Floating-point operations I

- The science of floating-point arithmetics
- IEEE standard
- Reference

  What every computer scientist should know about floating-point arithmetic, ACM computing survey, 1991

Why learn more about floating-point operations I

Example:

- A one-variable problem

$$\min_{x} f(x)$$
$$x \geq 0$$

- In your program, should you set an upper bound of x?
- x in your program may be wrongly increased to ∞

Why learn more about floating-point operations II

- What is the largest representable number in the computer ?
- Is there anything called infinity ?

Example:

- A ten-variable problem

$$\min f(x)$$
$$0 \le x_i, i = 1, \ldots, 10$$

Why learn more about floating-point operations III

- After the problem is solved, want to know how many are zeros ?
- Should you use

```
for (i=0; i < 10; i++)
    if (x[i] == 0) count++ ;
```

- People said: don't do floating-point comparisons

```
epsilon = 1.0e-12 ;
for (i=0; i < 10; i++)
    if (x[i] <= epsilon) count++ ;
```

How do you choose ϵ ?

Why learn more about floating-point operations IV

- Is this true ?

Floating-point Formats I

- We know float (single): 4 bytes, double: 8 bytes
  Why ?
- A floating-point system
  base β, precision p, significand (mantissa) $d.d \dots d$
- Example

$$
\begin{aligned}
0.1 &= \overbrace{1.00}^{3} \times 10^{-1} && (\beta = 10, p = 3) \\
&\approx \underbrace{1.1001}_{5} \times 2^{-4} && (\beta = 2, p = 5)
\end{aligned}
$$

  exponent: -1 and -4
- Largest exponent e_{max}, smallest e_{min}

Floating-point Formats II

- β^p possible significands, $e_{\max} - e_{\min} + 1$ possible exponents

$$\lceil \log_2(e_{\max} - e_{\min} + 1) \rceil + \lceil \log_2(\beta^p) \rceil + 1$$

  bits for storing a number

  1 bit for $\pm$
- But the practical setting is more complicated

  See the discussion of IEEE standard later
- Normalized: 1.00×10^{-1} (yes), 0.01×10^{1} (no)
- Now most used normalized representation

  $\Rightarrow$ cannot represent zero

Floating-point Formats III

the smallest number

$1.0 \times \beta^{e_{min}}$

- A natural way for 0: $1.0 \times \beta^{e_{\min}-1}$ preserve the ordering
- Will use $p = 3, \beta = 10$ for most later explanation

Relative Errors and Ulps I

3.14×10^0

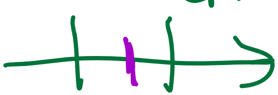
0.00159×10^0

- When $\beta = 10$, $p = 3$, 3.14159 represented as 3.14×10^0 0.159 unit

  $\Rightarrow$ error $= 0.00159 = 0.159 \times 10^{-2}$, i.e. 0.159 units in the last place

  10^{-2}: unit of the last place
- `ulps`: unit in the last place
- relative error $0.00159/3.14159 \approx 0.0005$
- For a number $d.d \ldots d \times \beta^e$, the largest error is

$d.d \ldots d \quad d.d \cdots (d+1)$



$$0.\underbrace{0 \ldots 0}_{p-1}\beta' \times \beta^e, \beta' = \beta/2$$

floating point number

Relative Errors and Ulps II

- Error $= \frac{\beta}{2} \times \beta^{-p} \times \beta^{e}$ ex. $1 \times 10^5 \leq \ ? < 9 \times 10^5$

$$1 \times \beta^{e} \leq \text{ original value } < \beta \times \beta^{e}$$

relative error between

$$\frac{\frac{\beta}{2} \times \beta^{-p} \times \cancel{\beta^{e}}}{\cancel{\beta^{e}}} \quad \text{and} \quad \frac{\frac{\beta}{2} \times \beta^{-p} \times \beta^{e}}{\beta^{e+1}}$$

so

$$\text{relative error } \leq \frac{\beta}{2} \beta^{-p} \tag{1}$$

- $\frac{\beta}{2}\beta^{-p} = \beta^{-p+1}/2$: machine epsilon

Relative Errors and Ulps III

The bound in (1)

- When a number is rounded to the closest, relative error $\underbrace{\text{bounded by}}$ ϵ

ulps and ϵ I

$x = 12.35$ $\tilde{x} = 12.4$

- $p = 3$, $\beta = 10$
- Example: $x = 12.35 \Rightarrow \tilde{x} = 1.24 \times 10^1$

  error $= 0.05 = 0.005 \times 10^1$
- ulps $= \boxed{0.01} \times 10^1$, $\epsilon = \frac{1}{2}10^{-2} = 0.005$
- error 0.5 ulps

  $$\begin{array}{r} 12.40 \\ 12.35 \\ \hline \end{array}$$

  last place $= 0.1 = 0.01 \times 10^1 =$ ulps

  error 0.05

  relative error $0.05/12.35 \approx 0.004 = 0.8\epsilon$
- $8x = 98.8, 8\tilde{x} = 9.92 \times 10^1$

  error $= \underline{4.0 \text{ ulps}}$

  relative error $= 0.4/98.8 = 0.8\epsilon.$
- ulps and ϵ may be used interchangeably

Guard Digits I

- $p = 3, \beta = 10$
- Calculate $2.15 \times 10^{12} - 1.25 \times 10^{-5}$:
- Compute and then round

$$
\begin{aligned}
x &= 2.15 \times 10^{12} \\
y &= 0.0000000000000000125 \times 10^{12} \\
x - y &= 2.1499999999999999875 \times 10^{12}
\end{aligned}
$$

round to 2.15×10^{12}

Here we assume that computation is exactly done

Guard Digits II

- Round and then compute

$$
\begin{aligned}
x &= 2.15 \times 10^{12} \\
y &= 0.00 \times 10^{12} \\
x - y &= 2.15 \times 10^{12}
\end{aligned}
$$

ulps $= 10^{-2} \times 10^{-1}$

  Answer is the same

  Reasonable as $x \approx x - y$

- Another example: $10.1 - 9.93 = 0.17$

1.70×10^{-1}

2.00×10^{-1}

Guard Digits III

- Round and then compute

$$10.1 - 9.93 = 1.01 \times 10^1 - 0.99 \times 10^1 = 0.02 \times 10^1$$

$$= 2.00 \times 10^{-1}$$

result

Correct

$$\text{error} = 2.00 \times 10^{-1} - 0.17 = 0.03$$

$$\text{ulps} = 0.01 \times 10^{-1} = 10^{-3}$$

ulps $= \beta^{-(p-1)} \cdot \beta^e$

$$\text{error} = 0.03 = 30\text{ulps}$$

Relative error

$$= 0.03/0.17 = 3/17$$

.

