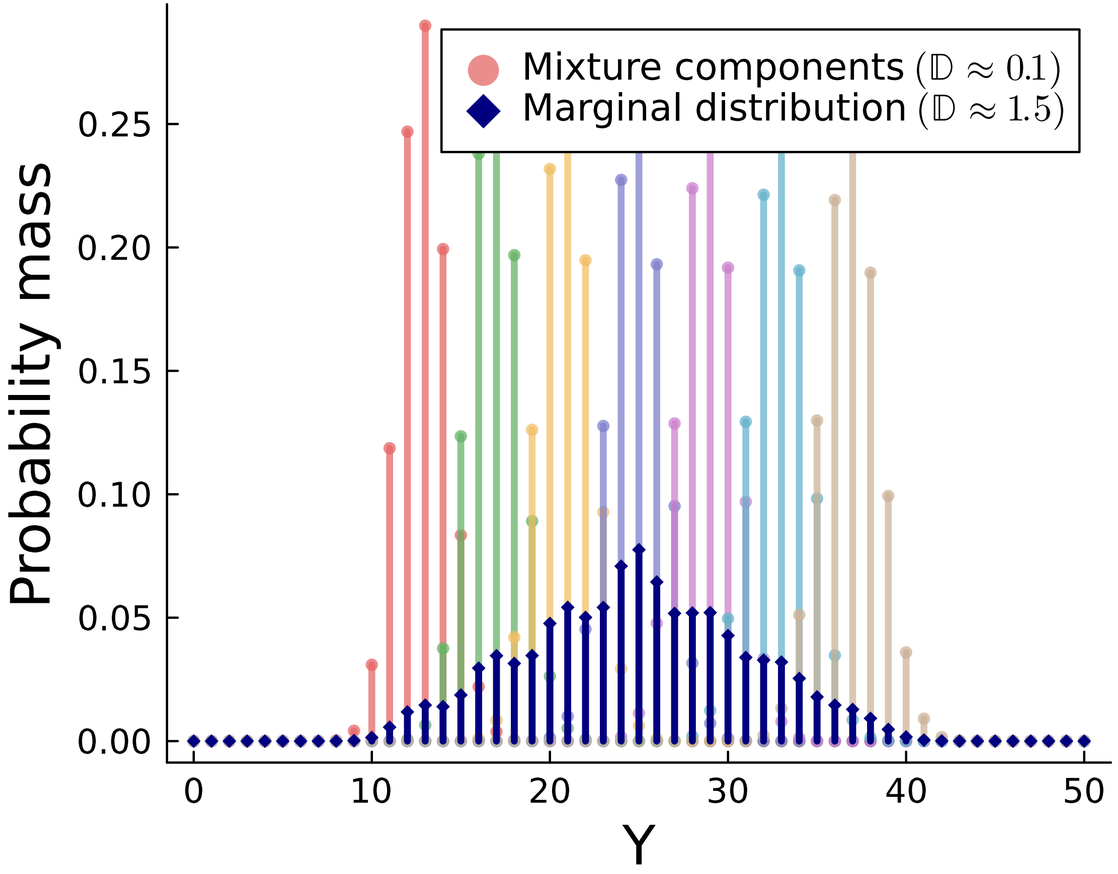
Modeling Latent Underdispersion with Discrete Order Statistics

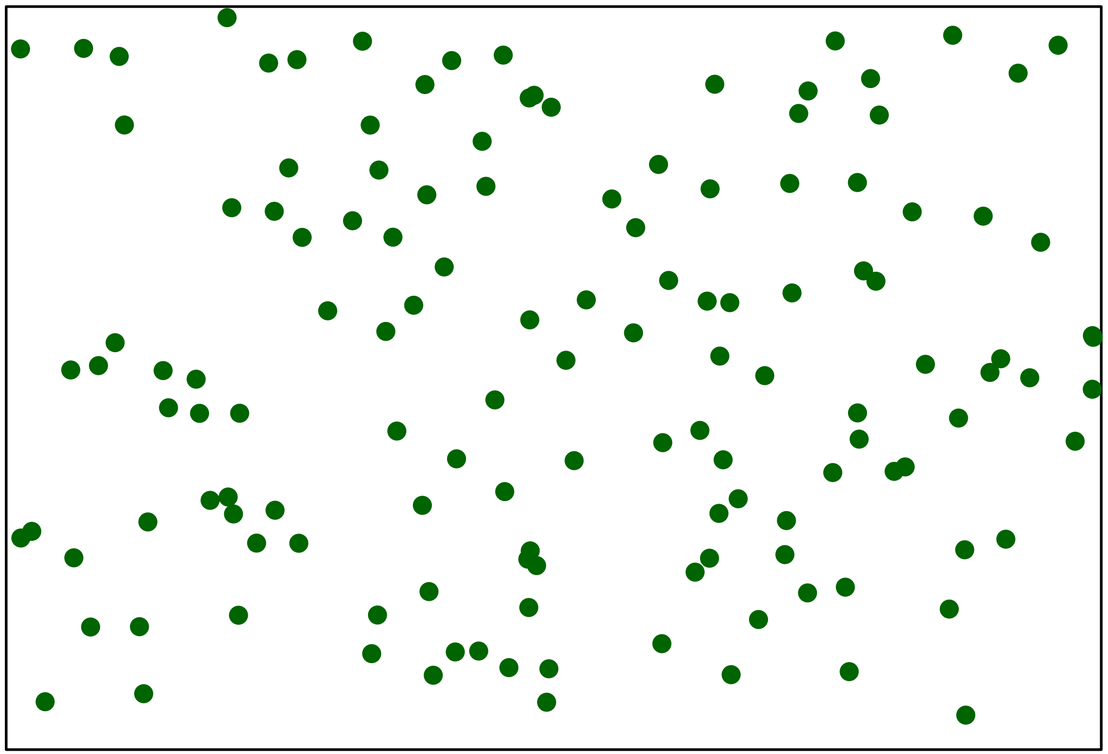


Jimmy Lederman, University of Chicago

Work with Aaron Schein

Fast and Curious 2: MCMC in Action

The Poisson process



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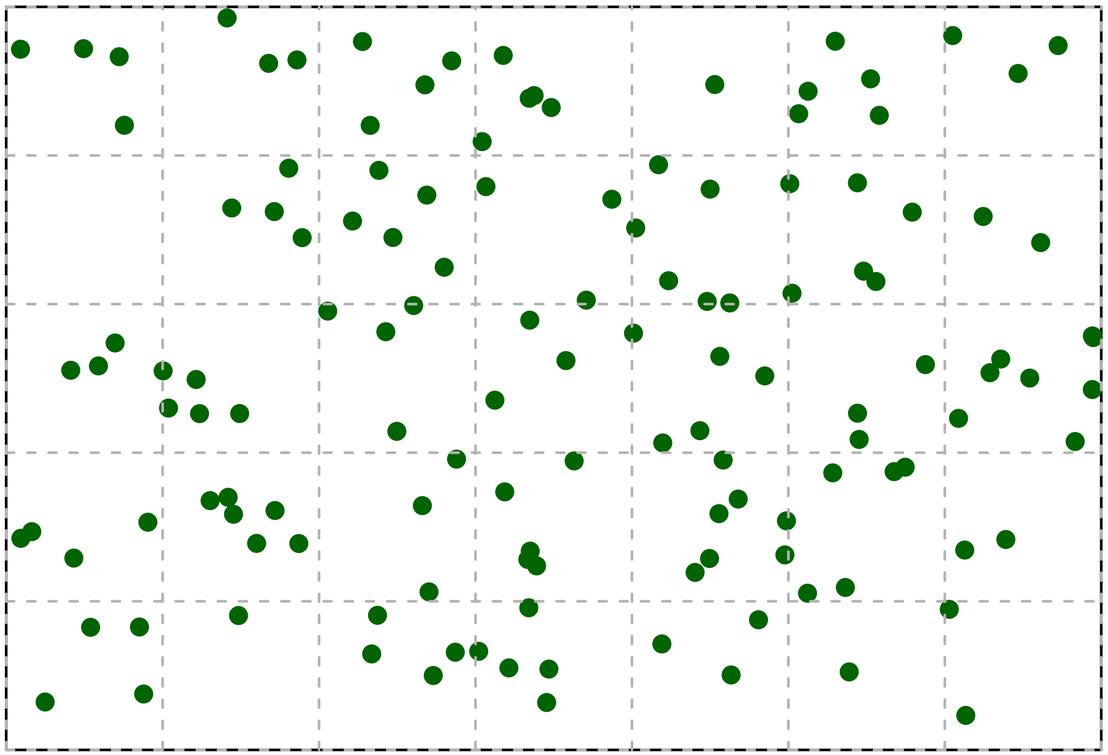
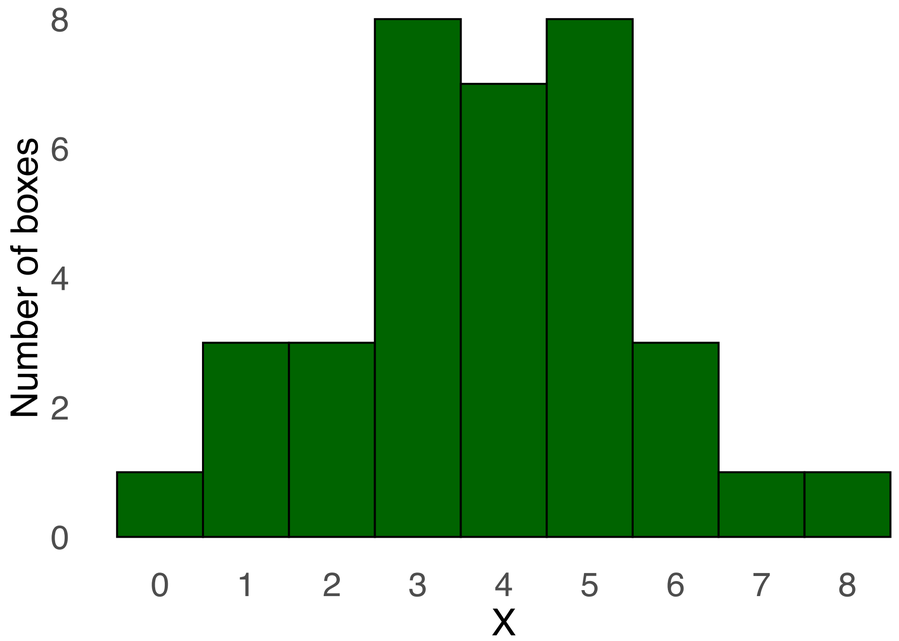


