* Solving Homogeneous Recurrence Relations: (First and Second order).

Let $f: N \to R$ and $a_n = f(n)$ for all $n \in N$.
Suppose, we know $a_1 \dots a_k$ and for
$a_n = F(a_{n-1}, \dots, a_{n-k})$ ← is a recursively defined sequence.

- A recurrence relation is linear if 'f' is linear
  or $a_n = f(a_{n-1}, \dots a_{n-k}) = s_1 a_{n-1} + s_2 a_{n-2} + \dots + s_k a_{n-k} + f(n)$
  where, s_i, $f(n)$ are real numbers.

- A linear recurrence relation is homogenous - if $f(n) = 0$.

- A recurrence is of order 'k' if $a_n = (a_{n-1}, \dots, a_{n-k})$

* First order linear recurrence relation (RR) [Degree 1]

Let $a_n = s_1 a_{n-1}$ — be 1st order R.R. with $a_1 = k$

Then
$$a_2 = s_1 a_1 = s_1 \cdot k$$

$$a_3 = s_1 a_2 = s_1 \cdot s_1 \cdot k = s_1^2 k$$

$$a_4 = s_1 a_3 = s_1 \cdot s_1^2 k = s_1^3 k$$

In general
$$\boxed{a_n = s_1^{n-1} k}$$ — Closed form solution

Ex: (1) If $a_1 = 4$ and $a_n = \frac{a_{n-1}}{2}$ for $n \geq 2$.

Solve for a_n.

$\Rightarrow$ $a_n - \frac{a_{n-1}}{2} = 0$.

Characteristic equation. $r - 1/2 = 0$

$r = 1/2$

$a_n = c_1 \left(\frac{1}{2}\right)^n$

$a_1 = 4$.

$c_1 \left(\frac{1}{2}\right)^1 = 4$.

$\boxed{c_1 = 8}$

$a_n = 8 \cdot \frac{1}{2^n} = 2^3 \cdot \frac{1}{2^n}$

$$\boxed{a_n = \frac{1}{2^{n-3}}}$$

OR

$a_n = (s_1)^{n-1} \cdot k$.

$= \left(\frac{1}{2}\right)^{n-1} \cdot 4$.

$$\boxed{a_n = \frac{1}{2^{n-3}}}$$

$\Rightarrow$ General steps to solve homogeneous recurrences.

1. find characteristic equation for recurrence relation
2. use algebra to find roots. Let roots are r_1 and r_2 (for 2nd order) & r (for first order)
3. $a_n = c_1 r_1^n + c_2 r_2^n$ (second order) $a_n = c_1 r^n$ (first order)
4. use initial conditions to find c_1 & c_2.

Ex (2) $a_n = 3a_{n-1}$ $a_1 = 2$

$r - 3 = 0$ $\boxed{r = 3}$

$a_n = c \cdot 3^n$

$a_1 = c \cdot 3^1 = 3c = 2$

$c = 2/3$

$a_n = \frac{2}{3} \cdot 3^n$

$\boxed{a_n = 2 \cdot 3^{n-1}}$

* Second order linear recurrence relation (RR) [Degree 2]

Let $a_n = s_1 a_{n-1} + s_2 a_{n-2}$ with $a_1 = k_1$ & $a_2 = k_2$

— be 2^{nd} order RR.

We take guess that solⁿ have form - $a_n = c \cdot r^n$

$c \cdot r^n = s_1 \cdot c \cdot r^{n-1} + s_2 \cdot c \cdot r^{n-2}$

dividing by r^{n-2} - $\boxed{r^2 - s_1 r - s_2 = 0}$

↳ characteristic equation

We have 3 possibilities for roots of this equation -

① If Roots are real and distinct (a, b) then.
solution will be - $a_n = c_1 a^n + c_2 b^n$.

