EE 131A
Probability
Instructor: Professor Roychowdhury

Homework Solution 1
Wednesday, September 30, 2015
Due: Thursday October 8, 2015

1. *Problem 2.2* A die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.

   (a) Find the sample space.

   (b) Find the set A corresponding to the event "the number of dots in first toss is not less than number of dots in second toss"

   (c) Find the set B corresponding to the event "number of dots in first toss is 6"

   (d) Does A imply B or does B imply A?

   (e) Find $A \cap B^c$ and describe this event in words.

   (f) Let C correspond to the event "number of dots in dice differs by 2". Find $A \cap C$

   Solution: (a) We denote the outcome of this experiment as a pair of numbers (x, y) where x is number of dots in the first toss and y is the number of dots in the second toss. Then sample space S is given by $S=$
   $\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)$
   $(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$
   $(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$
   $(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$
   $(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)$
   $(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

   (b) $A = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4)$
   $(5,1),(5,2),(5,3),(5,4),(5,5),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

   (c) $B = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

   (d) We have $B \subset A$, so B implies A.

   (e) $A \cap B^c = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4)$
   $(5,1),(5,2),(5,3),(5,4),(5,5)\}$ and the event means "The first toss cannot be less than the second toss, and the first toss cannot be six."

   (f) $A \cap C = \{(6,4),(5,3),(4,2),(3,1)\}$

2. *Problem 2.8* A number U is selected uniformly at random from the unit interval. Let the events A and B be

$A =$ "U differs from $\frac{1}{2}$ by more than $\frac{1}{4}$",
and $B =$ "$1 - U$ is less than $\frac{1}{2}$".

Find the events $A \cap B$, $A^c \cap B$, and $A \cup B$.
Solution:

$$A = \{U \in (0, \frac{1}{4})\} \cup \{U \in (\frac{3}{4}, 1)\}$$
$$B = \{U \in (\frac{1}{2}, 1)\}$$
$$A \cap B = \{U \in (\frac{3}{4}, 1)\}$$
$$A^c \cap B = \{U \in (\frac{1}{2}, \frac{3}{4})\}$$
$$A \cup B = \{U \in (0, \frac{1}{4})\} \cup \{U \in (0, \frac{1}{4})\}$$

3. *Problem 2.10* Use Venn diagrams to verify the set identities given in
Eqs.(2.2) $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
Eqs.(2.3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
You will need to use different colors or difference shadings to denote the various regions clearly.
Solution: Eqs.(2.2) is verified in Figure 1 and Eqs.(2.3) is verified in Figure 2

4. *Problem 2.12* Show that if $A \cup B = A$ and $A \cap B = A$ then $A = B$
Solution: If $A \cup B = A$, then $B \subseteq A$.
If $A \cap B = A$, then $A \subseteq B$.
Therefore, if $A \cup B = A$ and $A \cap B = A$, then $A = B$

5. *Problem 2.19* A random experiment has sample space $S = \{-1, 0, +1\}$.

(a) Find all the subsets of S.

(b) The outcome of a random experiment consists of pairs of outcomes from S where the elements of the pair cannot be equal. Find the sample space S' of this experiment. How many subsets does S' have?
Solution:

(a) Φ, S, $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$, $\{-1, 1\}$, $\{0, 1\}$

(b)
$$S' = \{(-1, 0), (0, -1), (-1, 1), (1, -1), (0, 1), (1, 0)\}$$

There are $2^6 = 64$ subsets.

