

Floating-point operations I

- The science of floating-point arithmetics
- IEEE standard
- Reference

What every computer scientist should know about floating-point arithmetic, ACM computing survey, 1991

Why learn more about floating-point operations I

Example:

- A one-variable problem

$$\begin{aligned} \min_x f(x) \\ x \geq 0 \end{aligned}$$

- In your program, should you set an **upper bound** of x ?
- x in your program may be wrongly increased to ∞

Why learn more about floating-point operations II

- What is the largest representable number in the computer ?
- Is there anything called infinity ?

Example:

- A ten-variable problem

$$\begin{aligned} & \min f(x) \\ & 0 \leq x_i, i = 1, \dots, 10 \end{aligned}$$

Why learn more about floating-point operations III

- After the problem is solved, want to know how many are zeros ?

- Should you use

```
for (i=0; i < 10; i++)
    if (x[i] == 0) count++ ;
```

- People said: don't do floating-point comparisons

```
epsilon = 1.0e-12 ;
```

```
for (i=0; i < 10; i++)
    if (x[i] <= epsilon) count++ ;
```

How do you choose ϵ ?

Why learn more about floating-point operations IV

- Is this true ?

Floating-point Formats I

- We know float (single): 4 bytes, double: 8 bytes
Why ?
- A floating-point system
base β , precision p , significand (mantissa) $d.d\dots d$
- Example

$$0.1 = \underbrace{1.00}_{\text{3}} \times 10^{-1} \quad (\beta = 10, p = 3)$$

$$\approx \underbrace{1.1001}_{\text{5}} \times 2^{-4} \quad (\beta = 2, p = 5)$$

exponent: -1 and -4

- Largest exponent e_{\max} , smallest e_{\min}

Floating-point Formats II

- β^p possible significands, $e_{\max} - e_{\min} + 1$ possible exponents

$$\lceil \log_2(e_{\max} - e_{\min} + 1) \rceil + \lceil \log_2(\beta^p) \rceil + 1$$

bits for storing a number

1 bit for \pm

- But the practical setting is **more complicated**
See the discussion of IEEE standard later
- Normalized: 1.00×10^{-1} (yes), 0.01×10^1 (no)
- Now ~~most used normalized representation~~
⇒ **cannot represent zero**

Floating-point Formats III

the smallest
number
 e_{\min}

$$1.0 \times \beta$$

- A natural way for 0: $1.0 \times \beta^{e_{\min}-1}$ preserve the ordering
- Will use $p = 3, \beta = 10$ for most later explanation

Relative Errors and Ulps I

$$\begin{array}{l} 3.14 \times 10^0 \\ 0.00159 \times 10^0 \end{array}$$

\Rightarrow

0.159 units

- When $\beta = 10, p = 3, 3.14159$ represented as 3.14×10^0
 \Rightarrow error $= 0.00159 = 0.159 \times 10^{-2}$, i.e. 0.159 units in the last place
 10^{-2} : unit of the last place
- ulps: unit in the last place
- relative error $0.00159/3.14159 \approx 0.0005$
- For a number $d.d\dots d \times \beta^e$, the largest error is

$$d.d\dots d + \underbrace{0.0\dots 0}_{p-1} \beta' \times \beta^e, \beta' = \beta/2$$

\nwarrow floating point number

Relative Errors and Ulps II

- Error = $\frac{\beta}{2} \times \beta^{-p} \times \beta^e$

Ex. $1 \times 10^5 \leq ? < 9 \times 10^5$

$$1 \times \underbrace{\beta^e}_{\text{original value}} \leq \text{original value} < \beta \times \underbrace{\beta^e}$$

relative error between

$$\frac{\frac{\beta}{2} \times \beta^{-p} \times \beta^e}{\beta^e} \text{ and } \frac{\frac{\beta}{2} \times \beta^{-p} \times \beta^e}{\beta^{e+1}}$$

so

$$\text{relative error} \leq \frac{\beta}{2} \beta^{-p} \quad (1)$$

- $\frac{\beta}{2} \beta^{-p} = \beta^{-p+1}/2$: machine epsilon

Relative Errors and Ulps III

The bound in (1)

- When a number is rounded to the closest, relative error bounded by ϵ



ulp and ϵ

$$x = 12.35 \quad \tilde{x} = 12.4$$

- $p = 3, \beta = 10$
- Example: $x = 12.35 \Rightarrow \tilde{x} = 1.24 \times 10^1$
error = $0.05 = 0.005 \times 10^1$
- ulps = 0.01×10^1 , $\epsilon = \frac{1}{2} \times 10^{-2} = 0.005$
- error 0.5 ulps \rightarrow last place = 0.1
relative error $0.05/12.35 \approx 0.004 = 0.8\epsilon$
 $= 0.01 \times 0.1$
 $= 1 \text{ ulps}$
- $8x = 98.8, 8\tilde{x} = 9.92 \times 10^1$
error = 4.0 ulps
relative error = $0.4/98.8 = 0.8\epsilon$.
- ulps and ϵ may be used interchangeably

Guard Digits I

- $p = 3, \beta = 10$
- Calculate $2.15 \times 10^{12} - 1.25 \times 10^{-5}$.
- Compute and then round

$$x = 2.15 \times 10^{12}$$

$$y = 0.000000000000000125 \times 10^{12}$$

$$x - y = 2.14999999999999875 \times 10^{12}$$

round to 2.15×10^{12}

Here we assume that computation is exactly done

Guard Digits II

- Round and then compute

$$x = 2.15 \times 10^{12}$$

$$y = 0.00 \times 10^{12}$$

$$x - y = 2.15 \times 10^{12}$$

Answer is the same

Reasonable as $x \approx x - y$

- Another example: $10.1 - 9.93 = 0.17$

$$\text{ulp} s = 10^2 \times 10^{-1}$$

$(.7 \downarrow) \times 10^{-1}$

2.00×10^{-1}

Guard Digits III

- Round and then compute

$$10.1 - 9.93 = 1.01 \times 10^1 - 0.99 \times 10^1 = 0.02 \times 10^1$$

$$= 2.00 \times 10^{-1}$$

$$\text{error} = \underbrace{2.00 \times 10^{-1}}_{\text{result}} - \underbrace{0.17}_{\text{Correct}} = 0.03$$

$$\text{ulp} = 0.01 \times 10^{-1} = 10^{-3}$$

$$\text{error} = 0.03 = 30\text{ulp}$$

$$\text{ulp} = \beta^{-(p-1)} \cdot \beta^e$$

Relative error

$$= 0.03/0.17 = 3/17$$

Guard Digits IV

The error is quite large

- Compute and round

$$10.1 - 9.93 = 0.17 = 1.7 \times 10^{-1}$$

error = 0

The problem: **cannot** compute and then round

Guard Digits V

$$S_{\text{small}} = \frac{0.00099}{0.90001}$$

- How big can the error be? (if round and then compute)

$$x \approx 1.000$$

$$\Rightarrow x - y = 0.90001$$

$$y = 0.999 \boxed{99}$$

$$x - y = 0.90100$$

Theorem

Using p digits with base β for $x - y$, the relative error can be as large as $\beta - 1$

Proof:

$$x = 1.0\dots 0, y = 0.\underbrace{\eta\dots\eta}_{p \text{ digits}}, \eta = \beta - 1$$

Correct solution $x - y = \beta^{-p}$

$$\text{Computed solution} = 1.0\dots 0 - 0.\underbrace{\eta\dots\eta}_{p-1 \text{ digits}} = \beta^{-p+1}$$

