sense that it lives in a two-dimensional world and, according to its local map, each movement refers to a direction along the $[3/2]$ and $[5/4]$ axes. Wherever he is, he has these axes in

[22]The equal importance of each tonality leads to the next paragraph on manifolds.

mind and moves according to them. He realizes, of course, that after 12 steps in the direction [3/2] he is back where he started from, just as the same happens after 3 steps in the direction [5/4]. He cannot, of course, perceive what we, living in three-dimensional space, easily perceive, namely that his world is on a torus surface. The best he can do to map his world is to draw local maps, where each map has points in common with each of its neighbors. One such local map, relating to the tonality point $[C]$, is the figure 5. The set of all local maps maps the surface of the torus as manifold. The relations with the four classes around C, as local, are precise. The relations of C with more distant classes, due to the topology of the torus, are distorted the more distant they are from $[C]$.

2.4 The missing tonality

The projection of the homomorphism of the hidden rhythmicity, as a hidden grouping of the 5 page, on the frequency spectrum is the hidden or lost tonality, or missing fundamental frequency. Each frequency chord produces a fundamental complex waveform that repeats with a frequency equal to the tonality of the chord. The repetition in the acoustic spectrum is taken as a frequency. *The tonality of a chord is always in use.* If its class corresponds to a frequency of the chord, then it is obvious. If the tonality class does not correspond to a frequency of the chord, then it is hidden or missing.

The tonality class of a chord is completely determined by the classes of frequencies that support it. In the equal tempered system, due to the exact proportional division of each octave, these frequencies themselves lose their significance and the ratio between them, i.e. the musical intervals, becomes important. This means that, to compute any chord tonality, we take the class of one of its frequencies as the unit and compute it, in terms of k intervals from it, from the ratios of the table 1. Because the associativity holds (page 7) the tonality of a chord is easily approximated by steps of two frequencies. It is therefore worth calculating the tonality of all intervals defined by two frequencies. Consider that all the ratios in the table 1 are irreducible fractions.

- $[1] + [\frac{a}{2^n}] = [1] + [a] = [1]$ so all intervals corresponding to a ratio with a denominator of power of 2 have as their tonality class the one corresponding to the reference frequency. For example, $[C]$, as the reference frequency, co-heared with $[C\#], [D], [E], [G], [B]$ has a tonality of $[C]$ itself. If we exclude $[C\#]$, we conclude that, given $[C]$, every chord subset of the frequencies of the chord C^9 has a tonality of $[C]$, to such a submultiple of C as corresponds to the highest denominator of the power of 2. Any resonance with a different frequency class results in a different tonality from $[C]$. The $[C\#] = [17/16]$ can be excluded for three reasons. First, with tonality corresponding to four octaves lower it can be put out of the acoustic spectrum; second, it is a very sensitive ratio. It is enough to take it equal to [16/15] in the just intonation system and it will no longer belong to a frequency that supports the $[C]$ class tonality; and thirdly, the almost perpetual use of the [16/15] ratio, in the just intonation system, has created such a habit as to make its acoustic acceptance unlikely.

- $[1] + [\frac{a}{b}] = [\frac{1}{b}]$, $b \neq 2^n$ so the tonality corresponds to the class $[\frac{1}{b}]$ and the submultiple $1/b$ of C. So,
  $[C] + [D\#] = [1] + [\frac{6}{5}] = [\frac{1}{5}] = [G\#]$ i.e. **the minor third has the tonality of the major chord that uses it**. Of course, the same result applies to the sixth major $[C] + [\mathrm{A}] = [1] + [\frac{5}{3}] = [\frac{1}{3}] = [F]$.
  $[C] + [\mathrm{A}b] = [1] + [\frac{8}{5}] = [\frac{1}{5}] = [Ab]$ i.e. the sixth minor has the tonality of the corresponding third major.
  $[C] + [F] = [1] + [\frac{4}{3}] = [\frac{1}{3}] = [F]$ that is, the perfect fourth has the tonality of the perfect fourth.
  $[C] + [\mathrm{B}b] = [1] + [\frac{16}{9}] = [\frac{1}{9}] = [\mathrm{B}b]$ that is, the minor seventh has the tonality of the minor seventh.
  $[C] + [F\#] = [1] + [\frac{17}{12}] = [\frac{1}{12}] = [\frac{1}{3}] = [F]$ i.e. the augmented fifth has the tonality of the perfect fifth, but more than three octaves below $[C]$.

In summary:

- The difference of $2, 4, 7, 11$ halftones supports a tonality class equal to the class of the reference frequency[23].

- The 3 halftone difference supports tonality equal to the class of the 4 halftones frequency *before* the reference frequency.

- The $5, 10$ difference of halftones supports tonality equal to the frequency class of the difference, i.e. the classes [5] and [10] respectively.

- The difference of 6 of halftones, i.e. of tritone, supports tonality equal to the 5 halftones frequency class *after* the reference frequency, equivalently 1 halftone less than the difference.

We see that in the presence of the $[C]$ frequency class, apart from itself, only the $[F] = [4/3], [G\#] = [8/5]$ and $[Bb] = [15/8]$ tonality classes can be supported.

Chords are set of co-heared frequencies. They can not but to only support one particular tonality. The tonality can be the simple trivial one, but a *chord* will be called **major** or full **if it is supported by the tonality class** [1] **and its two key tones** [3/2] **and** [5/4]. The tonality in the standard major chord $C - E - G$ is obvious, since $[C]$ is used in the act. Let us call the class $[C] = [1]$ the **root** of the chord with the sense of *intent* being the same as its tonality. Let us call the class $[G] = [3/2]$ the **source** of the chord in the sense of being the main source of support for the root of the chord as its intended tonality. Let us call the class $[E] = [5/4]$ **quality** of the chord in the sense of being the secondary source of support or not of the root of the chord, as its intended tonality. Since the source always supports the root as tonality, it is the quality

[23]Let us ignore the difference of 1, which supports the same tonality, for the reasons mentioned in its calculation.

of a chord that determines whether or not the *root* of the chord is indeed its apparent tonality. Note that the interval of three halftones, corresponding to the class [6/5], is always present, as $E - G$, in every major chord.
The tonality is approximated by combining the frequencies in pairs. Calculating the pitch of the major chord gives the result [1/4]. The class of the result is obviously [1] and the tonality is $2^2 = 4$ two octaves below C. If we also add $[B] = [15/8]$, obviously creating C^7, the result [1/4] is combined with $[B] = [15/8]$ and we find $[1/4] + [15/8] = [1/8]$. The seventh's class is again [1] and the tonality is $2^3 = 8$ three octaves below C, so easily within the acoustic spectrum.

Exact tonality calculation is needed when the tonality is estimated to be very distant, so it is not obvious which of $2^{k/12}$ best approximates it. For this reason I constructed the function $G(x, y, \ldots)$ in maxima as

```
G([arguments]):= block([i,x,y,marg,minmarg,sol,ratprint,ratepsilon],
ratprint:false,
ratepsilon:1e-2,
marg:mod(arguments,12),
minmarg:first(marg),
marg:mod(marg-minmarg,12),
y:apply('ezgcd,rat(bfloat(2^(args(marg)/12))))[1],
i:0,
while y*2^i < 1 do i:i+1,
sol:round(rhs(solve([2^(x/12)=2^i*y],[x])[1])),
[y,arguments,mod(sol+minmarg,12)]
)$
```

The $x, y, \ldots$ are the k involved in the chord. The result shows, with the first fraction, what submultiple of $[x]$ the tonality is in, which is shown as the last number of the result. One of the most interesting results is

```
(%i5) G(11,2,5,8);G(2,5,8,11);G(5,8,11,2);G(8,11,2,5);
(%o2)/R/ [1/60,[11,2,5,8],0]
(%o3)/R/ [1/60,[2,5,8,11],3]
(%o4)/R/ [1/60,[5,8,11,2],6]
(%o5)/R/ [1/60,[8,11,2,5],9]
```