Guard Digits IV

The error is quite large

- Compute and round

$$10.1 - 9.93 = 0.17 = 1.7 \times 10^{-1}$$

error $= 0$

The problem: cannot compute and then round

Guard Digits V

Small $= \frac{0.00099}{0.90001}$

- How big can the error be? (if round and then compute)

$x = 1.000$

$y = 0.099\boxed{99}$

$\rightarrow x - y = 0.90001$

$x - y = 0.90100$

Theorem

Using p digits with base β for $x - y$, the relative error can be as large as $\beta - 1$

Proof:

$$x = 1.0\ldots0,\ y = 0.\underbrace{\eta\ldots\eta}_{p \text{ digits}},\ \eta = \beta - 1$$

Correct solution $x - y = \beta^{-p}$

Computed solution $= 1.0\ldots0 - 0.\underbrace{\eta\ldots\eta}_{p-1 \text{ digits}} = \beta^{-p+1}$

Guard Digits VI

Relative error

$$\frac{|\beta^{-p} - \beta^{-p+1}|}{\beta^{-p}} = \beta - 1$$

- Example: $p = 3$, $\beta = 10$

  $x = 1.00, y = 0.999, x - y = 0.001 = 10^{-3}$

  Computed solution $= 1.00 \times 10^0 - 0.99 \times 10^0$

  $= 0.01 \times 10^0 = 0.01$

  Relative error

$$\frac{|0.01 - 0.001|}{0.001} = 9$$

Super large

Guard Digits VII

Such large errors occur if ~~x and y are close~~

- Single guard digit

  p increased by 1 in the device for addition and subtraction

  round and then compute

  $p=4$ $p=4$

  $\underbrace{1.010}$ $\times 10^1 - \underbrace{0.993} \times 10^1 = 0.017 \times 10^1$

  Note $\underbrace{0.017}_{p=4} \times 10^1 = \underbrace{1.70}_{p=3} \times 10^{-1}$ can be stored as $p = 3$

- That is, one additional digit in the process of subtraction. All values are still stored using $p = 3$

Guard Digits VIII

- So in the device for subtraction, we should put additional digits
- Another example:

$$\begin{array}{r} 1.100 \times 10^2 \\ - \; 0.0859 \times 10^2 \\ \hline 1.0141 \times 10^2 \end{array}$$

$$
\begin{aligned}
& 110 - 8.59 \\
= & 1.100 \times 10^2 - 0.085 \times 10^2 \\
= & \underbrace{1.015}_{p=4} \times 10^2 \approx \underbrace{1.02}_{p=3} \times 10^2
\end{aligned}
$$

round

Correct answer 101.41

Relative error around 0.006

Guard Digits IX

$$\epsilon = \frac{1}{2}\beta^{-p+1} = \frac{1}{2}10^{-2} = 0.005$$

Theorem

Using $p+1$ digits for $x-y \Rightarrow$ relative rounding error $< 2\epsilon$ (ϵ: machine epsilon)

Proof:

- Assume $x > y$
- Assume $x = x_0.x_1 \cdots x_{p-1} \times \beta^0$

  The proof is similar if it's not β^0
- If $y = y_0.y_1 \cdots y_{p-1}$ no error

Guard Digits X

$x_0 . x_1 \cdots x_{p-1}$
$0 . y_1 \cdots y_{p-1} y_p$

- If $y = \overbrace{0.y_1 \cdots y_p}^{p} \Rightarrow$ 1 guard digit, exact $x - y$ rounded to a closest number $\Rightarrow$ relative error $\leq \epsilon$
- In general $y = 0.\underbrace{0 \cdots 0}_{k} \underbrace{y_{k+1} \cdots y_{k+p}}_{\text{length} = p}$
  $\bar{y}$: y truncated to $p \underbrace{+1}_{\text{guard}}$ digits

$$
\begin{aligned}
|y - \bar{y}| &< (\beta - 1)(\beta^{-p-1} + \beta^{-p-2} + \cdots + \beta^{-p-k}) \\
&\leq \beta^{-p} \qquad (2)
\end{aligned}
$$

$-p-1$: we have $p+1$ digits now $\underset{1}{0} . \underbrace{0 0 \cdots y_{k+1}}_{p}$ truncated $\Downarrow$
(Think about $p = 3, \beta = 10$, first digit truncated $\leq 9 \times 0.0001 = 9 \times 10^{-4}$)

$< (\beta - 1)(\beta^{-(p+1)} + \cdots$
$= (\beta - 1) \cdot \beta^{-p-1} \cdot \frac{(1 - (\beta^{-1})^k}{}$

Guard Digits XI

$\leq (\beta - 1)\beta^{-p-1} \cdot \frac{\beta}{\beta-1}$

$= \beta^{-p}$

After y is truncated, we need to calculate

$$x - \bar{y}$$

It's rounded to

$$x - \bar{y} + \delta$$

$$|\delta| \leq (\beta/2)\beta^{-p} = \epsilon$$

The inequality comes from rounding a number of $p + 1$ digits

and round it to p digit

$$0.\underbrace{0\ldots0}_{\cancel{p}\text{ digits}}(\beta/2)\ldots$$

p digit

$\frac{\beta}{2} \cdot \beta^{-p}$

Guard Digits XII

max error

$$\text{error: } (x - y) - (x - \bar{y} + \delta) = \bar{y} - y - \delta$$

Guard Digits XIII

case 1: if $x - y \geq 1$,

$$|\bar{y} - y - \delta| \leq |\bar{y} - y| + |\delta|$$

relative error

$$\epsilon = \frac{\beta}{2}\beta^{-p}$$

$$2\epsilon = \frac{2\beta}{2}\beta^{-p}$$

$$
\begin{aligned}
&= \frac{|\bar{y} - y - \delta|}{x - y} \leq \frac{|\bar{y} - y - \delta|}{1} \\
&\leq \beta^{-p}[(\beta - 1)(\beta^{-1} + \cdots + \beta^{-k}) + \beta/2] \\
&= \beta^{-p}[(\beta - 1)\beta^{-k}(1 + \cdots + \beta^{k-1}) + \beta/2] \\
&= \beta^{-p}[(\beta - 1)\beta^{-k}\frac{1 - \beta^k}{1 - \beta} + \beta/2] \\
&= \beta^{-p}[(1 \boxed{- \beta^{-k}}) + \beta/2] \\
&< \beta^{-p}(1 + \beta/2) \leq 2\epsilon
\end{aligned}
$$