② If Roots are non-distinct (a, a) then
solution will be - $a_n = (c_1 + n \cdot c_2) a^n$

③ If Roots are complex conjugate $(a+bi, a-bi)$ then.
$a_n = c_1 (a+bi)^n + c_2 (a-bi)^n$

Ex ③ $a_n = 5a_{n-1} - 6a_{n-2}$ $a_0 = 1$ $a_1 = 0$

Characteristic equation -
$r^2 - 5r + 6 = 0$
$(r-3)(r-2) = 0$
$r = 3$ and $r = 2$

$\therefore a_n = c_1 3^n + c_2 2^n$

∴ prove that $2^n, 3^n$ is also the solution of this RR.

$a_0 = c_1 \cdot 3^0 + c_2 2^0 = c_1 + c_2 = 1$
$a_1 = c_1 \cdot 3^1 + c_2 2^1 = 3c_1 + 2c_2 = 0$

solving these we get

$\boxed{c_1 = -2}$ $\boxed{c_2 = +3}$

$\therefore \boxed{a_n = 3 \cdot 2^n - 2 \cdot 3^n}$

Ex (4) $a_n = 2a_{n-1} + 3a_{n-2}$ $a_1 = 3$ $a_2 = 7$

Characteristic eq- $r^2 - 2r - 3 = 0$

$r = 3, \quad r = -1$

$$a_n = c_1 3^n + c_2 (-1)^n$$

$a_1 = 3 \Rightarrow c_1 \cdot 3 + c_2(-1) = 3$

$a_2 = 7 \Rightarrow c_1 \cdot 3^2 + c_2(-1)^2 = 7$

$\left.\begin{array}{l} 3c_1 - c_2 = 3 \\ 9c_1 + c_2 = 7 \end{array}\right\}$ — Solving these two simultaneously

$\boxed{c_1 = \frac{5}{6}}$ and $\boxed{c_2 = -\frac{1}{2}}$

$$a_n = \frac{5}{6} 3^n - \frac{1}{2} (-1)^n$$

$$\boxed{a_n = \frac{5}{2} 3^{n-1} + \frac{1}{2} \cdot (-1)^{n-1}}$$

Ex (5) $a_n = 6a_{n-1} - 9a_{n-2}$ $a_1 = 2$ $a_2 = 5$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$ $r = 3, 3.$

$$a_n = (c_1 + n \cdot c_2) 3^n$$

$a_1 = (c_1 + 1 \cdot c_2) \cdot 3 = 2 \qquad 3c_1 + 3c_2 = 2$

$a_2 = (c_1 + 2 \cdot c_2)\, 3^2 = 5 \qquad 9c_1 + 18c_2 = 5$

$\boxed{c_1 = \frac{7}{9}}$ and $\boxed{c_2 = \frac{-1}{9}}$

$$a_n = \frac{7}{9} 3^n - \frac{1}{9} n \cdot 3^n$$

$$\boxed{a_n = 7\, 3^{n-2} - n \cdot 3^{n-2}}$$

* **Non-homogenous recurrence equation**

A recurrence relation -

$a_n = s_1 a_{n-1} + s_2 a_{n-2} + \dots s_k a_{n-k} + f(n)$

is non-homogeneous iff $f(n) \neq 0$

Terminology

The equation -

$a_n = s_1 a_{n-1} + s_2 a_{n-2} + \dots + s_k a_{n-k} + f(n)$

↳ NHRR [Non-Homogeneous Recurrence Relation]

$a_n = s_1 a_{n-1} + s_2 a_{n-2} + \dots + s_k a_{n-k}$

↳ AHRR [Associated Homogeneous Recurrence Relation]

General Solution of NHRR = General Solution of AHRR (a_n^h) + Particular solution of NHRR. (a_n^p)

[We know how to find] [In some limited cases we know how to find]