The result shows that the chord $B-D-F-Ab$ need not necessarily be considered as one of the four dominant ninths, which lack the root and end up in its corresponding tonic, but that it directly tends to end up in one of its four hidden tonalities, i.e. in one of the $[C], [Eb], [F\#], [A]$.
But I have to be sceptical. The submultiples are too small to accept that these are "close" frequency distances. I accept the result because of the symmetry of the chord. These are all four different classes of each horizontal line in the figure 5. However, I much more easily accept the result for the chord $G\# - E - C$, where it is all three different classes of the upper left diagonal.
In contrast to the previous acceptable results, an unacceptable result is the

```
(%i3) G(7,11,20,27);G(6,0,9,2);
(%o2)/R/ [1/80,[7,11,20,27],3]
(%o3)/R/ [1/60,[6,0,9,2],7]
```

This is dissonance which, according to page 324 of the book Theory of harmony[24], used by Mozart. The very small submultiple 1/80 undermines the result $3 = [D\#]$ of the first chord. The most prudent thing to do is to consider the lower frequency $7 = [G]$ as the intended tonality, but the next chord, while indicating $[G]$ as its tonality, again, as a distant tonality and not even through the more common $[F\#] - [A] - [C] - [Eb]$ but through the $[F\#] - [A] - [C] - [D]$ chord.

The minor chord is the deconstruction of the major chord, by removing its root, making its tonality hidden. It is essentially a deconstruction of the seventh major $C - E - G - B$. If we define the reference class and root of the minor chord as $[E] = [5/4]/[5/4] = [1]$, the tonality approximation yields a result of $[8/5]$, corresponding to 4 halftones before the reference frequency E, i.e. C. Thus, the frequency C is in use but not produced in practice. So the $[C]$ tonality is a hidden tonality of the $E - G - B$ minor chord, which together with the latter forms a deconstructed, by its root, C^7 chord. Obviously the E^7 seventh minor is the deconstructed, from its root, C^9 ninth major chord.

The first example came from my curiosity to examine the first six measures of the well-known moonlight sonata. In figure 6 added, with cello, the moments where the tonality changes, either hidden or obvious. The first six measures of piano are heard in the sound, which are repeated with the addition of all the tonalities in cello. Only the cello tonalities follow, and it ends by repeating the six piano cello measures together. Note that, in the presence of the root frequency, the alternation of tonalities in the arpeggio is because the chords are minor. In the major arpeggios, in the presence of the root frequency, the tonality remains constant. Also worthy of note is the ascending melody of the tonality at the end of the fourth measure, produced by the progression $i_4^6 - V^7$.

The second example answers the question of which different chords support the same tonality. The chord of the standard minor triad was determined to support a hidden tonality located at 1/10 of the reference frequency. In the query, such a constraint is imposed that the tonality of the requested chords is equal to or less than their reference frequency, relative to that of the minor chords. The first and second measures of the figure 7 reflect that without frequencies there is only rhythm, the same as if only the trivial simple tonality is heard. In the third measure, with the two-frequency chord, the reference frequency $[C]$ is accompanied by each of the frequencies of $[C^9]$, as we showed earlier. In the remaining measures, all combinations of chords of $1, 2, 3, 4$, and 5

[24] Search for the word "fellow" to see figure 233 of the book.

Figure 6: Hidden and obvious tonalities in the "Moonlight sonata".

frequencies that answer the question posed are developed. There are ultimately 23 such different chords. Note that the frequency classes $[F]$ and $[A]$, which do not belong to $[C^9]$, are not used by these chords. The numbers above the pentagram indicate what submultiples the $[C]$ tonality is in. In the sound of the figure the same frequency arrangement is repeated for the tonalities $[F]$ and $[G]$. We eventually hear the progression $I - IV - V - I$ with extended harmonies twice.

2.5 Tonality as a springboard for changes

Let us imagine that we are supratemporal observers and perceive time as a three-dimensional space. It is the perception of a vast empty space in which

Figure 7: Chords that support the [C] tonality.

events of short or long duration are scattered[25].

Non-existence is the average state in time and the starting point from which this state is changed to a state of "existence", only to return again to the average state of "non-existence". Let us perceive the performance of a musical composition from afar, as we would perceive the earth from the edge of the solar system. We would perceive it as a point in time, a dot in the vastness of space-time. Something was created and ended. There was a change in the otherwise empty structure of time. I describe the whole thing by saying that, from so far away, *the primordial tonality of any composition is its "non-existence"*. Any signifi-

[25] If an event existed forever, this would be tantamount to its "existence". It would fill all of time space, so our perception of it would change. We would perceive time as an infinite compact in which short or long-lived events of 'non-existence' are scattered. What is absolutely important is change, differentiation.

cance of the composition lies not in its tonality, that is in its non-existence, but in its change around it.

The universal tonality is best approached by getting closer and understanding the composition as a perfect sphere. We understand the tonality of the composition as the perfectly symmetrical structure of the surface of the sphere. This perfect surface itself, apart from not existing, could not describe anything creatively, because it does not contain the element of variation. If we call the surface of the sphere a C major, it makes absolutely no difference if we claim to have composition in C major. We have to get a little closer in order to distinguish ocean depths, valleys and mountain ranges. Then we can see the C major as the perfect, ideal and therefore non-existent, middle surface around which, de facto, the actual tonalities change, for example the G major surfaces of the deeps, the C major surfaces of the valleys and the F major surfaces of the mountain ranges. The composition is therefore in C major but, any significance of the composition lies not in its tonality, but in the variation of the tonalities in use around it.

Music consists exclusively of temporal events. Whether we are dealing with beats at equal intervals or with frequencies, which are patterns of air pressure at equal intervals as well, time alone is its fundamental essence. Whatever reference structure we accept exists and operates in the relationships between groupings, because of the fractal nature of time, the same reference structure will exist and operate at every level of magnification. Whatever we claim works in rhythmicities will work in tonalities, will work in tonalities of tonalities, and so on in every consideration of every level of magnification of the composition. The meaning of music, is based on the creation and synthesis of variations. If any composition is called a creation, it will most likely be the scale variation that will have meaning. If there is only one scale, it will be the chord variation that will be significant. If there is a monotonous repetition of chords, melody variation will matter. If there is only one melody, it will be the change in frequency of the melody that will matter. If there is only one frequency, the rhythm remains, so it will be rhythm variation that will matter. If there is only one rhythm, it will be the change in the meaning of lyrics relative to the average state of everyday life that will matter. If it is lyrics of the day, something else will matter, otherwise the characterization that it is a musical creation is obviously wrong[26].

The individual tonalities become perceptible by approaching the surface even more. At mid-sea level, the C major valley is not perfect. It consists of changing tonalities of chords that move away from it, forming hills and gullies.

[26]Let me remind you that these are personal opinions based on mathematical intuition, since I have no knowledge of music theory. In fact, my views on tonality changed during the writing of the present, because it is one thing to think you understand something and another to try, without easily perceived contradictions, to write what you think you understand.

Even these individual ones, if we get closer, we can see that they are created by frequencies of soil, bushes, trees that change and interact with each other.

Major scale tones emerge from the specific homonymous tonality reference class and indicate the nearest departures from and arrivals to it, according to the principle that tonalities, if any, are variable. Each frequency class, because of the duality of tones and tonalities mentioned on page 14, is interwoven with the tonality class of the homonymous major chord, because the latter uses both basic tones from which this duality originated. Of the two basic tones, the tone [3/2] is the one that corresponds to the closest tonality change. As far as departure from tonality $[C]$ is concerned, the closest energetic progression to it is achieved by considering tonality $[C]$ as the source of the new major chord supporting tonality $[F]$. The departure, by active progression, to the nearest tonality is therefore $C \to F$[27]. By exactly the same logic, the arrival, with active progression, from the nearest tonality is $G \to C$. The tones of the reference major scale are the tones of the major chords of arrival, departure and reference tonality, in this case for $[C]$ the tones of the chords $GBD \to CEG \to FAC$. Thus, the frequency classes of the $[C]$ major scale are $\{[C], [D], [E], [F], [G], [A], [B]\}$. It can be considered that *the major scale is a mixture of the nominal reference tonality, and the tonality of the two extremes before and after it.* It differs from the scales of its extremes only by one frequency class. It uses $[F]$ instead of $[F\#]$, used by the $[G]$ scale, and $[B]$ instead of $[Bb]$, used by the $[F]$ scale.

