EE131A: Practice Problems for Final Exam
The exam will have 6 questions
More than 6 problems have been provided for practice purposes

Problem 1. (*25 pts*)
(a) (10 pts) Students taking a test have to answer 4 out of 5 questions, not in any specific order. The questions are numbered 1–5. Assuming that a student chooses questions randomly, what is the probability that she will answer questions 2–5 (again, not in any specific order)?

(b) (15 pts) Now suppose that the questions are multiple choice, and each question has 4 possible answers, out of which **exactly one** choice is the correct answer. For each correct answer, students get 10 points, but for each incorrect answer they lose 2 points. One of the students has not studied at all and he is just guessing his answers. What is the probability that he will score at least 20 points on the 4 questions he answers?

Sol.
(a) P(problem 2,3,4,5 is choosen)=$\frac{1}{\binom{5}{4}} = \frac{1}{5}$

(b) If the students gives 0 correct answer, he gets $(-2)(4) = -8$ points; else if he gives 1 correct answer, he gets $10 + (-2)(3) = 4$ points; else if he gives 2 correct answer, he gets $20 - 4 = 16$ points; else if he gives 3 correct answer, he gets $30 - 2 = 28$ points...

So, P(he scores at least 20 points) = P(he gives at least 3 correct answer) =$\binom{4}{3}\, p^3(1-p) + p^4 = 4(0.25^3)(0.75) + 0.25^4 = 0.05$

Problem 2. (25 pts)
(a) (15 pts) A continuous random variable X that takes on only positive values (i.e., $P[X \leq 0] = 0$) has cdf $F_X(x)$ and pdf $f_X(x)$. Find the cdf and pdf of the random variable $Y = \ln X$ in terms of $F_X(x)$ and $f_X(x)$.

(b) (10 pts) In a large hotel it is known that each guest has a 1% chance of losing his key. If a key is lost, the guest must pay a fine of \$20. A group of 200 UCLA students checks into the hotel for a meeting. **Using the Poisson approximation**, compute the probability that the hotel will collect at least \$60 in lost key fines from the group.

Sol.
(a) $F_Y(y) = P(Y \leq y) = P(\ln X \leq y) = P(X \leq e^y) = F_X(e^y)$
$f_Y(y) = \frac{dF_Y(y)}{dy} = F'_X(e^y)\frac{d}{dy}e^y = f_X(e^y)e^y$
(b) Let X be the the number lost keys by all the 200 students. Then $\alpha = E[X] = 200 \times 0.01 = 2$. For the hotel to collect at least \$60, at least three students must lose their keys, or $X \geq 3$. Hence,

$$P(X \geq 3) = e^{-2}\sum_{k=3}^{\infty}\frac{2^k}{k!} = 1 - e^{-2}\sum_{k=0}^{2}\frac{2^k}{k!} = 1 - 5 \times e^{-2} = 0.32332$$

Note that if no approximation is made, then we have to use Binomial law, and the calculations will go as follows:

$$P(\text{at least \$60 in fines}) = P(X \geq 3) = 1 - \sum_{i=0}^{2}\binom{200}{i}(0.01)^i(0.99)^{200-i},$$

which equals (accurate up to 5 decimal places) 0.32332. Hence, the Poisson approximation is excellent for this situation.

Problem 3. (*25 pts*)
A communications channel is used to transmit bits of information. Of the transmitted bits, 60% are 1 and 40% are 0. When the transmitter sends a 1, the input voltage is $v = 5$, and when a 0 is sent the input voltage is $v = -5$. The noise N added by the channel is uniformly distributed on the interval $[-6, 6]$. The decision threshold of the receiver is at $T = 0$. This means that the receiver at the other end decides that a 1 was sent if the received voltage $v + N$ is positive ($v + N > 0$), and that a 0 was sent if $v + N \leq 0$.

(a) (5 pts) Find the probability that the receiver decides a 1 was sent, given that the transmitter sent a 0.

(b) (10 pts) Find the **total** probability of receiver error.

(c) (10 pts) Find the value of the decision threshold T which minimizes the total probability of receiver error. (**Hint:** If you move the decision threshold from 0 to 0.1, does the total probability of receiver error decrease or increase from what you found in (b)? You could also try computing the probability of error in terms of the threshold T and then find the value of T that minimizes the error.)

