EE 131A
Probability
Instructor: Professor Roychowdhury

Homework Solution 4
Wednesday, October 21, 2015
Due: Thursday, October 29, 2015

1. Two players, A and B, play a series of points in a game with player A winning each point with probability p and player B winning each point with probability $q = 1 - p$. The first player to win n points wins the game. What is the probability that the player A wins?
Solutions:
This problem is a classic in the development of probability theory. Pascal and Fermat had correspondences over this problem. One can view this problem as a one player game with success probability p and probability of failure $q = 1-p$ at each round. Here we look at a more general case that the event of interest is that n successes occur before m failures. Denote this event by $P_{n,m}$. Due to the recursive nature of the problem, by conditioning on the outcome of the first trial,

$$P_{n,m} = pP_{n-1,m} + qP_{n,m-1}$$

In order to solve this equation, naturally one should continue conditioning on the following trials and note that $P_{0,m} = 1$ and $P_{n,0} = 0$. Solving this recursive formula is not an easy task.

In the following, we consider a solution not involving conditional probabilities due to Fermat. Suppose that the game keeps going on until round $m + n - 1$, even after reaching n successes or m failures. In order for n successes to occur before m failures, it is necessary and sufficient that there be at least n successes in the first $m + n - 1$ trials.(Argue why?) Now, we know that the probability of exactly k successes in $m + n - 1$ trials is $\binom{m+n-1}{k} p^k q^{m+n-1-k}$. Thus, the probability of interest is

$$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k q^{m+n-1-k}$$

In the special case of our problem where $m = n$, the answer is

$$P_{n,n} = \sum_{k=n}^{2n-1} \binom{2n-1}{k} p^k q^{2n-1-k}$$

2. *Problem 2.125* Compare the binomial probability law and the hypergeometric law introduced in Problem 2.54 as follows.

(a) Suppose a lot has 20 items of which five are defective. A batch of ten items is tested without replacement. Find the probability that k are found defective for k = 0,..., 10. Compare this to the binomial probabilities with $n = 10$ and $p = 5/20 = .25$.

(b) Repeat but with a lot of 1000 items of which 250 are defective. A batch of ten items is tested without replacement. Find the probability that k are found

defective for k = 0,..., 10. Compare this to the binomial probabilities with n = 10 and $p = 5/20 = .25$.
(*E2.54* A lot of 100 items contains k defective items. M items are chosen at random and tested. (a) What is the probability that m are found defective? This is called the hypergeometric distribution. (b) A lot is accepted if 1 or fewer of the M items are defective. What is the probability that the lot is accepted?)
Solutions:

(a)

$$P_H = \begin{cases} \frac{\binom{5}{k}\binom{15}{10-k}}{\binom{20}{10}} & \text{k} \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$$P_B = \binom{10}{k} 0.25^k 0.75^{10-k}$$

By comparing values of P_H and P_B for $k = 0, ..., 10$, we can see that they are very different.

(b)

$$P_H = \frac{\binom{250}{k}\binom{750}{10-k}}{\binom{1000}{10}}$$

$$P_B = \binom{10}{k} 0.25^k 0.75^{10-k}$$

By comparing values of P_H and P_B for $k = 0, ..., 10$, we can see that they are similar.

3. *Problem 2.126* Suppose that in Example 2.43, computer A sends each message to computer B simultaneously over two unreliable radio links. Computer B can detect when errors have occurred in either link. Let the probability of message transmission error in link 1 and link 2 be q_1 and q_2 respectively. Computer B requests retransmissions until it receives an error-free message on either link.

   (a) Find the probability that more than k transmissions are required.

   (b) Find the probability that in the last transmission, the message on link 2 is received free of errors.
   (*E2.43* Computer A sends a message to computer B over an unreliable radio link.The message is encoded so that B can detect when errors have been introduced into the message during transmission.)
   Solution:

   (a)

$$P[\{m > k\}] = (q_1 q_2)^k$$

   (b)

$$P[\text{b is error free}] = \frac{(1 - q_2)}{1 - q_1 q_2}$$

4. A fair coin is thrown n times. Show that the conditional probability of a head on any specified trial, given a total of k heads over the n trials, is $\frac{k}{n}$.
Solutions:
In conditional probability questions, it is crucial to properly define the events that you are dealing with. Let A be the event that k heads are flipped over n trials, and let B be the event that on a specified trial a head is flipped. Therefore, we are interested in evaluating $P(B|A)$. Using Bayes' formula,