Harmonic progress refers, in my opinion, to the progress of the tonalities. *Progression must always be active.* We must develop tonalities by building on what we have already built. Theoretically, passive progression XY is allowed, in a sequence XYZ, if the result of the next progression YZ looks like an active progression XZ overall. But, even in this theoretical acceptance, I encountered exceptions. So, I take it that passive progress is allowed, as long as we are aware that we are using it. Just as we would not let our house deconstruct and collapse, but would soon proceed with repair work in case of partial deconstruction, so we should not let the tonality, built by us, deconstruct with passive progressions and collapse, but should soon proceed with repair work of active progressions to restore it. The change in the tonality of a chord is always referred to in relation to the tonality of the next chord. This is a first axis of reference. Chords, however, refer to two axes.

The chord, as a function, is the second axis of reference. The chord function refers to the major or minor scale to which, according to composer, it belongs. We must accept axiomatically that each sequence of chords $xyzw\ldots$ has a single result. It does not matter if this can be determined objectively, which it obviously cannot. What matters is that the result is only one. In the absence of any objective determination of it, let us call it A, that is, $A = xyzw\ldots$.

[27] The tonality $[F]$ of the chord $F - A - C$, by deconstructing $[F]$ and $[A]$, is changed to the tonality $[C]$. Therefore the progression $F \to C$ is a passive progression and $C \to F$ is an active progression.

Each subset of the chord sequence can be considered a function that, with variable values equal to the other chords, gives the given result. For example, the function $yw()$ can be defined such that $yw(x, z, \ldots) = xyzw \ldots$. The simplest case is to assume that the results are produced by a sequence of only two chords. Then the set of functions is transformed into a dual of chord space. This is because, if the function $x()$ is defined such that, with a variable equal to the chord y, it yields the result $x(y) = xy$, the roles of chord functions can be mutually interchanged without affecting the result. That is, it could just as well be that the function $y()$ could be defined such that, with variable equal to the chord x, it gives the same result $y(x) = xy$. Because functions refer to whatever scale is in use at any given time, they are represented by Roman numbers, conceptually accompanied by the function name that characterizes them, e.g. V(dominant). Mathematically, one could subjectively catalog every possible unique result xy and thus define each function as either $x()$ or $y()$. Despite the subjectivism, everyone seems to agree on the result of the sequence $V - I$ which they call cadence. From this result, which produces a sense of finality, two of the most important functions, $V()$ as dominant and $I()$ as tonic, can, according to the above, be defined. Thus, dominant is defined as the function-chord $V()$ which, when followed by the tonic chord I, results in a finality with a cadence $V(I) = V - I$. Tonic is defined as the function-chord $I()$ which, when preceded by the dominant chord V, results in the ending with cadence $I(V) = V - I$.

Scale change occurs when the scale to which the chord functions refer is changed. If there is a desire for an extended stay on a chord function, that function can be considered as a temporary scale region and subsequent functions refer to it. Then we say, for example, if we wish to dwell on the dominant V, that we rove in the region[28] of the dominant and the results of the progressions

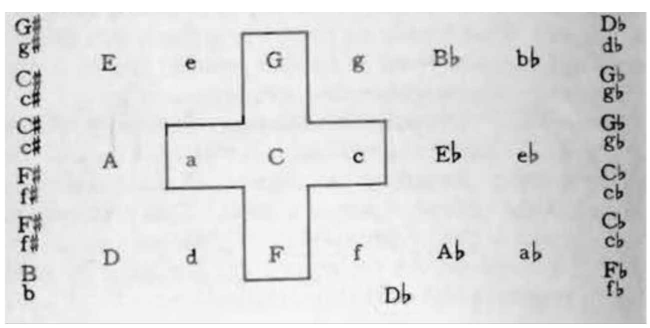

Figure 8: Part of page 20 from the book Structural functions of harmony by Arnold Schoenberg

from chord to chord are listed in it. If the dwell is of sufficiently long duration then we say that we changed the scale to V.

[28]The Schoenberg regions are shown in the figure 8, where in Roman numerals $C = I$, $F = IV$, $G = V$ etc.

2.6 Melodies

Each melody consists of frequency variations and each frequency is identified with its simple trivial tonality. Since each change in trivial tonality is considered an energetic progression, is it possible for these frequencies to be classified under and support a particular tonality, so that we can say that we have a melody in, say, C major? I haven't dwelt much on this question, but I believe so. But we have to make the following assumption.

The standard progression, in the nearest tonality, is *unidirectional* and corresponds to the musical intervals *tone-tone-halftone*[29]. Therefore, a melody around tonality $[C]$ will use tones of standard progression from the reference tonality $[C]$ to $[F]$, i.e. $[C]-[D]-[E]-[F]$, and tones of standard progression from $[G]$ to the reference tonality $[C]$, i.e. $[G]-[A]-[B]-[C]$.

The unidirectionality of the standard progression could be justified as an attempt to resemble the progression of major chords, considering the intermediate tones of the standard quartet as merely passing tones from one chord to another. Then the unidirectionality of the standard progression would correspond to the strong recommendation, *to almost always use energetic progressions*, expressed in terms of melodies. In any case, the above justification is incomplete. It answers why, but leaves open the question of how unaccompanied and harmonically accompanied melodies are equated. Nor does it answer the question of how the middle trivial tonalities, of unaccompanied standard melodic progression, lose this property and are counted as passing tones, that is, something outside the structure in which they participate.

Leading-tone is the tone before the halftone of the standard progression. The hearing of leading-tone by the listener is perceived as an announcement of the end of the wandering, resulting in the anticipation of a return to the "safety" of the middle tonal reference. In the above examples of standard progression, *provided that the reference tonality $[F]$ or $[C]$ has been previously established*, the hearing of the frequency, $[E]$ or $[B]$ respectively, announces the end of the melodic wanderings and strongly evokes the anticipation of $[F]$ or $[C]$ respectively, for confirmation.

The establishment of the reference tonality cannot be achieved only by using the leading tone. Sufficient use of the scale tones is required to establish a sense of the beginning of an adventure away from the reference tonality and a safe return to it. The first two measures of the figure 9 are, in my opinion, a strong indication that standard progression and the use of leading-tones, $[B]$ and $[E]$, are not enough to establish a sense of integration. It "does not sound" like the standard descending scale is sufficient either. But in the third measure, which is the ascending C major scale, we move away with the standard progression

[29] At least this is convenient as far as the Ionian mode of the diatonic scale.

Figure 9: Scale C major

sufficiently to touch the tonality $[F]$ and are approached by the equally distant tonality $[G]$, so that the leading-tone $[B]$ marks the end of the wandering and evokes the anticipation of the reference tonality $[C]$, which we, as listeners, accept with a sense of justification.
The class $[B] = [15/8]$ is the most distant class that can support the reference class $[C]$, but it is called *under* leading-tone if the frequency $[Db] = [C\#] = [17/16]$, which we excluded from the supporting tones of $[C]$ in the 17 page, is also used as *upper* leading-tone.
Note that continued use of the standard progression causes a change in scale or tonality reference according to the cycle of fifths. By placing $[C]$ between two standard progressions, the sequence $[G - A - B - (C] - D - E - F)$ ought to be treated roughly like a circle, nearly identifying its ends, as would be the case in a spiral. To establish the tonality $[C]$ there must be a sense that after the standard $C - D - E - F$ distance progression, we are not going off-scale to the $[Bb]$ major, but approaching through the standard $G - A - B - C$ approximating progression. The use of the leading-tone $[B]$, of the standard $G - A - B - C$ progression of the $[C]$ scale, protects against the assumption that $[F]$ can become the reference tonality. It does not, however, prohibit $[G]$ from being considered as the reference tonality, in the sequence $[D - E - F\# - (G] - A - B - C)$. For this reason I believe that, in addition to the leading-tone $[B]$, the use of $[F]$ can also play an important role in establishing the $[C]$ scale.

