Probability
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Suppose we have $A_1, A_2, ..., A_n$ independent events. We know the probability of these events happening at the same time will be

$$P(A_1 \cap A_2 \cap ... \cap A_3) = P(A_1)P(A_2)...P(A_n)$$

Now assume we know that these events are not independent, or that we think they may not be independent. For two such events A and B we can write

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

We can use the same approach recursively to get the probability of $A_1, ..., A_n$

$$\begin{aligned}
P(A_1 \cap A_2 \cap ... \cap A_n) &= P(A_1 \cap A_2 \cap ... \cap A_{n-1}|A_n)P(A_n) \\
&= P(A_1 \cap A_2 \cap ... \cap A_{n-2})P(A_{n-1}|A_n)P(A_n) \\
&= P(A_1 \cap A_2 \cap ... \cap A_{n-3})P(A_{n-2}|A_{n-1} \cap A_n)P(A_{n-1}|A_n)P(A_n) \\
&. \\
&. \\
&. \\
&= P(A_1|A_2 \cap A_3 \cap ... \cap A_n)P(A_2|A_3 \cap ... \cap A_n)P(A_{n-1}|A_n)P(A_n)
\end{aligned}$$

This means that in order to calculate the probability of n we can have a step-by-step method, where we calculate probability of the event A_n, then given A_n we calculate the probability of event A_{n-1}, then given A_n and A_{n-1} we calculate the probability of A_{n-2} and so on. The exercises you see next show how to use this equation.

Exercises:

1. Imagine a bag with two balls in it, one white and one red. We randomly take a ball out of the bag and look at its color. Then we put the ball back in and add another ball of the same color to the bag. We define event W_i = "Event that the i_{th} pick is a white ball". Now let's calculate the probability that the firs k balls are white:

$$\begin{aligned} P(W_1 \cap W_2 \cap ... \cap W_k) &= P(W_1)P(W_2|W_1)...P(W_k|W_{k-1}...W_2W_1) \\ &= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times ... \frac{k-1}{k} \times \frac{k}{k+1} \\ &= \frac{1}{k+1} \end{aligned} \tag{1}$$

   Notice that the probability of picking the same color over and over again is higher than the case of only having the first two balls in the bag (When the probability of picking the white one k times would have been $(\frac{1}{2})^k$), since we are adding another ball of the same color each time. We can use the same approach to calculate the probability of any particular sequence. e.g. 1st pick being white, 2nd pick red, 3rd pick red,..., k_{th} pick white or etc.

2. There are two genes in each person's body controlling the color of their eyes. If both of these are blue genes then the eye color will be blue, but if any of them are brown the eyes will be brown-colored (The brown gene is dominant). The child gets one gene from the father and one from the mother, both randomly. So if the fathers genes are $g_1^f g_2^f$ and the mothers genes are $g_1^m g_2^m$, there will be 4 possible gene combinations for the child:

$$(g_1^f g_1^m), (g_1^f g_2^m), (g_2^f g_1^m), (g_2^f g_2^m)$$

   and each of the above combinations will have a probability of $\frac{1}{4}$.
   Therefore, if we name genes responsible for brown and blue eyes as Br and Bl and the parents' genes are $Br^f Bl^f$ and $Br^m Bl^m$ both of the parents will have brown eyes. The possible gene combinations for the child would be:

$$(Br^f Br^m), (Br^f Bl^m), (Bl^f Br^m), (Bl^f Bl^m)$$

   Each one with $\frac{1}{4}$ probability. In the first three cases the child will have brown eyes and only in the last case the child will be blue-eyed, so the probability of the child having brown eyes would be $\frac{3}{4}$ and the probability of the child having blue eyes would be $\frac{1}{4}$.
   Now consider Mr. S and Ms. F are brothers and sisters and their parents are both Brown-Eyed. Mr. S is Brown-Eyed while Ms. F is Blue Eyed. We want to calculate the probability of Mr. S's genes being the combination of (Br, Bl):

$P(\text{Mr. S is } (Br, Bl)|$ all the information$) = ?$
Given that the parents have a Blue-Eyed child we know that they should each carry the blue gene. Since they are both Brown-Eyed they must be carrying the brown gene as well, so the parents both have the gene combination (Br, Bl) and their offspring can be any of the 4 cases $(Br^f Br^m), (Br^f Bl^m), (Bl^f Br^m), (Bl^f Bl^m)$ with equal probability. Since we know that Mr. S is Brown-Eyed the last case is eliminated, and the probability of Mr. S having (Br, Bl) combination will be equal to $\frac{2}{3}$.
Another way of calculating the probability would be using the following equation:

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
A: The event that Mr. S is (Br, Bl)
B: The event that Mr. S is Brown-Eyed

$P(A) = \dfrac{2}{4}$
$P(B) = \dfrac{3}{4}$
$=> P(A|B) = \dfrac{\frac{2}{4}}{\frac{3}{4}} = \dfrac{2}{3}$
Now considering Mr. S has a Blue-Eyed wife, we want to calculate the probability of their first child being Blue-Eyed:

Mr. S probability for each case: $(Br, Bl) : \frac{2}{3}$, $(Bl, Bl) : \frac{1}{3}$
Mother : (Bl, Bl)
$P[\text{child is} (Bl, Bl)] =?$

Using conditional probability for each case of gene combinations for Mr. S we get:

$$\begin{aligned}
P[\text{child is } (Bl, Bl)] &= P[\text{child is } (Bl, Bl)|\text{Mr. S is } (Br, Bl)]P[\text{Mr. S } (Br, Bl)] \\
&+ P[\text{child is } (Bl, Bl)|\text{Mr. S is } (Br, Br)]P[\text{Mr. S } (Br, Br)] \\
&= \frac{1}{2} \times \frac{2}{3} + 0 = \frac{1}{3}
\end{aligned}$$

Now suppose we know the first child of Mr. S and his wife is Blue-Eyed. What would then be the probability of the second child being Blue-Eyed?