Sol.

(a) $P(v + N > 0|v = -5) = \frac{P(N>5)P(v=-5)}{P(v=-5)} = \frac{1}{12}$

(b) $P(e) = P(v + N > 0|v = -5)P(v = -5) + P(v + N \leq 0|v = 5)P(v = 5) = \frac{1}{12}\frac{2}{5} + \frac{1}{12}\frac{3}{5} = \frac{1}{12}$

(c)

$$P(e) = P(v + N > T|v = -5)P(v = -5) + P(v + N \leq T|v = 5)P(v = 5) \Rightarrow$$

$$P(e) = \begin{cases} \frac{1-T}{12}\frac{2}{5} + \frac{T+1}{12}\frac{3}{5}, & \text{if } -1 \leq T \leq 1 \\ 0 + \frac{T+1}{12}\frac{3}{5}, & \text{if } 1 \leq T \leq 11 \\ \frac{1-T}{12}\frac{2}{5} + 0, & \text{if } -11 \leq T \leq -1 \\ \frac{2}{5}, & \text{if } T \leq -11 \\ \frac{3}{5}, & \text{if } T \geq 11 \end{cases}$$

It is clear that when $T = -1$, we have the minimum $P^*(e) = \frac{1}{15}$.

Problem 4. (*25pts*)
At the Olympic Games, 20 athletes enter the men's long jump competition. Each athlete gets 3 jumps, and if any of his jumps is longer than 8, he advances to the finals. The jump length of these athletes follows a Gaussian distribution with mean $m = 7.5$ and variance $\sigma^2 = 0.25$. You are given that $Q(1) = 0.159$. **Assume** that the three jumps of any athlete are independent of each other; similarly, jumps made by different athletes are independent of each other.

(a) (6 pts) Find the probability of a specific jump being longer than 8.

(b) (5 pts) Find the probability that the sum of the two jumps by an athlete is longer than 17.

(c) (7 pts) Find the probability of a specific athlete advancing to the finals.

(d) (7 pts) Find the probability of at least 4 athletes advancing to the finals.

Sol.

(a) $P(X_i > 8) = P(\frac{X_i - 7.5}{0.5} > \frac{8-7.5}{0.5}) = Q(1) = 0.159$

(b) Note that the sum of two independent Gaussian RV is again a Gaussian RV, and hence, $Y = X_1 + X_2 \approx N(m = 15, \sigma^2 = \frac{1}{2})$.

$P(Y > 17) = P(\frac{Y-15}{\sqrt{0.5}} > \frac{17-15}{\sqrt{0.5}}) = Q(2\sqrt{2})$

(c) $p = P(X_1 > 8 \cup X_2 > 8 \cup X_3 > 8) = 1 - P^3(X_i < 8) = 1 - (1 - 0.159)^3 = 0.4$

(d) P(at least 4 athletes advancing to the finals)= $\sum_{k=4}^{20} \begin{pmatrix} 20 \\ k \end{pmatrix} (p)^k (1-p)^{20-k}$, where p is given in part (c).

Problem 5. (*25 pts*) Suppose a manufacturing process designed to produce rods of length 1 inch in fact produces rods whose length, X, has a distribution with probability density function

$$f_X(x) = \begin{cases} (x - 0.9)C & \text{if } 0.9 \leq x \leq 1 \\ (1.1 - x)C & \text{if } 1.0 \leq x \leq 1.1 \\ 0 & \text{otherwise,} \end{cases}$$

where x is in inches. For quality control, the manufacturer scraps all rods except those with length between 0.92 and 1.08 inches before he offers them to buyers.

(a) (*10 pts*) Find the constant C, and find the mean of X.
Hint: Sketch $f_X(x)$.

(b) (*10 pts*) What proportion of output is scrapped?

(c) (*5 pts*) Suppose you need a rod of length between 0.95 and 1.05 inches. What is the probability that you get what you need if you buy it from the manufacturer ?

Sol.