ignore

$$\cancel{(\beta-1)}\beta^{-k}\frac{\beta^k - 1}{\cancel{\beta - 1}} = 1 - \beta^{-k}$$

$$= \frac{2+\beta}{2} \leq \frac{\beta+\beta}{2} \quad (\beta \geq 2)$$

Guard Digits XIV

$x - y < 1 \rightarrow$ or $x - \bar{y} \leq 1$ / $x - \bar{y} > 1$

case 2: $x - \bar{y} \leq 1$: enough digits $\delta = 0$

the smallest $x - y$: (smallest x - largest y)

$$1.0 - 0.\underbrace{0 \ldots 0}_{k}\rho \ldots \rho > (\beta - 1)(\beta^{-1} + \cdots + \beta^{-k})$$

k zeros, p ρ's, $\rho = \beta - 1$, from (2) the relative error

$$\leq \frac{|\bar{y} - y - \delta|}{(\beta - 1)(\beta^{-1} + \cdots + \beta^{-k})}$$

$$< \frac{(\beta - 1)\beta^{-p}(\beta^{-1} + \cdots + \beta^{-k})}{(\beta - 1)(\beta^{-1} + \cdots + \beta^{-k})} = \beta^{-p} < 2\epsilon$$

$\frac{2\beta}{2}\beta^{-p}$ $=$ 2ϵ

case 3: $x - y < 1$ but $x - \bar{y} > 1$

We already prove : $|x-y| \geq 1$

We will then prove $|x-y| \leq 1$

$|x-y|<1$ $\rightarrow$ $x-\bar{y} \leq 1 \rightarrow$ Case 2

$\rightarrow$ $x-\bar{y} > 1 \rightarrow$ Case 3

Case 2 : $|x-y|<1$ and $x-\bar{y} \leq 1$

After $y \rightarrow \bar{y}$, $x-\bar{y} \Rightarrow \delta = 0$

p+1 digits $\overbrace{x_0 . x_1 x_2 \cdots x_{p-1} 0}^{p+1\ \text{digit}}$

1.0000...

if $x-\bar{y}=1 \Rightarrow \delta = 0$ $\quad \underbrace{? . ? \cdots ?}_{p+1\ \text{digits}}$

if $x-\bar{y}<1 \Rightarrow$ we can normalize this number so that $\delta = 0$

ex. $0.9999 \Rightarrow 9.999$

Guard Digits XV

We show that this situation is impossible

If $x - \bar{y} = 1.\underbrace{0\cdots 1}_{p} \Rightarrow x - y \ge 1$: a contradiction

Why $x - y$ must be ≥ 1: $\begin{array}{r} 1.00\cdots1 \\ -0.00\cdots1 \\ \hline 1.000000 \end{array}$

The diffence between $y, \bar{y}$ is $0.\underbrace{0\cdots1}_{p}$

$-\beta^{-p} < y - \bar{y} < \beta^{-p}$ $\quad |y - \bar{y}| < \beta^{-p} = 0.\underbrace{0\cdots 1}_{p}$

$\Rightarrow -\bar{y} - \beta^{-p} < -y \Rightarrow \boxed{x - \bar{y} - \beta^{-p}} < x - y$

$x - y > 1$ #

- Conclusion: adding some guard digits can reduce the error

  Especially when subtracting two nearby numbers
- Cost: the adder is one bit wider (cheap)

  Most modern computers have guard digits

Cancellation I

- Catastrophic cancellation and benign cancellation
- Catastrophic cancellation :

  $b = 3.34, a = 1.22, c = 2.28$, $b^2 - 4ac = 0.0292$

  $b^2 \approx 11.2, 4ac \approx 11.1 \Rightarrow \text{answer} = 0.1$

  $\text{error} = 0.1 - 0.0292 = 0.0708$

  $\text{answer} = 0.0292 = 2.92 \times 10^{-2}$

  $\text{ulps} = 0.01 \times 10^{-2} = 10^{-4}$

  $0.0708 \approx 708 \text{ ulps}$
- Happens when subtracting two close numbers

Cancellation II

- Benign cancellation: subtracting exactly known numbers, by guard digits
- $\Rightarrow$ small relative error
- In the example, b^2 and $4ac$ already contain errors

Avoid Catastrophic Cancellation I

- By rearranging the formula
- Example

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

- If $b^2 \gg 4ac \Rightarrow$ no cancellation when calculating $b^2 - 4ac$ and $\sqrt{b^2 - 4ac} \approx |b|$
  Then $-b + \sqrt{b^2 - 4ac}$ has a catastrophic cancellation if $b > 0$

Avoid Catastrophic Cancellation II

- Multiplying $-b - \sqrt{b^2 - 4ac}$, if $b > 0$

$$\frac{2c}{-b - \sqrt{b^2 - 4ac}} \tag{4}$$

- Use (3) if $b < 0$, (4) if $b > 0$
- Difficult to remove all catastrophic cancellations, but possible to remove most by reformulations
- Another example: $x^2 - y^2$

  Assume $x \approx y$

  $(x - y)(x + y)$ is better than $x^2 - y^2$

Avoid Catastrophic Cancellation III

x^2, y^2 may be rounded $\Rightarrow x^2 - y^2$ may be a catastrophic cancellation

$x - y$ by guard digit

- A catastrophic cancellation is replaced by a benign cancellation

  Of course x, y may have been rounded and $x - y$ is still a catastrophic cancellation.

  Again, difficult to remove all catastrophic cancellations, but possible to remove some

Avoid Catastrophic Cancellation IV

- Calculating area of a triangle

$$A = \sqrt{s(s-a)(s-b)(s-c)}, s = \frac{a+b+c}{2} \quad (5)$$

a, b, c: length of three edges

If $a \approx b + c, s = (a+b+c)/2 \approx a$, $s - a$ may have a catastrophic error

Example: $a = 9.00, b = c = 4.53$

$s = 9.03, A = 2.342$

Computed solution: $A = 3.04$, error ≈ 0.7

ulps $= 0.01$, error $= 70$ ulps

Avoid Catastrophic Cancellation V

- A new formulation by Kahan [1986] , $a \geq b \geq c$

$$A = \frac{\sqrt{(a+(b+c))(c-(a-b))(c+(a-b))(a+(b-c))}}{4} \tag{6}$$

  $A \approx 2.35$, close to 2.342

- HW 1-1: Calculate $A = 3.04$ using (5) and $A = 2.35$ using (6)

Avoid Catastrophic Cancellation VI

Note: to get $A = 3.04$ you need to calculate s by

$$s = \frac{a + (b + c)}{2}$$

Note that for multiplication and square root we assume that exact calculation can be done and results are rounded.