Note :-

		Non-Homogeneous term $f(n)$	Trial Solution a_n^p
Constant	1)	$f(n) = a$	$a_n^p = a$
Polynomial	2)	$f(n) = n$	$a_n^p = An + B$
		$f(n) = n + a$	$a_n^p = An + B$
		$f(n) = n^2 + n + a$	$a_n^p = An^2 + Bn + C$
Exponential	3)	$f(n) = 2^n$	$a_n^p = A \cdot 2^n$
		$f(n) = -5^n$	$a_n^p = A \cdot 5^n$
		$f(n) = e^n$	$a_n^p = A \cdot e^n$
Combination	4)	$f(n) = n + 2^n$	$a_n^p = (An+B) + C2^n$
		$f(n) = (n+a)2^n$	$a_n^p = (An+B) \cdot 2^n$
		$f(n) = n + \sin an$	$a_n^p = (An+B) + C \cdot \cos an + D \sin an$
Trigonometric	5)	$f(n) = \sin an$	$a_n^p = A \cos an + B \sin an$
		$f(n) = \cos an$	$a_n^p = A \cos an + B \sin an$
		$f(n) = \sin an + \cos an$	$a_n^p = A \cos an + B \sin an$

If $a_n^p = a \Rightarrow$ fails

then $a_n^p = n \cdot a$. If this fails too

$a_n^p = n^2 a$.

Ex (1) $a_n = 2a_{n-1} + n - 7$. Solution with $a_1 = -3$

Step 1 find General solution of AHRR.

$$a_n = 2a_{n-1}$$

$$r - 2 = 0$$

$$r = 2$$

$$a_n^h = C_1 \cdot 2^n$$

Step 2 find particular solution of NHRR.

$f(n) = n - 7$.

Reasonable Trial solution is $An + B$.

$$An + B = 2\left(A(n-1) + B\right) + n - 7.$$

$$An + B = 2An - 2A + \cancel{2B}^{\,2B} + n - 7$$

$$An + B = (2A + 1)n - (2A - 2B + 7)$$

Equating 'n' & constant Term.

$$2A + 1 = A \implies \boxed{A = -1}$$

$$B = -2A + 2B - 7 \rightarrow \boxed{B = 5}$$

Thus $a_n^p = -n + 5$ → is particular solution of NHRR.

∴ General solution is.

$$a_n = a_n^h + a_n^p$$

$$= C_1 \cdot 2^n + -n + 5$$

$a_1 = -3$ $\quad -3 = C_1 \cdot 2^1 - 1 + 5 \Rightarrow \boxed{C_1 = -7}$

$$\boxed{a_n = \frac{-7}{2} 2^n - n + 5} \quad \text{or} \quad \boxed{a_n = -7 \cdot 2^{n-1} - n + 5}$$

Verification.

R.H.S $= 2a_{n-1} + n - 7$

$= 2\left[-\frac{7}{2} \cdot 2^{n-1} - (n-1) + 5 \right] + n - 7.$

$= -\frac{7}{2} 2^{n} - 2n + 2 + 10 + n - 7$

$= -\frac{7}{2} 2^{n} - n + 5 = a_n.$

$=$ L.H.S

Ex (2) $a_n = 2a_{n-1} - a_{n-2} + 1$ $a_0 = 2$, $a_1 = 5$

Step 1

$a_n = 2a_{n-1} - a_{n-2}$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r = 1, 1$

$a_n^h = (c_1 + c_2 n) 1^n$

Step 2

$a_n^p = A.$ — put in eq.

$A = 2A - A + 1$

$0 = 1 \longrightarrow$ fails.

$a_n^p = n \cdot A.$

$n \cdot A = 2(n-1)A - (n-2)A + 1$

$\cancel{nA} = 2\cancel{nA} - 2A - \cancel{nA} + 2A + 1$

$0 = 1 \longrightarrow$ fails.

