EE 131A
Probability
Instructor: Professor Roychowdhury

Homework Set 6
Thursday, November 6 , 2015
Due: Wednesday, November 12, 2015

1. *Problem 4.12* The cdf of the random variable X is given by:

$$F_x(x) = \begin{cases} 0 & x < -1 \\ 0.5 & -1 \leq x \leq 0 \\ (1+x)/2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

(a) Plot the cdf and identify the type of random variable.

(b) Find $P[X \leq -1]$,$P[X = -1]$,$P[X < 0.5]$,$P[-0.5 < X < 0.5]$,$P[X > -1]$,$P[X \leq 2]$,$P[X > 3]$.

Solution:

(a) The plot is given below

(b) From the cdf plot, we can see,

$$P[X \leq -1] = 0.5$$
$$P[X = -1] = 0.5$$

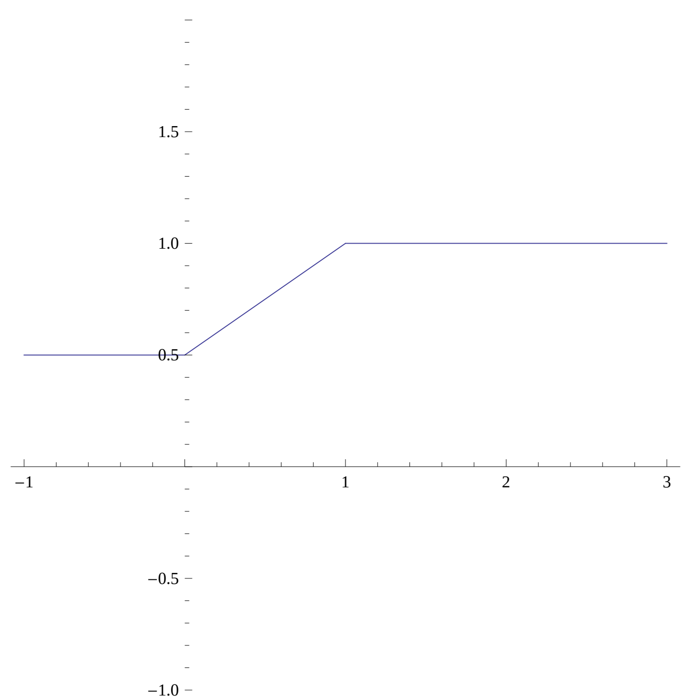

Figure 1: 4.12

$$P[X < 0.5] = 0.75$$
$$P[-0.5 < X < 0.5] = 0.25$$
$$P[x > -1] = 0.5$$
$$P[x \leq 2] = 1$$
$$P[x > 3] = 0$$

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2. *Problem 4.13* A random variable X has cdf,

$$F_x(x) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1}{4}e^{-2x} & x \geq 0 \end{cases}$$

(a) Plot the cdf and identify the type of random variable.

(b) Find $P[X \leq 2]$,$P[X = 0]$,$P[X < 0]$,$P[2 < X < 6]$,$P[x > 10]$.

Solution:

(a) X is an exponential random variable and the plot of the cdf function is given below