The general recommendation for melodies is that, ascending or descending, small changes of tone or half-tone are preferred to larger changes. Greater detail is a matter of voice leading, which I have not dealt with adequately, but also seems a little too remote in relation to the considerations developed herein. However, an attempt to link tonalities and melodies, since they too consist of trivial tonalities, is shown in figure 10. The first beat of the first measure is a

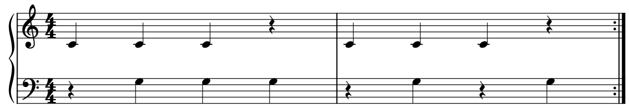

Figure 10: Simple tonality and melody connection.

melody (trivial tonality) C. In the second and third beats the trivial tonality becomes a chordal tonality but remains the same, so there is no tonality progression. On the fourth beat we have *deconstruction* of the chord, with a change in tonality from C to trivial melody tonality G, so passive progression occurs. In the second measure the situation is different. From the second beat to the third, the tonality does not change but becomes trivial melody tonality. Thus, from the third beat to the fourth, the tonality progression is considered active, since it is a trivial tonality progression. Given that chords contain major and minor third intervals, the above perspective would recommend tone or half-tone melody steps to avoid passive progressions, such as that of the first measure of the figure. In any case, the fact that the presence of a rest in the bass of the third beat seems to transform a passive progression into an active one is a warning alarm regarding the contrast between harmonic and melodic progressions.

The contradiction between harmonic and melodic progressions probably needs tools of dialectical philosophy to be approached. We have to accept the existence of opposites axiomatically. Chords desire leaps of tonality but, conversely, when they degenerate into trivial tonalities they desire steps of second. Unaccompanied melodies desire second steps but, on the contrary, when they are composed in chords, they desire, at the same time, leaps in the tonalities they create. The synthesis of the opposite desired progressions is the composer's creation. Its main tool is the use of first inversions which is palpated on the 39 page. The whole idea of harmonic progression is a synthesis of opposites. We don't want our chords to be deconstructed, and we consider any non-passive progression to be active, so we only want to add frequencies, but usually no more than three so that the tonality of the chord is not taken away from them. Any continuation with a "foreign" chord is by definition an active progression, but we limit ourselves not only to using frequencies in our scale but usually in close relation to the previous chord.

2.7 Chords and scales

2.7.1 Umbilic torus

The umbilic torus will eventually become, as a special circumstance of the torus, the object on whose surface the classes of tones and tonalities are placed. Let us lift the $G - C - F$ axis slightly from the plane of the paper in the figure 5 and plot the support of the counterparts of $[E]$ by the counterparts of $[G\#]$, respecting the identifications of $[= E]$ and $[= G\#]$. The twelve tonality classes (or notes if you like) are shown as points on the prism of the figure 11. The prism appears to have three flat surfaces, with all the tones of the major scale (page 24) belonging exclusively to one of them. It is worth noting that the major triads intersect the prism vertically, their classes support each other, and each "side" belongs to a different surface(?), hence to a different scale[30]. More importantly, it is worth noting that the upper three classes $[Bb], [F\#], [D]$ are identical to the lower ones. If we imagine the prism to be very flexible, we can bend the upper part of it, so that the upper side is placed opposite the lower side, and by turning it, by one side, the same classes, the upper and the lower, are identified. Then the twelve tonality classes, the notes if you like of the equal

[30]This is why they are particularly useful for scale changes. For the same reason, because each "side" belongs to a different scale, diminished sevenths are also useful.

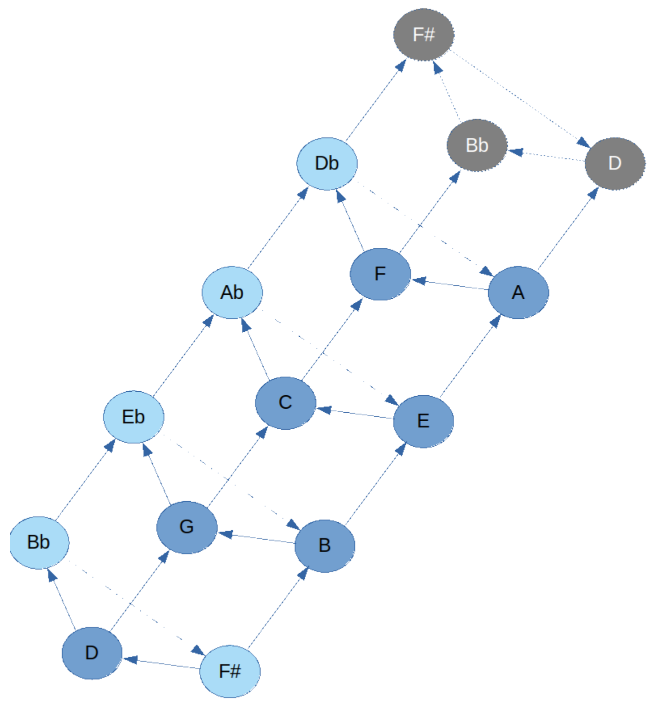

Figure 11: Prism of tones & tonalities (Developing a torus)

temperament system, are twelve points on the surface of this three-dimensional solid, and *we need no further identification*.

The remarkable thing is that there are no three surfaces that could differentiate the scales into three categories, depending on which surface of the three they belong to. *The curved prism*, the torus we assumed at the beginning of the chapter, has only one surface. Imagine that you are on the surface of the $[C]$ major to the right of the prism. Moving upwards you will exit from the edge $Bb - D$, that is, from below, to $Bb - D$ on the left side of the prism, without crossing its longitudinal edges[31]. If you continue upwards you will exit from edge $F\# - Bb$, that is, from below, to $F\# - Bb$ of the lower side of the prism,

[31]The edges, from a mathematical point of view, are not required. But it is more descriptive to refer to a pure triangular intersection.

without again crossing its longitudinal edges. Continue upwards, for the last time, and you will exit from $F\# - D$, that is, from below, to $F\# - D$ of the right side of the prism from which you started. So all *scales are equivalent and belong to a single surface*, no matter that the prism - torus of tones and pitches, more like an illusion, seems to have three. This ring-like prism is called Umbilic torus. Its surface is described as *manifold*, as mentioned on the 16 page. The action of any set of tone classes is relative[32] and depends on the reference tonality with which the composer wants to associate them. For each tonality there is a local map, for a small area around it, which is precise, such as the one in the figure 15. However, it cannot be used for more distant relationships unless combined with neighbouring local maps. The best I have been able to surmise about the laws governing the union of local maps, covering the entire surface of the ring, is the function I quoted on page 19. The torus is shown in the figures 12 and

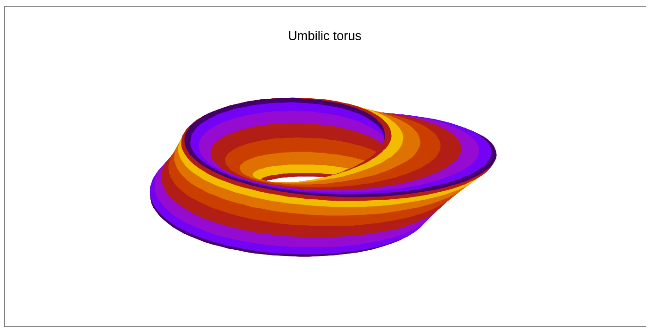

Figure 12: Umbilic torus (upper view)

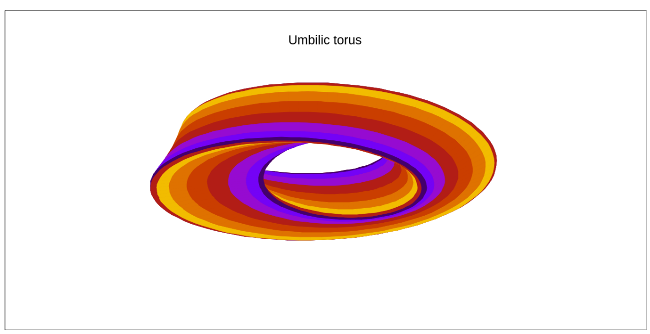

Figure 13: Umbilic torus (bottom view)