Guard Digits VI

Relative error

$$\frac{|\beta^{-p} - \beta^{-p+1}|}{\beta^{-p}} = \beta - 1$$

- Example: $p = 3, \beta = 10$

$$x = 1.00, y = 0.999, x - y = 0.001 = 10^{-3}$$

$$\begin{aligned}\text{Computed solution} &= 1.00 \times 10^0 - 0.99 \times 10^0 \\ &= 0.01 \times 10^0 = 0.01\end{aligned}$$

Relative error

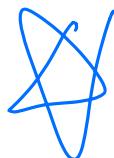
$$\frac{|0.01 - 0.001|}{0.001} = 9$$

Super large

Guard Digits VII

Such large errors occur if x and y are close

- Single guard digit



p increased by 1 in the device for addition and subtraction

round and then compute

$$p=4 \quad 1.010 \times 10^1 - 0.993 \times 10^1 = 0.017 \times 10^1$$

Note $0.017 \times 10^1 = 1.70 \times 10^{-1}$ can be stored as
 $p = 3$ $\overbrace{1.70}^{p=3} \times 10^{-1}$

- That is, one additional digit in the process of subtraction. All values are still stored using $p = 3$

Guard Digits VIII

- So in the device for subtraction, we should put additional digits
- Another example:

$$\begin{aligned} & 110 - 8.59 \\ & = 1.100 \times 10^2 - 0.085 \times 10^2 \\ & = 1.015 \times 10^2 \approx \underline{1.02} \times 10^2 \end{aligned}$$

$\underbrace{}_{P=4} \quad \underbrace{}_{\text{round}} \quad P=3$

Correct answer 101.41

A handwritten diagram for floating-point subtraction. It shows two numbers in scientific notation: 1.100×10^2 and 0.0859×10^2 . The numbers are aligned by their decimal points. The result of the subtraction is 1.0141×10^2 . Blue annotations include 'X10^2' next to each number and the result, and a bracket under the result labeled 'P=4'. A blue arrow labeled 'round' points to the fourth digit of the result.

Relative error around 0.006

Guard Digits IX

$$\epsilon = \frac{1}{2}\beta^{-p+1} = \frac{1}{2}10^{-2} = 0.005$$

Theorem

Using $p + 1$ digits for $x - y \Rightarrow$ relative rounding error
 $< 2\epsilon$ (ϵ : machine epsilon)

Proof:

- Assume $x > y$
- Assume $x = x_0.x_1 \cdots x_{p-1} \times \beta^0$
The proof is similar if it's not β^0
- If $y = y_0.y_1 \cdots y_{p-1}$ no error

Guard Digits X

x_0, x_1, \dots, x_{p-1}
 $0, y_1, \dots, y_{p-1}, y_p$

- If $y = 0.y_1 \dots y_p \Rightarrow$ 1 guard digit, exact $x - y$
 rounded to a closest number \Rightarrow relative error $\leq \epsilon$
- In general $y = 0.0 \dots 0 \underbrace{y_{k+1} \dots y_{k+p}}$
 \bar{y} : y truncated to $p + 1$ digits
 length = p

$$\begin{aligned} |y - \bar{y}| &< (\beta - 1)(\beta^{-p-1} + \beta^{-p-2} + \dots + \beta^{-p-k}) \\ &\leq \beta^{-p} \end{aligned} \quad (2)$$

$-p - 1$: we have $p + 1$ digits now

(Think about $p = 3, \beta = 10$, first digit truncated
 $\leq 9 \times 0.0001 = 9 \times 10^{-4}$)

$$\approx (\beta - 1) \cdot \beta^{-p-1} \cdot \left(1 - \left(\frac{1}{\beta} \right)^{p+1} \right) < (\beta - 1) \left(\beta^{-p-1} + \dots \right)$$

Guard Digits XI

$$\leq (\cancel{\beta^{-1}}) \beta^{-p-1}.$$

After y is truncated, we need to calculate

$$x - \bar{y} = \beta^{-p}$$

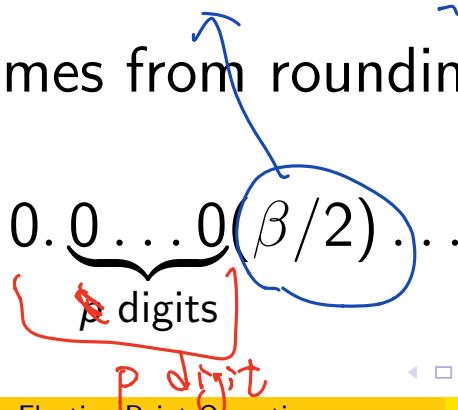
It's rounded to

$$x - \bar{y} + \delta$$

$$|\delta| \leq (\beta/2)\beta^{-p} = \epsilon$$

The inequality comes from rounding a number of $p+1$ digits

and round it to p digit



$$\frac{\beta}{\sum} \cdot \beta^{-p}$$

Guard Digits XII

max error



$$\text{error: } (x - y) - (\underbrace{x - \bar{y}}_{\delta}) = \bar{y} - y - \delta$$

Guard Digits XIII

case 1: if $x - y \geq 1$,

relative error

$$\begin{aligned}
 \epsilon &= \frac{\beta^{-p}}{\sum \beta^{-i}} = \frac{|\bar{y} - y - \delta|}{x - y} \leq \frac{|\bar{y} - y - \delta|}{1} \\
 &\leq \beta^{-p} [(\beta - 1)(\beta^{-1} + \dots + \beta^{-k}) + \beta/2] \\
 2\epsilon &= \frac{2\beta^{-p}}{\sum \beta^{-i}} = \beta^{-p} [(\beta - 1)\beta^{-k}(1 + \dots + \beta^{k-1}) + \beta/2] \\
 &= \beta^{-p} [(\beta - 1)\beta^{-k} \frac{1 - \beta^k}{1 - \beta} + \beta/2] \\
 &= \beta^{-p} [(1 - \beta^{-k}) + \beta/2] \quad \text{ignore } \beta^{-k} \\
 &< \beta^{-p} (1 + \beta/2) \leq 2\epsilon \\
 &= \frac{2 + \beta}{2 - \beta} \quad (\beta \geq 2)
 \end{aligned}$$

$$|\bar{y} - y - \delta| \leq |\bar{y} - y| + |\delta|$$

$$\beta^{-p} [(\beta - 1)(\beta^{-1} + \dots + \beta^{-k}) + \beta/2]$$

$$\beta^{-p} [(\beta - 1)\beta^{-k}(1 + \dots + \beta^{k-1}) + \beta/2]$$

$$\beta^{-p} [(\beta - 1)\beta^{-k} \frac{1 - \beta^k}{1 - \beta} + \beta/2]$$

$$\beta^{-p} [(1 - \beta^{-k}) + \beta/2]$$

$$\beta^{-p} (1 + \beta/2) \leq 2\epsilon$$

Guard Digits XIV

$$x - y < 1 \rightarrow \begin{cases} x - \bar{y} \leq 1 \\ x - \bar{y} > 1 \end{cases}$$

case 2: $x - \bar{y} \leq 1$: enough digits $\delta = 0$

the smallest $x - y$: (smallest x - largest y)

$$1.0 - 0.0 \dots 0 \rho \dots \rho > (\beta - 1)(\beta^{-1} + \dots + \beta^{-k})$$