X = number of points in each box

$X \sim \text{Poisson}(\mu)$



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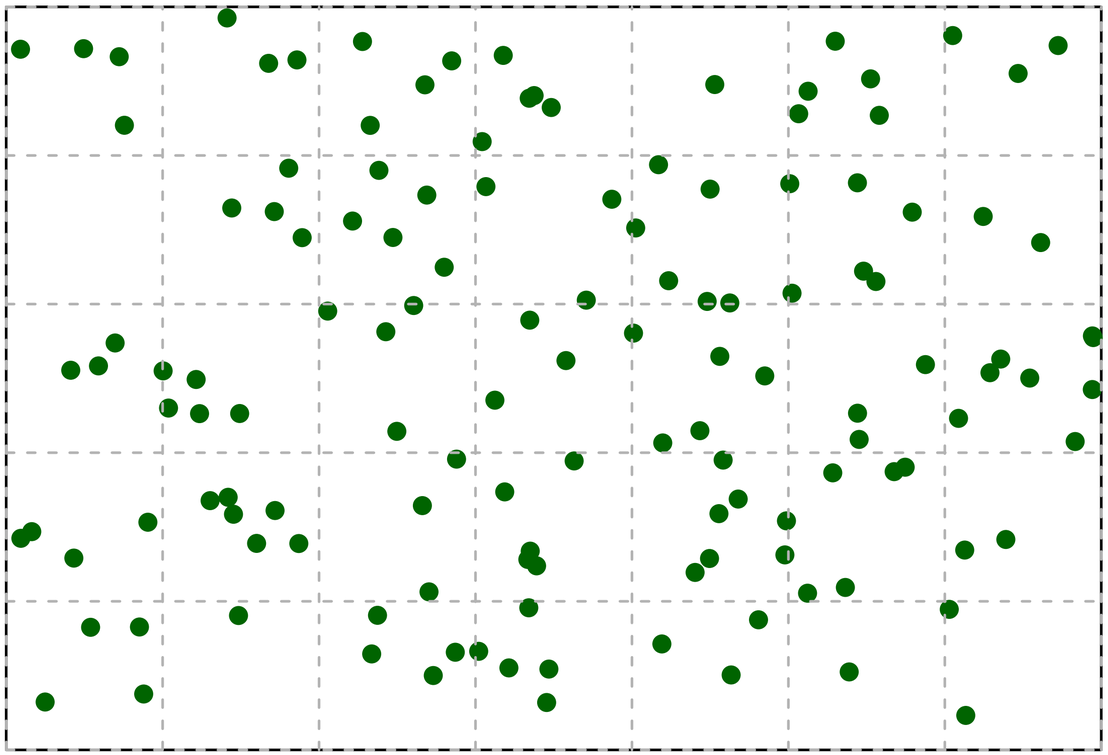

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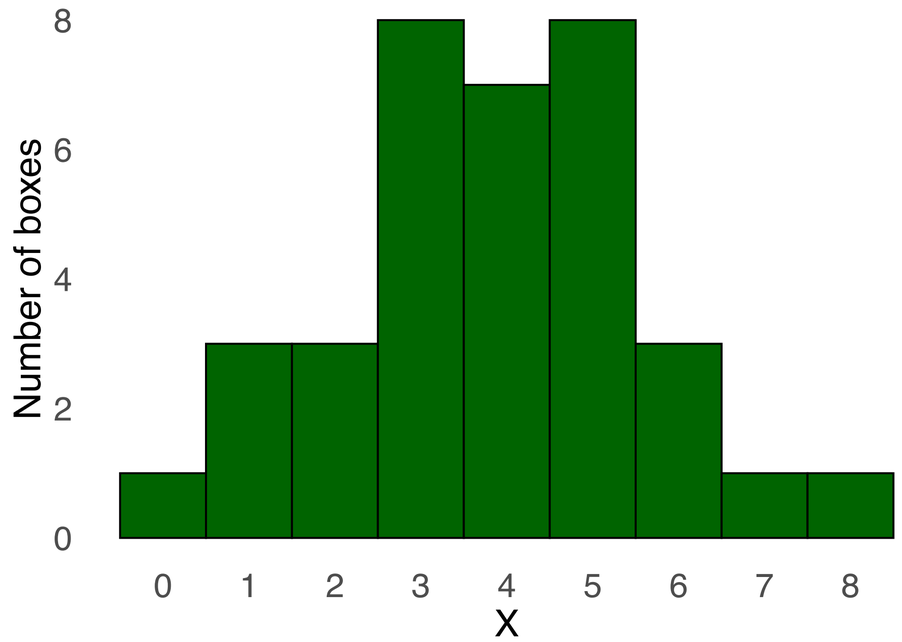
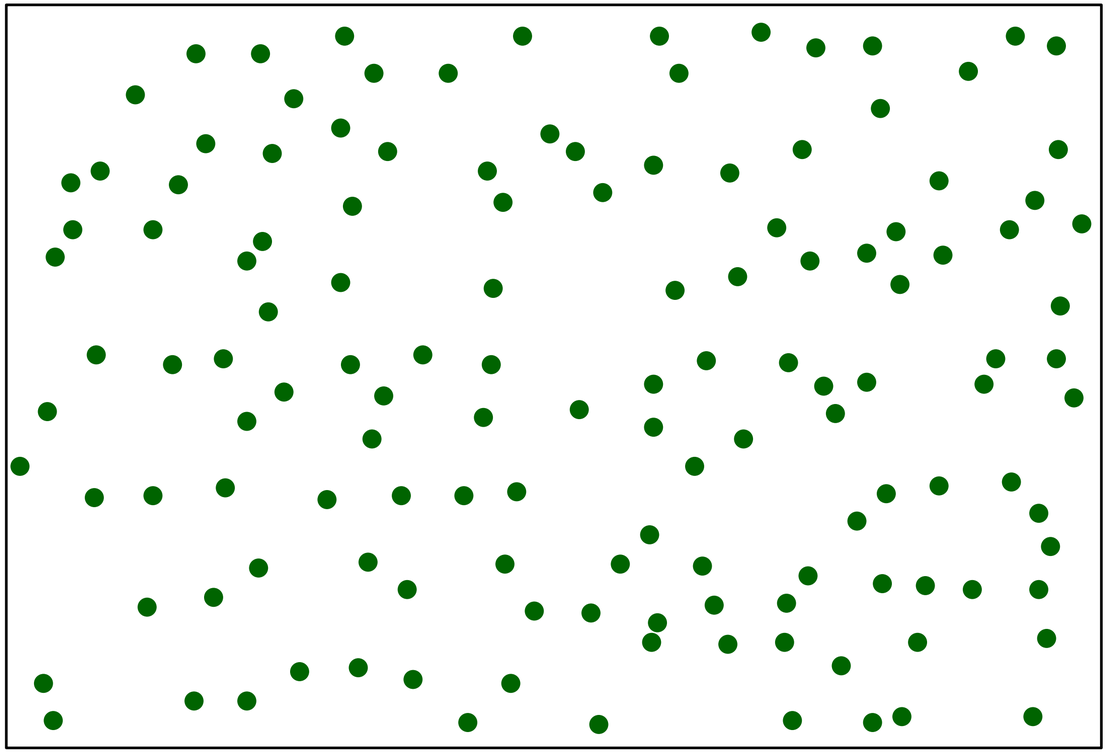