- Conclusion: sometimes a formula can be rewritten to have higher accuracy using benign cancellation
- Only works if guard digit is used; most computers use guard digits now

Avoid Catastrophic Cancellation VII

- But reformulation is difficult!!

  You may think that you will never need to do this
- Two real cases:
- Line 213-216 of tron.cpp of LIBLINEAR version 2.11

  `http://www.csie.ntu.edu.tw/~cjlin/liblinear/oldfiles`

  HW1-2: Check Eq. (13) of the paper

  `http://www.csie.ntu.edu.tw/~cjlin/papers/logistic.pdf`

  and explain how we avoid catastrophic cancellations

Avoid Catastrophic Cancellation VIII

We do not consider the latest version of LIBLINEAR because some more complicated settings have been used

- Probability outputs of LIBSVM

  HW1-3: Repeat the experiment on page 5, line 12 of the paper

  `http://www.csie.ntu.edu.tw/~cjlin/papers/plattprob.pdf`

  Discuss what you found

Exactly Rounded Operations I

- Round then calculate $\Rightarrow$ may not be very accurate
- Exactly rounded: compute exactly then rounded to the nearest $\Rightarrow$ usually more accurate
- The definition of rounding
- 12.5 $\Rightarrow$ 12 or 13 ?
- Rounding up: 0, 1, 2, 3, 4 $\Rightarrow$ down, 5, 6, 7, 8, 9 $\Rightarrow$ up

  Why called "rounding up"? Always up for 5
- Rounding even:

  the closest value with even least significant digit

Exactly Rounded Operations II

50% probability up, 50% down

example: $12.5 \Rightarrow 12$; $11.5 \Rightarrow 12$

- Reiser and Knuth [1975] show rounding even may be better

Theorem

Let $x_0 = x, x_1 = (x_0 \ominus y) \oplus y, \ldots, x_n = (x_{n-1} \ominus y) \oplus y$*, if* $\oplus$ *and* $\ominus$ *are exactly rounded using rounding even, then* $x_n = x, \forall n$ *or* $x_n = x_1, \forall n \geq 1$*.*

$x \ominus y$: computed solution

- Consider rounding up,

Exactly Rounded Operations III

$\beta = 10, p = 3, x = 1.00, y = -0.555$

$x - y = 1.555, x \ominus y = 1.56, (x \ominus y) + y =$
$1.56 - 0.555 = 1.005, x_1 = (x \ominus y) \oplus y = 1.01$

$x_1 - y = 1.565, x_1 \ominus y = 1.57, (x_1 \ominus y) + y =$
$1.57 - 0.555 = 1.015, x_2 = (x_1 \ominus y) \oplus y = 1.02$

Increased by 0.01 until $x_n = 9.45$

- Rounding even:

$x - y = 1.555, x \ominus y = 1.56, (x \ominus y) + y =$
$1.56 - 0.555 = 1.005, x_1 = (x \ominus y) \oplus y = 1.00$

$x_1 - y = 1.555, x_1 \ominus y = 1.56, (x_1 \ominus y) + y =$
$1.56 - 0.555 = 1.005, x_2 = (x_1 \ominus y) \oplus y = 1.00$

Exactly Rounded Operations IV

- How to implement "exactly rounded operations"?
  We can use an array of words or floating-points
  But you don't have an infinite amount of spaces
- Goldberg [1990] showed that using 3 guard digits the result is the same as using exactly rounded operations

IEEE standard I

- IEEE 754 during 80s, now standard everywhere
- Two IEEE standards:

  754: specify $\beta = 2, p = 24$ for single, $\beta = 2, p = 53$ for double

  854 ($\beta = 2$ or 10, does not specify how floating-point numbers are encoded into bits)
- Why IEEE 854 allows $\beta = 2$ or 10 but not other numbers:

  10 is the base we use

  smaller β causes smaller relative error

IEEE standard II

smaller β: more precision. For example,

$$\beta = 16, p = 1 \text{ versus } \beta = 2, p = 4$$

- 4 bits for significand

$$\epsilon = \frac{16}{2} 16^{-1} = 1/2, \epsilon = \frac{2}{2} 2^{-4} = 1/16$$

We can see that ϵ of $\beta = 2, p = 4$ is smaller

- However, IBM/370 uses $\beta = 16$. Why? Two possible reasons:

First,

IEEE standard III

a number: 4 bytes = 32 bits

$\beta = 16, p = 6$, significand: $4 \times 6 = 24$ bits,

exponents: $32 - 24 - 1 = 7$ bits (1 bit for sign), 16^{-2^6} to $16^{2^6} = 2^{2^8}$

for $\beta = 2 \Rightarrow 9$ bits (-2^8 to $2^8 = 2^9$) for exponents, $\Rightarrow 32 - 9 - 1 = 22$ for significand

Same exponents, less significand for $\beta = 2$ (24 vs. 22)

Second,

Shifting: $\beta = 16$, less frequently to adjust exponents when adding or subtracting two numbers

IEEE standard IV

For modern computers, this saving is not important

- Single precision: $\beta = 2, p = 24$ (23 bits as normalized), exponent 8, 1 bit for sign $(32 = 23 + 8 + 1)$
- An example: $176.625 = 1.0101100101 \times 2^7$

0	10000110	01011001010000000000000

  1 of $1.\cdots$ is not stored (normalized)
- Biased exponent (described later in detail)

  $10000110 = 128 + 4 + 2 = 134, 134 - 127 = 7$

  Note that we have negative exponent

IEEE standard V

- Use rounding even

Binary	rounded	reason
10.00011	10.00	($<$ 1/2, down)
10.00110	10.01	($>$ 1/2, up)
10.11100	11.00	(1/2, up)
10.10100	10.10	(1/2, down)

  This example is from `http://www.cs.cmu.edu/afs/cs/academic/class/15213-s12/www/lectures/04-float-4up.pdf`