k zeros, p ρ 's, $\rho = \beta - 1$, from (2) the relative error

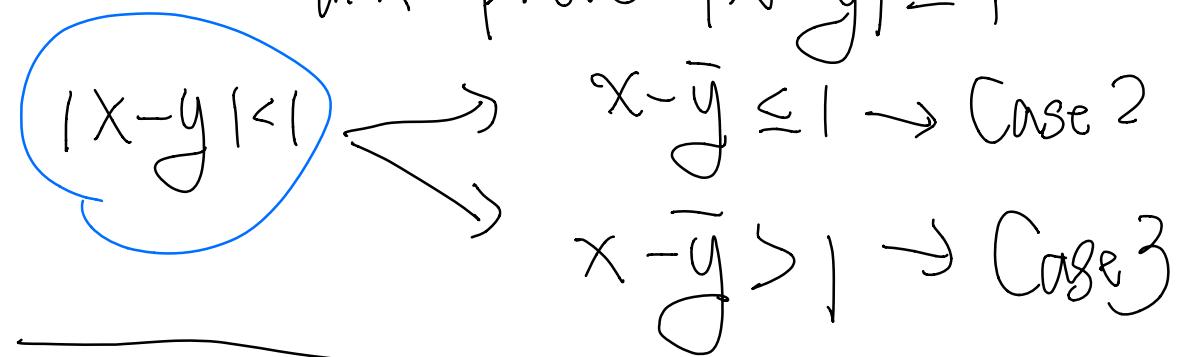
$$\begin{aligned} &\leq \frac{|\bar{y} - y - \delta|}{(\beta - 1)(\beta^{-1} + \dots + \beta^{-k})} \\ &< \frac{(\beta - 1)\beta^{-p}(\beta^{-1} + \dots + \beta^{-k})}{(\beta - 1)(\beta^{-1} + \dots + \beta^{-k})} = \beta^{-p} < 2\epsilon \end{aligned}$$

$\frac{2\beta}{\sum \beta^{-p}} \parallel$

case 3: $x - y < 1$ but $x - \bar{y} > 1$

We already prove: $|x-y| \geq |$

We will then prove $|x-y| \leq |$



Case 2: $(x-y) < 1$ and $x - \bar{y} \leq |$

After $y \rightarrow \bar{y}$, $x - \bar{y} \Rightarrow f = 0$

p+1 digits

$\overline{x_0.x_1x_2\dots x_{p-1}0}$

1.0000...

? . ? ----- ?
p+1 digits

if $x - \bar{y} = 1 \Rightarrow f = 0$

if $x - \bar{y} < 1 \Rightarrow$ We can normalize
this number

so that $f = 0$

ex. $0.\underline{9999} \Rightarrow \underline{1.999}$

Guard Digits XV

We show that this situation is impossible

If $x - \bar{y} = 1.\underbrace{0\cdots 1}_p \Rightarrow x - y \geq 1$: a contradiction

Why $x - y$ must be ≥ 1 :

$$-\beta^{-p} < y - \bar{y} < \beta^{-p} \quad |y - \bar{y}| < \beta^{-p} \neq 0.0\cdots 1$$
$$\Rightarrow -\bar{y} - \beta^{-p} < -y \Rightarrow x - y - \beta^{-p} < x - y$$

The difference between y and \bar{y} is $0.0\cdots 1_p$

- Conclusion: adding some guard digits can reduce the error

Especially when subtracting two nearby numbers

- Cost: the adder is one bit wider (cheap)

Most modern computers have guard digits

Cancellation I

$$1.12 \times 10^1$$

$$(1.11x10)^1$$

- Catastrophic cancellation and benign cancellation
- Catastrophic cancellation :

$$b = 3.34, a = 1.22, c = 2.28, b^2 - 4ac = 0.0292$$

$$b^2 \approx 11.2, 4ac \approx 11.1 \Rightarrow \text{answer} = 0.1$$

$$\text{error} = 0.1 - 0.0292 = 0.0708$$

$$\text{answer} = 0.0292 = 2.92 \times 10^{-2}$$

$$\text{ulps} = 0.01 \times 10^{-2} = 10^{-4}$$

$$0.0708 \approx 708 \text{ ulps} \rightarrow \text{huge!}$$

- Happens when subtracting two close numbers

Cancellation II

- Benign cancellation: subtracting exactly known numbers, by guard digits
- \Rightarrow small relative error
- In the example, b^2 and $4ac$ already contain errors

Avoid Catastrophic Cancellation I

- By rearranging the formula
- Example

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (3)$$

- If $b^2 \gg 4ac \Rightarrow$ no cancellation when calculating $b^2 - 4ac$ and $\sqrt{b^2 - 4ac} \approx |b|$
Then $-b + \sqrt{b^2 - 4ac}$ has a catastrophic cancellation if $b > 0$

Avoid Catastrophic Cancellation II

- Multiplying $-b - \sqrt{b^2 - 4ac}$, if $b > 0$

$$\frac{2c}{-b - \sqrt{b^2 - 4ac}} \quad (4)$$

- Use (3) if $b < 0$, (4) if $b > 0$ *if $b > 0 \Rightarrow -(\overbrace{b + \sqrt{b^2 - 4ac}}^{\Rightarrow \text{almost}})$*
- Difficult to remove all catastrophic cancellations, but possible to remove most by reformulations *no error incurs*
- Another example: $x^2 - y^2$

Assume $x \approx y$

$(x - y)(x + y)$ is better than $x^2 - y^2$

Avoid Catastrophic Cancellation III

x^2, y^2 may be rounded $\Rightarrow x^2 - y^2$ may be a catastrophic cancellation

$x - y$ by guard digit

$x - y$ must be better than $x^2 - y^2$

- A catastrophic cancellation is replaced by a benign cancellation

Of course x, y may have been rounded and $x - y$ is still a catastrophic cancellation.

Again, difficult to remove all catastrophic cancellations, but possible to remove some

Avoid Catastrophic Cancellation IV

- Calculating area of a triangle

$$A = \sqrt{s(s-a)(s-b)(s-c)}, s = \frac{a+b+c}{2} \quad (5)$$

a, b, c : length of three edges

If $a \approx b + c$, $s = (a + b + c)/2 \approx a$, $\underline{s - a}$ may have a catastrophic error

Example: $a = 9.00, b = c = 4.53$

$s = 9.03, A = 2.342$

Computed solution: $A = 3.04$, error ≈ 0.7

ulps = 0.01, error = 70 ulps

Avoid Catastrophic Cancellation V

- A new formulation by Kahan [1986] , $a \geq b \geq c$

$$A =$$

$$\frac{\sqrt{(a + (b + c))(c - (a - b))(c + (a - b))(a + (b - c))}}{4} \quad (6)$$

$A \approx 2.35$, close to 2.342

- HW 1-1: Calculate $A = 3.04$ using (5) and $A = 2.35$ using (6)

$$(5): \quad a = 9.00 \quad b = c = 4.53$$

$$S = \frac{9.00 + (4.53 + 4.53)}{2}$$

$$4.53 + 4.53 = 9.06$$

$$\begin{aligned} 9.06 + 9.00 &= 18.06 \\ &= 1.81 \end{aligned}$$

$$1.81 / 2 = 0.905 = 9.05 = S$$

$$S-a = 9.05 - 9.00 = 0.05$$

$$S-b = 9.05 - 4.53 = 4.52$$

$$S-c = 9.05 - 4.53 = 4.52$$

$$S \cdot (S-a) \cdot (S-b) \cdot (S-c) \quad 9.05$$

$$\begin{aligned} &= 0.4525 \times 20.4304 \quad \times 0.05 \\ &= (4.53 \times 10^{-1}) \times (2.04 \times 10^1) \overline{0.4525} \end{aligned}$$

$$= 9.24$$

$$\begin{array}{r} (S-b) \cdot \\ (S-c) \end{array} \begin{array}{r} 4.52 \\ \times 4.52 \\ \hline 9.04 \end{array}$$