Dispersion $\longrightarrow$ $\mathbb{D}[X] = \dfrac{\mathbb{V}[X]}{\mathbb{E}[X]} = 1$

Underdispersion: Spruce trees



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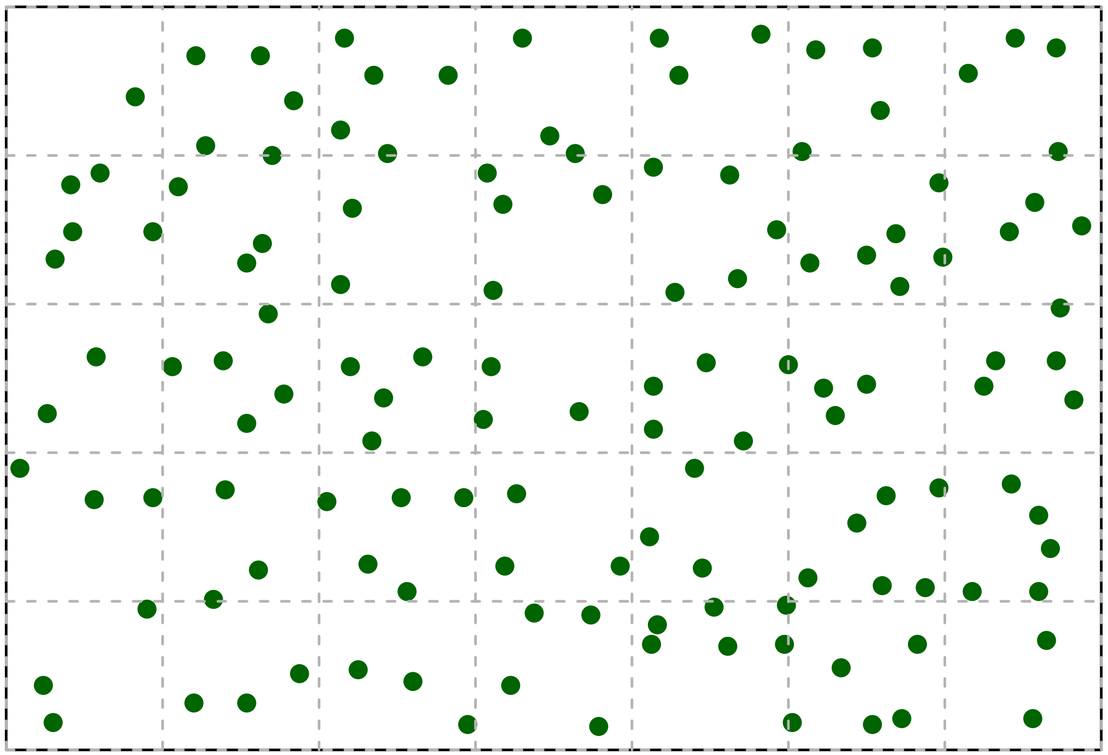

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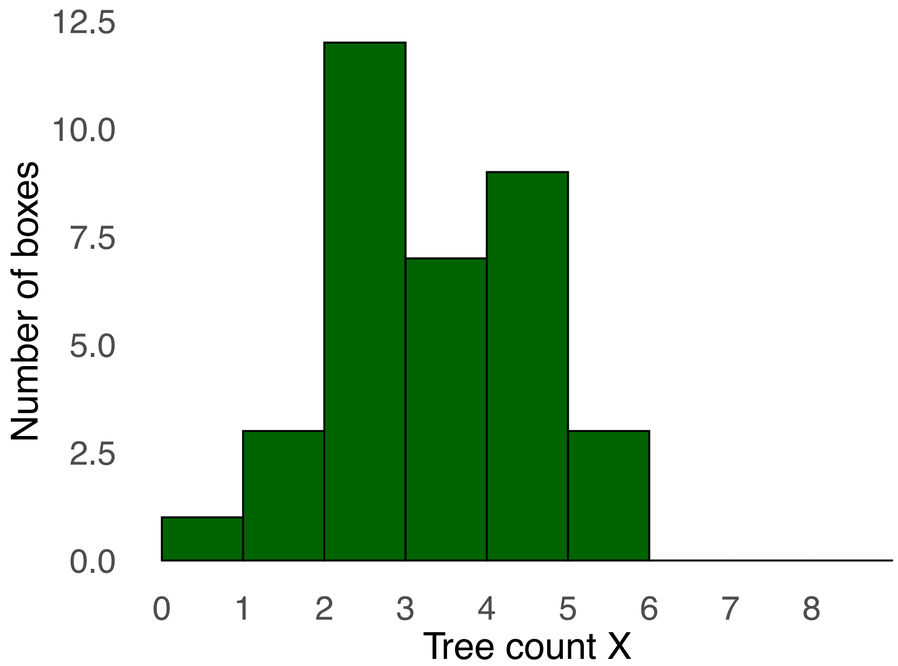

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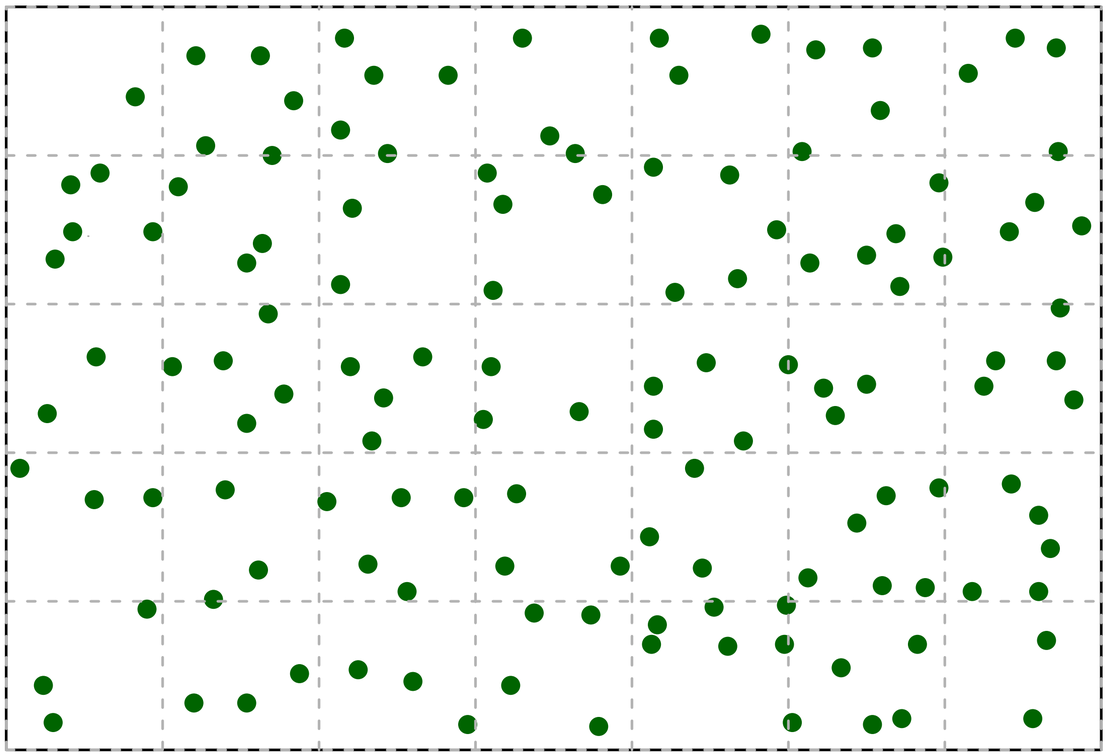