(b)We have,

$$P[X \leq 2] = 1 - \frac{1}{4}e^{-2 \cdot 2} = 0.9954$$

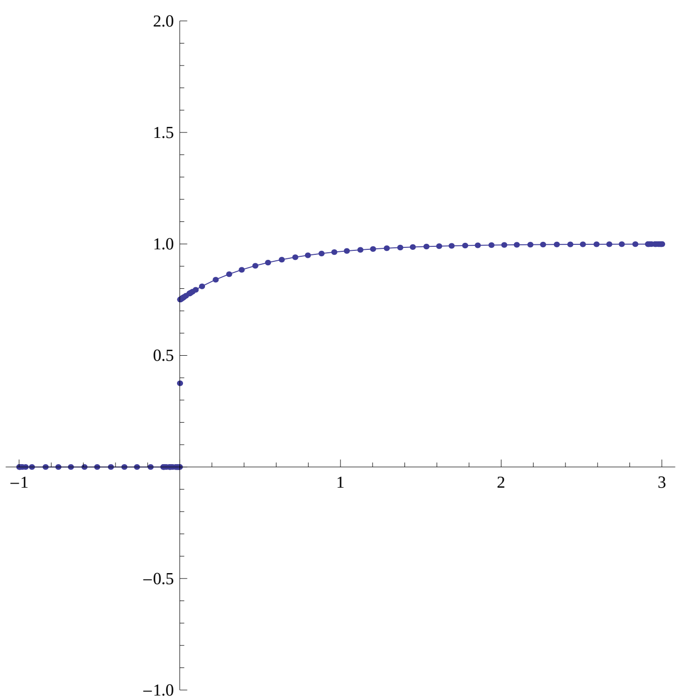

Figure 2: 4.13

$$P[X = 0] = 1 - \frac{1}{4}e^{-2 \cdot 0} = 0.75$$

$$P[X < 0] = 0$$

$$P[2 < X < 6] = 1 - \frac{1}{4}e^{-2 \cdot 6} - 1 + \frac{1}{4}e^{-2 \cdot 2} = 0.0046$$

$$P[x > 10] = 1 - 1 + \frac{1}{4}e^{-2 \cdot 10} = 5.15 \cdot 10^{-10}$$

3. *Problem 4.16* A random variable X has cdf,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 0.5 + c\sin^2(\pi x/2) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

   (a) What values can c assume?

   (b) Plot the cdf.

   (c) Find $P[X > 0]$.

Solution:

(a) Since the CDF function is right continuous and non-decreasing, c can only take $c = 1/2$, because $F_X(1) = 1$.

(b) The plot is give for the case $c = 1/2$

(c) We can see,

$$F_X(0) = \frac{1}{2}$$

as a result,

$$P[X > 0] = 1 - 0.5 = 0.5$$

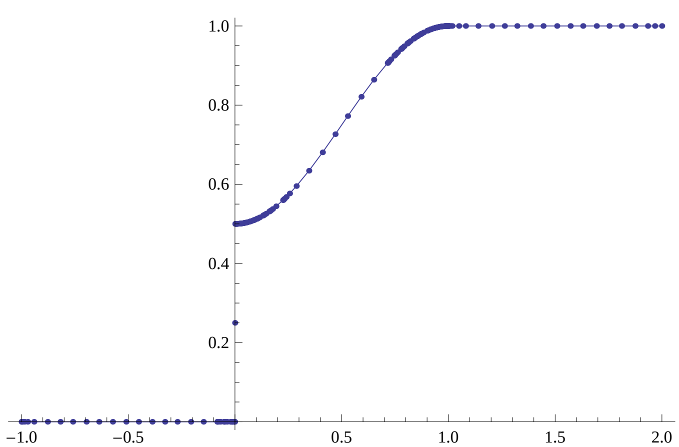

Figure 3: 4.16

4. *Problem 4.17* A random variable X has pdf,

$$F_x(x) = \begin{cases} c(1 - x^2) & -1 \le x \le 1 \\ 0 & elsewhere. \end{cases}$$

(a) Find c and plot the pdf.

(b) Plot the cdf of X.

(c) Find $P[X = 0]$,$P[0 < X < 0.5]$,$P[| X - 0.5 |< 0.25]$.

Solution:

(a)

$$1 = c \int_{-1}^{1} (1 - x^2) dx \rightarrow c = \frac{3}{4}$$

(b)

(c)

$$P[X = 0] = 0$$

$$P[0 < X < 0.5] = \frac{11}{32}$$

$$P[| X - 0.5 |< 0.25] = P[0.25 < X < 0.75] = \frac{3}{4}[(\frac{3}{4} + 1) - \frac{1}{3}((\frac{9}{4})^3 + 1)] = 0.2734$$

5. *Problem 4.38* A binary transmission system sends a 0 bit using a -1 voltage signal and a 1 bit by transmitting a $+1$, The received signal is corrupted by noise N that has a Laplacian distribution with parameter α. Assume that 0 bits and 1 bits are equiprobable.

(a) Find the pdf of the received signal $Y = X + N$, where X is the transmitted signal, given that a 0 was transmitted; that a 1 was transmitted.

