Problem 1. (Total: 20pts) From a group of 5 women and 7 men, a committee is to be formed by picking **5 members** randomly from this group of 12 individuals:

(a) ***(5pts)*** What is the probability that the committee has **exactly** 2 women and 3 men?

(b) ***(10pts)*** Two men, Mr. Jones and Mr. Chen, are sworn enemies. What is the probability that the committee has two women and 3 men, but **not both** Mr Jones and Mr. Chen?

(c) ***(5pts)*** Given that the randomly picked committee has 2 women and 3 men, what is the **conditional probability** that the committee does not have **both** Mr Jones and Mr. Chen?

Problem 2 (Total: 25 pts) A bin contains three different types of disposable flashlisghts. The probability that a type 1 flashlisght will give over 100 hours of use is 0.7; the corresponding probabilities for type 2 and and 3 flashlisghts being 0.4 and 0.3, respectively. Suppose that 20% of the flashlights in the bin are type 1, 30% are type 2, and 50% are type 3.

(a) ***(10pts)*** What is the probability that a randomly chosen flashlight will give more than 100 hours of use?

(b) ***(15 pts)*** Given that a randomly picked flashlight lasts over 100 hours, what is the conditional probability that that it is of type "j" flashlight for j= 1, 2, and 3?

Problem 3. (Total: 15pts) A laboratory blood test is 95% effective in detecting a certain disease when it is in fact present. (That is, if a person with the disease is tested then with probability 0.95 the test will come out positive). However, the test also yields a "false" positive result for 1% of the healthy persons tested. (That is, if a healthy person is tested, then with probability 0.01, the test will imply he or she has the disease, i.e., the test will be positive.)

If 0.5% of the population actually has the disease, what is the probability a randomly tested person actually has the disease given that the test result is positive.

Problem 4. (Total: 30 pts) The game of craps is played as follows:

A player **rolls two fair dice**. If the **sum** of the dice is **either 2, 3, or 12,** the player **loses**; if the sum is either a **7 or an 11**, he or she **wins**.

If the outcome is anything else, the player continues to roll the two dice until he or she rolls either the initial outcome or a 7. If the 7 comes first, the player loses; whereas if the initial outcome reoccurrs before the 7, the player wins.

(a) ***(5 pts)*** What is the probability of winning on the first roll of the two dice?

(b) ***(5 pts)*** What is the probability of losing on the first roll of the two dice?

(c) ***(8 pts)*** Suppose that the first roll of the two dice yields a sum of 5. What is the probability that the player wins? [**Hint:** use a geometric law]

(d) ***(7 pts)*** Suppose that the first roll of the two dice yields a sum of 6. What is the probability that the player wins? [**Hint:** use a geometric law]

(e) ***(***5 pts) Provide a strategy (not necessarily a complete solution) for computing the winning probability of aa player at the game of craps.

Problem 5. (Total: 15 pts) A binary information source (e.g.,a document scanner) generates very long strings of 0's followed by occasional 1's. Suppose that symbols are independent and that p = P[symbol = 0] is very close to one. Consider the following scheme for encoding the run *X* of 0's between consecutive 1's:

1) If X = n, express *n* as a multiple of an integer M = 2^m and a remainder *r*, that is, find *k* and *r* such that $n = kM + r$, where $0 \leq r < M - 1$;
2) The binary codeword for *n* then consists of a prefix consisting of *k* 0's followed by a 1, and a suffix consisting of the *m*-bit representation of the remainder *r*. The decoder can deduce the value of *n* from this binary string.
   a. Find the probability that the prefix has *k* zeros, assuming that $p^M = 1/2$.
   b. Find the average codeword length when $p^M = 1/2$.
   c. Find the compression ratio, which is defined as the ratio of the average run length to the average codeword length when $p^M = 1/2$.