13. Its design in maxima will allow you interactive observation if you wish. To draw it simply run the commands

```
--> svn:7$ thr:3$ two:2$ two2:3$
```

[32]in correspondence with the theory of relativity

```
--> funx(u,v):=sin(u)*(svn+cos(u/thr-two*v)+two2*cos(u/thr+v));
--> funy(u,v):=cos(u)*(svn+cos(u/thr-two*v)+two2*cos(u/thr+v));
--> funz(u,v):=sin(u/thr-two*v)+two2*sin(u/thr+v);
--> draw3d(title          = "Umbilic torus",
xu_grid        = 100,
yv_grid        = 25,
axis_3d = false,
colorbox = false,
font = "Arial",
font_size = 20,
view           = [35,35],
surface_hide   = true,
enhanced3d =  3*(2+v/%pi)/2-floor(3*(2+v/%pi)/2),
parametric_surface(funx(u,v),funy(u,v),funz(u,v),u,-%pi,%pi,v,-%pi,%pi))$
```

If you happen to make it on a 3D printer[33] and you want to glue the twelve classes on it, cut a piece of rope as long as the dark area, shown as an edge in the picture. Allocate the classes onto the rope, according to the circle of fifths or in the order you see them in the figure 11, with the direction upwards, and glue it to the dark area where you measured it from, so that the edges of the rope will coincide. The classes in the rope will be distributed roughly as in the figure 14. For the

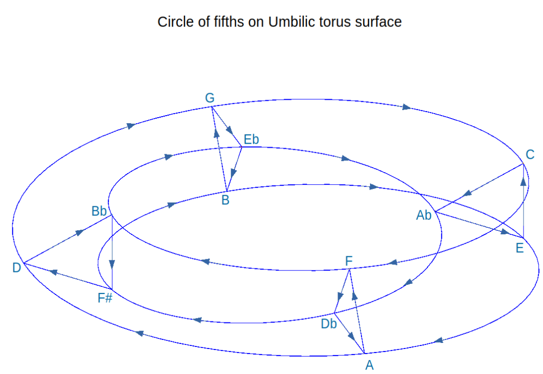

Figure 14: The circle of fifths on umbilic torus

remaining supports of the prism, join the classes of the four augmented triads

[33] It would be much more supervisory if there is and you chose transparent printing material. In theory, because the surface expansion is a long rectangle, it can be made with a strip of paper, but it might prove difficult to papercraft and unsupervisory because of its opacity. Temporarily glue the edge $BbF\#$ to a central guide cylinder and try to wrap the paper strip around it, turning it at the same time as in the figure.

with four ropes.[34].Let us remember that the arrows show the path of support for tonalities, from their tones to them, which is the path of the recommendation for active progression of tonalities. We are building on the foundations we have already established, and we are not, without sufficient reason, allowing what we have built to be deconstructed, by passive progression. Active progress, in this particular view of the figure 14, revealed clockwise. The triangular paths of the augmented triads are connected to the unique path of energetic progression of the cycle of fifths, according to right-hand rule. An incredible amount is revealed about harmonic progressions by mentally visualizing the path by which the *EC* edge creates the unique surface of the ring as it passes through the *AF*, *DBb*, *GEb*, *CAb*, *FDb*, *BbF#*, *EbB*, *AbE*, *DbA*, *F#D*, *BG* to end up in *EC*, where it started. Indicatively, the active progress of an animated major seventh, with the pitch supports in red, is shown in major7.gif[35].
The 14 figure seems to fulfill the goal of recording thoughts in this article, since it seems to be the means of visualizing mnemonic rules for harmonic progressions.

2.7.2 Minor chords

Before moving on to harmonic progressions, let us complete our thoughts on the place of minor scales and chords, explaining why there was no need for their participation in the structures we have examined so far. The ring structure was derived from the figure 4 generated from the tree structure, which was mentioned in the page 14. The smaller local map, which describes precisely the two branches of the main tree of the whole structure, is shown in figure 15. The tree was either constructed by starting the two unique branches *from C*, towards the tonalities *Ab* and *F*, which *C* supports, or constructed from the two unique supporting branches *G* and *E*, *to C*. In both cases the same local map results. Upon completion of all connections, in the first case, two additional branches supporting *C* will have been created, and, in the second case, two additional branches to the tonalities that *C* supports will have been created. The upper and lower triads of tonality classes are the minor and major three-tone chords.

The triads of tones are due to our tendency, because of the information economy mentioned on page 6, to project simple patterns onto the data we perceive. Just as we perceive familiar figures in clouds or assign names to constellations, so in the figure 15 we may perceive various triangles, squares, hourglasses and more, without any of them actually being designed or existing. The absolutely necessary drawn arrows are the existing connections only. From the five tonalities of the local map, one to five classes can be co-heared. If we refer to the whole ring, from one to twelve classes can be co-heared. The notes

[34]Every two opposing triads belong to the same plane and the two planes are perpendicular to each other.
[35]The violet line in the gif file, separating minor from major triad, is a supervisory line and does not correspond to a support.

of each chord that sounds can be represented on the ring as shining stars[36]. And that's all it is. A set of shining stars. They interact through various physical laws. The fact that we perceive constellation formation, equivalent to a chord sounding, is irrelevant to the physical laws of star interaction. Astrological charts are based on idealized representations with simple patterns and are of little help in overseeing the natural laws at work. If I had the misfortune to draw, either the figure 4 or the 5, with a division into minor and major chord triangles, the extra information of the ideal lines might have prevented my perception from recognizing the surface of the ring as a, constructible in three-dimensional space, modeling object.

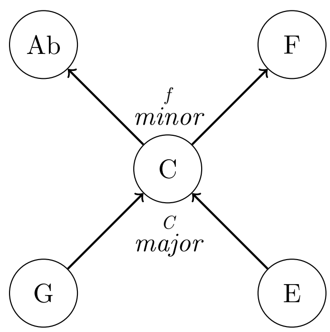

Figure 15: Minimum local map of tonality C

minor chords *are a direct result of the creation of major chords* and, equivalently, *major chords are a direct result of the creation of minor chords.* Major and minor chords are aspects of the same structure, and no additional information or conclusion can be drawn from the simultaneous use of their concepts. For this reason, in any calculus in which the simultaneous use of their concepts is deemed necessary, care is required to avoid the logical fallacy of circular reasoning. In the figure 15 are marked the upper and lower positions, where we can imagine the existence of the major and minor triads, as constellations of triangles. The left and right triangles are not known to be used.

2.7.3 Relative scales

In the figure 14, and more clearly in the related gif file, it is shown that we need three rotations to return to the starting point. This is because, before returning, we pass through all twelve major scales. If a composition makes extensive use of this path, it is probably classified as an atonal composition. We can imagine that it gives importance to "travelling", to changes. The points of safety or rest

[36]If I were to develop a ring-based equalizer, in addition to the stars, I would choose to illuminate the "heteroluminescent" hidden tonalities with less glow.

are temporary and are themselves subject to change[37]. In tonal music we impose the constraint of returning to the reference tonality after a single rotation.

Tonic music defines a tonality of reference, both as the starting point of the journey, and as the point of rest and end. We can of course temporarily change tonality regions, considering any intermediate tonality as a temporary starting tonality, but at the end the rest will be realized in the original reference tonality. In the figure 14 four *stations* of rest are distinguished at the positions of the intersections of the augmented triads. Referring to the viewpoint of this figure, I will name them according to the points on the horizon. *The means* of realizing the clockwise, energetic progression, travel of the tonalities are the chords. The simplest are the major and minor triads, which is why all other chords are, almost always, named with reference to some triad appropriate to the situation. The moment of arrival, of any chord, at one of the four stations is significant in the journey as it touches it with some frequency of her.

