EE 131A | Homework Set 7
Probability | Friday, November 13, 2015
Instructor: Professor Roychowdhury | Thursday, November 19, 2015

1. *Problem 4.61* Let X be an exponential random variable with parameter λ.

   (a) For $d > 0$ and k a nonnegative integer, find $P[kd < X < (k+1)d]$.

   (b) Segment the positive real line into four equiprobable disjoint intervals.

   Solution:

   (a) The cdf of an exponential random variable with parameter λ is described as:

$$F_x(d) = 1 - e^{-\lambda d}$$

   Then we have:

$$P(kd < X < (k+1)d) = F_X((k+1)d) - F_X(kd) = e^{-\lambda kd}(1 - e^{-\lambda d})$$

   (b) To segment the positive real line into four equiprobable disjoint intervals, we find three segmentation points x_k for $k = 1, 2, 3$ such that $F_X(x_k) = \frac{k}{4}$.

$$x_1 = \frac{ln\frac{4}{3}}{\lambda}$$

$$x_2 = \frac{ln2}{\lambda}$$

$$x_3 = \frac{ln4}{\lambda}$$

2. *Problem 4.62* The rth percentile, $\pi(r)$, of a random variable X is defined by $P[X \leq \pi(r)] = r/100$.

   (a) Find the 90%, 95% and 99% percentiles of the exponential random variable with parameter λ.

   (b) Repeat part a for the Gaussian random variable with parameters $m = 0$ and σ^2.

   Solution:

   (a) For exponential random variable, r percentiles are computed as follows:

$$\begin{aligned} P[X &\leq x] = 1 - e^{-\lambda x} = \frac{r}{100} \\ \Rightarrow \quad x &= -\frac{1}{\lambda}\ln(1 - \frac{r}{100}) \end{aligned}$$

   then

$$\pi(90) \approx \frac{2.3}{\lambda} \quad \pi(95) \approx \frac{3}{\lambda} \quad \pi(99) \approx \frac{4.6}{\lambda}$$

(b) For Gaussian random variable, r percentiles are computed as follows:

$$\begin{aligned} P[X &\leq x] = 1 - Q(\frac{x}{\sigma}) = \frac{r}{100} \\ \Rightarrow x &= Q^{-1}(1 - \frac{r}{100}) \end{aligned}$$

then

$$\pi(90) = 1.3\sigma \quad \pi(95) \approx 1.6\sigma \quad \pi(99) \approx 2.3\sigma$$

3. *Problem 4.67* A binary transmission system transmits a signal X (-1 to send a "0" bit; +1 to send a "1" bit). The received signal is $Y = X + N$ where noise N has a zero-mean Gaussian distribution with variance σ^2. Assume that "0" bits are three times as likely as "1" bits.

   (a) Find the conditional pdf of Y given the input value: $f_Y(y|X = +1)$ and $f_Y(y|X = -1)$.

   (b) The receiver decides a "0" was transmitted if the observed value of y satisfies

$$f_Y(y|X = -1)P[X = -1] > f_Y(y|X = +1)P[X = +1]$$

and it decides a "1" was transmitted otherwise. Use the results from part a to show that this decision rule is equivalent to: If $y < T$ decide "0"; if $y \geq T$ decide "1".

   (c) What is the probability that the receiver makes an error given that a +1 was transmitted? a -1 was transmitted? Assume $\sigma^2 = 1/16$.

   (d) What is the overall probability of error?

Solution:

(a) The conditional pdf of Y given the input is found as follows:

$$f_Y(y|X = +1) = f_N(1 + n \leq y) = f_N(y - 1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-1)^2}{2\sigma^2}}$$

$$f_Y(y|X = -1) = f_N(-1 + n \leq y) = f_N(y + 1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y+1)^2}{2\sigma^2}}$$

(b) The receiver decides "0" was transmitted if the observed value of y satisfies:

$$\begin{aligned} &f_Y(y|X = -1)P[X = -1] > f_Y(y|X = +1)P[X = +1] \\ &\Longrightarrow \tfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y+1)^2}{2\sigma^2}} \tfrac{3}{4} > \tfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-1)^2}{2\sigma^2}} \tfrac{1}{4} \\ &\Longrightarrow 3 > e^{-\frac{(y-1)^2}{2\sigma^2} + \frac{(y+1)^2}{2\sigma^2}} \\ &\Longrightarrow \ln 3 > -\tfrac{2y}{\sigma^2} \\ &\Longrightarrow y < \tfrac{\ln 3}{2}\sigma^2 \end{aligned}$$

Therefore, it is equivalent to: if $y < T$ decide "0"; if $y \geq T$,decide "1" where $T = \frac{\ln 3}{2}\sigma^2$.

(c) The conditional probability of error is given by:

$$P(error|X=+1) = P[1+N<T] = P[N<T-1] = \Phi(\frac{T-1}{\sigma}) = 5.6076\times 10^{-5}$$

$$P(error|X=-1) = P[-1+N>T] = P[N>T+1] = 1-\Phi(\frac{T+1}{\sigma}) = 1.7569\times 10^{-5}$$

(d) The overall probability of error is given by:

$$\begin{aligned} P(error) &= P(error|X=+1)P(X=+1) + P(error|X=-1)P(X=-1) \\ &= \frac{1}{4}\Phi(\frac{T-1}{\sigma}) + \frac{3}{4}[1-\Phi(\frac{T+1}{\sigma})] \\ &= 2.7196\times 10^{-5} \end{aligned}$$

Note: the error probability is close to zero.