(a) First sketch the pdf and note that the pdf when sketched is a triangular region with vertices at: $(0, 0.9)$, $(0, 1.1)$, and $(0.1C, 1)$. Hence, the area under the pdf is [(base x height)/ 2] $= 0.01C = 1$. Hence, $C = 100$.

(b) Prob[an item is scrapped] $=$ Prob$[X > 1.08]$ $+$ Prob$[X < 0.92]$. Now, Prob$[x > 1.08]$ $=$ Prob$[x < 0.92]$ $=$ the area of a triangle with height = $0.02C$, and base = 0.02. Hence, Prob[an item is scrapped] $= 0.04$.

(c) We need to calculate the following conditional probability:

$$\text{Prob}[.95 \leq X \leq 1.05 | .92 \leq X \leq 1.08] = \frac{\text{Prob}[\{.95 \leq X \leq 1.05\} \cap \{0.92 \leq X \leq 1.08\}]}{\text{Prob}[0.92 \leq X \leq 1.08]}$$

$$= \frac{\text{Prob}[.95 \leq X \leq 1.05]}{\text{Prob}[0.92 \leq X \leq 1.08]} = \frac{1 - 2(0.5)(0.5)}{1 - 2(0.2)(0.2)} = \frac{1 - .25}{1 - 0.04} \approx 0.78$$

Problem 6. (*25pts*)

A test is designed so that student's scores are iid (independent and identically distributed) random variables with mean $m = 65$ and standard deviation $\sigma = 25$. How many students would have to take the test so that the average score would be between 60 and 70 with at least 98% probability? Use the central limit theorem. (**Hint:** $Q(2.3) = 0.01$.)

Sol.

Let A_n be the average score.

$E[A_n] = \frac{1}{n}E[\sum X_i] = E[X_i] = m$
$V[A_n] = E[(\sum X_i/n)^2] - (E[\sum X_i/n])^2 = \frac{1}{n}(E[X_i^2] - (E[X_i])^2) = \frac{V[X_i]}{n}$
$P(\frac{60-65}{25/\sqrt{n}} \leq \frac{A_n-65}{25/\sqrt{n}} \leq \frac{70-65}{25/\sqrt{n}}) = 1 - 2 * Q(\frac{\sqrt{n}}{5}) \geq 0.98$. Equivalently, we have $Q(\frac{\sqrt{n}}{5}) \geq 0.01$. Given that $Q(2.3) = 0.01$, we get $\frac{\sqrt{n}}{5} \geq 2.3$, or $n \geq 132.25$.

So, we need at least 133 students to take the test.

Problem 7. (*15 pts*)
Gear key pins are manufactured to have an expected length (μ) of $10mm$, and a variance (σ^2) of $0.01mm$. Suppose, N of these key pins are picked at random and their average length, $\overline{L_N}$, is computed, i.e.,

$$\overline{L_N} = \frac{1}{N}\sum_{i=1}^{N} L_i \ ,$$

where L_i is the length of the i-th pin.
(i) (*10 pts*) Using the Central Limit Theorem, calculate the probability $P_{\text{exceed}} = P[|\overline{L_N} - \mu| \geq 0.1\sigma]$. (Leave your answer in terms of the $Q()$ function).
(ii) (*5 pts*) How large should N be so that $P_{\text{exceed}} \leq 0.0456$? (**Hint**: $Q(2) = 0.0228$)

Sol.
(i) $P[|\overline{L_N} - \mu| \geq 0.1\sigma] = P[\frac{|\overline{L_N}-\mu|}{\sigma} \geq 0.1] = P[\frac{|\overline{L_N}-\mu|}{\sigma/\sqrt{N}} \geq 0.1\sqrt{N}] = 2Q(0.1\sqrt{N})$
(ii) $P[|\overline{L_N} - \mu| \geq 0.1\sigma] \leq 0.0456$. Hence, we get
$Q(0.1\sqrt{N}) \leq 0.028 \Rightarrow$
$\Rightarrow 0.1\sqrt{N} \geq 2 \Rightarrow$
$\Rightarrow N \geq 400$

Problem 8. (*25 pts*)
A communications channel is used to transmit bits of information. Of the transmitted bits, 60% are 1 and 40% are 0. When the transmitter sends a 1, the input voltage is $v = 5$, and when a 0 is sent the input voltage is $v = -5$. **The noise N depends on the input voltage**: it is uniformly distributed on $[-8, 8]$ when $v = 5$ (i.e., a 1 is sent), and on $[-5, 5]$ when $v = -5$ (i.e., a 0 is sent). The decision threshold of the receiver is at a certain voltage T. This means that the receiver at the other end decides that a 1 was sent if the received voltage $v + N$ is greater than T ($v + N > T$), and that a 0 was sent if $v + N \leq T$.