The major triad touches the arrival station with two tones, those which, on the page 18, we called root and quality tones of the chord. The figure 16 shows, with reference to the C major scale, a sequence of major triads from station to station. Each arrival is considered complete and stable, since the

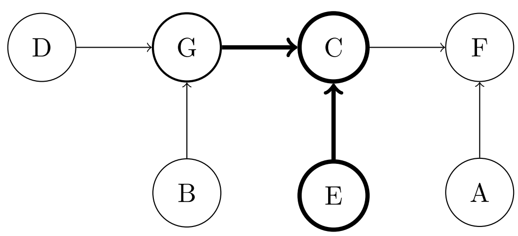

Figure 16: Energetic major triad progress

obvious tonality of the chord sounds as its frequency. Because the journey of major scales ends in a major triad, we can imagine major scales as someone who, however wondrous their journeys, prefer the stability and security of home.

> Ring reading hint:
> Only one major triad CEG arrives at each vertex C. Its plane $CEBG$ contains the edge $\vec{EC}$ of the station leading to it.

The minor triad touches the arrival station with only one tone, the one we called the root of the chord. The figure 17 shows, with reference to the A minor scale, a sequence of minor triads from station to station. Each arrival is considered incomplete and unstable, since the hidden tonality of the chord does

[37] I know nothing of officially accepted music theory, let alone atonal music. I am making assumptions

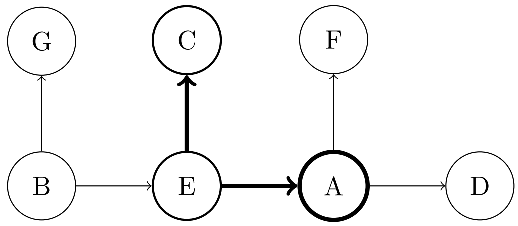

Figure 17: Energetic minor triad progress

not sound as its frequency. It is given the impression that, instead of ending in a major chord of obvious tonality, it stopped halfway, reaching its destination tonality, but realizing it only as a hidden tonality. Because the journey of minor scales ends in a minor triad, we can imagine minor scales as someone who, no matter how stable and secure their home, prefer to travel and therefore, upon their return, are always on the move.

> Ring reading hint:
> At each vertex A only one minor triad ACE arrives. Its plane $AFCE$ contains the edge $\vec{AF}$ of the station away from it.

Unitary move I consider to be an active move from one station to the next, along the basic 4/3 branch. Let us call it motion R, in the sense that the **R**oot of a chord becomes the source of the next chord. **Half move** I consider either the movement of a minor chord that, staying at the same station, completes its hidden tonality, thus becoming a major chord, or the movement of a major chord to the next station, but losing its obvious tonality, thus it becomes a minor[38]. Let us call it motion Q, in the sense that the **Q**uality of a chord, through a unitary movement, becomes the source of the next chord. The move from minor to major at the next station is *hyper energetic move*, because it requires a unit move to the next minor and half a move to convert it to major[39]. In *almost* all cases it is true that $R = Q \circ Q$ and that the hyper energetic motion is equal to $R \circ Q = Q \circ R = Q \circ Q \circ Q$[40]. For passive progressions, the opposites of the above are defined, so that their composition results in the identity $(-Q) \circ Q = I$.

[38] Perhaps the name half a move seems unfortunate, since it contradicts the fact that the connection corresponding to the R move is the smallest possible, in terms of ordering. But it shows that the hidden tonality would be obvious if its frequency were not acoustically missing.

[39] Of course it can be seen as a half move, a conversion to a major of the same station, followed by a whole major move.

[40] The exception is due to the adherence to the reference scale, expressed using only unaltered tones. For the C major for example, by necessity the move ($FAC \rightarrow BDF$) will be used instead of ($FAC \rightarrow BbDF$) or the ($BDF \rightarrow EGB$) instead of ($BDF\# \rightarrow EGB$). For the same reason, a consistent algebra cannot be constructed with the composition of movements. If it could, it would always hold $R \circ R \circ R \circ Q = I$, but unfortunately this depends on the order of movements, which should be such that we do not go out of scale.

The major surface is defined by the constraint to return to the reference tonality with a single rotation. With reference to the figure 18, let us define C major, of the eastern station, as the reference tonality, mainly because it gives the impression that the $\vec{EC}$ connection belongs to the half-cylinder surface, starting from the $\vec{BG}$ connection, of the northern station, and, passing through $\vec{EC}$, ending at $\vec{AF}$ of the southern station. In order not to change tonality, hence region or surface, the clockwise energetic progression must appear to belong to the surface of this cylinder, so that the "journey" lasts only one rotation. The simplest clockwise progression that gives this impression is to use three steps. From the eastern station to start for the southern, and thence to jump directly to the northern, from which we shall return to the eastern, from which we started. It's as if we've only used half a cylinder. We cannot cross the, at the opposite side from the start, western station with a major triad because every connection on this station has one end outside the C major scale. The only frequency class within the scale is $[D]$. To access the western station we have to use the minor triad DFA of the "up" surface $DBbFA$ and from vertex D we have to go to the vertex of the major triad GBD of the "down" surface $DGBF\#$. The sequence of moves Q, together with the necessary discontinuity at vertex D, expressed in Roman numerals, is as follows:

$$I - vi - IV - ii - V - iii - I$$

Notice that the chord vii^o does not belong to the surface through which the return to the tonic I is accomplished.
Lest we forget that the ring refers to the sets of frequencies that occur as chords and not to the constellations of triangles formed by the triads, let us note how identical and almost consonant are the sevenths of the tonic $\mathrm{I}^7 = C^7$ and the subdominant $\mathrm{I}V^7 = F^7$. They distributed their frequencies to adjacent stations. In addition, let us note the contrast of the seventh of the dominant $V^7 = G^7$, which distributes its frequencies over three stations, with the frequency $[F]$ being at the opposite station from that of its root[41].

The cadence is the most essential and perhaps the most objective element of the structure of tonality. By tradition it is interwoven with the use of the Leading tone. In major scales, such as the C major, the presence of $[B]$ in the dominant chord G consolidates the scale, against $[Bb]$, but it is not this property that causes the feeling of cadence in the $V - I$ movement. It is the property of $[B]$ as a Leading tone that causes this feeling. Thus, with reference to the A minor scale, the presence of $[B]$ in the E minor establishes the scale but, it is not sufficient to create the feeling of cadence with the $V - i$ movement. For this, we need the Leading tone $[G\#]$ that converts v to V major, i.e. E major.

[41] The tonality of G^7 seems to be the distant one, $(1/36)$, $[F]$. But the tonality progression $V^7 - I$, which is the same as the tonality progression $IV - I = F - C$, is not passive, because the chord C is not a strict subset of G^7.

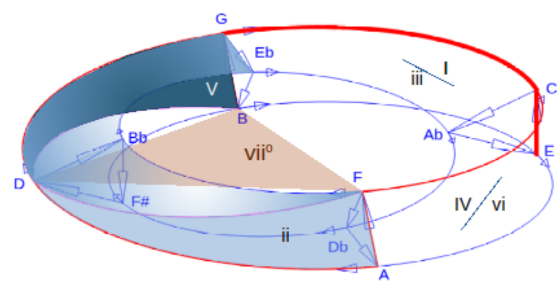

Figure 18: Relative C major and A minor scales

The minor surface is obviously the surface of the major C scale relative to it. But there is the difficulty that, after its establishment as a reference scale, for the realization of the cadence, a major chord for a dominant, which does not belong to the surface of the relevant major C, must be used from the A major. From the surface of the minor scale we have to "slip" temporarily on the surface of the parallel of the homonymous major and, instead of the major tonic of the latter, return to the minor tonic. The easiest way, after establishing e.g. A minor as a reference scale, is to continue from the south station, as if we were in A major, following the motion of the edge $A\vec{D}_b = A\vec{C}_\#$, doing at the vertex $[B]$, the opposite north station, what we did at $[D]$ of C major. From there to move to the edge $E\vec{A}_b = E\vec{G}_\#$ of $[E]$ major and, from this dominant, to arrive at the requested cadence, the A minor chord. The sequence of Q movements, one rotation to establish the minor and another, with as many R movements as possible, to use the major in the cadence, expressed in Roman numerals, is as follows:

$$i - VI - iv - VII - v - III - i\text{-IV-ii-V-i}$$

2.8 Harmony progressions

I believe I am now able to formulate simple mnemonic rules for harmony progressions, as I conceived them.