- A summary

IEEE standard VI

IEEE	Fortran	C	Bits	Exp.	Mantissa
Single	REAL*4	float	32	8	24
Single-extended			44	≤ 11	32
Double	REAL*8	double	64	11	53
Double-extended	REAL*10	long double	≥ 80	≥ 15	≥ 64

$$32 = 8 + 24 \text{ but } 44 \neq 11 + 32$$

- $44 \neq 11 + 32$:

  Hardware implementation of extended precision normal don't use a hidden bit

  (Remember we normalized each number so 1 is not stored)

IEEE standard VII

- It seems everyone is using double now

  But single is still needed sometime (if memory is not enough)
- Minimal normalized positive number

$$1 \times 2^{-126} \approx 1.17 \times 10^{-38}$$

$e_{\text{min}} = -126$

- 8 bits for exponent: 0 to 255. IEEE uses biased approach exponent

$$(0 \text{ to } 255) - 127 = -127 \text{ to } 128$$

IEEE standard VIII

- Why $e_{\min} = -126$ but $e_{\max} = 127$?

  reasons: $1/2^{e_{\min}}$ not overflow, $1/2^{e_{\max}}$ underflow, but less serious
- Thus, -127 for 0 and denormalized numbers (discussed later), -126 to 127 for exponents, 128 for special quantity
- Motivation for extended precision: from calculator, display 10 digits but 13 internally

  Some operations benefit from using more digits internally

IEEE standard IX

Example: binary-decimal conversion (Details not discussed here)

- Operations: IEEE standard requires results of addition, subtraction, multiplication and division exactly rounded.
- Exactly rounded: an array of words or floating-point numbers, expensive
- Goldberg [1990] showed using 3 guard digits the result is the same as using exactly rounded operations

  Only little more cost

IEEE standard X

- Reasons to specify operations

  run on different machines $\Rightarrow$ results the same
- HW 2-1: write the binary format of -250 as a double floating-point number
- IEEE: square root, remainder, conversion between integer and floating-point, internal formats and decimal are correctly rounded (i.e. exactly rounded operations)
- Binary to decimal conversion

  Think about reading numbers from files

IEEE standard XI

When writing a binary number to a decimal number and read it back, can we get the same binary number?

- Writing 9 digits is enough for short

  Though $10^8 > 2^{24}$, 8 digits are not enough
- 17 for double precision (proof not provided).

Example:

numbers in a data set from Matrix market:

IEEE standard XII

```
> tail s1rmq4m1.dat
  8.2511736085618438E+01  2.513452865992495
 -6.0042951255041466E+00  8.659944220661552
  1.0026197619563723E+01 -1.313683766184450
 -1.5108331040361231E+01  5.142317399695508
 -1.1690286345961363E+03  1.625072665580781
  8.2511736074473220E+01  1.510833104036122
```

- Matrix market:

  `http://math.nist.gov/MatrixMarket/`

  A collection of matrix data

IEEE standard XIII

- Transcendental numbers:

  e.g., exp, log
- IEEE does not require transcendental functions to be exactly rounded

  Cannot specify the precision because they are arbitrarily long

Special quantities I

- On some computers (e.g., IBM 370) every bit pattern is a valid floating-point number
- For IBM 370, $\sqrt{-4} = 2$ and it prints an error message

  IEEE : `NaN`, not a number

  why $\sqrt{-4} = 2$ on IBM 370 $\Rightarrow$ every pattern is a number
- Special value of IEEE:

  $+0, -0$, denormalized numbers, $+\infty$, $-\infty$, NaNs (more than one NaN)

Special quantities II

- A summary

Exponent	significand	represents
$e = e_{\min} - 1$	$f = 0$	$+0, -0$
$e = e_{\min} - 1$	$f \neq 0$	$0.f \times 2^{e_{\min}}$
$e_{\min} \leq e \leq e_{\max}$		$1.f \times 2^{e}$
$e = e_{\max} + 1$	$f = 0$	$\pm\infty$
$e = e_{\max} + 1$	$f \neq 0$	NaN

- Why IEEE has NaN

  Sometimes even $0/0$ occurs, the program can continue

Special quantities III

- Example: find $f(x) = 0$, try different x's, even $0/0$ happens, other values may be ok.
- If $b^2 - 4ac < 0$

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

returns NaN

$-b+$ NaN should be NaN

In general when a NaN is in an operation, result is NaN

- Examples producing NaN:

Special quantities IV

Operation	NaN by
$+$	$\infty + (-\infty)$
$\times$	$0 \times \infty$
$/$	$0/0, \infty/\infty$
REM	x REM 0, ∞ REM y
$\sqrt{}$	$\sqrt{x}$ when $x < 0$

Infinity I

- $\beta = 10, p = 3, e_{\max} = 98, x = 3 \times 10^{70}$,
  x^2 overflow and replaced by 9.99×10^{98}??
  In IEEE, the result is ∞
- Note $0/0 = \text{NaN}$, $1/0 = \infty, -1/0 = -\infty$
  $\Rightarrow$ nonzero divided by 0 is ∞ or $-\infty$
  Similarly, $-10/0 = -\infty$, and $-10/-0 = +\infty$
  (± 0 will be explained later)
- $3/\infty = 0, 4 - \infty = -\infty, \sqrt{\infty} = \infty$
- How to know the result?
  replace ∞ with x, let $x \to \infty$

Infinity II

Example:

$$3/\infty : \quad \lim_{x \to \infty} 3/x = 0$$

If limit does not exist $\Rightarrow$ NaN

- $x/(x^2+1)$ vs $1/(x+x^{-1})$
  $x/(x^2+1)$: if x is large, x^2 overflow, $x/\infty = 0$ but not $1/x$.
  $1/(x+x^{-1})$: x large, $1/x$ ok
  $1/(x+x^{-1})$ looks better but what about $x = 0$?
  $x = 0$, $1/(0+0^{-1}) = 1/(0+\infty) = 1/\infty = 0$
- If no infinity arithmetic, an extra instruction needed to test if $x = 0$, may interrupt the pipeline

Signed zero I

- Why do we have $+0$ and -0 ?
  First, it is available (1 bit for sign)
  if no sign, $1/(1/x)) = x$ fails when $x = \pm\infty$
  $x = \infty, 1/x = 0, 1/0 = +\infty$
  $x = -\infty, 1/x = 0, 1/0 = +\infty$
- Compare $+0$ and -0: `if (x == 0)`
  IEEE defines $+0 = -0$
- IEEE: $3 \times (+0) = +0$, $+0/(-3) = -0$