(a) (*5 pts*) Find the probability that the receiver decides a 1 was sent, given that the transmitter sent a 0 **for the case**, where $T = 0$.

(b) (*10 pts*) Find the **total** probability of receiver error for $T = 0$.

(c) (*10 pts*) Find the value of the decision threshold T which minimizes the total probability of receiver error.

Sol.

1. Denote R to be ‘receive’ and S to be ‘send’.

$$P[R = 1|S = 0] = P[V + N_2 \geq 0|V = -5] = P[N_2 \geq 5] = 0,$$

since N_2 is uniformly distributed in the interval $[-5,\ 5]$.

2.

$$\begin{aligned} P[error] &= P[R=1|S=0]P[S=0] + P[R=0|S=1]P[S=1] \\ &= 0 + P[N_1 \leq -5]P[S=1] \\ &= 0 + 3/16 \cdot 3/5 = 9/80 \end{aligned}$$

Now N_1 is uniformly distributed in the interval $[-8,\ 8]$.

3.

$$P[R=1|S=0] = P[N_2 \geq T+5] = \begin{cases} 1, & if\ T < -10 \\ -T/10 & if\ -10 \leq T \leq 0 \\ 0, & if\ T > 0 \end{cases}$$

$$P[R=0|S=1] = P[N_1 \leq T-5] = \begin{cases} 0, & if\ T < -3 \\ (T+3)/16 & if\ -3 \leq T \leq 13 \\ 1, & if\ T > 13 \end{cases}$$

$$\begin{aligned} P[error] &= P[R=1|S=0]P[S=0] + P[R=0|S=1]P[S=1] \\ &= 0.4P[R=1|S=0] + 0.6P[R=0|S=1] \\ &= \begin{cases} 0.4 & if\ T < -10 \\ -T/25 & if\ -10 \leq T < -3 \\ -T/25 + 0.6(T+3)/16 & if\ -3 \leq T < 0 \\ 0.6(T+3)/16 & if\ 0 \leq T \leq 13 \\ 0.6 & if\ T > 13 \end{cases} \end{aligned}$$

It is easy to see that the overall minimum of the error happens when $T = 0$.

Problem 9. (*25 pts*)
Fifty-two percent of the residents of New York are in favor of outlawing cigarette smoking in publicly owned places. A pollster randomly picks n people and determines what percentage of them are in favor of banning smoking. How large should n be such that with at least 95% probability, greater than 50 percent of people polled would favor banning smoking in public places?
Hint: You can consider each person's yes/no answer to be a Bernoulli random variable. The event you are concerned with is where more than half of the polled people say "yes" to the ban. Also, $\Phi(1.645) = 0.95$. Use central limit theorem.

Sol.
Bernoulli R.V. X_i with $p = 0.52, q = 0.48$.
$E[X_i] = 0.52, \sigma^2_{X_i} = pq$.
Using CLT $S_n = \sum_{i=1}^{n} X_i \sim \mathcal{N}(np, npq)$

$$P[S_n \geq 0.5n] \geq 0.95 \Rightarrow$$

$$P[\frac{S_n - np}{\sqrt{npq}} \geq \frac{0.5 - np}{\sqrt{npq}}] \geq 0.95 \Rightarrow$$

$$Q(-\frac{0.02n}{\sqrt{n0.52 \times 0.48}}) \geq 0.95 \Rightarrow$$

$$\Phi(\frac{0.02n}{\sqrt{n0.52 \times 0.48}}) \geq 0.95 \Rightarrow$$
$$\frac{0.02\sqrt{n}}{\sqrt{0.52 \times 0.48}} \geq 1.645 \Rightarrow$$
$$n \geq (\frac{1.645\sqrt{0.52 \times 0.48}}{0.02})^2 \Rightarrow$$
$$n \geq 1689$$