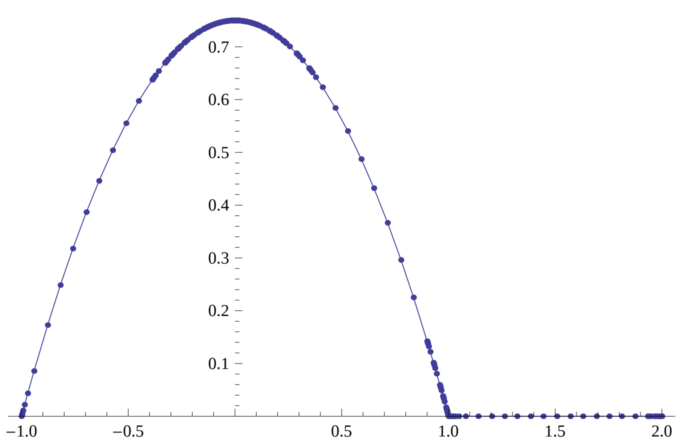

Figure 4: 4.17 pdf

(b) Suppose that the receiver decides a 0 was sent if $Y < 0$, and a 1 was sent if $Y \geq 0$. What is the probability that the receiver makes an error given that a $+1$ was transmitted? a -1 was transmitted?

(c) What is the overall probability of error?

Solution:

(a)

$$\begin{aligned} f_Y(x|B_0) &= & F_N(x+1) \\ &= & \frac{1}{2}\alpha e^{-\alpha|x+1|} \\ f_Y(x|B_1) &= & F_N(x-1) \\ &= & \frac{1}{2}\alpha e^{-\alpha|x-1|} \\ f_Y(x) &= & \frac{1}{2}(F_N(x+1) + F_N(x-1)) \\ &= & \frac{1}{4}\alpha[e^{-\alpha|x+1|} + e^{-\alpha|x-1|}] \end{aligned}$$

(b)

$$P[Y < 0|B_1] = P[N < -1] = \frac{1}{2}e^{-\alpha}$$

$$P[Y \geq 0|B_0] = P[N \geq 1] = \frac{1}{2}e^{-\alpha}$$

(c)

$$P_{err} = 0.5 \cdot P[Y < 0|B_1] + 0.5 \cdot P[Y \geq 0|B_0] = \frac{1}{2}e^{-\alpha}$$

6. *Problem 4.66* Let X be a Gaussian random variable with mean m and variance σ^2.

(a) Find $P[X \leq m]$.

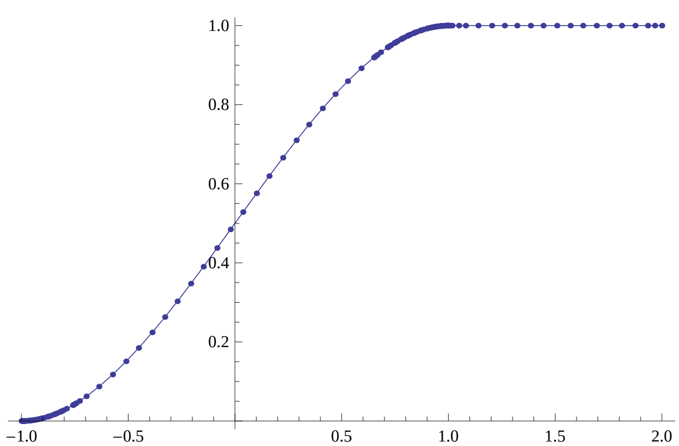

Figure 5: 4.17 cdf

(b) Find $P[|X-m| < k\sigma]$, for $k = 1, 2, \cdots, 6$.

(c) Find the value of k for which $Q(k) = P[X > m + k\sigma] = 10^{-j}$, for $j = 1, 2, \cdots, 6$.

Solution:

(a) $P[X \leq m] = 1/2$

(b) $P[|X-m| < k\sigma] = P[\frac{|X-m|}{\sigma} < k] = P[-k < \frac{X-m}{\sigma} < k] = 2\Phi(k) - 1$.

Looking up the table for Φ function, $\Phi(1) = 0.8413$, $\Phi(2) = 0.9772$, $\Phi(3) = 0.9987$, the rest are very close to 1.

(c) looking up the table for Q-function $Q(k) = P[\frac{X-m}{\sigma} > k]$, taking the value 10^{-1}, yields $k = 1.23$.

Likewise, $Q(k) = 10^{-2}$ gives $k = 2.3$, and we have $Q(3.08) = 10^{-3}$, $Q(3.7) = 10^{-4}$, $Q(4.27) = 10^{-5}$, and $Q(4.73) = 10^{-6}$.