$a_n^p = n^2 A.$

$n^2 A = 2(n-1)^2 A - (n-2)^2 A + 1$

$\cancel{n^2 A} = 2n^2 A - \cancel{4nA} + 2A - \cancel{n^2 A} + \cancel{4nA} - 4A + 1$

$2A = 1$

$\boxed{A = 1/2}$

$a_n = a_n^h + a_n^p.$

$= (C_1 + C_2 n) 1^n + \frac{1}{2} n^2$

$a_0 = C_1 + C_2 \cdot 0 + 0 = 2 \quad \boxed{C_1 = 2}$

$a_1 = (C_1 + C_2) \cdot 1^1 + 1/2 = 5 \Rightarrow \boxed{C_2 = 5/2}$

$$\boxed{a_n = \left(2 + \frac{5n}{2}\right) 1^n + \frac{n^2}{2}}$$

Ex (3) $a_n = 2a_{n-1} + n^2 - 12.$

Step 1 :- $a_n = 2a_{n-1}$

$r - 2 = 0$

$r = 2$

$a_n^h = C_1 \cdot 2^n$

Step 2 : $a_n^p = An^2 + Bn + C$

$An^2 + Bn + C = 2[A(n-1)^2 + B(n-1) + C] + n^2 - 12$

$An^2 + Bn + C = (2A+1) n^2 - 4An + 2A + 2Bn - 2B + 2C - 12$

$= (2A+1)n^2 + (-4A + 2B) n - 2B + 2C + 2A - 12$

Equating n^2, n & constant terms.

$A = 2A + 1 \Rightarrow \boxed{A = -1}$

$B = -4A + 2B \Rightarrow \boxed{B = -4}$

$C = 2A - 2B + 2C + -12 \Rightarrow \boxed{C = 6}$.

$a_n^p = -n^2 - 4n + 6$

$\therefore \boxed{a_n = C_1 \cdot 2^n - n^2 - 4n + 6.}$

Ex 4 $\quad a_n = 2a_{n-1} + 2^n$

Step 1 $\quad a_n^h = C_1 \cdot 2^n$

Step 2 $\quad a_n^p = A 2^n$

$A \cancel{2^n} = 2A \cancel{2^{n-1}} + 2^n$

$0 = 2^n \rightarrow$ fails as we cannot solve for A.

$a_n^p = A \cdot n \cdot 2^n$

$A \cdot n \cdot 2^n = 2\left[A(n-1) 2^{n-1}\right] + 2^n$

$A n \cdot 2^n = A(n-1) 2^n + 2^n$

$A\cancel{n} = A\cancel{n} - A + 1$

$\boxed{A = 1}$

$\boxed{a_n = C_1 \cdot 2^n + n \cdot 2^n}$

Ex ③ $a_n = 2a_{n-1} + (n^2 - 12)\cdot 5^n$.

Step 1 $a_n^h = C_1 \cdot 2^n$.

Step 2 $a_n^p = (An^2 + Bn + C)\cdot 5^n$

$$(An^2+Bn+C)5^n = 2\left[A(n-1)^2 + B(n-1) + C\right]\cdot 5^{n-1} + (n^2-12)\cdot 5^n$$

⋮

$$(An^2 + Bn + C) \cancel{5^n} = \left(\frac{2}{5}A + 1\right)n^2 + \left(-\frac{4}{5}A + \frac{2}{5}B\right)n + \left(\frac{2A}{5} - \frac{2B}{5} + \frac{2C}{5} - 12\right)$$

$A = \frac{2A}{5} + 1 \implies \boxed{A = 5/3}$

$B = -\frac{4A}{5} + \frac{2B}{5} \implies \boxed{B = -20/9}$

$C = \frac{2A}{5} - \frac{2B}{5} + \frac{2C}{5} - 12 \implies \boxed{C = -470/27}$

$$\boxed{a_n = C_1\cdot 2^n + \left(\frac{5n^2}{3} - \frac{20n}{9} - \frac{470}{27}\right)\cdot 5^n}$$

Imp. Lemma 1 :-

If $t_1(n)$ is a particular solution of
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f_1(n)$

and

If $t_2(n)$ is a particular solution of -
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f_2(n)$

then -

$t_1(n) + t_2(n)$ is a particular solution of -
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f_1(n) + f_1(n)$