Problem 10. (*25 pts*)
A student is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. She figures that if she has an "on" day then each of her examiners will pass her independently of each other, with probability 0.8; whereas, if she has an "off" day, this probability will be reduced to 0.4.
Suppose that the student will pass the exam if a majority of the examiners pass him. If the student feels that he is twice as likely to have an off day as he is to have an on day, should he request an examination with three examiners or with 5 examiners? Note that all the oral exams take place on the same day.

Sol.
$P[ON] = \frac{1}{3}, P[OFF] = \frac{2}{3}$.

$$P[\text{at least 3 out of 5 examiners pass her}] =$$
$$= P[\text{at least 3 out of 5 examiners pass her}|ON]P[ON]+$$
$$+P[\text{at least 3 out of 5 examiners pass her}|OFF]P[OFF] =$$
$$= \frac{1}{3}\sum_{k=3}^{5}\binom{5}{k}0.8^k 0.2^{5-k} + \frac{2}{3}\sum_{k=3}^{5}\binom{5}{k}0.4^k 0.6^{5-k}.$$

Similarly,

$$P[\text{at least 2 out of 3 examiners pass her}] =$$
$$= \frac{1}{3}\sum_{k=2}^{3}\binom{3}{k}0.8^k 0.2^{3-k} + \frac{2}{3}\sum_{k=2}^{3}\binom{3}{k}0.4^k 0.6^{3-k}.$$

Decision: Choose the one which provides greater probability to pass.

Problem 11. (25 pts)
The random vector variable (X, Y) has the joint **pdf** (note that the pdf is 0 outside of the region given below):

$$f_{XY}(x, y) = kxy \quad \text{for} \quad 0 < x \leq y < 1\ .$$

(i) (5 pts.) Find k.
(ii) (5 pts.) Find the marginal pdf of X and Y.
(iii) (5 pts.) Find $P[0 < X < 0.5,\ 0.5 < Y < 1]$.

(iv) (10 pts) Find the joint CDF $F_{XY}(x, y)$.

Sol.

1.
$$\int_{y=0}^{y=1} \int_{x=0}^{x=y} kxy\, dxdy = k/8 = 1 \Rightarrow$$
$$\Rightarrow k = 8.$$

2.
$$f_X(x) = \int_x^1 8xy\, dy = 4x - 4x^3.$$
$$f_Y(y) = \int_0^y 8xy\, dx = 4y^3.$$

3.
$$P[0 < X < 0.5, 0.5 < Y < 1] = \int_{0.5}^1 \int_0^{0.5} 8xy\, dxdy = 3/8.$$

Note that the entire region of integration lies inside the non-zero probability region of $f_{XY}(x, y)$, thus we did not need to worry about the boundaries.

4. Let u, v denote the dummy variable on x, y axis, respectivily. Then
$$F_{XY}(x, y) = \int_{u=0}^{u=x} \int_{v=u}^{v=y} 8uv\, dvdu = 2x^2y^2 - x^4 \qquad if \;\; 0 < x \leq y < 1.$$
$$F_{XY}(x, y) = F_{XY}(x, 1) = 2x^2 - x^4 \qquad if \;\; 0 < x < 1, y \geq 1.$$
$$F_{XY}(x, y) = F_{XY}(y, y) = y^4 \qquad if \;\; 0 < y < 1, y < x.$$
$$F_{XY}(x, y) = 0 \qquad otherwise.$$

Problem 12.
A point (X,Y) is selected at random inside a triangle defined by $\{(x,y) : 0 \leq y \leq x \leq 1\}$. Assume the point is equally likely to fall anywhere in the triangle.

(a) Find the joint cdf of X and Y.

(b) Find the marginal cdf of X and of Y.

(c) Find the probabilities of the following events in terms of the joint cdf:
$A = \{X \leq 1/2, Y \leq 3/4\}$, $A = \{1/4 \leq X \leq 3/4, 1/4 \leq Y \leq 3/4\}$.
Sol.