Signed zero II

- ± 0 useful in the following situations:

$$\log x \equiv \begin{cases} -\infty & x = 0 \\ \text{NaN} & x < 0 \end{cases}$$

A small underflow negative number $\Rightarrow \log x$ should be NaN

x underflow $\Rightarrow$ round to 0, if no sign, $\log x$ is $-\infty$ but not NaN

Signed zero III

- With ± 0, we have

$$\log x = \begin{cases} -\infty & x = +0 \\ \text{NaN} & x = -0 \\ \text{NaN} & x < 0 \end{cases}$$

  Positive underflow $\Rightarrow$ round to $+0$

- Very useful in complex arithmetic

  $\sqrt{1/z}$ and $1/\sqrt{z}$

  $z = -1, \sqrt{1/-1} = \sqrt{-1} = i, 1/\sqrt{-1} = 1/i = -i$

  $\Rightarrow \sqrt{1/z} \neq 1/\sqrt{z}$

Signed zero IV

- This happens because square root is multi-valued.
  $i^2 = (-i)^2 = -1$
- However, by some restrictions (or ways of calculation), they can be equal
- $z = -1 = -1 + 0i$,
  $1/z = 1/(-1 + 0i) = -1 + (-0)i$
  so $\sqrt{1/z} = \sqrt{-1 + (-0)i} = -i$
  $\Rightarrow -0$ is useful
- Disadvantage of $+0$ and -0:
  $x = y \Leftrightarrow 1/x = 1/y$ is destroyed
  $x = 0, y = -0 \Rightarrow x = y$ under IEEE

Signed zero V

$$1/x = +\infty, 1/y = -\infty, +\infty \neq -\infty$$

- There are always pros and cons for floating-point design

HW 2-2 I

- If

```
if (a < 0)
```

always holds and b is neither too large nor too small, how do we guarantee

```
if a/max(b, 0.0) < 0
```

always holds

- If `max(b,0.0)` returns -0.0, then it may not hold
- For the max function, should we use

```
(x>y)?  x:y
```

or

HW 2-2 II

```
(x<y)?  y:x
```

- Your max need to return $+0.0$ but not -0.0
- How to specifically assign $+0.0$ and -0.0?
- How to use subroutines to get the sign of a number?
- In a regular program, if you write 0.0, is it $+0.0$ or -0.0?

  Find the statement in the manual saying that 0.0 means $+0.0$
- Do some experiments to check your arguments
- Use Java but not other systems

Denormalized number I

- $\beta = 10, p = 3, e_{\text{min}} = -98$, $x = 6.87 \times 10^{-97}$, $y = 6.81 \times 10^{-97}$
- x, y are ok but $x - y = 0.6 \times 10^{-98}$ rounded to 0, even though $x \neq y$
- How important to preserve

$$x = y \Leftrightarrow x - y = 0$$

- `if (x` $\neq$ `y) {z = 1/(x-y);}`
  The statement is true, but z becomes ∞
  Tracking such bugs is frustrating

Denormalized number II

- IEEE uses denormalized numbers

  Guarantee $x = y \Leftrightarrow x - y = 0$

  Details of how this is done are not discussed here
- Most controversial part in IEEE standard

  It caused long delay of the standard
- If denormalized number is used, 0.6×10^{-98} is also a floating-point number
- Remember we do not store 1 of $1.d \cdots d$
- How to represent denormalized numbers ?

  Recall for valid value, $e \geq e_{\min}$ and we have $1.d \cdots d \times 2^e$

Denormalized number III

- For denormalized numbers, we let $e = e_{\min} - 1$ and the corresponding value be

$$0.d \cdots d \times 2^{e+1} = 0.d \cdots d \times 2^{e_{\min}}$$

- Why not

$$1.d \cdots d \times 2^{e_{\min}-1}$$

  can't represent

$$0.0x \ldots x \times 2^{e_{\min}}$$

- $6.87 \times 10^{-97} - 6.81 \times 10^{-97} \Rightarrow$ underflow due to cancellation

Denormalized number IV

Underflow: smaller than the smallest floating-point number

- An example of using denormalized numbers

$$
\begin{aligned}
\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\
&= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i
\end{aligned}
$$

If c or $d > \sqrt{\beta}\beta^{e_{\max}/2} \Rightarrow$ overflow

Denormalized number V

overflow: larger than the maximal floating-point number

- Smith's formula

$$\frac{a+bi}{c+di} = \begin{cases} \frac{a+b(d/c)}{c+d(d/c)} + \frac{b-a(d/c)}{c+d(d/c)} i & \text{if } (|d| < |c|) \\ \frac{b+a(c/d)}{d+c(c/d)} + \frac{-a+b(c/d)}{d+c(c/d)} i & \text{if } (|d| \geq |c|) \end{cases}$$

avoid overflow

- However, using Smith's formula, without denormalized numbers

Denormalized number VI

If

$$a = 2 \times 10^{-98}, b = 1 \times 10^{-98}, c = 4 \times 10^{-98},$$
$$d = 2 \times 10^{-98}$$

then

$$d/c = 0.5, c + d(d/c) = 5 \times 10^{-98},$$
$$b(d/c) = 1 \times 10^{-98} \times 0.5 = 0$$
$$a + b(d/c) = 2 \times 10^{-98}$$

Solution $= 0.4$, wrong

Denormalized number VII

If denormalized numbers are used, 0.5×10^{-98} can be stored,

$$a + b(d/c) = 2.5 \times 10^{-98} \Rightarrow 0.5$$

the correct answer

- Usually hardware does not support denormalized numbers directly

  Using software to simulate
- Programs may be slow if a lot of underflow

Exception, Flags, Trap handlers I

- We have mentioned things like overflow, underflow
  What are other exceptional situations ?
- Motivation: usually when exceptional condition like $1/0$ happens, you may want to know
- IEEE requires vendors to provide a way to get status flags
- IEEE defines five exceptions: overflow, underflow, division by zero, invalid operation, inexact
- overflow: larger than the maximal floating-point number

Exception, Flags, Trap handlers II

Underflow: smaller than the smallest floating-point number

- Invalid:

  $\infty + (-\infty), 0 \times \infty, 0/0, \infty/\infty,$
  x REM $0, \infty$ REM y, $\sqrt{x}, x < 0$, any comparison involves a NaN