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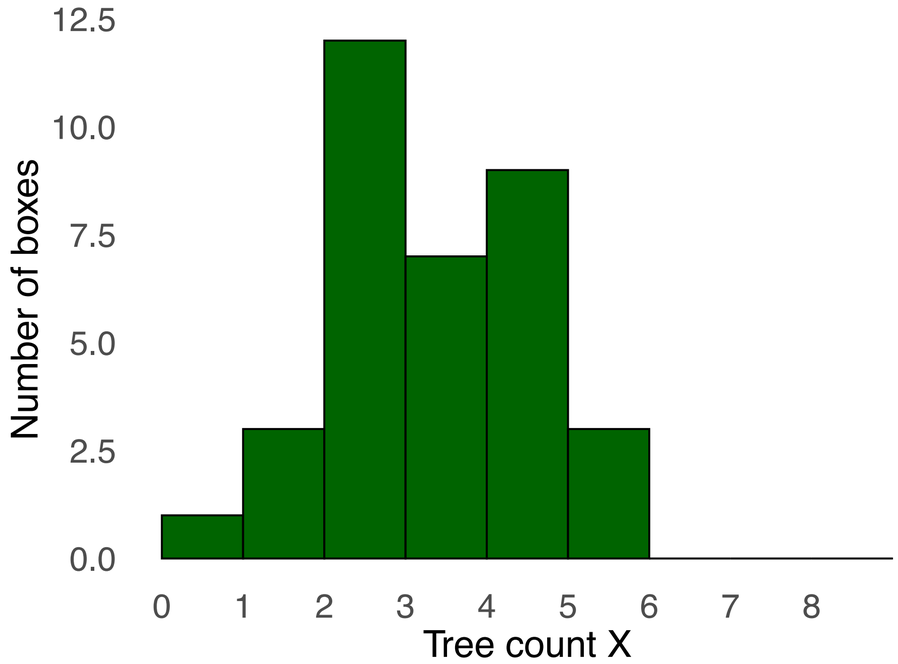

Dispersion $\longrightarrow$ $\mathbb{D}[X] = \dfrac{\mathbb{V}[X]}{\mathbb{E}[X]} \approx .41$

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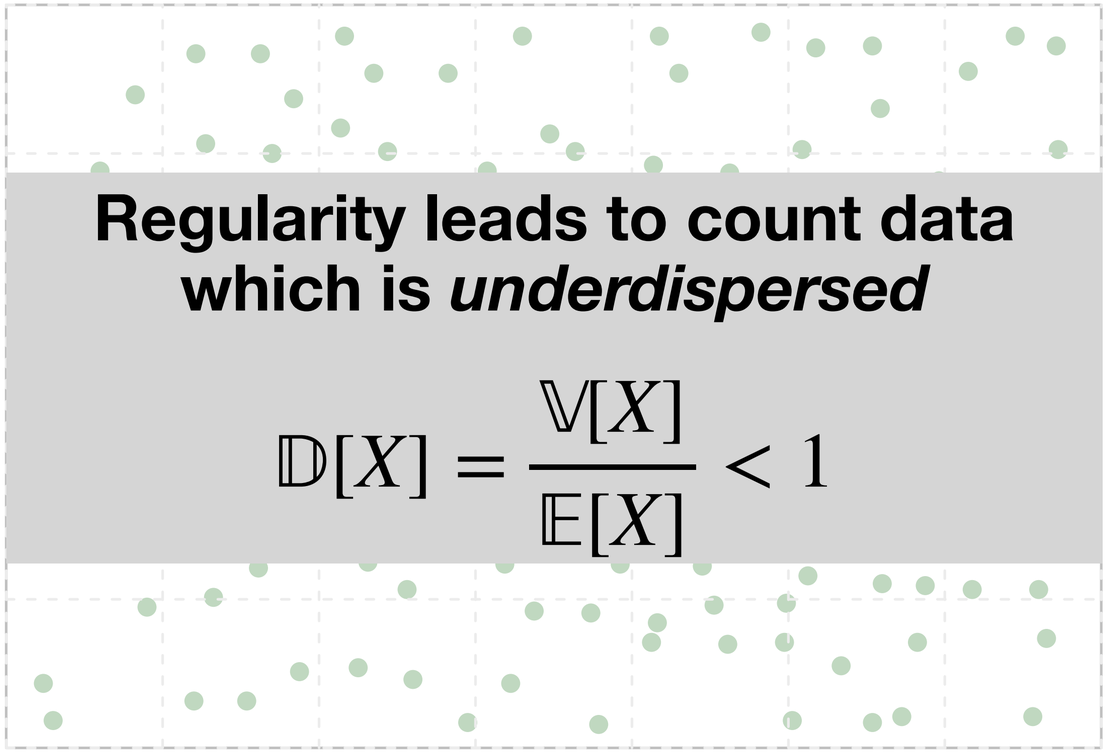
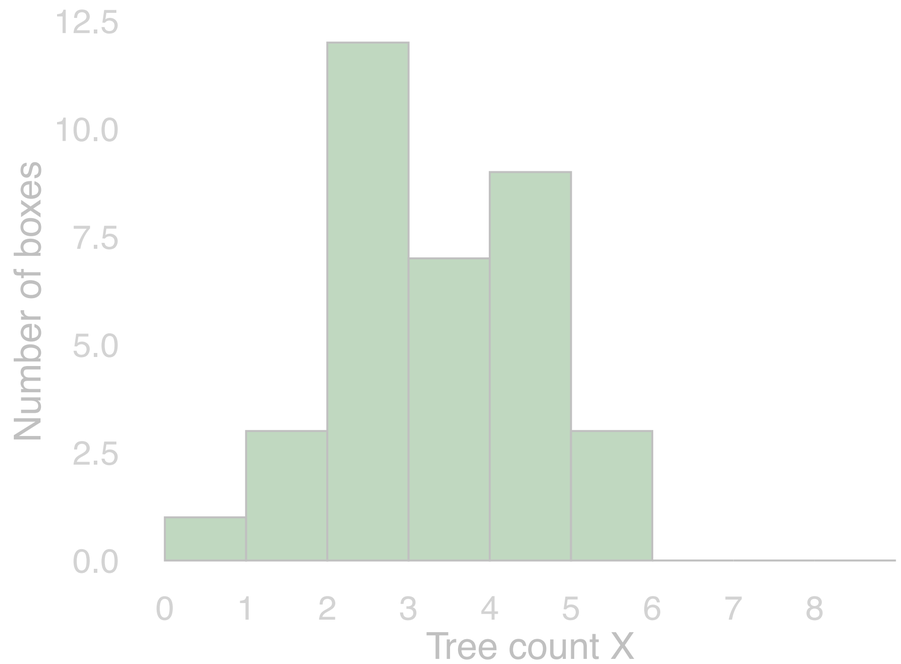
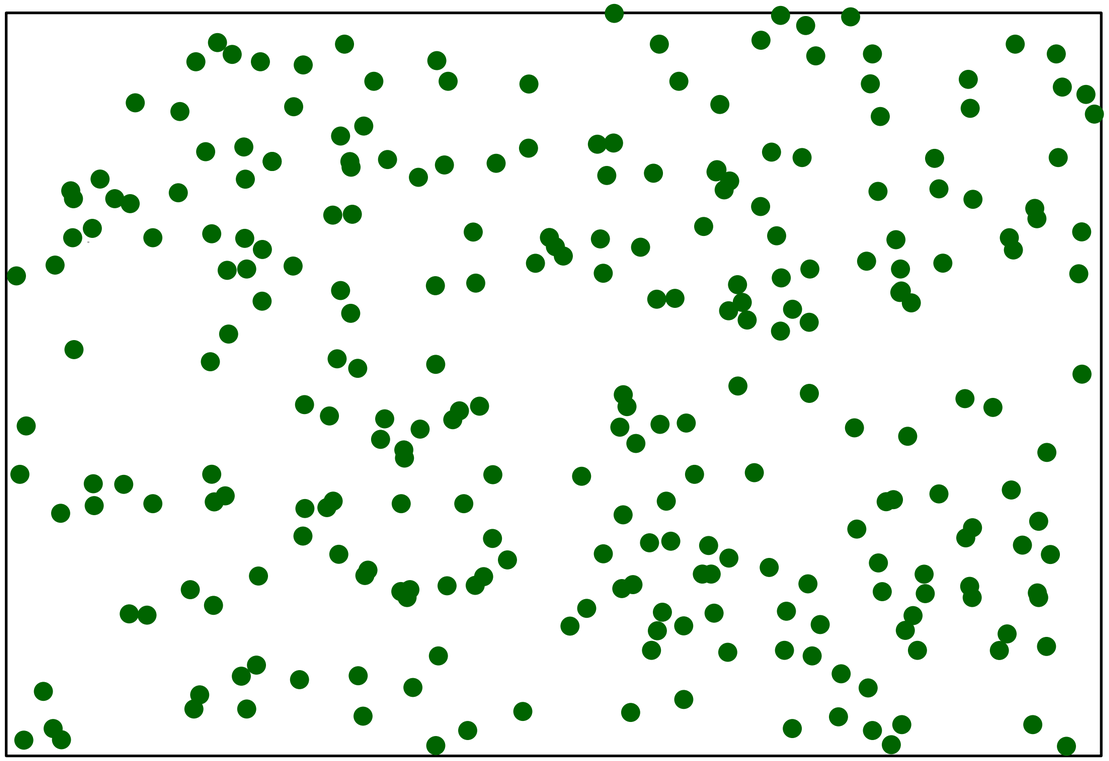