- As a principle, we always choose energetic progressions, unless passive progess serves a specific conscious purpose.
- *Non-change of station*, in my personal opinion, *is not clearly part of progression*, because tonality, whether hidden or obvious, is always present. If I had to characterize, as progression, the movement at the same station, I would characterize the movement $(-Q)$, from major to minor, as passive,

like $I - iii$ or $IV - vi$, and the movement (Q), from minor to major, as active, like $iii - I$ or $vi - IV$. Such moves can be made for various reasons, as in $V - iii - I$ because we want to weaken the feeling of cadence. To prolong our stay in tonality with $I - iii - I$. To reduce the speed of the movement, as in $I - vi - IV$, or to satisfy the need to complete an imperfect arrival by revealing its tonality. Whatever reasonable, in this context, we can think of.

- The angular speed of rotation is up to the composer's intentions. Sometimes there will be a desire to remain in a certain tonality or in one of the four stations, sometimes fast passages will be chosen, sometimes jumps to the opposite station, sometimes a retreat.

- *Scale progress is always active.* Scales are not chordal co-hearing, so they can only be assigned to trivial tonalities.

- The imitation of the movements of the universally accepted cadence is almost always logical, especially if we consider the appropriate tonality for the imitation of the movement as a temporary reference tonality, as in the case of secondary dominant chord .

- *Dissonances* are usually created either by delaying a melody to follow the harmonic change or, more rarely, by delaying the harmony to follow the advancing melody. This is usually how they are prepared. Let the dissonance be considered a mismatch of the homonymous tonality of the chord to which it seems to have been added. It is usually resolved in a consonant chord of the next station, as in the case $V^7 - I$. Despite the scepticism of the very small submultiple case of the 19 function, several times the resolution remains in the tonality that the dissonant chord seems to create. Without skepticism, the latter claim may lead to irregular resolution, as in the case $V^7 - IV$ [42] or the $vii^o - I$ [43].

Inversions do not affect tonality, either obvious or hidden. Theoretically, however, in the first inversion X^6, the root of the chord is not at the lowest frequency, as would be expected. In the second inversion X^6_4 nor its quality is lower than the source. Normally with the inversions we deal only at the *voice leading* level, but it doesn't seem to hurt if we classify them as intermediate states mainly of arrival at a station. I therefore classify them as $\mathrm{X}^6_4 \to \mathrm{X}^6 \to \mathrm{X}$, without any other stronger argument in its favour.
As mentioned in the limits of self-similarity, on page 9, the supporting role is not consistent with the position of the lower frequency of the chord. The bass frequency fits the role of the supported obvious tonality. Let us therefore

[42] The chord $GBDF = G(7, 11, 2, 5) = [1/36, [7, 11, 2, 5], 5]$, with the minor submultiple 1/36 of $7 = G$, has tonality $5 = F$.

[43] The chord $BDF = G(11, 2, 5) = [1/60, [11, 2, 5], 0]$, with the small submultiple 1/60 of $11 = B$, has tonality $0 = C$.

assume that the bass frequency of the inversions heralds the transition to it[44], especially the bass frequency of the second inversion. We must have realized, by now, that, in any scale, the chords that have obvious tonalities are only the three major chords. With reference to the related C major and A minor scales, these chords are G, C and F. We move, from station to station, always energetically, as vehicles always move forward. Before moving backwards, we put the car in back gear and the back lights come on forewarningly. I like to imagine that this function, forewarning passive transition, is especially the function of the second inversion. Let's look at the two inversions.

The second inversion X_4^6 minor chords (such as EAC), because it would announce the movement $(-R)$, instead of towards the normal minor chords (such as EGC), towards the obvious tonality major chords (such as $EG\#C$), is not used, because the latter are out of scale. Passive progression is not consistent with an effect as active as scale change. The same is true of the second inversion of the dominant. The second inversion of the tonic, however, can be used to extend the cadence, as $I_4^6 - V - I$[45]. and of the subdominant, for an extended stay in the tonic region, such as $I-(IV_4^6)-I$.

The first inversion is not used to announce passive progress for two reasons. The pre-announcement of passive progression $(-Q)$ from major ACF to major $AC\#E$ is not valid because the latter does not belong to the scale and, at the same time, the progression $(-Q)$ from minor CEA to major CEG, with two notes in common between the chords, is short and does not need pre-announcement of one of them. The first inversion is mainly used for the dialectical synthesis of the contrast between harmonic and melodic progressions, mentioned on page 27. For example, in the progression $IV-ii^6 - V$ the desired energetic progression with tonality jumps is combined with the also desired melodic progression of the bass by a step of minor or major second.

The scales and chords on the ring structure are revealed in figure 19. The twelve notes are distributed, in groups of three, in the four stations.
Let's focus on the lateral surface of the ring, as we see it developed by the motion of the support arrow $\vec{EC}$. The surface generated by the arrow corresponds to the relative C major and a minor scales.

The arrival of major chords at a station, as from the G major of the north station to the C major of the east station, characterized by the synchronous arrival of both ends of the arrow $\vec{EC}$ and the upper movement curve $\vec{GC}$. Because the notes are interwoven with the apparent tonalities and these with the major chords, **major chords are mapped to a section of a unit movement curve from a previous station to the homonymous tonality of the arrival station**, thus the C major chord corresponds to the $\vec{GC}$ curve. Because

[44]It does not make sense to announce openly a tonality that will be realized as hidden. Whether Leading voice reasons refute the claim, I don't know.

[45]Using I_4^6 in this way is often denoted by Cad_4^6, such as $Cad_4^6 - V - I$

the scales are mapped to their tonal chord, *the major scales are mapped to the same part of a unit-movement curve* as well.

The arrival of minor chords at a station, as from the E minor of the eastern station to the A minor of the southern station, characterized by the arrival of only the A end of the lower $\vec{EA}$ movement curve. But the curve already corresponds to the A major chord. But we have seen, on the 35 page, that we need half a Q move in order for the root A of the minor chord to move to the root F of the major chord. This move corresponds to the support arrow $\vec{AF}$. Thus, we assign the minor chord to half the movement corresponding to the station edge, thus **minor chords are assigned to a station side arrow which starts from the homonymous tonality**. In the example, the a minor chord corresponds to the arrow $\vec{AF}$. Because scales are mapped to their tonic chord, *the minor scales are assigned to the same arrow of the arrival station* also.

In the figure 19 were marked: the minor chords next to the corresponding edges of the stations, the twelve tonalities at their vertices and the major chords were omitted, as the self-evident parts of curves ending at the homonymous vertices. In this way the structure appears complete.

The structure of the umbilic torus contains

the twelve notes, as the vertices of the four triangular arrival/departure stations,

the twelve major chords, corresponding to all twelve major scales, as sections of a unit movement curve between stations, and

the twelve minor chords, corresponding to the twelve minor scales as edges of the four triangular arrival/departure stations.

The structure reading depends on our object of interest. For the progression and linking of chords of the same tonality we may prefer to read them from the figure 18, where "noted" is the progression of Q major-scale movement sequences, with the corresponding minor-scale progression simply stated in the text as self-evident. For scale change and linking, as for the progression of chords that "flirt" with scale change, we might prefer the scheme 19. The following emerges from the reading:

- Each tonality ends with the homonymous major and the relative minor scale or chord *of the previous station*, e.g. the tonality F ends with the F major ($\vec{CF}$) and the a minor ($\vec{AF}$) [46].