$$\sqrt{9.24} = 3.039$$

$$\begin{array}{r} 2260 \\ 1808 \\ \hline 20.4304 \end{array}$$

$$\begin{array}{r} \approx 3.04 \\ \hline (S-S-a) \\ (S-b)(S-c) \end{array} \begin{array}{r} 4.53 \\ \times 2.04 \\ \hline 1812 \\ 9060 \\ \hline 9.2412 \end{array}$$

$$\begin{array}{r} \bar{(6)} \quad - \quad - \quad - \quad - \quad 9.2412 \\ a + (b+c) = 9.00 + (4.53 + 4.53) \\ = 9.00 + (9.06) \\ = 18.06 \approx 1.81 \times 10^1 \end{array}$$

$$\begin{array}{r} C - (a-b) = 4.53 - (9.00 - 4.53) \\ = 4.53 - (4.47) \\ = 0.06 \end{array}$$

$$C + (a-b) = 4.53 + (9.00 - 4.53)$$

$$= 4.53 + 4.47$$

$$= 9.00$$

$$a + (b-C) = 9.00 + (4.53 - 4.53)$$

$$= 9.00$$

$$1.81 \times 0.06 = 1.086 \approx 1.09$$

$$9.00 \times 9.00 = 81.00 = 8.10 \times 10^1$$

$$\begin{array}{r} 1.09 \\ \times 8.1 \\ \hline 8.829 \end{array}$$

$$\sqrt{8.83 \times 10^1} = 9.3968$$

≈ 9.40

$$9.40 / 4 \quad \begin{array}{c} \textcircled{=} \\ \hline \end{array} \quad 2.35$$

$\begin{array}{r} 235 \\ \hline 9.40 \\ -8 \\ \hline 140 \\ -12 \\ \hline 20 \end{array}$

Avoid Catastrophic Cancellation VI

Note: to get $A = 3.04$ you need to calculate s by

$$s = \frac{a + (b + c)}{2}$$

Note that for multiplication and square root we assume that exact calculation can be done and results are rounded.

- Conclusion: sometimes a formula can be rewritten to have higher accuracy using benign cancellation
- Only works if guard digit is used; most computers use guard digits now

Avoid Catastrophic Cancellation VII

- But reformulation is difficult!!
You may think that you will never need to do this
- Two real cases:
- Line 213-216 of tron.cpp of LIBLINEAR version 2.11
$$\frac{b^2 - 4ac}{\dots}$$

HW1-2: Check Eq. (13) of the paper
<http://www.csie.ntu.edu.tw/~cjlin/papers/logistic.pdf>
and explain how we avoid catastrophic cancellations

Avoid Catastrophic Cancellation VIII

We do not consider the latest version of LIBLINEAR because some more complicated settings have been used

- Probability outputs of LIBSVM

HW1-3: Repeat the experiment on page 5, line 12 of the paper

<http://www.csie.ntu.edu.tw/~cjlin/papers/plattprob.pdf>

Discuss what you found

HW 1-2:

Let $\alpha = \gamma$

$$|S + \alpha d| = \Delta_k - (13)$$

$$(S + \alpha d)^T (S + \alpha d) = \Delta_k^2$$

$$S^T S + \alpha S^T d + \alpha S^T d + \alpha^2 d^T d$$

$$\alpha^2 \boxed{d^T d} + 2\alpha \boxed{S^T d} + \boxed{S^T S - \Delta_k^2} = 0$$

(I think $\alpha > 0$, so take +)

$$\alpha = \frac{-2b + \sqrt{4b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - ac}}{a}$$

(reformulation)

$$\begin{aligned} X &= \frac{-b + \sqrt{b^2 - ac}}{a} \cdot \frac{(-b - \sqrt{b^2 - ac})}{(-b - \sqrt{b^2 - ac})} \\ &= \frac{b^2 - (b^2 - ac)}{-ab - a\sqrt{b^2 - ac}} \\ &= \frac{ac}{-ab - a\sqrt{b^2 - ac}} \end{aligned}$$

$$\begin{aligned} &= -\frac{c}{b + \sqrt{b^2 - ac}} \\ &= \left[\frac{\Delta_k^2 - S^T S}{S^T D + \text{rad}} \right] \cancel{\boxed{}}$$

Exactly Rounded Operations I

- Round then calculate \Rightarrow may not be very accurate
- Exactly rounded: compute exactly then rounded to the nearest \Rightarrow usually more accurate
- The definition of rounding
- $12.5 \Rightarrow 12$ or 13 ?
- Rounding up: $0, 1, 2, 3, 4 \Rightarrow$ down, $5, 6, 7, 8, 9 \Rightarrow$ up

Why called “rounding up”? Always up for 5

- Rounding even:
the closest value with even least significant digit

Exactly Rounded Operations II

50% probability up, 50% down

example: $12.5 \Rightarrow 12$; $11.5 \Rightarrow 12$

- Reiser and Knuth [1975] show rounding even may be better

Theorem

Let $x_0 = x$, $x_1 = (x_0 \ominus y) \oplus y, \dots, x_n = (x_{n-1} \ominus y) \oplus y$, if \oplus and \ominus are exactly rounded using rounding even, then $x_n = x, \forall n$ or $x_n = x_1, \forall n \geq 1$.

$x \ominus y$: computed solution

- Consider rounding up,

Exactly Rounded Operations III

$$\beta = 10, p = 3, x = 1.00, y = -0.555$$

$$x - y = 1.555, x \ominus y = 1.56, (x \ominus y) + y = \\ 1.56 - 0.555 = 1.005, x_1 = (x \ominus y) \oplus y = 1.01$$

$$x_1 - y = 1.565, x_1 \ominus y = 1.57, (x_1 \ominus y) + y = \\ 1.57 - 0.555 = 1.015, x_2 = (x_1 \ominus y) \oplus y = 1.02$$

Increased by 0.01 until $x_n = 9.45$

- Rounding even:

$$x - y = 1.555, x \ominus y = 1.56, (x \ominus y) + y = \\ 1.56 - 0.555 = 1.005, x_1 = (x \ominus y) \oplus y = 1.00$$

$$x_1 - y = 1.555, x_1 \ominus y = 1.56, (x_1 \ominus y) + y = \\ 1.56 - 0.555 = 1.005, x_2 = (x_1 \ominus y) \oplus y = 1.00$$

Exactly Rounded Operations IV

- How to implement “exactly rounded operations”?
 - We can use an array of words or floating-points
 - But you don’t have an infinite amount of spaces
 - Goldberg [1990] showed that using **3 guard digits** the result is the same as using exactly rounded operations

IEEE standard I

- IEEE 754 during 80s, now standard everywhere
- Two IEEE standards:
 - 754: specify $\beta = 2, p = 24$ for single, $\beta = 2, p = 53$ for double
 - 854 ($\beta = 2$ or 10 , does not specify how floating-point numbers are encoded into bits)
- Why IEEE 854 allows $\beta = 2$ or 10 but not other numbers:
 - 10 is the base we use
 - smaller β causes smaller relative error

IEEE standard II

smaller β : more precision. For example,

$$\beta = 16, p = 1 \text{ versus } \beta = 2, p = 4$$

- 4 bits for significand

$$\epsilon = \frac{16}{2} 16^{-1} = 1/2, \epsilon = \frac{2}{2} 2^{-4} = 1/16$$

We can see that ϵ of $\beta = 2, p = 4$ is smaller

- However, IBM/370 uses $\beta = 16$. Why? Two possible reasons:

First,

IEEE standard III

a number: 4 bytes = 32 bits

$\beta = 16, p = 6$, significand: $4 \times 6 = 24$ bits,

exponents: $32 - 24 - 1 = 7$ bits (1 bit for sign),

16^{-2^6} to $16^{2^6} = 2^{2^8}$

for $\beta = 2 \Rightarrow 9$ bits (-2^8 to $2^8 = 2^9$) for exponents,
 $\Rightarrow 32 - 9 - 1 = 22$ for significand

Same exponents, less significand for $\beta = 2$ (24 vs. 22)

Second,

Shifting: $\beta = 16$, less frequently to adjust
exponents when adding or subtracting two numbers

IEEE standard IV

For modern computers, this saving is not important

- Single precision: $\beta = 2, p = 24$ (23 bits as normalized), exponent 8, 1 bit for sign ($32 = 23 + 8 + 1$)
- An example: $176.625 = 1.0101100101 \times 2^7$

0 10000110 010110010100000000000000

1 of 1. . . is not stored (normalized)

- Biased exponent (described later in detail)

$$10000110 = 128 + 4 + 2 = 134, 134 - 127 = 7$$

Note that we have **negative** exponent

IEEE standard V



- Use rounding even

Binary	rounded	reason
10.00011	10.00	(< 1/2, down)
10.00110	10.01	(> 1/2, up)
10.11100	11.00	(1/2, up)
10.10100	10.10	(1/2, down)

round
even
on the
boundary

This example is from http:

//www.cs.cmu.edu/afs/cs/academic/class/
15213-s12/www/lectures/04-float-4up.pdf

- A summary

IEEE standard VI

IEEE	Fortran	C	Bits	Exp.	Mantissa
Single	REAL*4	float	32	8	24
Single-extended			44	≤ 11	32
Double	REAL*8	double	64	11	53
Double-extended	REAL*10	long double	≥ 80	≥ 15	≥ 64

$$32 = 8 + 24 \text{ but } 44 \neq 11 + 32$$

- $44 \neq 11 + 32$:

Hardware implementation of extended precision
normal don't use a hidden bit

(Remember we normalized each number so 1 is not stored)

IEEE standard VII

- It seems everyone is using double now
But single is still needed sometime (if memory is not enough)
- Minimal normalized positive number

$$1 \times 2^{-126} \approx 1.17 \times 10^{-38}$$

$$e_{\min} = -126$$

- 8 bits for exponent: 0 to 255. IEEE uses biased approach exponent

$$(0 \text{ to } 255) - 127 = -127 \text{ to } 128$$

IEEE standard VIII

- Why $e_{\min} = -126$ but $e_{\max} = 127$?
reasons: $1/2^{e_{\min}}$ not overflow, $1/2^{e_{\max}}$ underflow, but less serious
- Thus, -127 for 0 and denormalized numbers (discussed later), -126 to 127 for exponents, 128 for special quantity
- Motivation for extended precision: from calculator, display 10 digits but 13 internally
Some operations benefit from using more digits internally

IEEE standard IX

Example: binary-decimal conversion (Details not discussed here)

- Operations: IEEE standard requires results of addition, subtraction, multiplication and division exactly rounded.
- Exactly rounded: an array of words or floating-point numbers, expensive
- Goldberg [1990] showed using 3 guard digits the result is the same as using exactly rounded operations

Only little more cost

IEEE standard X

- Reasons to specify operations
 - run on different machines \Rightarrow results the same
- HW 2-1: write the binary format of -250 as a double floating-point number
- IEEE: square root, remainder, conversion between integer and floating-point, internal formats and decimal are correctly rounded (i.e. exactly rounded operations)
- Binary to decimal conversion
 - Think about reading numbers from files

IEEE standard XI

When writing a binary number to a decimal number and read it back, can we get the same binary number?

we might think $10^8 > 2^{24} \leftarrow P$

- Writing 9 digits is enough for short but do not forget Though $10^8 > 2^{24}$, 8 digits are not enough
- 17 for double precision (proof not provided).
Example:

numbers in a data set from Matrix market:

IEEE standard XII

```
> tail s1rmq4m1.dat
 8.2511736085618438E+01  2.5134528659924950
 -6.0042951255041466E+00 8.6599442206615524
 1.0026197619563723E+01 -1.3136837661844502
 -1.5108331040361231E+01 5.1423173996955084
 -1.1690286345961363E+03 1.6250726655807816
 8.2511736074473220E+01 1.5108331040361227
```

- Matrix market:

<http://math.nist.gov/MatrixMarket/>

A collection of matrix data

IEEE standard XIII

- Transcendental numbers:
e.g., \exp , \log
- IEEE does not require transcendental functions to be exactly rounded

Cannot specify the precision because they are arbitrarily long

→ cannot compute "exactly" and then rounded

Special quantities I

- On some computers (e.g., IBM 370) every bit pattern is a valid floating-point number
- For IBM 370, $\sqrt{-4} = \underline{2}$ and it prints an error message
IEEE : NaN, not a number
why $\sqrt{-4} = 2$ on IBM 370 \Rightarrow every pattern is a number
- Special value of IEEE:
 $+0, -0, \text{denormalized numbers}, +\infty, -\infty, \text{NaNs}$
(more than one NaN)

Special quantities II

numbers that are smaller than

1,000.... $2^{e_{\min}}$

- A summary

Exponent	significand	represents
$e = e_{\min} - 1$	$f = 0$	$+0, -0$
$e = e_{\min} - 1$	$f \neq 0$	$0.f \times 2^{e_{\min}}$
$e_{\min} \leq e \leq e_{\max}$		$1.f \times 2^e$
$e = e_{\max} + 1$	$f = 0$	$\pm\infty$
$e = e_{\max} + 1$	$f \neq 0$	NaN

- Why IEEE has NaN

Sometimes even $0/0$ occurs, the program can continue

Special quantities III

- Example: find $f(x) = 0$, try different x 's, even $0/0$ happens, other values may be ok.
- If $b^2 - 4ac < 0$

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

returns NaN

$-b + \text{NaN}$ should be NaN

In general when a NaN is in an operation, result is NaN

- Examples producing NaN:

Special quantities IV

Operation	NaN by
+	$\infty + (-\infty)$
\times	$0 \times \infty$
/	$0/0, \infty/\infty$
REM	$x \text{ REM } 0, \infty \text{ REM } y$
\sqrt{x}	$\sqrt{x} \text{ when } x < 0$



remainder after division

Infinity I

- $\beta = 10, p = 3, e_{\max} = 98, x = 3 \times 10^{70}$,
 x^2 overflow and replaced by 9.99×10^{98} ??
- In IEEE, the result is ∞
- Note $0/0 = \text{NaN}, 1/0 = \infty, -1/0 = -\infty$
⇒ nonzero divided by 0 is ∞ or $-\infty$
Similarly, $-10/0 = -\infty$, and $-10/-0 = +\infty$
(± 0 will be explained later)
- $3/\infty = 0, 4 - \infty = -\infty, \sqrt{\infty} = \infty$
- How to know the result?
replace ∞ with x , let $x \rightarrow \infty$

Infinity II

Example: $\lim_{x \rightarrow \infty} \frac{3}{x}$

$$3/\infty : \lim_{x \rightarrow \infty} 3/x = 0$$

If limit does not exist \Rightarrow NaN

- $x/(x^2 + 1)$ vs $1/(x + x^{-1})$

$x/(x^2 + 1)$: if x is large, x^2 overflow, $x/\infty = 0$ but not $1/x$.

$1/(x + x^{-1})$: x large, $1/x$ ok

$1/(x + x^{-1})$ looks better but what about $x = 0$?