Regularity leads to count data which is *underdispersed*

$$\mathbb{D}[X] = \frac{\mathbb{V}[X]}{\mathbb{E}[X]} < 1$$

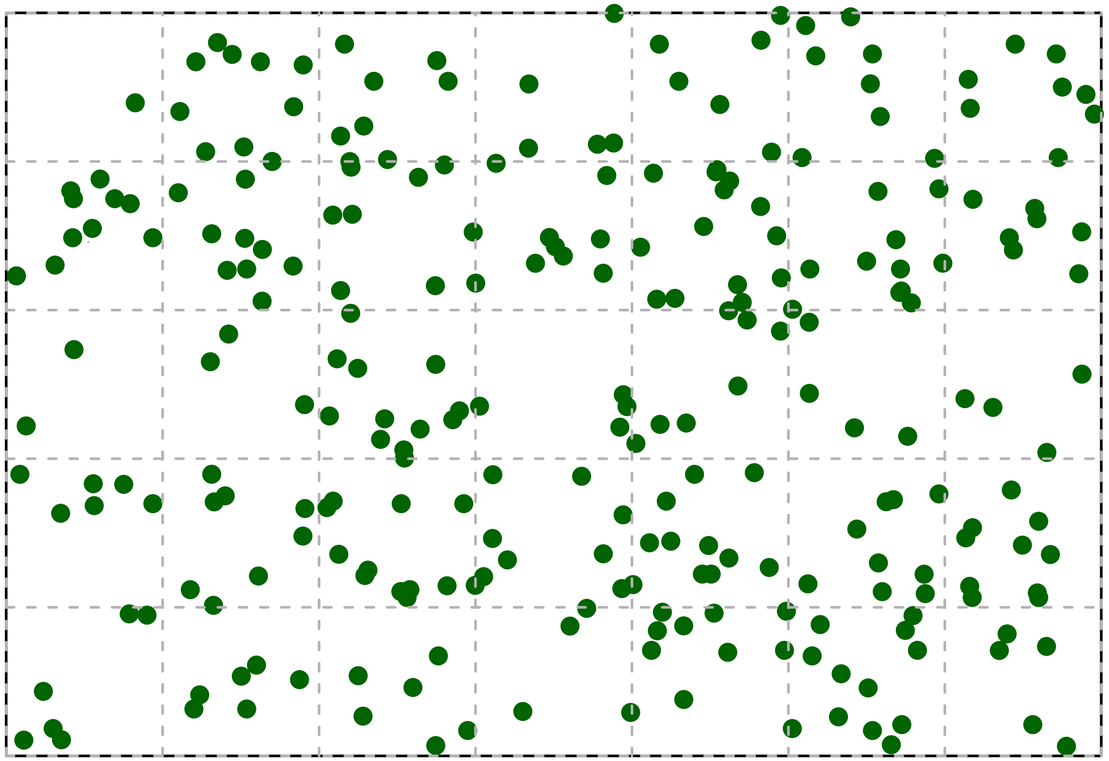


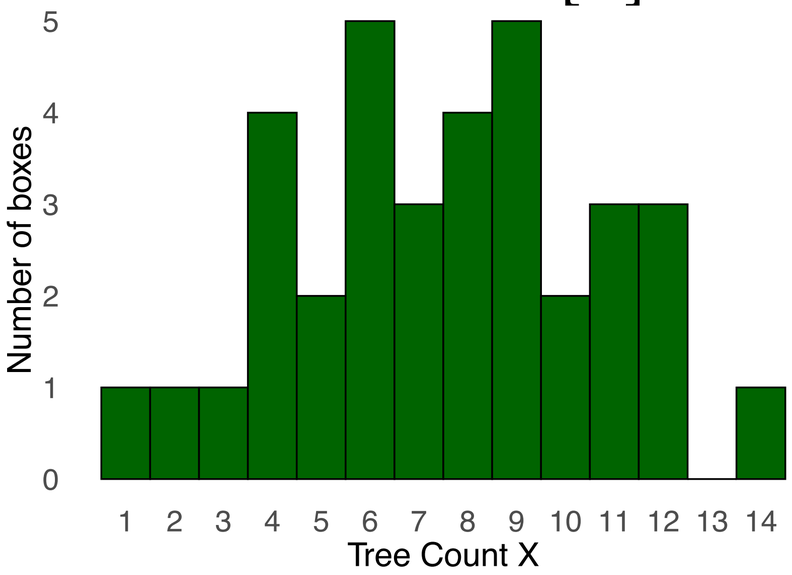
Conditional underdispersion



Conditional underdispersion

$$\mathbb{D}[X] = \frac{\mathbb{V}[X]}{\mathbb{E}[X]} \approx 1.29$$





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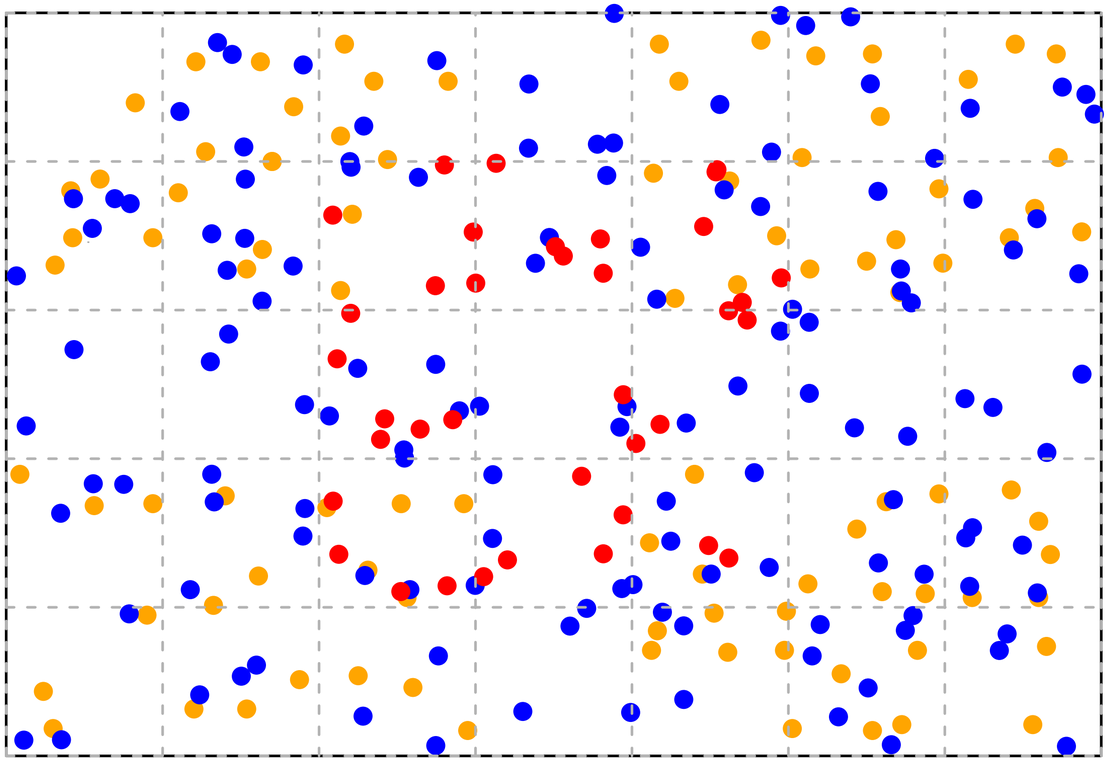

