EE 131A
Probability
Instructor: Professor Roychowdhury

Homework Set 5
Thursday, October 29, 2015
Due: NA

1. *Problem 3.6* An information source produces binary triplets $\{000, 111, 010, 101, 001, 110, 100, 011\}$, corresponding probabilities $\{1/4, 1/4, 1/8, 1/8, 1/16, 1/16, 1/16, 1/16\}$, A binary code assigns a codeword of length $-\log_2 p_k$ to triplet k. Let X be the length of the string assigned to the output of the information source.

   (a) Show the mapping from S to S_x, the range of X.

   (b) Find the probabilities for the various values of X.

   Solution:

   (a) The mapping is

$$
\begin{array}{ccc}
000 & \rightarrow & 2 \\
111 & \rightarrow & 2 \\
010 & \rightarrow & 3 \\
101 & \rightarrow & 3 \\
001 & \rightarrow & 4 \\
110 & \rightarrow & 4 \\
100 & \rightarrow & 4 \\
011 & \rightarrow & 4
\end{array}
$$

   (b)

$$P[x = 2] = P[\{000, 111\}] = \frac{1}{2}$$
$$P[x = 3] = P[\{010, 101\}] = \frac{1}{4}$$
$$P[x = 4] = P[\{001, 110, 100, 011\}] = \frac{1}{4}$$

2. *Problem 3.13* Let X be a random variable with pmf $p_k = c/k^2$ for $k = 1, 2, ...$

   (a) Estimate the value of c numerically. Note that the series converges.

   (b) Find $P[X > 4]$.

   (c) Find $P[6 \leq X \leq 8]$.

   Solution:

   (a)

$$
\begin{aligned}
1 &= \sum_{k=1}^{\infty} \frac{c}{k^2} \\
&= c \sum_{k=1}^{\infty} \frac{1}{k^2} \\
&= c \frac{\pi^2}{6}
\end{aligned}
$$

We have $c \approx 0.608$.

(b)

$$P[x > 4] = 1 - c(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}) = 1 - 0.8656 = 0.344$$

(c)

$$P[6 \leq X \leq 8] = c(\frac{1}{36} + \frac{1}{49} + \frac{1}{64}) = 0.039$$

3. *Problem 3.49* Let X be binomial random variable that results from performance of n Bernoulli trials with probability of success p.

(a) Suppose that $X = 1$. Find the probability that the single event occurred in the kth Bernoulli trial.

(b) Suppose that $X = 2$. Find the probability that the two events occurred in the jth and kth Bernoulli trials where $j < k$

(c) In light of your answers to part a and part b in what sense are the successes distributed "completely at random" over the n Bernoulli trials?

Solution: (a) Let I_k denote the outcome of the kth Benoulli trials. The probability that the single event occurred in the kth trial is:

$$\begin{aligned} P[I_k = 1 | X = 1] &= \frac{P[I_k = 1 \text{ and } I_j = 0 \text{ for all } j \neq k]}{P[X = 1]} \\ &= \frac{p(1-p)^{n-1}}{\binom{n}{1} p(1-p)^{n-1}} \\ &= \frac{1}{n} \end{aligned}$$

Note this probablity is not dependent on k, equally likely to occur in any of n trials

(b) The probability that the two events occurred in the jth and kth Bernoulli trials is given by:

$$\begin{aligned} P[I_j = 1, I_k = 1 | X = 2] &= \frac{P[I_j = 1, I_k = 1 \text{ and } I_m = 0 \text{ for all } m \neq j, k]}{P[X = 2]} \\ &= \frac{p^2(1-p)^{n-2}}{\binom{n}{2} p^2(1-p)^{n-2}} \\ &= \frac{2}{n(n-1)} \end{aligned}$$

Note this probability is not dependent on j or k, equally likely to occur in any of $\binom{n}{2}$ trials

(c) From part a and part b, we can see the location of sucess are randomly distributed over $\binom{n}{k}$ trials.

4. *Problem 3.52* A sequence of characters is transmitted over a channel that introduces errors with probability $p = 0.01$

   (a) What is the pmf of N, the number of error-free characters between erroneous characters?

   (b) What is $E[N]$?

   (c) Suppose we want to be 99% sure that at least 1000 characters are received correctly before a bad one occurs. What is the appropriate value of p?

   Solution:(a) The pmf of N is given by:

$$P[N = k] = (1 - p)^k p \qquad k = 0, 1, 2, ...$$

(b) Geometric random variable has the mean:

$$E(X) = \frac{1 - p}{p}$$

(c)

$$\begin{aligned} P &= \sum_{k=1000}^{\infty} (1 - p)^k p \\ &= (1 - p)^{1000} = 0.99 \end{aligned}$$

Therefore, the appropriate value of p is given by $p = 1.005 \times 10^{-5}$.

5. *Problem 3.59* The number of page requests that arrive at a Web server is a Poisson random variable with an average of 6000 requests per minute.

   (a) Find the probability that there are no requests in a 100-ms period.

   (b) Find the probability that there are between 5 and 10 requests in a 100-ms period.

   Solution: λ=6000 reqs/mins = 100 reqs/ second, then $\alpha = \lambda t = 10$ for Poisson random variable.

(a) the probability no requests in a 100-ms period

$$P[N = 0] = e^{-10} = 4.54 \times 10^{-5}$$

(b) the probability of 5 to 10 requests in a 100-ms period

$$P[5 \leq N \leq 10] = \sum_{k=5}^{10} \frac{10^k}{k!} e^{-10} = 0.554$$

6. *Problem 3.66* A data center has 10,000 disk drives. Suppose that a disk drive fails in a given day with probability 10^{-3}.

   (a) Find the probability that there are no failures in a given day.

(b) Find the probability that there are fewer than 10 failures in two days.

(c) Find the number of spare disk drives that should be available so that all failures in a day can be replaced with probability 99%.

Solution:

a) $\lambda = 10^{-3} \cdot 10^4$disks/day $=$ 10disks/ day, then $\alpha = \lambda t = 10$ for Poisson random variable.

$$P[N = 0] = e^{-10} = 4.54 \times 10^{-5}$$

(b) the probability of fewer than 10 failures in a 2-day period

$$P[N < 10] = \sum_{k=0}^{9} \frac{20^k}{k!} e^{-20} = 4.995 \times 10^{-3}$$

(c) let $f(N) = \sum_{k=0}^{N} \frac{10^k}{k!} e^{-10}$, we have,

$$f(17) = 0.986 < 0.99$$

and,

$$f(18) = 0.993 > 0.99$$

So at least 18 disks should be available.