Problem 6. (Total: *20 points*) In a game of poker five cards are picked at random from a deck of 52 cards. Note that a deck of cards has 13 **denominations** (or **kinds**) (namely, they are ordered as Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King) and 4 **suits** (namely, Hearts, Spades, Clubs, and Diamonds). What is the probability of being dealt

(a) (*10 points*) A **full house**? That is, when the cards have *denominations* **a, a, a, b, b, where a** and **b** are **distinct**. So, 10 of Spades, 10 of Hearts, 10 of Diamonds, Queen of Hearts and Queen of Clubs will comprise a full house.

(b) (*10 points*) A **Straight**? This occurs when the cards have **distinct conescutive** *denominations* but **not all** of the same *suit*. So Ace of Spades, 2 of Hearts, 3 of Spades, 4 of Spades, and 5 of Spades will comprise a Straight. Note that Ace can be regarded as **both** the least or the greatest value, so 10 of Clubs, Jack of Clubs, Queen of Diamonds, King of Diamonds, and Ace of Diamonds will also comprise a Straight.

Problem 7. (Total: *25 Points*) Suppose you roll two *fair* dice. If the sum is **greater than or equal to 10** you **stop**, but **if** the sum is **less than 10** you roll the two dice one more time and then **stop** (so you either roll once or twice). If you stop the game with a sum less than 10, *you lose $10*, but if you stop the game with a sum greater than or equal to 10, *you win 3 times the amount of the sum in dollars*.

For example, (i) in one scenario in your first roll you might get a sum of 11, then you stop and you win $33. (ii) in another scenario, in your first roll you might get a sum of 3, then you are forced to roll again, and say you get a sum of 12 in your second roll; then you win $36, (iii) in yet another scenario, you might get a sum of 5 in your first roll, then roll again, and might get a sum of 4; then you lose $10.

(a) *(6 points)* What is the probability of stopping after the **first** roll. What about the probability of having to roll **twice**?
Hint: You may first want to calculate the probability of getting a sum of "**x**" in any roll of two fair dice, for **x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.**

(b) *(5 points)* What is the probability of stopping with a **sum** of **11**? What is the probability of stopping with a **sum** of **3**?

(c) *(5 points)* If a player stops with a **sum** of **11**, what is the probability they have rolled **twice**? (**Hint:** Use Baye's Law).

(d) *(3 points)* What is the probability the player has rolled **twice** if they stopped with a **sum** of **3**?

(e) *(6 points)* What is the **expected value** of your **money winnings** in this game?

Problem 8. (Total: 25 points) A man has 5 coins in his pocket. **Two** are *double-headed*, **one** is *double-tailed*, and **two** are *fair coins*. The coins cannot be distinguished unless one examines both sides of the coin. For example, if the upper side of a coin is a Head, then the coin can be either one of the two double-headed coins or one of the two fair coins.
Overall Hint: Once the coin is randomly picked in part (a) it is never changed. All parts follow the same randomly picked coin.

(a) *(6 points)* The man shuts his eyes, chooses a coin at *random*, and tosses it. What is the probability that the lower face of the coin is heads? (**Note that his eyes are still shut**!)
Hint: Use the Total probability Theorem.

(b) *(7 points)* He opens his eyes and **sees** that the upper face of the coin is a head. What is the probability that the lower face is a head.

(c) *(6 points)* He shuts his eyes again, **picks up the same coin**, and tosses it again. What is the probability that the lower face is a head?

(d) *(6 points)* He opens his eyes and sees that the upper face is a head. What is the probability that the lower face is a head?

Problem 9. (Total: 20 points) A Christmas fruitcake has **Poisson-distributed** independent numbers of sultana raisins, irridescent red cherry bits, and radioactive green cherry bits with respective **averages** of 48, 24, 12 per cake. The bits are spread randomly throughout the cake. Suppose you get a 1/12 slice of the cake.

(a) *(6 points)* What is the probability that you get lucky and get **no green bits** in your slice?

(b) *(7 points)* What is the probability that you get really lucky and get **no green bits <u>and</u> two or fewer red bits** in your slice?

(c) *(7 points)* What is the probability that you get extremely lucky and get **no green bits, no red bits, <u>and</u> more than five raisins** in your slice?