[46] The greatest difficulty in this fact is the contrasts of progress Q from a major to its relative, next-station, minor scale, which while changing station (so there is removal) does not change key (so there is not so much removal), and progress from a minor to the same-station, major scale, which while not changing station (so there is not so much removal) changes key (so there is removal).

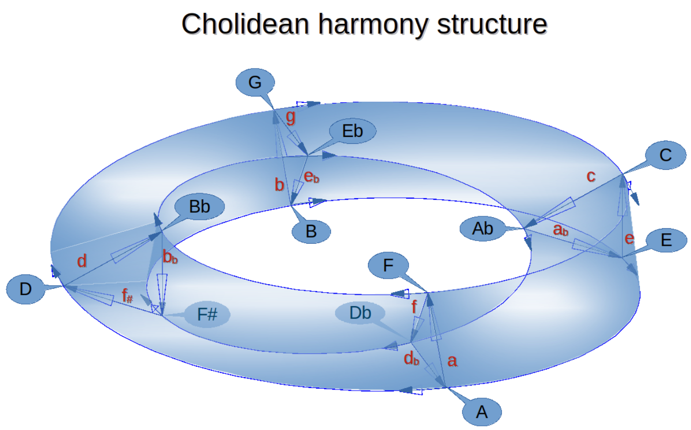

Figure 19: Cholidean harmony structure

- Major arrow connections are made lengthwise, e.g. from C major ($\overrightarrow{CF}$) to F major ($\overrightarrow{FB_b}$).
- The minor arrow connections are made in parallel, e.g. from a minor ($\overrightarrow{AF}$) to d minor ($\overrightarrow{DB_b}$).
- The half move Q is vertically left, e.g. from a minor ($\overrightarrow{AF}$) to F major ($\overrightarrow{FB_b}$) or from C major ($\overrightarrow{CF}$) to a minor ($\overrightarrow{AF}$).
- The half move $(-Q)$ is done according to the pattern $(-R) \circ Q$ e.g. from a minor ($\overrightarrow{AF}$) to C minor ($\overrightarrow{CF}$) or from F minor FB_b to a minor ($\overrightarrow{AF}$).
- The connection of major scales to their parallel minor scale is done vertically right, e.g. from C major ($\overrightarrow{CF}$) to c minor $\overrightarrow{CA_b}$ and vice versa for connecting minor to parallel major. This follows from the role of the common, necessarily major, dominant of the parallel scales. This is not the case in chord connections. The difference lies in the fact that in scale conjunctions, the preparation for the use of the common dominant is inserted by chordal succession, whereas chordal conjunctions are understood to mean the simple succession of chords.
- As a reading summary we can consider the following short progressions: From the C vertex $\overrightarrow{GC}$ of the major $\overrightarrow{GC}$ scales we go to the minor c and e

My point is that in, always energetic, scale progressions it is not the comparison of arrival and destination that matters, but the path we choose. Thus, although it is true that $QQ = R$, the path of two half-moves $Q \circ Q$ is larger than the unitary move R.

of the surfaces on either side of the $\vec{GC}$ curve. From the arrows $\vec{AF}$ of the minor scale surfaces a we pass to the major curves $A = \vec{EA}$ and $F = \vec{CF}$ of the edges of a.

The use of symmetrical chords is useful in switching scales. Two cases can be distinguished.

The use of augmented triads is synchronous arrival on all three edges of the station. The arrival on edge, however, characterizes the arrival of a major chord. Thus, the arrival CEA_b is regarded as an arrival on any of the three majors corresponding to the vertices, with the result that, from this station, we may, depending on our choice, continue on any of its three surfaces or vertices.

The use of diminished sevenths considers that there is synchronous arrival at all stations. If we remember that, given the importance of obvious tonality, even minor chords can be seen as rootless seventh-major chords, each vertex of the diminished seventh is seen as a quality major chord that arrived at the vertex station. Thus, $BDF_{\#}A$ is regarded as arriving at any of the four majors whose quality corresponds to any of the given vertices, so that, e.g., from the station with B as the quality of GBD, we continue as if we were on the surface of the relevant C majors and a minors and have just arrived at G majors [47].

Examples of scale or chord progression I will ultimately give very little, because I think that when and if I decide to reread harmony theory, it is worthwhile to do so with the visualization aid of the figure 19.

The intermediate scales is a useful tool for their transition. Let's look at an example. From the C major to the d minor scale the shortest way seems to be via F or a. Notice that $\vec{a}$ and $\vec{d}$ are "parallel". I mention this example because of the possibility of using intermediate scales on the path to the scale we are targeting. Intermediate scales do not need to be fully established, rather the opposite, since they are simply used as an aid in our journey. Noting only the scales, the short transitions in our example are $C - F - d$ or $C - a - d$.

The Neapolitan Sixth could be described as an elaborate rotation - return to the C major, performed by a continuous alternation of major curves and minor surfaces, such as $C - F|f - D_b - F_{\#}|f_{\#} - b - G - C$. The notation $F|f$ is used to show that if C is taken as the common dominator of parallel scales, we can arrive at any of them. From the arrow of the minor $\vec{f}$ we immediately realize that the D_b chord is the VI of the f minor. The f minor scale, however, as an intermediate scale, does not need to be established. It can be used simply

[47]This consideration coincides with the result of the $G()$ function of the 19 page, which counts $[C]$ as one of the tonalities of $BDF_{\#}A$, so the diminished seventh is considered an alternative to the dominant of the C major scale.

to glue - progress $V - VI$ of its degrees, without its tonic appearing. Thus our sequence becomes $C - D_b^6 - F_\# | f_\# - b - G - C$ [48]. As C is the dominant of $F|f$ so D_b is the dominant of $f_\#$ minor. Since our purpose is to pass to G as the dominant of C, the note $[F_\#]$, as the leading tone of the G major, would strongly oppose its role as dominant. The $V - (i) - iv$ progression of $f_\#$ minor scale, with $[b]$ as its subdominant, seems reasonable, but $[f_\#]$ is also present in the subdominant. The solution is to compose the progression with a movement Q, which does not change station or tonality anyway, since $[G]$ is a hidden tonality of the chord b minor. Thus, with reference to the $f_\#$ minor scale, the sequence $D_b^6 - F_\# | f_\# - b - G$ becomes $V - (i - iv \circ Q) = V - bII = D_b^6 - G$.
D_b^6 is called the Neapolitan sixth of the C major scale.
If I observe D_b, with reference to the C major scale, and compare G, with reference to the $f_\#$ minor scale, I reasonably assume that, without inversion, *the G chord is the Neapolitan chord of the $f_\#$ minor scale* [49].

Epilogue

Although the purpose of writing the present was to systematically record, before I forgot, what I thought I had understood about musical harmonic progress, I have finally completed a wonderful, for me, mathematical journey. It is striking how, from two basic supporting branches, the minimal approximations of the frequencies of the twelve-tone system and the manifold, which is a tool used extensively in general relativity, arose the three-dimensional surface on which the twelve frequencies and twenty-four chords and scales can be placed, visualizing their basic relationships.
Even if the claims concerning the relationships between chords and scales are questioned, and indeed it would be much more useful to question them, I believe that the three-dimensional surface of the harmonic structure will remain an objective visualization tool for any musical theory on the twelve-tone system.
In the above sense, it is not necessary to read the whole article, but only the construction of the surface and the notation of the twelve tones on it. Reading this will be of interest to those who are curious as to how the manifold came about from scratch, almost on its own, and those who understand that changing the basic conditions of its creation, such as using three basic branches instead of two, would probably also bring about the need to modify it, perhaps into a three-dimensional object with more stations and/or edges of each station, than the two-dimensional surface constructed.

Thanks in advance to all readers for their interest.
Dimitrios Cholidis

[48] The first inversion of D_b was used to emphasize the role of f, whose tonic was not used.

[49] It doesn't make clear sense, but it seems that the sequence $C - D_b^6 - G$ is a succession of two Neapolitan chords, first the C major, in first inversion, and then the "antidiametric" $f_\#$ minor scale, without inversion.