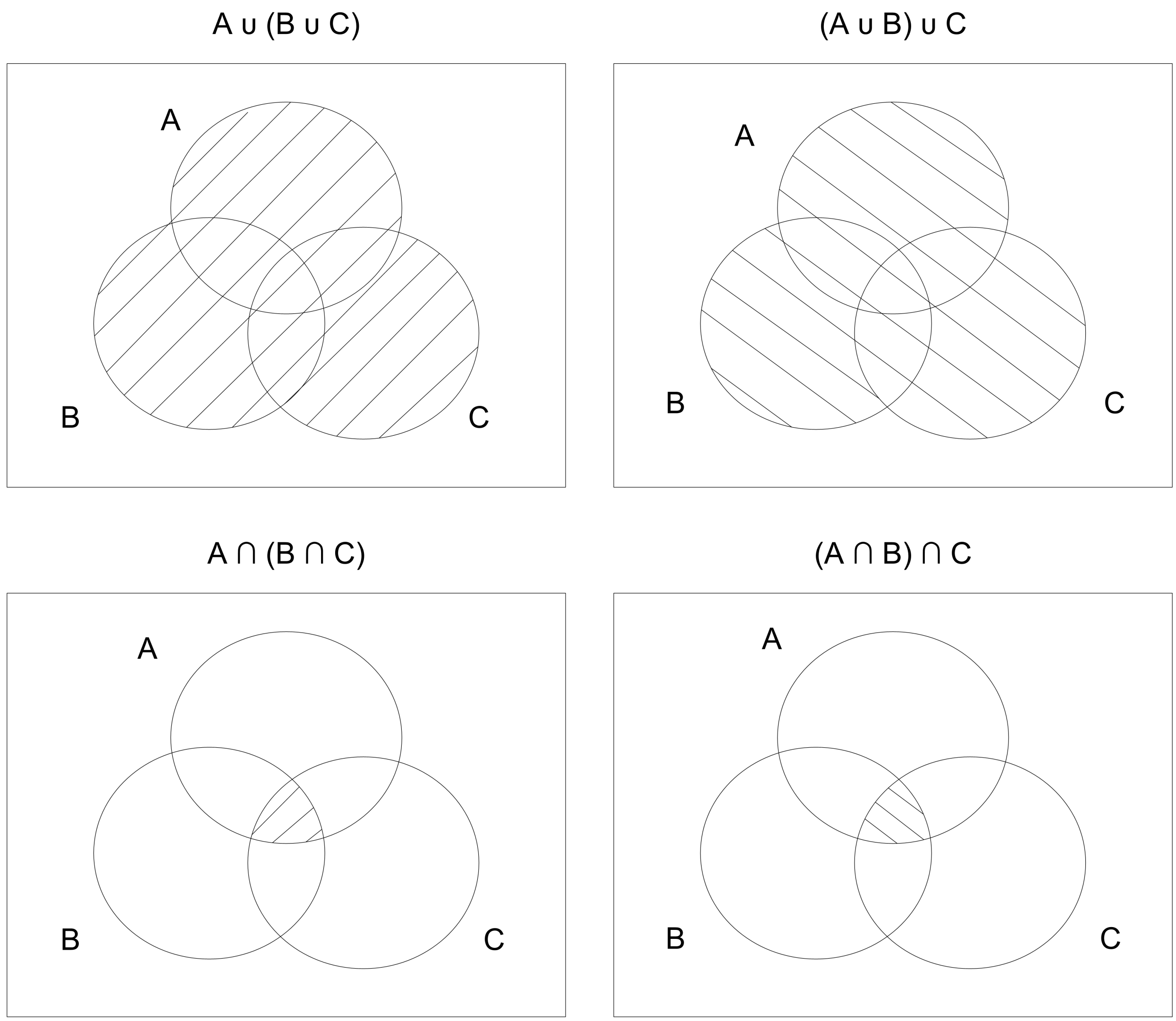

Figure 1: Venn diagrams for associative property

6. *Problem 2.34* A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$,$B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$

   (a) Find the probability of A,B, $A \cap B$, $A \cap C$.

   (b) Find the probability of $A \cup B$,$A \cup C$,$A \cup B \cup C$, first, by directly evaluating the sets and then their probabilities, and second, by using the appropriate axioms or corollaries.

   Solution:

   (a)