- Invalid returns NaN; NaN may not be from invalid operations
- Inexact: the result is not exact

  $\beta = 10, p = 3, 3.5 \times 4.2 = 14.7$ exact,
  $3.5 \times 4.3 = 15.05 \Rightarrow 15.0$ not exact

Exception, Flags, Trap handlers III

inexact exception is raised so often, usually we ignore it

Exception	when trap disabled	argument to handler
overflow	$\pm\infty$ or $\pm 1.1\cdots 1 \times 2^{e_{\max}}$	$\text{round}(x2^{-\alpha})$
underflow	$0, \pm 2^{e_{\min}}$, or denormal	$\text{round}(x2^{\alpha})$
division by zero	∞	operands
invalid	NaN	operands
inexact	$\text{round}(x)$	$\text{round}(x)$

- Trap handler: special subroutines to handle exceptions

Exception, Flags, Trap handlers IV

You can design your own trap handlers

- In the above table, "when trap disabled" means results of operations if trap handlers not used
- $\alpha = 192$ for single, $\alpha = 1536$ for double reason: you cannot really store x
- Examples of using trap handlers described later

Compiler Options I

- Compiler may provide a way so the program stops if an exception occurs
- Easy for debugging
- Example: SUN's C compiler (I learned this on an old machine)
- Reason: gcc doesn't have this to explicitly detect exceptions
- **-ftrap=t**

Compiler Options II

- t: %all, %none, common, [no%]invalid, [no%]overflow, [no%]underflow, [no%]division, [no%]inexact.
- common: invalid, division by zero, and overflow.
- The default is -ftrap=%none.
- Example: -ftrap=%all,no%inexact means set all traps, except inexact.
- If you compile one routine with -ftrap=t, compile all routines of the program with the same -ftrap=t option

  otherwise, you can get unexpected results.

Compiler Options III

- Example: on the screen you will see

```
Note: IEEE floating-point exception flags raise
   Inexact;  Underflow;
See the Numerical Computation Guide, ieee_flag
```

- gcc:
- -fno-trapping-math: default -ftrapping-math
  Setting this option may allow faster code if one relies on "non-stop" IEEE arithmetic
- -ftrapv

Compiler Options IV

Generates traps for signed overflow on addition, subtraction, multiplication

Trap Handler I

- Example:

```
do  {
      ....
} while {not x >= 100;}
```

  If $x = \text{NaN}$, an infinite loop
  Any comparison involving NaN is wrong
- A trap handler can be installed to abort it
- Example:

Trap Handler II

Calculate $x_1 \times \cdots \times x_n$ may overflow in the middle (the total may be ok!):

```
for (i = 1; i <= n; i++)
    p = p * x[i] ;
```

- $x_1 \times \cdots \times x_r, r \leq n$ overflow but $x_1 \times \cdots \times x_n$ may be in the range
- $e^{\sum \log(x_i)} \Rightarrow$ a solution but less accurate and costs more
- A possible solution

Trap Handler III

```
for (i = 1; i <= n; i++) {
    if (p * x[i] overflow) {
        p = p * pow(10,-a);
        count = count + 1 ;
    }
    p = p * x[i] ;
}
p = p * pow(10, a*count) ;
```

An Example of Handlers I

- Example using SUN's numerical computation guide
  Again, old. Reason of not using existing systems such a glibc: so you can have HW
- standard math library libm.a
  exp, pow, log, ...
- On SUN machines, there are additional math library: libsunmath.a
  exp2, exp10, ..., ieee_flags, ieee_handler, ieee_retrospective
- A program:

An Example of Handlers II

```
#include <stdio.h>
#include <sys/ieeefp.h>
#include <sunmath.h>
#include <siginfo.h>
#include <ucontext.h>

void handler(int sig, siginfo_t *sip,
             ucontext_t *uap)
{
    unsigned    code, addr;

    code = sip->si_code;
```

An Example of Handlers III

```
    addr = (unsigned) sip->si_addr;
    fprintf(stderr, "fp exception %x at
        address %x \n", code, addr);
}
int main()
{
    double  x;

    /* trap on common floating point
       exceptions */
    if (ieee_handler("set", "common", handler)
        != 0)
```

An Example of Handlers IV

```
    printf("Did not set exception
            handler \n");

/* cause an underflow exception (not
   reported) */
x = min_normal();
printf("min_normal = %g \n", x);
x = x / 13.0;
printf("min_normal / 13.0 = %g \n", x);

/* cause an overflow exception
   (reported) */
```

An Example of Handlers V

```
    x = max_normal();
    printf("max_normal = %g \n", x);
    x = x * x;
    printf("max_normal * max_normal = %g \n",
            x);

    ieee_retrospective(stderr);
    return 0;
}
```

- Result:

An Example of Handlers VI

```
min_normal = 2.22507e-308
min_normal / 13.0 = 1.7116e-309
max_normal = 1.79769e+308
fp exception 4 at address 10d0c
max_normal * max_normal = 1.79769e+308
 Note: IEEE floating-point exception flags raise
    Inexact;  Underflow;
 IEEE floating-point exception traps enabled:
    overflow; division by zero; invalid operati
 See the Numerical Computation Guide, ieee_flag
ieee_handler(3M)
```

An Example of Handlers VII

- invalid, division, and overflow sometimes called common exceptions here

  ieee_handler("set", "common", handler) means handlers used for common exceptions
- min_normal / 13.0: using denormalized numbers

  handler: subroutines to handle exceptions
- HW 3-1: regenerate this example using GNU C library

An Example of Handlers VIII

- How to find GNU C library information: on linux, type
  % info libc
  check the category of “Arithmetics” and “Signal Handling”

The Use of Flags: An Example I

- Calculate x^n, n : integer

```
double pow(double x, int n)
{
        double tmp = x, ret = 1.0;

        for(int t=n; t>0; t/=2)
        {
                if(t%2==1) ret*=tmp;
                tmp = tmp * tmp;
        }
        return ret;
```