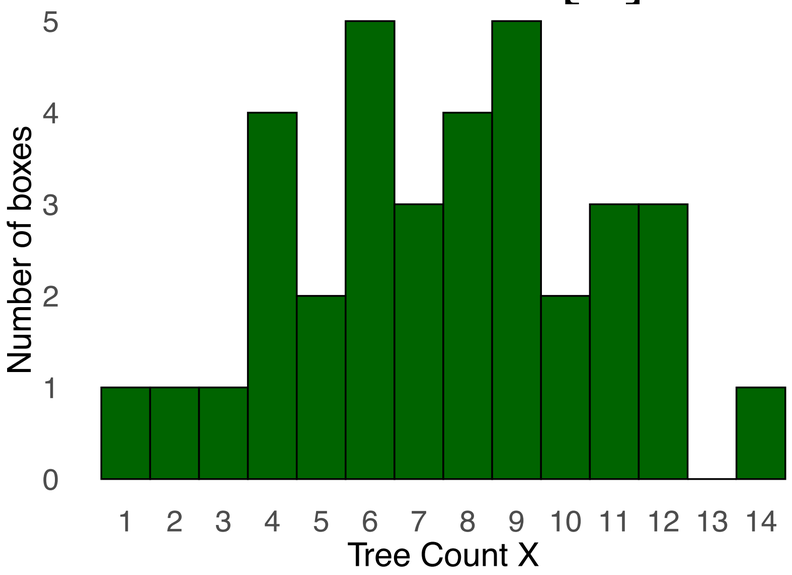

Color denotes the covariate species

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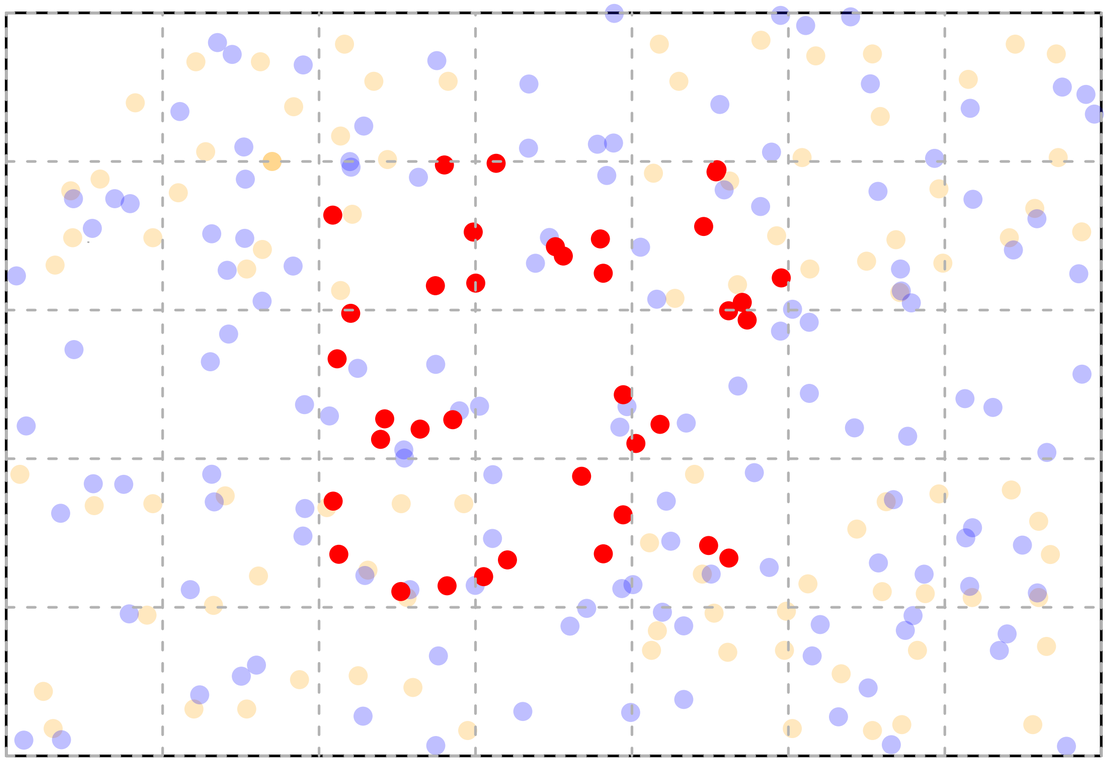

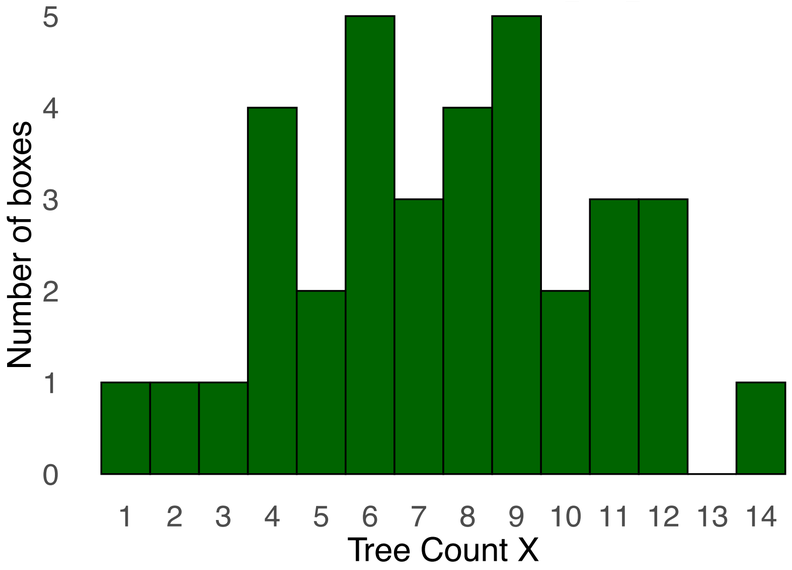

Color denotes the covariate species

$\mathbb{D}[X \mid \text{Species} = \text{red}] \approx .70$

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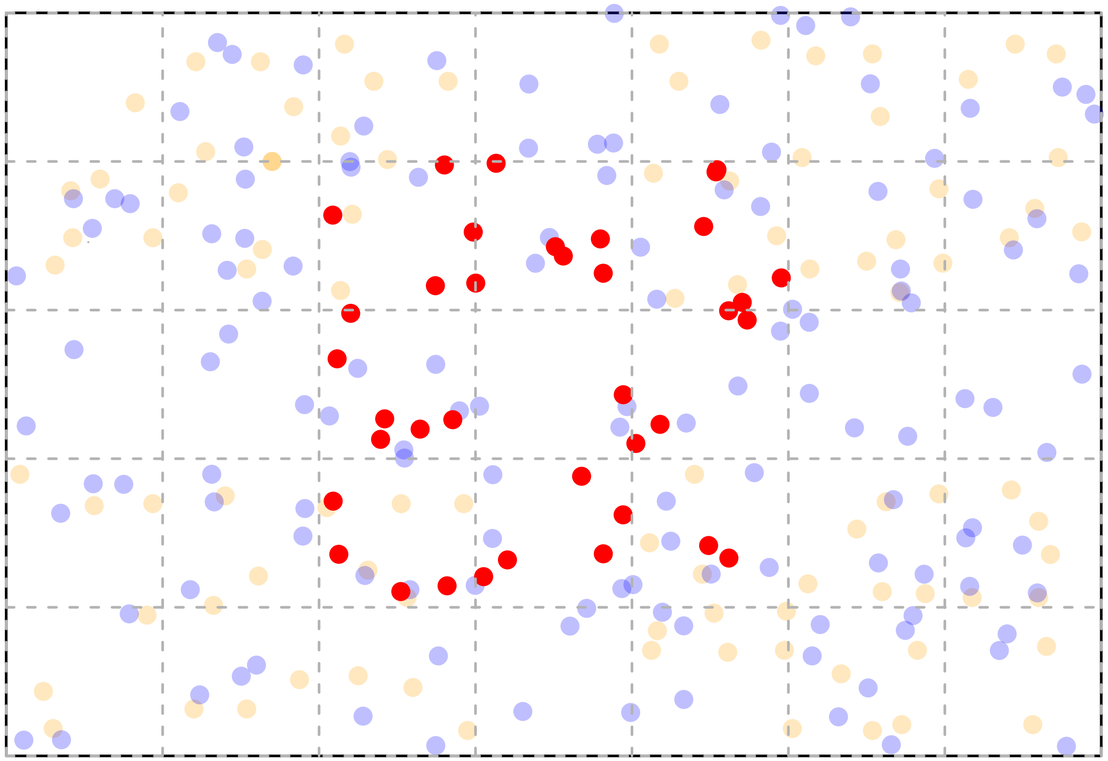