$P[\text{2nd child is } (Bl, Bl)|\text{1st child is } (Bl, Bl)] =?$

Using Bayes' Theorem we have:

$P[\text{Mr. S is } (Br, Bl)|\text{1st child is } (Bl, Bl)] = \dfrac{P[\text{1st child is } (Bl, Bl)|\text{Mr. S is } (Br, Bl)]}{P[\text{1st child is } (Br, Bl)]}$
which leads to:

$$P[\text{Mr. S is } (Br, Bl)|\text{1st child is } (Bl, Bl)] = \frac{\dfrac{1}{2} \times \dfrac{2}{3}}{\dfrac{1}{3}} = 1$$

This verifies the notion that if the first child has blue eyes, Mr. S has to be a carrier of blue genes and cannot have a gene combination of (Br, Br). Therefore, knowing that Mr. S has a gene combination of (Br, Bl) and his wife's gene combination is (Bl, Bl), the probability of the second child being Blue-Eyed is now equal to:

$P[\text{2nd child is } (Bl, Bl)|\text{1st child is } (Bl, Bl)] = \frac{1}{2}$

3. Suppose you have three colored cards. The first one is colored Red on both sides, the second one is colored Red on one side and Blue on the other side, and the third one is colored Blue on both sides. We randomly pick a card and look at it's color on one side (without seeing the other cards). If the side we see is Red, what is the probability that the other side is also red? In other words:
P[lower side of a randomly picked card is Red | upper side of the card is Red] =?

   We can name sides of the cards as following: $(R_1, R_2), (R_3, B_1), (B_2, B_3)$
   Any of these sides have equal probability of showing up in the first pick:
   $S = \{R_1, R_2, R_3, B_1, B_2, B_3\}$
   So if the first pick is red side it can be any of R_1, R_2, R_3 with equal probability $\frac{1}{3}$. In these cases the second side would be for $R_1 : R_2$, for $R_2 : R_1$,and for $R_3 : B_1$. Therefore the probability of the lower side being red is equal to $\frac{2}{3}$. Another way of writing it is:
   A: The event that the lower side of a randomly picked card is red
   B: The event that the upper side of a randomly picked card is red

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

4. n people leave their car keys on a table and then we randomly distribute these keys among them, each person getting one. What is the probability that k specific people get their own keys?
   We define A_i as the event that person i get their own key. Each person gets a key randomly with equal probability, therefore we have:
   $P(A_i) = \dfrac{1}{n}$

   the probability that k specific people get their own keys will be equal to:
   $P[A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}] = \dfrac{(n-k)!}{n!}$

   since the total number of ways to distribute the rest of the keys, assuming these k people are getting theirs, is equal to $(n-k)!$.

   Now what is the probability that none of them gets their own key?

   $P(A_1^c \cap A_2^c \cap ... \cap A_n^c) =?$

   We can calculate this using the probability that 'the first person does not get the right key' and then the conditional probability that 'the rest do not get the right key given that the first person got a wrong key':

$P(A_1^c \cap A_2^c \cap ... \cap A_n^c) = P(A_1^c)P(A_2^c \cap A_3^c \cap ... \cap A_n^c | A_1^c)$
$P(A_1^c) = \dfrac{n-1}{n}$

Now given that the first person does not get the right key, $n-1$ people are choosing out of $n-1$ keys, one of which belongs to none of them. We need to consider two cases where the person whose key went to the 1st person now gets 1st person's key, and the case where he does not. If we define $P_n = P(A_1^c \cap A_2^c \cap ... \cap A_n^c)$ (The probability of n people all getting the wrong key out of their own n keys), now we have:
$P(A_2^c \cap A_3^c \cap ... \cap A_n^c | A_1^c) = \dfrac{1}{n-1}P_{n-2} + P_{n-1}$
since the first case is equivalent to $n-2$ people choosing out of the $n-2$ keys belonging to them, when the extra person got the first person's key with $\frac{1}{n-1}$ probability. The second case will be the probability of $n-1$ people not getting their own key out of $n-1$ keys belonging to them, imagining 1st person's key belongs to that extra person. Therefore, we have:

$$P_n = \frac{n-1}{n}(\frac{1}{n-1}P_{n-2} + P_{n-1}) = \frac{1}{n}P_{n-2} + \frac{n-1}{n}P_{n-1}$$

Knowing that for one and two people we have $P_1 = 0$ and $P_2 = \frac{1}{2}$ you can show that:

$$P_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ... + \frac{(-1)^n}{n!}$$