The Use of Flags: An Example II

```
}
```

$x^{16} = (x^2)^8 = \cdots$, $x^{15} = x(x^2)^7$, treat x^2 as the new x

$x^{15} = x(x^2)^7 = x(x^2)(x^4)^3 = x(x^2)(x^4)(x^8)^1$

- If $n < 0$, we need to use

$$x^n = (1/x)^{-n} = 1/(x)^{-n}$$

pow$(1/x, -n)$ less accurate, $1/$pow$(x, -n)$ is better

There is already error on $1/x$

The Use of Flags: An Example III

Example: $2^{-5} = (1/2)^5$ and $1/(2^5)$

- A small problem on using $1/\text{pow}(x, -n)$:

  if $\text{pow}(x, -n)$ underflow (i.e. when $x < 1, n < 0$), either underflow trap handler or underflow status flag set $\Rightarrow$ incorrect

  x^{-n} underflow, x^n overflow or be in range $(e_{\min} = -126, 2^{-e_{\min}} = 2^{126} < 2^{127} = 2^{e_{\max}})$

- Turn off overflow & underflow trap enable bits, save overflow & underflow status bits

  Compute $1/\text{pow}(x, -n)$

The Use of Flags: An Example IV

If neither overflow nor underflow status is set $\Rightarrow$ restore them

If one is set, restore & calculate pow($1/x, -n$), which causes correct exception to occur

- Practically the calculation of pow() is more complicated

  e.g. google e_pow.c and e_log.c

- In glibc-2.17/sysdeps/ieee754/dbl-64, e_pow.c has 420 lines

The Use of Flags: An Example V

- Another example: calculate arccos using arctan

$$\arccos x = 2 \arctan \sqrt{\frac{1-x}{1+x}}$$

$$\cos\theta = x = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{x+1}{2}}, \sin\frac{\theta}{2} = \sqrt{\frac{1-x}{2}}, \tan\frac{\theta}{2} = \sqrt{\frac{1-x}{1+x}}$$

Hence

$$\arccos x = 2 \arctan \sqrt{\frac{1-x}{1+x}}$$

The Use of Flags: An Example VI

- Consider $x = -1$

  $\arctan(\infty) = \pi/2 \Rightarrow \arccos(-1) = \pi$

- A small problem:

  $\frac{1-x}{1+x}$ causes the divide-by-zero flag set though $\arccos(-1)$ not exceptional

- Solution: save divide-by-zero flag, restore after arccos computation

A Real Study I

- Let's start with a simple example

```
#include <stdio.h>

int main()
{
float a = 123.123;
printf("%.10f\n", a);
printf("%.10f\n", a*a);

a = 123.125;
printf("%.10f\n", a);
```

A Real Study II

```
printf("%.10f\n", a*a);

}
```

- Results are

```
$gcc test.c;./a.out
123.1230010986
15159.2734375000
123.1250000000
15159.7656250000
$gcc -m32 test.c;./a.out
123.1230010986
```

A Real Study III

```
15159.2733995339
123.1250000000
15159.7656250000
```

- -m 32 generates code for a 32-bit environment (because we don't have a 32-bit machine)
- That is, same code gives different results under 32 and 64-bit environments
- Why?

A Real Study IV

- On 32 bit, 387 floating-point coprocessor is used. From gcc manual, "The temporary results are computed in 80-bit precision instead of the precision specified by the type, resulting in slightly different results compared to most of other chips."
- In other words, they somehow violate IEEE standard
- But 123.123 has infinite digits after transformed to binary

A Real Study V

- Compiler options can help to make things more consistent. For example, -ffloat-store: "Do not store floating-point variables in registers, and inhibit other options that might change whether a floating-point value is taken from a register or memory."

```
$gcc -ffloat-store test.c;./a.out
123.1230010986
15159.2734375000
123.1250000000
15159.7656250000
$gcc -ffloat-store -m32 test.c;./a.out
123.1230010986
```

A Real Study VI

```
15159.2734375000
123.1250000000
15159.7656250000
```

- Note that other issues such as order of operations can also affect results.
- Consider running a real example using a machine learning software LIBSVM
- 64 bit:

A Real Study VII

```
$ ./svm-train -c 100 -e 0.00001 heart_scale
........*..*
optimization finished, #iter = 2872
nu = 0.148045
obj = -2526.925470, rho = 1.145512
nSV = 107, nBSV = 9
Total nSV = 107
```

- 32bit:

A Real Study VIII

```
$  ./svm-train -c 100 -e 0.00001 heart_scale
........*..*
optimization finished, #iter = 2819
nu = 0.148045
obj = -2526.925470, rho = 1.145515
nSV = 107, nBSV = 9
Total nSV = 107
```

- They are different
- Adding `-ffloat-store -mfpmath=387` is not enough
- 64 bit:

A Real Study IX

```
$ make clean; make
$ ./svm-train -c 100 -e 0.00001 heart_scale
rm -f *~ svm.o svm-train svm-predict svm-sca
g++ -Wall -Wconversion -O3 -fPIC -ffloat-sto
g++ -Wall -Wconversion -O3 -fPIC -ffloat-sto
g++ -Wall -Wconversion -O3 -fPIC -ffloat-sto
g++ -Wall -Wconversion -O3 -fPIC -ffloat-sto
........*..*
optimization finished, #iter = 2863
nu = 0.148045
obj = -2526.925470, rho = 1.145512
nSV = 107, nBSV = 9
```

A Real Study X

```
Total nSV = 107
```

- We also need to disable all optimization
- 64bit:

```
$ make clean; make
$ ./svm-train -c 100 -e 0.00001 heart_scale
rm -f *~ svm.o svm-train svm-predict svm-sca
g++ -ffloat-store -mfpmath=387 -c svm.cpp
g++ -ffloat-store -mfpmath=387 svm-train.c s
g++ -ffloat-store -mfpmath=387 svm-predict.c
g++ -ffloat-store -mfpmath=387 svm-scale.c -
........*...*
```

A Real Study XI

```
optimization finished, #iter = 3051
nu = 0.148045
obj = -2526.925470, rho = 1.145515
nSV = 107, nBSV = 9
Total nSV = 107
```

- 32 bit:

```
$ make clean; make
$ ./svm-train -c 100 -e 0.00001 heart_scale
rm -f *~ svm.o svm-train svm-predict svm-sca
g++ -m32 -ffloat-store -mfpmath=387 -c svm.c
g++ -m32 -ffloat-store -mfpmath=387 svm-trai
```

A Real Study XII

```
g++ -m32 -ffloat-store -mfpmath=387 svm-pred
g++ -m32 -ffloat-store -mfpmath=387 svm-scal
........*...*
optimization finished, #iter = 3051
nu = 0.148045
obj = -2526.925470, rho = 1.145515
nSV = 107, nBSV = 9
Total nSV = 107
```