$x = 0, 1/(0 + 0^{-1}) = 1/(0 + \infty) = 1/\infty = 0$

- If no infinity arithmetic, an extra instruction needed to test if $x = 0$, may interrupt the pipeline

Signed zero |

- Why do we have $+0$ and -0 ?

First, it is available (1 bit for sign)

if no sign, $1/(1/x) = x$ fails when $x = \pm\infty$

$x = \infty, 1/x = 0, 1/0 = +\infty$

$x = -\infty, 1/x = 0, 1/0 = +\infty$

- Compare $+0$ and -0 : if $(x == 0)$

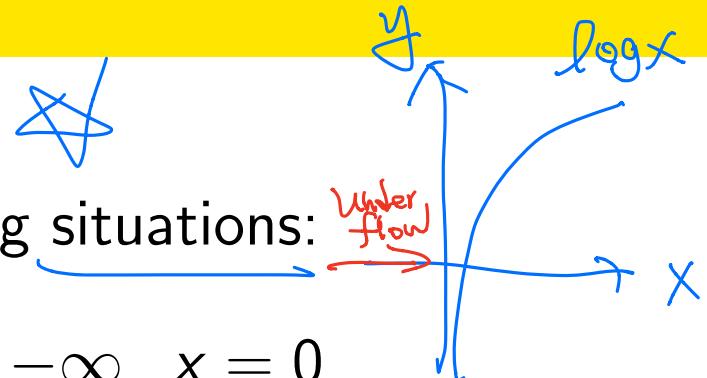
IEEE defines $+0$ = -0

- IEEE: $3 \times (+0) = +0, +0 / (-3) = -0$

Signed zero II

- ± 0 useful in the following situations:

$$\log x \equiv \begin{cases} -\infty & x = 0 \\ \text{NaN} & x < 0 \end{cases}$$



A small underflow negative number $\Rightarrow \log x$ should be NaN

x underflow \Rightarrow round to 0, if no sign, $\log x$ is $-\infty$ but not NaN

Signed zero III

- With ± 0 , we have

$$\log x = \begin{cases} -\infty & x = +0 \\ \text{NaN} & \underline{x = -0} \\ \text{NaN} & x < 0 \end{cases}$$

Positive underflow \Rightarrow round to $+0$

- Very useful in complex arithmetic

$\sqrt{1/z}$ and $1/\sqrt{z}$

$$z = -1, \sqrt{1/-1} = \sqrt{-1} = i, 1/\sqrt{-1} = 1/i = -i$$

$$\Rightarrow \sqrt{1/z} \neq 1/\sqrt{z}$$

Signed zero IV

- This happens because square root is multi-valued.

$$i^2 = (-i)^2 = -1$$

- However, by some restrictions (or ways of calculation), they can be equal

- $z = -1 = -1 + 0i,$

$$1/z = 1/(-1 + 0i) = -1 + (-0)i$$

$$\text{so } \sqrt{1/z} = \sqrt{-1 + (-0)i} = -i$$

$\Rightarrow -0$ is useful

See: Every CS book

- Disadvantage of $+0$ and -0 : P184

$x = y \Leftrightarrow 1/x = 1/y$ is destroyed

$x = 0, y = -0 \Rightarrow x = y$ under IEEE

$$\begin{aligned} & \frac{\sqrt{}}{-1+0i} \\ & \frac{-1-0i}{(-1+0i)(-1-0i)} \end{aligned}$$

$$-1-0i$$

Signed zero V

$$1/x = +\infty, 1/y = -\infty, +\infty \neq -\infty$$

- There are always pros and cons for floating-point design

HW 2-2 |

- If

if ($a < 0$)

always holds and b is neither too large nor too small, how do we guarantee

if $a/\max(b, 0.0) < 0$

always holds

- If $\max(b, 0.0)$ returns -0.0 , then it may not hold
- For the max function, should we use

$$(x>y)? \quad x:y$$

or

$$\frac{a(<0)}{\max(b, 0.0)} \rightarrow \text{Assume } D.O = +0.0$$

$$\Rightarrow ① b > 0 \Rightarrow \frac{-}{+} \Rightarrow - [0k)$$

$$② b < 0 \Rightarrow \frac{-}{0.0} \Rightarrow \lim_{x \rightarrow 0^+} \frac{-}{x}$$

$$③ b = +0.0 \Rightarrow -$$

or $\lim_{x \rightarrow 0^-} \frac{-}{x}$

$$\underbrace{\max(+0.0, 0.0)}$$

\Rightarrow depend on f.o.D
or -0.0

\hookrightarrow we want it to output f_{0,0}

$$④ b = -0.0$$

$$\Rightarrow \frac{-}{\max(-0.0, 0.0)}$$

\hookrightarrow we want to output +0.0

I think ($x > y$)? $x = y$ is better because

$\frac{+0.0}{-0.0} > +0.0$ is always false, and we want it to output f_{0,0}!

HW 2-2 II

$(x < y) ? \quad y : x$

- Your max need to return $+0.0$ but not -0.0 (Java)
- How to specifically assign $+0.0$ and -0.0 ?
double a=+0.0
double b=-0.0
- How to use subroutines to get the sign of a number?
- In a regular program, if you write 0.0, is it $+0.0$ or -0.0 ?

Find the statement in the manual saying that 0.0 means +0.0

- Do some experiments to check your arguments
- Use Java but not other systems

Denormalized number I

- $\beta = 10, p = 3, e_{\min} = -98, x = \underline{6.87} \times 10^{-97},$
 $y = 6.81 \times 10^{-97}$
- x, y are ok but $x - y = 0.6 \times 10^{-98}$ rounded to 0,
even though $x \neq y$
- How important to preserve

$$0.06 \times 10^{-97}$$

$$= 0.6 \times 10^{-98}$$

$$x = y \Leftrightarrow x - y = 0$$

$6.87 \times 10^{-97} \neq 6.81 \times 10^{-97}$, but $6.87 - 6.81 = 0$

\Rightarrow I think

- if $(x \neq y) \{z = 1/(x-y);\}$ \Rightarrow not good; round to 1.0×10^{-98}
The statement is true, but z becomes ∞
Tracking such bugs is frustrating

Denormalized number II

- IEEE uses denormalized numbers

Guarantee $x = y \Leftrightarrow x - y = 0$

Details of how this is done are not discussed here

- Most controversial part in IEEE standard
 - It caused long delay of the standard
- If denormalized number is used, 0.6×10^{-98} is also a floating-point number
- Remember we do not store 1 of $1.d \dots d$
- How to represent denormalized numbers ?

Recall for valid value, $e \geq e_{\min}$ and we have

$1.d \dots d \times 2^e$

Denormalized number III

Sort understand
why CJ use
 $e_{min}-1$ here

- For denormalized numbers, we let $e = e_{min} - 1$ and the corresponding value be

$$0.d \dots d \times 2^{e+1} = 0.d \dots d \times 2^{e_{min}}$$

denormalize!