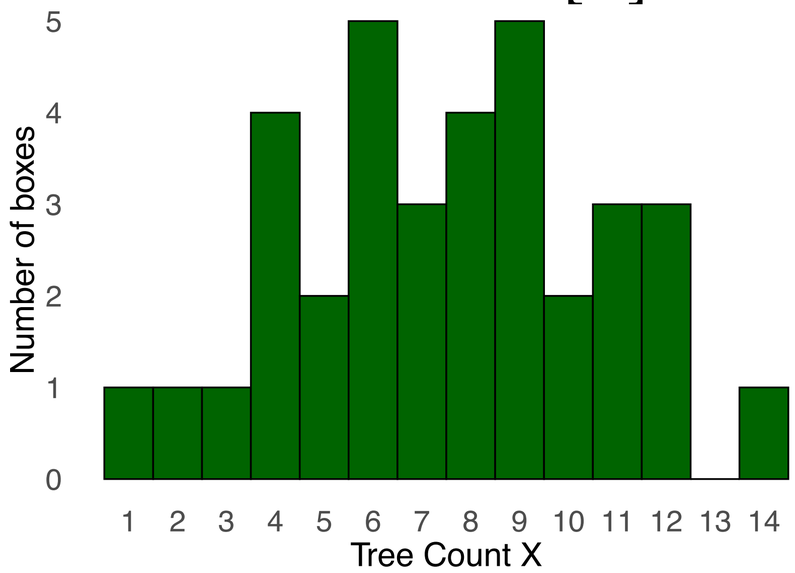

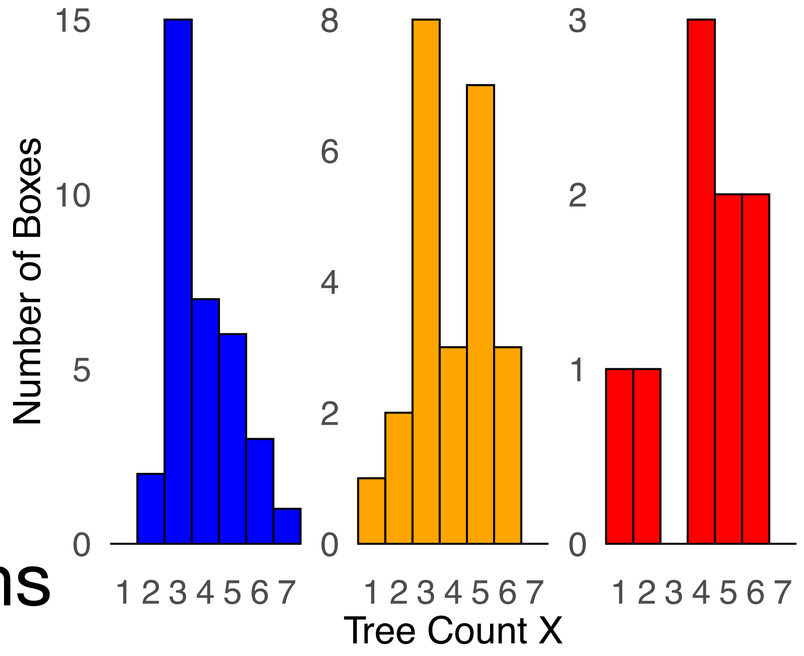

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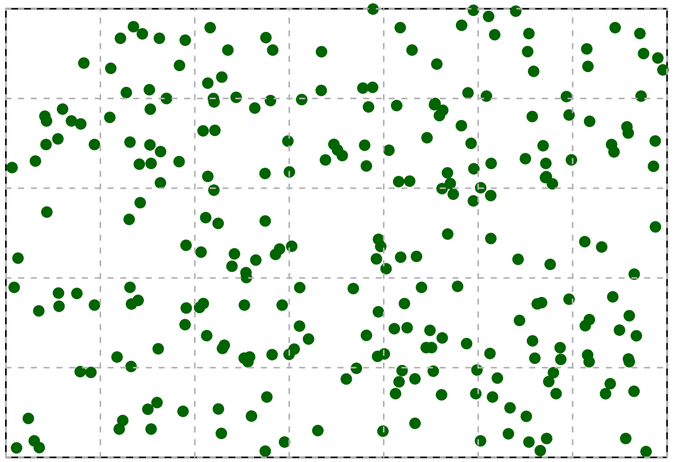
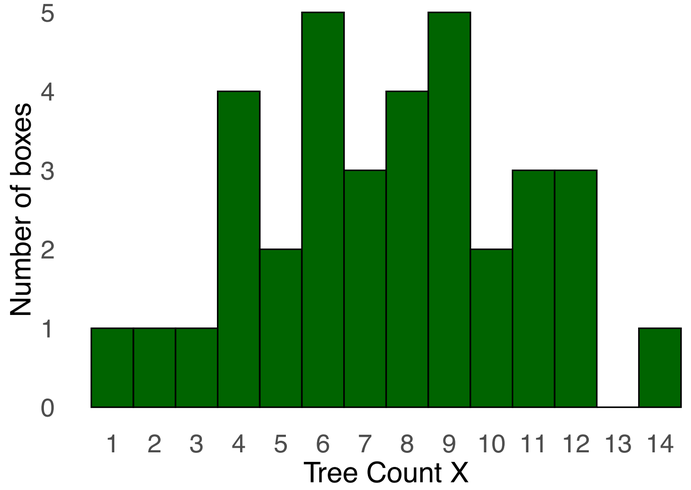
Latent structure can reveal more regular patterns

Underdispersion might be latent



Marginal view

Likely overdispersed



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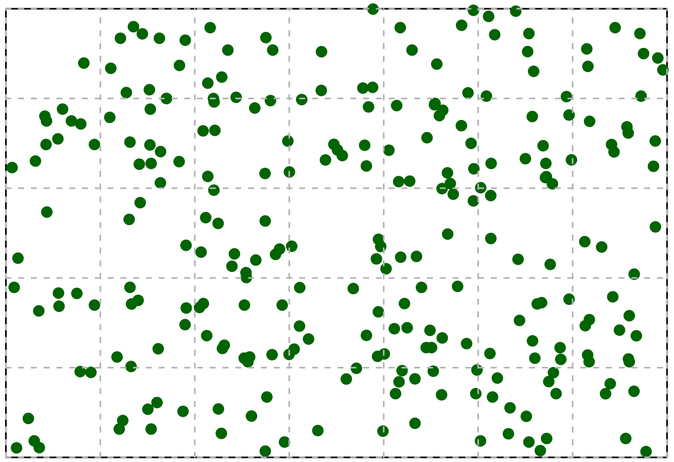

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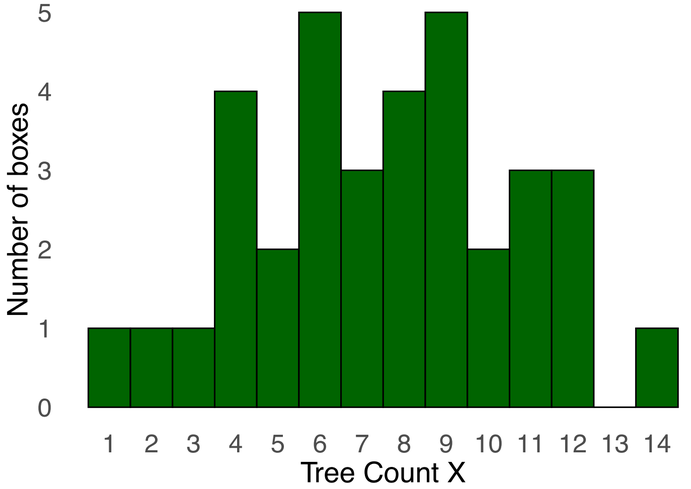
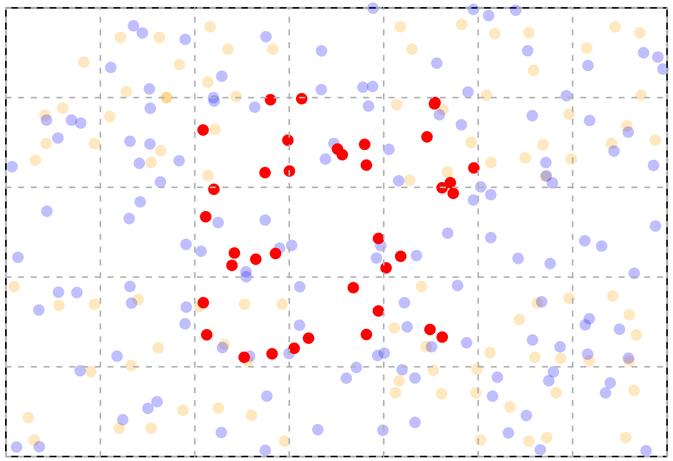
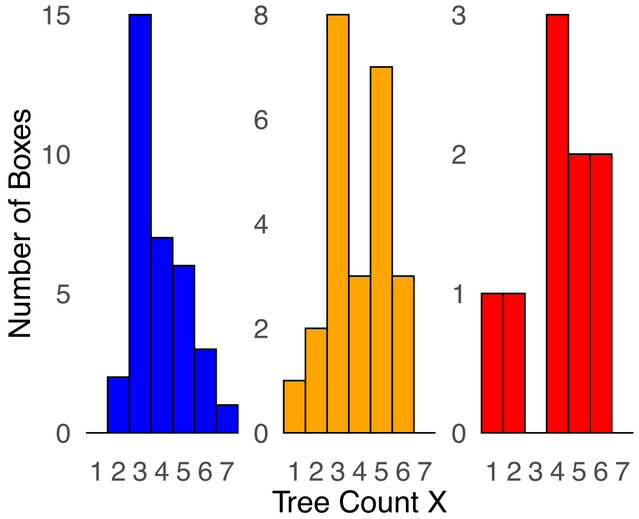




