EE 131A — Homework Solution 2
Probability — Wednesday, October 7, 2015
Instructor: Professor Roychowdhury — Due: Thursday, October 15, 2015

1. *Problem 2.35* A number x is selected at random in the interval $[-1, 2]$. Numbers from the subinterval $[0, 2]$ occur half as frequently as those from $[-1, 0)$.

   a) Find the probability assignment for an interval completely within $[-1, 0)$; completely within $[0, 2]$; and partly in each of the above intervals.

   b) Repeat *Problem 2.34* with this probability assignment.

   Solution:

   a) There are two possible ways to interpret the problem.

   - When the probability of choosing a point from an interval of given length lying in $[-1, 0)$ is twice as much as the probability of an interval of the same size lying in $[0, 2]$ and probability mass is distributed uniformly over each interval.

     If the probability distribution over the whole interval $[-1, 2]$ was uniform, the solution was straightforward. You can deal with uneven probabilities by enlarging the intervals proportional to the weights and treat the problem as an evenly distributed weight problem. Therefore, for any interval I,

     $$P[I] = k|I \cap [0, 2]| + 2k|I \cap [-1, 0)|,$$

     where k is the probability of an interval of unit size lying in $[0, 2]$. Summing the mass over the whole interval yields

     $$1 = P([-1, 2]) = P([-1, 0)) + P([0, 2]) = 2k + 2k = 4k,$$

     hence $k = \frac{1}{4}$. Therefore

     $$P[I \in [-1, 0)] = \frac{1}{2}|I|$$

     $$P[I \in [0, 2]] = \frac{1}{4}|I|$$

     $$P[I \in [-1, 2]] = \frac{|I \cap [0, 2]| + 2|I \cap [-1, 0)|}{4}$$

   - When the probability of choosing a point from the interval $[-1, 0)$ is twice as much as choosing a point from $[0, 2]$ and probability mass is distributed uniformly over each interval.

     In this case, one can write

     $$1 = P([-1, 2]) = P([-1, 0)) + P([0, 2]) = 3P([0, 2]),$$

hence $P([0,2]) = \frac{1}{3}$. Therefore

$$P[I \in [-1,0)] = \frac{2}{3}|I|$$

$$P[I \in [0,2]] = \frac{1}{3}|I|$$

$$P[I \in [-1,2]] = \frac{|I \cap [0,2]| + 2|I \cap [-1,0)|}{3}$$

b)

$$P[A] = \frac{1}{2}$$

$$P[B] = \frac{1}{4}$$

$$P[A \cap B] = 0$$

$$P[A \cap C] = 0$$

$$P[A \cup B] = \frac{3}{4}$$

$$P[A \cup C] = \frac{13}{16}$$

$$P[A \cup B \cup C] = 1$$

2. *Problem 2.43* A Web site require that users create a password with the following specifications:

   - Length of 8 to 10 characters
   - Includes at least one special characters {!,@,#,$,%,^ ,&,*,(,),+,=,{,},|,<,>,\,_,-,[,],/,?}
   - No spaces
   - May contain numbers {0-9}, lower and upper case letters(a-z,A-Z).
   - Is case-sensitive.

   How many passwords are there? How long would it take to try all passwords if a password can be tested in a microsecond?

   Solution:

   Number of password: 8-character password:

$$(24 + 26 + 26 + 10)^8 - (26 + 26 + 10)^8$$

9-character password:

$$(24 + 26 + 26 + 10)^9 - (26 + 26 + 10)^9$$

10-character password:

$$(24 + 26 + 26 + 10)^{10} - (26 + 26 + 10)^{10}$$

Total:

$$86^8 + 86^9 + 86^{10} - 62^8 - 62^9 - 62^{10} = 2.15 \times 10^{19}$$

Time to try all password:

2.15×10^{19} microsecond $= 2.15 \times 10^{13}$ second $\approx 6.8 \times 10^5$ years.

3. *Problem 2.46* Ordering a "deluxe" pizza means you have four choices from 15 available toppings. How many combinations are possible if toppings can be repeated? If they cannot be repeated? Assume that the order in which the toppings are selected does not matter.

   Solution: Apparently the order does not matter. When toppings can be repeated, it is like you have replacement and in fact the problem is equivalent to having 15 different categories that the 4 items should be assigned to. Therefore if a_i represents the number of times topping i is used, we are seeking the number of solutions to

$$a_1 + \cdots + a_{15} = 4,$$

   which is $\binom{4+15-1}{4}$.

   If the toppings cannot be repeated, it is like choosing 4 objects out of 15 distinct objects which can be done in $\binom{15}{4}$ ways.

4. *Problem 2.50* Five balls are placed at random in five buckets, What is the probability that each bucket has a ball?

   Solution:

   Each arrangement of the balls in the buckets corresponds to a 5-tuple with elements from the set $\{1, 2, 3, 4, 5\}$, which represents the tags of the buckets. There are 5^5 different 5-tuples and there are $\binom{5}{1,1,1,1,1} = 5!$ ways to put one ball in each bucket. Thus, the answer is

$$\frac{5!}{5^5}.$$

   There is a delicate point to be noted here. Since the balls are not different, there is only one possible arrangement for 5 balls in 5 buckets without leaving any bucket empty and the number of possible ways to put 5 balls in 5 buckets is $\binom{5+5-1}{5}$. However, not all these arrangements are equally likely to happen.

5. *Problem 2.56* A lot of 50 items has 40 good items and 10 bad items.

   (a) Suppose we test five samples from the lot, with replacement, Let X be the number of defective items in the sample. Find P[X=k].

(b) Suppose we test five samples from the lot, without replacement. Let Y be the number of defective items in the sample, Find P[Y=k].

Solution:

a)

$$P[X = k] = B(5, k) = \binom{5}{k} 0.2^k 0.8^{(5-k)}$$

b)

$$P[X = k] = \frac{\binom{10}{k}\binom{40}{(5-k)}}{\binom{50}{5}}$$

6. *Problem 2.57* How many distinct permutations are there of four red balls, two while balls, and three black balls.

Solution:
Number of different permutation:

$$\frac{9!}{2!3!4!}$$

7. *Problem 2.59* Find the probability that in a class of 28 students exactly four were born in each of the seven days of the week.

Solution:
Suppose for each student it is equally likely to be born in any day. So there are 7^{28} possibilities for the days of birth, noting that each sequence of days of birth corresponds to a 28-tuple with elements from the set $\{M, T, W, R, F, S, U\}$, which represents the days of the week. The number of such sequence with exactly 4 elements of each kind is

$$\binom{28}{4, 4, 4, 4, 4, 4, 4} = \frac{28!}{(4!)^7}.$$

Thus, $P[4 \text{ students at each day}] = \frac{28!}{(4!)^7 7^{28}}$.