(a)

$$F_{XY}(x,y) = 2\left(xy - y^2/2\right) \qquad if\, 0 \leq y \leq x \leq 1$$
$$F_{XY}(x,y) = x^2 \qquad if\, 0 \leq x \leq 1, y \geq x$$
$$F_{XY}(x,y) = F_{XY}(1,y) = 2\left(y - y^2/2\right) \qquad if\, x \geq 1, 0 \leq y \leq 1$$
$$F_{XY}(x,y) = 1 \qquad if\, x \geq 1, y \geq 1$$
$$F_{XY}(x,y) = 0 \qquad otherwise$$

(b)

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = x^2 \qquad if\, 0 \leq x \leq 1$$
$$F_X(x) = 0 \qquad if\, x < 0$$
$$F_X(x) = 1 \qquad if\, x > 1$$

$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y) = 2\left(y - y^2/2\right) \qquad if\, 0 \leq y \leq 1$$
$$F_Y(y) = 0 \qquad if\, y < 0$$
$$F_Y(y) = 1 \qquad if\, y > 1$$

(c)

$$P\left[X \leq \frac{1}{2}, Y \leq \frac{3}{4}\right] = F_{XY}\left(\frac{1}{2}, \frac{3}{4}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\begin{aligned} P\left[\frac{1}{4} \leq X \leq \frac{3}{4}, \frac{1}{4} \leq Y \leq \frac{3}{4}\right] &= F_{XY}\left(\frac{3}{4}, \frac{3}{4}\right) - F_{XY}\left(\frac{3}{4}, \frac{1}{4}\right) - F_{XY}\left(\frac{1}{4}, \frac{3}{4}\right) + F_{XY}\left(\frac{1}{4}, \frac{1}{4}\right) \\ &= \left(\frac{3}{4}\right)^2 - 2\left(\frac{3}{4}.\frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2\right) - \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 \\ &= \frac{1}{4} \end{aligned}$$

Problem 13.
Let X and Y be independent exponential random variables with parameters 2 and 10, respectively. Let $Z = X + Y$.

(a) Find the characteristic function of Z.

(b) Find the pdf of Z from the characteristic function found in part a.

Sol.

(a)

$$\Phi_Z(\omega) = (\frac{\alpha}{\alpha - j\omega})(\frac{\beta}{\beta - j\omega})$$

(b) $\Phi_Z(\omega) = \frac{a}{\alpha - j\omega} + \frac{b}{\beta - j\omega}$ where $b = \frac{\alpha-\beta}{\alpha\beta}$,$a = \frac{\beta-\alpha}{\alpha\beta}$

$$f_Z(t) = \frac{\beta - \alpha}{\alpha\beta}e^{-\alpha t} - \frac{\beta - \alpha}{\alpha\beta}e^{-\beta t}$$

Problem 14.
From past experience a professor knows that the test score of a student taking her final examination is a random variable with a mean of 75 and standard deviation of 8. How many students would have to take the examination to ensure, with probability at least 0.95 that the class average would be at least 73?
Sol.
Let $\bar{X}_n$ denote the average test score based on n students. From the CLT we know that if the number of students is sufficiently large, then $\frac{\bar{X}_n - 75}{\frac{8}{\sqrt{n}}}$ converges to a standard Normal random variable in distribution; i.e. $\bar{X}_n \sim \mathcal{N}(75, \frac{64}{n})$. We need to find n such that $P(\bar{X}_n \geq 73) \geq 0.95$. We use the CLT approximation to calculate the probability and find the critical value of n.

$$P(\bar{X}_n \geq 73) = P(\frac{\bar{X}_n - 75}{\frac{8}{\sqrt{n}}} \geq \frac{73 - 75}{\frac{8}{\sqrt{n}}}) \simeq \phi(\frac{-\sqrt{n}}{4})$$

Thus, in order to make sure that this event happens with probability at least 0.95, we only need look up the table for the corresponding number a whose Q-function value is greater than 0.95 and pick n such that $\frac{-\sqrt{n}}{4} \leq a$. Noting that $a = -1.645$, it turns out that $n \geq 44$.