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Apparently, since the coin is fair $P(B) = \frac{1}{2}$. Moreover, due to binomial rule, the probability that we flip k heads in n trials is $P(A) = \binom{n}{k}(\frac{1}{2})^n$. Now, we need to calculate $P(A|B)$, the probability of flipping k heads in n trials given that we flipped a head on a specified trial. We need to calculate the probability of event A given that we know there's already (at least) one head in the sequence of flips in that specified trial, no matter where it is along the sequence of trials. Given this, we need to calculate the probability that we flip the remaining $k-1$ heads in the remaining $n-1$ flips. Thus,

$$\binom{n-1}{k-1}(\frac{1}{2})^{k-1}(\frac{1}{2})^{n-1-(k-1)}$$

Plugging all these expressions back into the Bayes' formula for $P(B|A)$, the result follows

5. *Problem 2.100* Each of n broadcasts a message in a given time slot with probability p.

   (a)Find the probability that exactly one terminal transmits so the message is received by all terminals without collision.

   (b) Find the value of p that maximizes the probability of successful transmission in the previous part.

   (c) Find the asymptotic value of the probability of successful transmission as n becomes large.

   Solutions:

   (a)

$$P(\text{only one of } n \text{ broadcasts transmits}) = \binom{n}{1}p(1-p)^{n-1}$$

   (b) Take derivatives from the left hand side of the above equation with respect to p and setting it to zero, we obtain

$$n(1-p)^{n-1} - n(n-1)(1-p)^{n-2}p = 0,$$

which gives $p = \frac{1}{n}$.

(c) Plugging in $p = \frac{1}{n}$ in the formula in part (a) and taking the limit as n tends to infinity,

$$\lim_{n\to\infty} P(\text{successful transmission}) = \lim_{n\to\infty}(1-\frac{1}{n})^{n-1} = \frac{1}{e}$$

6. *Problem 3.13* Let X be a random variable with pmf $p_k = c/k^2$ for $k = 1, 2, ...$

(a) Estimate the value of c numerically. Note that the series converges.

(b) Find $P[X > 4]$.

(c) Find $P[6 \leq X \leq 8]$.

Solution:

(a)

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} \frac{c}{k^2} \\ &= c\sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= c\frac{\pi^2}{6} \end{aligned}$$

We have $c \approx 0.608$.

(b)

$$P[x > 4] = 1 - c(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}) = 1 - 0.8656 = 0.344$$

(c)

$$P[6 \leq X \leq 8] = c(\frac{1}{36} + \frac{1}{49} + \frac{1}{64}) = 0.039$$

7. *Problem 3.49* Let X be binomial random variable that results from performance of n Bernoulli trials with probability of success p.

(a) Suppose that $X = 1$. Find the probability that the single event occurred in the kth Bernoulli trial.

(b) Suppose that $X = 2$. Find the probability that the two events occurred in the jth and kth Bernoulli trials where $j < k$

(c) In light of your answers to part a and part b in what sense are the successes distributed "completely at random" over the n Bernoulli trials?

Solution: (a) Let I_k denote the outcome of the kth Benoulli trials. The probability that the single event occurred in the kth trial is:

$$\begin{aligned} P[I_k = 1 | X = 1] &= \frac{P[I_k = 1 \text{ and } I_j = 0 \text{ for all } j \neq k]}{P[X = 1]} \\ &= \frac{p(1-p)^{n-1}}{\binom{n}{1}p(1-p)^{n-1}} \\ &= \frac{1}{n} \end{aligned}$$

Note this probablity is not dependent on k, equally likely to occur in any of n trials

(b) The probability that the two events occurred in the jth and kth Bernoulli trials is given by:

$$
\begin{aligned}
P[I_j = 1, I_k = 1 | X = 2] &= \frac{P[I_j = 1, I_k = 1 \text{ and } I_m = 0 \text{ for all } m \neq j, k]}{P[X = 2]} \\
&= \frac{p^2(1-p)^{n-2}}{\binom{n}{2}p^2(1-p)^{n-2}} \\
&= \frac{2}{n(n-1)}
\end{aligned}
$$

Note this probability is not dependent on j or k, equally likely to occur in any of $\binom{n}{2}$ trials

(c) From part a and part b, we can see the location of sucess are randomly distributed over $\binom{n}{k}$ trials.