4. *Problem 4.69* Passengers arrive at a taxi stand at an airport at a rate of one passenger per minutes. The taxi driver will not leave until seven passengers arrive to fill his van. Suppose that passenger interarrival times are exponential random variable, and let X be the time to fill a van. Find the probability that more than 10 minutes elapse until the van is full.
Solution: As we know the time elapsed between two successive occurrences of the event has an exponential distribution and it is independent of previous occurrences, then the total number of occurrences of the event has a Possion distribution. Then the probability less than 7 people in 10 minutes is given by:

$$\begin{aligned} P[X] &= 1-P[X\leq 10] \\ &= \sum_{k=0}^{6}\frac{(\lambda t)^k}{k!}e^{-\lambda t} \\ &= \sum_{k=0}^{6}\frac{10^k}{k!}e^{-10} \\ &= 0.1301 \end{aligned}$$

5. *Problem 4.74*

(a) Find the cdf of the m-Erlang random variable by integration of the pdf. Hint: Use integration by parts.

(b) Show that the derivative of the cdf given by

$$F_{S_m}(t) = 1-\sum_{k=0}^{m-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

gives the pdf of an m-Erlang random variable.
Solution:

(a) The cdf is given by the integral

$$F_X(x) = \frac{\lambda^{m-1}}{(m-1)!}\int_0^x y^{m-1}\lambda e^{-\lambda y}dy$$

Since $\Gamma(m) = (m-1)!$ Integrate by parts using $u = y^{m-1}$and $dv = \lambda e^{-\lambda y}dy$ so that $du = (m-1)y^{m-2}dy$ and $v = -e^{-\lambda y}$:

$$\begin{aligned} F_X(x) &= \frac{\lambda^{m-1}}{(m-1)!}\{-y^{m-1}e^{-\lambda y}|_0^x + \int_0^x (m-1)y^{m-2}e^{-\lambda y}dy\} \\ &= -\frac{(\lambda x)^{m-1}}{(m-1)!}e^{-\lambda x} + \frac{\lambda^{m-2}}{(m-2)!}\int_0^x y^{m-2}\lambda e^{-\lambda y}dy \end{aligned}$$

The integral on the right hand side is identical to the equation for $F_X(x)$ with $m-1$ replaced by $m-2$. We can therefore repeatly perform integration by parts to obtain the cdf of X:

$$\begin{aligned} F_X(x) &= -\frac{(\lambda x)^{m-1}}{(m-1)!}e^{-\lambda x} - \frac{(\lambda x)^{m-2}}{(m-2)!}e^{-\lambda x} - ... + \int_0^x e^{-\lambda y}dy \\ &= 1 - \sum_{k=0}^{m-1}\frac{(\lambda x)^k}{k!}e^{-\lambda x} \end{aligned}$$

(b) From part (a), we know

$$F_X(x) = 1 - \sum_{k=0}^{m-1}\frac{(\lambda x)^k}{k!}e^{-\lambda x} = \frac{\lambda^{m-1}}{(m-1)!}\int_0^x y^{m-1}\lambda e^{-\lambda y}dy$$

therefore:

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{\lambda^{m-1}}{(m-1)!}y^{m-1}\lambda e^{-\lambda y}$$

6. *Problem 4.90* A voltage X is a Gaussian random variable with mean 1 and variance 2. Find the pdf of the power dissipated by an R-ohm resistor $P = RX^2$
Solution: Note $X = \pm\sqrt{P/R}$ and $\frac{\mathrm{d}X}{\mathrm{d}P} = \pm\frac{1}{2}\frac{1}{\sqrt{RP}}$

$$\begin{aligned} f_P(p) &= [f_X(x) + f_X(-x)]|\frac{\mathrm{d}X}{\mathrm{d}P}| \\ &= [f_X(\sqrt{P/R}) + f_X(-\sqrt{P/R})]\frac{1}{2\sqrt{RP}} \\ &= \frac{1}{2\sqrt{\pi}}[e^{-(\sqrt{\frac{P}{R}}-1)^2/4} + e^{-(-\sqrt{\frac{P}{R}}-1)^2/4}]\frac{1}{2\sqrt{RP}} \\ &= \frac{1}{4\sqrt{PR\pi}}[e^{-(\sqrt{\frac{P}{R}}-1)^2/4} + e^{-(\sqrt{\frac{P}{R}}+1)^2/4}] \end{aligned}$$

7. Let $Y = e^X$, find the pdf of Y when X is Gaussian random variable. In this case Y is said to be a lognormal random variable.
Solution: For $y > 0$, $P[Y \leq y] = P[e^X \leq y] = P[X \leq \ln y] = F_X(\ln y)$

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ F_X(\ln y) & y \geq 0 \end{cases}$$

Therefore, for $y > 0$ we have

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{dy}} F_Y(y) = \frac{1}{y} f_x(\ln y)$$

If X is a Gaussian random variable, then

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{e^{-(\ln y - m)^2/2\sigma^2}}{y\sqrt{2\pi}\sigma} & y \geq 0 \end{cases}$$