EE 131A Homework Solution 3
Probability Wednesday, October 15, 2015
Instructor: Professor Roychowdhury Due: Wednesday, October 22, 2015

1. A deck of 52 playing cards is shuffled and the cards turned up one at a time until the first ace appears. Is the next card -that is, the card following the first ace- more likely to be the ace of spades or the two of clubs? What is the probability of the ace of spades showing up right after the first ace? What is the probability of the two of clubs showing up right after the first ace?

   Solutions: If we pick cards one by one, we know that there are 52! possible permutations of the cards. Now we want to find those where the ace of spades is showing up right after the first ace. To count all such cases we find all possible permutations of the other cards which is 51! and in any one of these cases insert the ace of spades right after the first ace. This will give us all the possible permutations where the ace of spades shows up after the first ace. The probability of this event is $\frac{51!}{52!} = \frac{1}{52}$.
   We can calculate the probability of the two of clubs showing right after the first ace in the same way, so it will be the same and equal to $\frac{1}{52}$.

2. A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 possible regions. Let $1 - \beta_i$ denote the probability that the plane will be found upon a search of the ith region when the plane is, in fact, in that region, $i =$ 1,2,3. (The constants β_i are called overlook probabilities because they represent the probability of overlooking the plane; they are generally attributable to the geographical and environmental conditions of the regions.) What is the conditional probability that the plane is in the ith region, given that a search of region 1 is unsuccessful, $i =$ 1,2,3?

   Solutions: Let R_i be the event that the plane is in region i and E the event that search in region 1 was unsuccessful. Using Bayes' formula:

$$\begin{aligned}
P(R_1|E) &= \frac{P(E|R_1)P(R_1)}{P(E)} \\
&= \frac{P(E|R_1)P(R_1)}{\sum_{i=1}^{3} P(E|R_i)P(R_i)} \\
&= \frac{\beta_1 . \frac{1}{3}}{\beta_1 . \frac{1}{3} + 1 . \frac{1}{3} + 1 . \frac{1}{3}} \\
&= \frac{\beta_1}{\beta_1 + 2}
\end{aligned}$$

   For $i = 2, 3$:

$$
\begin{aligned}
P(R_i|E) &= \frac{P(E|R_i)P(R_i)}{P(E)} \\
&= \frac{1.\frac{1}{3}}{\beta_1.\frac{1}{3}+1.\frac{1}{3}+1.\frac{1}{3}} \\
&= \frac{1}{\beta_1+2}
\end{aligned}
$$

3. Consider two boxes, one containing 1 black and 1 white marble, the other 2 black and 1 white marble. A box is selected at random, and a marble is drawn from it at random. What is the probability that the marble is black? What is the probability that the first box was the one selected given that the marble is white?

   Solutions: Let B and W be the events that the marble is black and white, respectively, and let B_i be the event that box i is chosen. Then, $P(B) = P(B|B_i)P(B_i) + P(B|B_2)P(B_2) = \frac{1}{2}.\frac{1}{2} + \frac{2}{3}.\frac{1}{2}$

   $P(B_1|W) = \frac{P(W|B_1)P(B_1)}{P(W)} = \frac{\frac{1}{2}.\frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}$

4. There is a 50-50 chance that the queen carries the gene for hemophilia. If she is a carrier, then each prince has a 50-50 chance of having hemophilia. If the queen has had three princes without the disease, what is the probability that the queen is a carrier? If there is a fourth prince, what is the probability that he will have hemophilia?

   Solutions: Let C be the event that the queen is a carrier and E the event that she has 3 children without hemophilia:

$$
\begin{aligned}
P(C|E) &= \frac{P(E|C)P(C)}{P(E)} \\
&= \frac{P(E|C)P(C)}{P(E|C)P(C)+P(E|C^c)P(C^c)} \\
&= \frac{(1/2)^3.(1/2)}{(1/2)^3.(1/2)+1.(1/2)} = \frac{1}{9}
\end{aligned}
$$

   Now if we name the event that the fourth child has hemophilia H, we have:

$$
\begin{aligned}
P(H|E) &= P(H|C,E)P(C|E) + P(H|C^c,E)P(C^c|E) \\
&= (1/2).(1/9) + 0.(8/9) = \frac{1}{18}
\end{aligned}
$$

   In other words, from the information we have, we calculate the probability that the queen is a carrier, and then from there the probability that the new child would have the disease.

5. A and B are involved in a duel. The rules of the duel are that they are to pick up their guns and shoot at each other simultaneously. If one or both are hit, then the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each shot of A will hit B with probability p_A, and each shot of B will hit A with probability p_B. What is

   (a) the probability that A is not hit?

   (b) the probability that both duelists are hit?

   (c) the probability that the duel ends after the nth round of shots?

   (d) the conditional probability that the duel ends after the nth round of shots given that A is not hit?

   (e) the conditional probability that the duel ends after the nth round of shots given that both duelists are hit?

   Solutions: Consider the final round of the duel. Let $q_x = 1 - p_x$

   (a)

$$P\{\text{A not hit}\} = P\{\text{A not hit, B hit}\}/P\{\text{at least one is hit}\} = \frac{q_B p_A}{(1 - q_A q_B)}$$

   (b)

$$P\{\text{both hit}\} = P\{\text{both hit}\}/P\{\text{at least one hit}\} = \frac{p_B p_A}{(1 - q_A q_B)}$$

   (c)

$$(q_A q_B)^{n-1}(1 - q_A q_B)$$

   (d)

$$P\{\text{n rounds}|\text{A unhit}\} = \frac{(q_A q_B)^{n-1} p_A q_B}{q_B p_A/(1 - q_A q_B)} = (q_A q_B)^{n-1}(1 - q_A q_B)$$

   (e)

$$P\{\text{n rounds}|\text{both hit}\} = \frac{(q_A q_B)^{n-1} p_A p_B}{p_B p_A/(1 - q_A q_B)} = (q_A q_B)^{n-1}(1 - q_A q_B)$$