$$P[A] = \frac{1}{3} length([-1, 0]) = \frac{1}{3}$$

$$P[B] = \frac{1}{3} length([0, 1]) = \frac{1}{3}$$

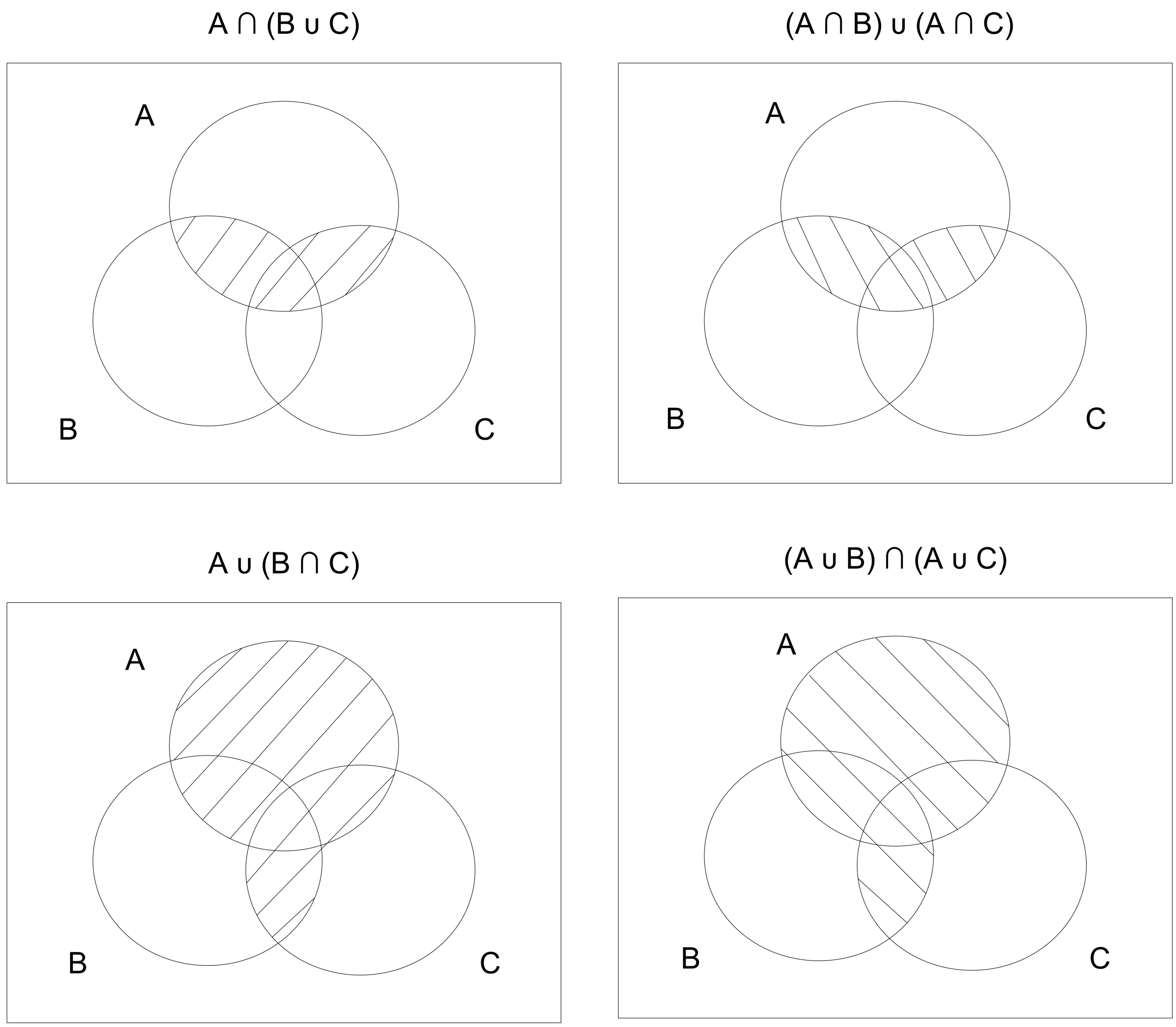

Figure 2: Venn diagrams for distributive property

$$P[A \cap B] = \frac{1}{3} length(\Phi) = 0$$

$$P[A \cap C] = \frac{1}{3} length(\Phi) = 0$$

(b)

$$P[A \cup B] = \frac{1}{3} length([-1, 1]) = \frac{2}{3}$$

$$P[A \cup C] = \frac{1}{3} length([-1, 0] \cup [0.75, 2]) = \frac{3}{4}$$

Using axioms:

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = \frac{2}{3}$$

$$P[A \cup C] = P[A] + P[C] - P[A \cap C] = \frac{3}{4}$$

7. *Problem 2.36* The lifetime of a device behaves according to the probability law $P[(t, \infty)] = 1/t$ for $t > 1$. Let A be the event " lifetime is greater than 4, " and B the event "lifetime is greater than 8."

   (a) Find the probability of $A \cap B$ and $A \cup B$

   (b) Find the probability of the event "lifetime is greater than 6 but less than or equal to 12."

   Solution:

   (a):

$$P[A \cap B] = P[B] = \frac{1}{8}$$

$$P[A \cup B] = P[A] = \frac{1}{4}$$

   (b):

$$P = P[(6, 12)] = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$