- Why not

can't represent

$$\frac{1.d \dots d \times 2^{e_{min}-1}}{= 0.1dd \dots \times 2^{e_{min}}} \\ 0.0x \dots x \times 2^{e_{min}}$$

a 1

- $6.87 \times 10^{-97} - 6.81 \times 10^{-97} \Rightarrow$ underflow due to cancellation

Denormalized number IV

Underflow: smaller than the smallest floating-point number

- An example of using denormalized numbers

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i\end{aligned}$$

If c or $d > \sqrt{\beta} \beta^{e_{\max}/2} \Rightarrow$ overflow

Denormalized number V

overflow: larger than the maximal floating-point number

- Smith's formula

$$\frac{a + bi}{c + di} = \begin{cases} \frac{a+b(d/c)}{c+d(d/c)} + \frac{b-a(d/c)}{c+d(d/c)}i & \text{if } (|d| < |c|) \\ \frac{b+a(c/d)}{d+c(c/d)} + \frac{-a+b(c/d)}{d+c(c/d)}i & \text{if } (|d| \geq |c|) \end{cases}$$

avoid overflow

- However, using Smith's formula, without denormalized numbers

Denormalized number VI

If

$$\frac{(2+1i)}{4+2i} = \frac{(2+1i)(4-2i)}{16+4} = \frac{1}{20} (8+2+4i-4i) \approx \frac{1}{20} = 0.5$$

$$a = 2 \times 10^{-98}, b = 1 \times 10^{-98}, c = 4 \times 10^{-98},$$

$$d = 2 \times 10^{-98}$$

$$|d| < |c|$$

then

$$d/c = 0.5, \overbrace{c + d(d/c)}^{= 1} = 5 \times 10^{-98},$$

$$b(d/c) = 1 \times 10^{-98} \times 0.5 = 0$$

$$a + b(d/c) = 2 \times 10^{-98}$$

$$\frac{a+b(d/c)}{c+d(d/c)} = \frac{2 \times 10^{-98}}{5 \times 10^{-98}} = 0.4$$

Solution = 0.4, wrong

Denormalized number VII

If denormalized numbers are used, 0.5×10^{-98} can be stored,

$$a + b(d/c) = 2.5 \times 10^{-98} \Rightarrow 0.5$$

the correct answer

- Usually hardware does not support denormalized numbers directly
Using software to simulate
- Programs may be slow if a lot of underflow

Exception, Flags, Trap handlers I

- We have mentioned things like overflow, underflow
What are other exceptional situations ?
- Motivation: usually when exceptional condition like 1/0 happens, you may want to know
- IEEE requires vendors to provide a way to get status flags
- IEEE defines five exceptions: overflow, underflow, division by zero, invalid operation, inexact
- overflow: larger than the maximal floating-point number

Exception, Flags, Trap handlers II

Underflow: smaller than the smallest floating-point number

- Invalid:
 $\infty + (-\infty)$, $0 \times \infty$, $0/0$, ∞/∞ ,
 $x \text{ REM } 0$, $\infty \text{ REM } y$, \sqrt{x} , $x < 0$, any comparison involves a NaN
- Invalid returns NaN; NaN may not be from invalid operations
- Inexact: the result is not exact
 $\beta = 10, p = 3, 3.5 \times 4.2 = 14.7$ exact,
 $3.5 \times 4.3 = 15.05 \Rightarrow 15.0$ not exact

Exception, Flags, Trap handlers III

inexact exception is raised so often, usually we ignore it

Exception	when trap disabled	argument to handler
overflow	$\pm\infty$ or $\pm1.1\dots1 \times 2^{e_{\max}}$	$\text{round}(x2^{-\alpha})$
underflow	$0, \pm2^{e_{\min}}$, or denormal	$\text{round}(x2^{\alpha})$
division by zero	∞	operands
invalid	NaN	operands
inexact	$\text{round}(x)$	$\text{round}(x)$

- Trap handler: special subroutines to handle exceptions

Exception, Flags, Trap handlers IV

You can design your own trap handlers

- In the above table, “when trap disabled” means results of operations if trap handlers not used
- $\alpha = 192$ for single, $\alpha = 1536$ for double
reason: you cannot really store x
- Examples of using trap handlers described later

Compiler Options I

- Compiler may provide a way so the program stops if an exception occurs
- Easy for debugging
- Example: SUN's C compiler (I learned this on an old machine)
- Reason: gcc doesn't have this to explicitly detect exceptions
- **-ftrap=t**

Compiler Options II

- t: %all, %none, common, [no%]invalid, [no%]overflow, [no%]underflow, [no%]division, [no%]inexact.
- common: invalid, division by zero, and overflow.
- The default is -ftrap=%none.
- Example: -ftrap=%all,no%inexact means set all traps, except inexact.
- If you compile one routine with -ftrap=t, compile all routines of the program with the same -ftrap=t option
otherwise, you can get unexpected results.

Compiler Options III

- Example: on the screen you will see

Note: IEEE floating-point exception flags raised
Inexact; Underflow;

See the Numerical Computation Guide, ieee_flags

- gcc:
- -fno-trapping-math: default -ftrapping-math
Setting this option may allow faster code if one
relies on “non-stop” IEEE arithmetic
- -ftrapv

Compiler Options IV

Generates traps for signed overflow on addition, subtraction, multiplication

Trap Handler I

- Example:

```
do  {  
    . . .  
} while {not x >= 100;}
```

If $x = \text{NaN}$, an infinite loop

Any comparison involving NaN is wrong

- A trap handler can be installed to abort it
- Example:

Trap Handler II

Calculate $x_1 \times \cdots \times x_n$ may overflow in the middle
(the total may be ok!):

```
for (i = 1; i <= n; i++)
    p = p * x[i] ;
```

- $x_1 \times \cdots \times x_r, r \leq n$ overflow but $x_1 \times \cdots \times x_n$ may be in the range
- $e^{\sum \log(x_i)} \Rightarrow$ a solution but less accurate and costs more
- A possible solution

Trap Handler III

```
for (i = 1; i <= n; i++) {  
    if (p * x[i] overflow) {  
        p = p * pow(10,-a);  
        count = count + 1 ;  
    }  
    p = p * x[i] ;  
}  
p = p * pow(10, a*count) ;
```

An Example of Handlers I

- Example using SUN's numerical computation guide
Again, old. Reason of not using existing systems such a glibc: so you can have HW
- standard math library libm.a
exp, pow, log, ...
- On SUN machines, there are additional math library: libsunmath.a
exp2, exp10, ..., ieee_flags, ieee_handler, ieee_retrospective
- A program:

An Example of Handlers II

```
#include <stdio.h>
#include <sys/ieeefp.h>
#include <sunmath.h>
#include <siginfo.h>
#include <ucontext.h>

void handler(int sig, siginfo_t *sip,
             ucontext_t *uap)
{
    unsigned      code, addr;

    code = sip->si_code;
```

An Example of Handlers III

```
addr = (unsigned) sip->si_addr;
fprintf(stderr, "fp exception %x at
address %x \n", code, addr);
}

int main()
{
    double x;

    /* trap on common floating point
       exceptions */
    if (ieee_handler("set", "common", handler)
        != 0)
```

An Example of Handlers IV

```
printf("Did not set exception
       handler \n");

/* cause an underflow exception (not
   reported) */
x = min_normal();
printf("min_normal = %g \n", x);
x = x / 13.0;
printf("min_normal / 13.0 = %g \n", x);

/* cause an overflow exception
   (reported) */
```

An Example of Handlers V

```
x = max_normal();  
printf("max_normal = %g \n", x);  
x = x * x;  
printf("max_normal * max_normal = %g \n",  
      x);  
  
ieee_retrospective(stderr);  
return 0;  
}
```

- Result:

An Example of Handlers VI

```
min_normal = 2.22507e-308
min_normal / 13.0 = 1.7116e-309
max_normal = 1.79769e+308
fp exception 4 at address 10d0c
max_normal * max_normal = 1.79769e+308
```

Note: IEEE floating-point exception flags raised:
Inexact; Underflow;
IEEE floating-point exception traps enabled:
overflow; division by zero; invalid operation
See the Numerical Computation Guide, ieee_flags
ieee_handler(3M)

An Example of Handlers VII

- invalid, division, and overflow sometimes called common exceptions here
`ieee_handler("set", "common", handler)` means handlers used for common exceptions
- `min_normal / 13.0`: using denormalized numbers
handler: subroutines to handle exceptions
- HW 3-1: regenerate this example using GNU C library

An Example of Handlers VIII

- How to find GNU C library information: on linux, type
% info libc
check the category of “Arithmetics” and “Signal Handling”

The Use of Flags: An Example I

- Calculate x^n , n : integer

```
double pow(double x, int n)
{
    double tmp = x, ret = 1.0;

    for(int t=n; t>0; t/=2)
    {
        if(t%2==1) ret*=tmp;
        tmp = tmp * tmp;
    }
    return ret;
```

The Use of Flags: An Example II

}

$x^{16} = (x^2)^8 = \dots, x^{15} = x(x^2)^7$, treat x^2 as the new x

$$x^{15} = x(x^2)^7 = x(x^2)(x^4)^3 = x(x^2)(x^4)(x^8)^1$$

- If $n < 0$, we need to use

$$x^n = (1/x)^{-n} = 1/(x)^{-n}$$

$\text{pow}(1/x, -n)$ less accurate, $1/\text{pow}(x, -n)$ is better
There is already error on $1/x$

The Use of Flags: An Example III

Example: $2^{-5} = (1/2)^5$ and $1/(2^5)$

- A small problem on using $1/\text{pow}(x, -n)$:
if $\text{pow}(x, -n)$ underflow (i.e. when $x < 1, n < 0$),
either underflow trap handler or underflow status
flag set \Rightarrow incorrect
 x^{-n} underflow, x^n overflow or be in range
($e_{\min} = -126, 2^{-e_{\min}} = 2^{126} < 2^{127} = 2^{e_{\max}}$)
- Turn off overflow & underflow trap enable bits, save
overflow & underflow status bits
Compute $1/\text{pow}(x, -n)$

The Use of Flags: An Example IV

If neither overflow nor underflow status is set \Rightarrow restore them

If one is set, restore & calculate $\text{pow}(1/x, -n)$, which causes correct exception to occur

- Practically the calculation of $\text{pow}()$ is more complicated
 - e.g. google `e_pow.c` and `e_log.c`
- In `glibc-2.17/sysdeps/ieee754/dbl-64`, `e_pow.c` has 420 lines

The Use of Flags: An Example V

- Another example: calculate \arccos using \arctan

$$\arccos x = 2 \arctan \sqrt{\frac{1-x}{1+x}}$$

$$\cos \theta = x = 2 \cos^2 \frac{\theta}{2} - 1 = 1 - 2 \sin^2 \frac{\theta}{2}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{x+1}{2}}, \sin \frac{\theta}{2} = \sqrt{\frac{1-x}{2}}, \tan \frac{\theta}{2} = \sqrt{\frac{1-x}{1+x}}$$

Hence

$$\arccos x = 2 \arctan \sqrt{\frac{1-x}{1+x}}$$

The Use of Flags: An Example VI

- Consider $x = -1$
 $\arctan(\infty) = \pi/2 \Rightarrow \arccos(-1) = \pi$
- A small problem:
 $\frac{1-x}{1+x}$ causes the divide-by-zero flag set though
 $\arccos(-1)$ not exceptional
- Solution: save divide-by-zero flag, restore after
 \arccos computation

A Real Study I

- Let's start with a simple example

```
#include <stdio.h>
```

```
int main()
{
    float a = 123.123;
    printf("%.10f\n", a);
    printf("%.10f\n", a*a);
```

```
a = 123.125;
printf("%.10f\n", a);
```

A Real Study II

```
printf("%.10f\n", a*a);
```

```
}
```

- Results are

```
$gcc test.c;./a.out
```

```
123.1230010986
```

```
15159.2734375000
```

```
123.1250000000
```

```
15159.7656250000
```

```
$gcc -m32 test.c;./a.out
```

```
123.1230010986
```

A Real Study III

15159.2733995339

123.1250000000

15159.7656250000

- -m 32 generates code for a 32-bit environment
(because we don't have a 32-bit machine)
- That is, same code gives different results under 32 and 64-bit environments
- Why?

A Real Study IV

- On 32 bit, 387 floating-point coprocessor is used.
From gcc manual, “The temporary results are computed in 80-bit precision instead of the precision specified by the type, resulting in slightly different results compared to most of other chips.”
- In other words, they somehow **violate** IEEE standard
- But 123.123 has **infinite digits** after transformed to binary

A Real Study V

- Compiler options can help to make things more consistent. For example, `-ffloat-store`: “Do not store floating-point variables in registers, and inhibit other options that might change whether a floating-point value is taken from a register or memory.”

```
$gcc -ffloat-store test.c; ./a.out
```

```
123.1230010986
```

```
15159.2734375000
```

```
123.1250000000
```

```
15159.7656250000
```

```
$gcc -ffloat-store -m32 test.c; ./a.out
```

```
123.1230010986
```

A Real Study VI

15159.2734375000

123.1250000000

15159.7656250000

- Note that other issues such as order of operations can also affect results.
- Consider running a real example using a machine learning software LIBSVM
- 64 bit:

A Real Study VII

```
$ ./svm-train -c 100 -e 0.00001 heart_scale
....*...
optimization finished, #iter = 2872
nu = 0.148045
obj = -2526.925470, rho = 1.145512
nSV = 107, nBSV = 9
Total nSV = 107
```

- 32bit:

A Real Study VIII

```
$ ./svm-train -c 100 -e 0.00001 heart_scale
.....*...
optimization finished, #iter = 2819
nu = 0.148045
obj = -2526.925470, rho = 1.145515
nSV = 107, nBSV = 9
Total nSV = 107
```

- They are **different**
- Adding `-ffloat-store -mfpmath=387` is **not enough**
- 64 bit:

A Real Study IX

```
$ make clean; make
$ ./svm-train -c 100 -e 0.00001 heart_scale
rm -f *~ svm.o svm-train svm-predict svm-scale
g++ -Wall -Wconversion -O3 -fPIC -ffloat-store
.....
optimization finished, #iter = 2863
nu = 0.148045
obj = -2526.925470, rho = 1.145512
nSV = 107, nBSV = 9
```

A Real Study X

Total nSV = 107

- We also need to **disable all optimization**
- 64bit:

```
$ make clean; make
$ ./svm-train -c 100 -e 0.00001 heart_scale
rm -f *~ svm.o svm-train svm-predict svm-scale
g++ -ffloat-store -mfpmath=387 -c svm.cpp
g++ -ffloat-store -mfpmath=387 svm-train.c
g++ -ffloat-store -mfpmath=387 svm-predict.c
g++ -ffloat-store -mfpmath=387 svm-scale.c -
.....*....*
```

A Real Study XI

```
optimization finished, #iter = 3051
nu = 0.148045
obj = -2526.925470, rho = 1.145515
nSV = 107, nBSV = 9
Total nSV = 107
```

- 32 bit:

```
$ make clean; make
$ ./svm-train -c 100 -e 0.00001 heart_scale
rm -f *~ svm.o svm-train svm-predict svm-scale
g++ -m32 -ffloat-store -mfpmath=387 -c svm.o
g++ -m32 -ffloat-store -mfpmath=387 svm-train.o
```

A Real Study XII

```
g++ -m32 -ffloat-store -mfpmath=387 svm-pred
g++ -m32 -ffloat-store -mfpmath=387 svm-scal
.....*...
optimization finished, #iter = 3051
nu = 0.148045
obj = -2526.925470, rho = 1.145515
nSV = 107, nBSV = 9
Total nSV = 107
```

l