Conditional view

Potentially underdispersed



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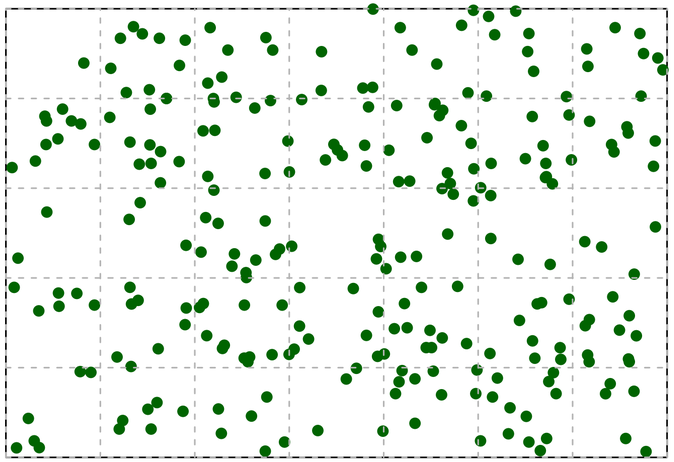

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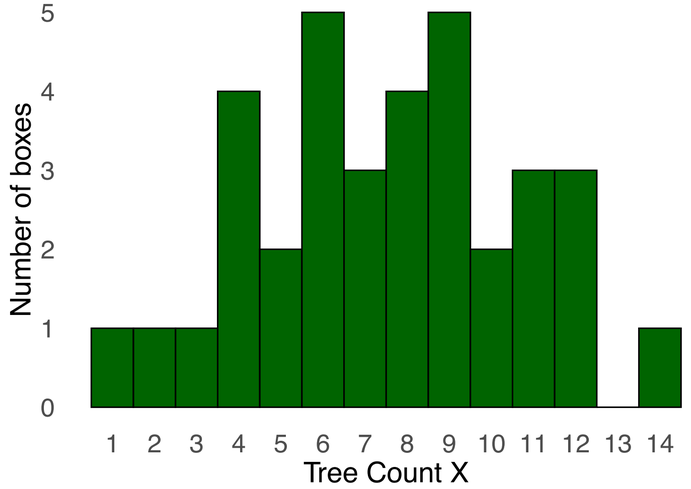
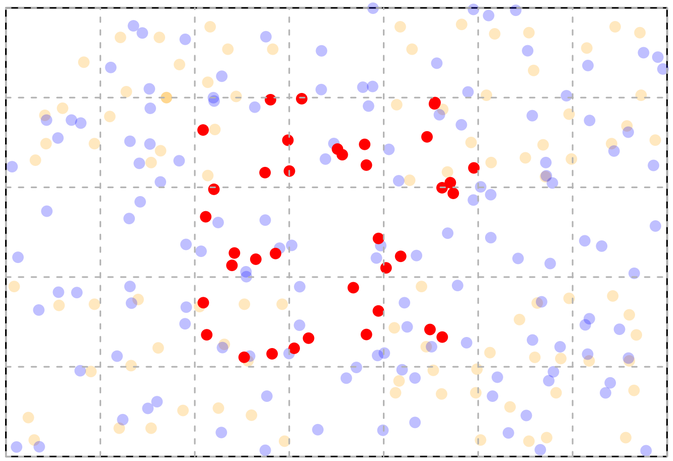


Goal: build **latent variable models** which allow for **conditional underdispersion**



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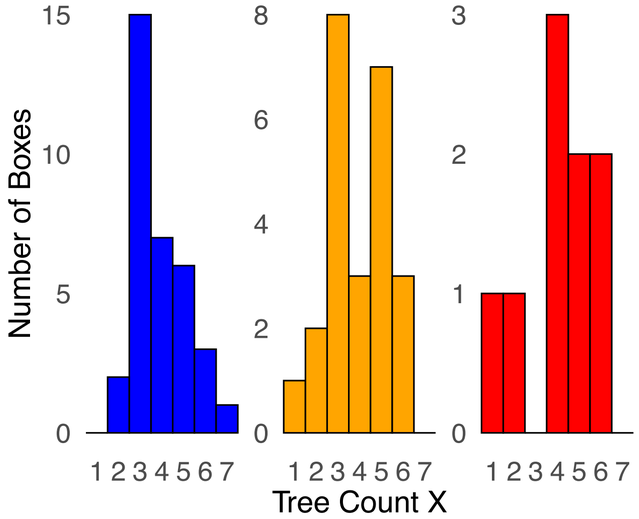

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Count distributions which allow for underdispersion include:

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Our solution: build models around a distribution which

1. is capable of attaining underdispersion
2. can be written in terms of latent Poisson random variables

Poisson order statistics for underdispersed counts

Consider a count-valued datapoint $Y \in \mathbb{N}_0$ assumed to be a Poisson order statistic

$$Y \sim \text{Pois}_{\mu}^{(r,D)} \iff Y = Z^{(r,D)} \text{ where } Z_1, \ldots, Z_D \sim \text{Pois}(\mu)$$

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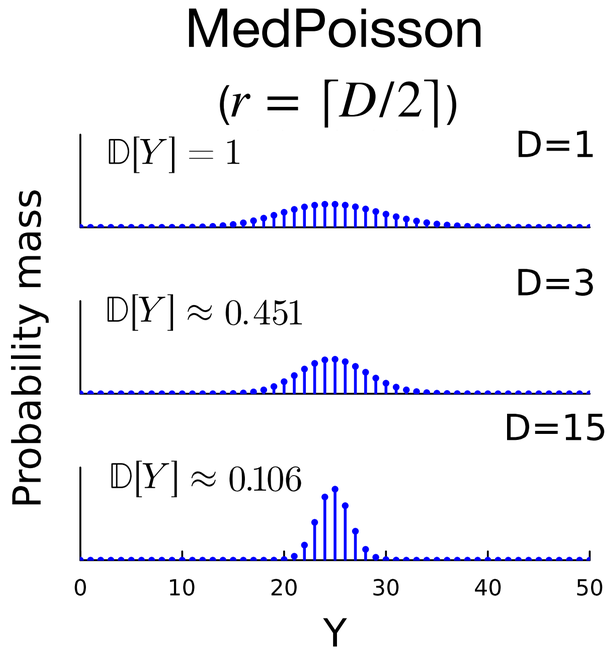

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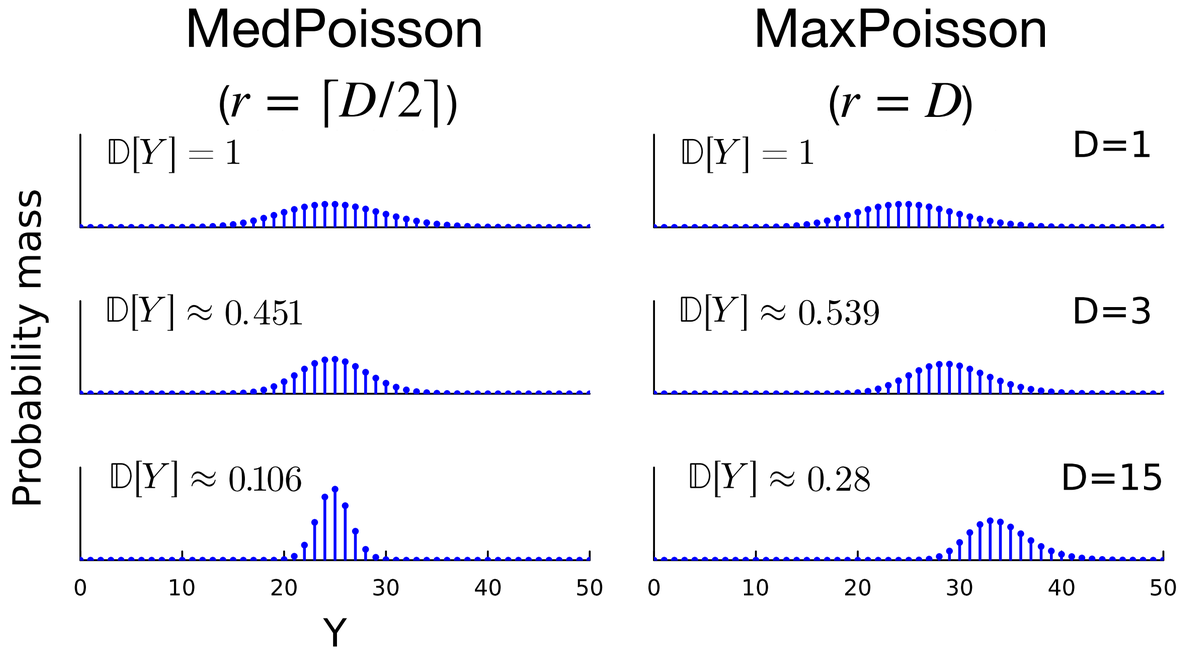

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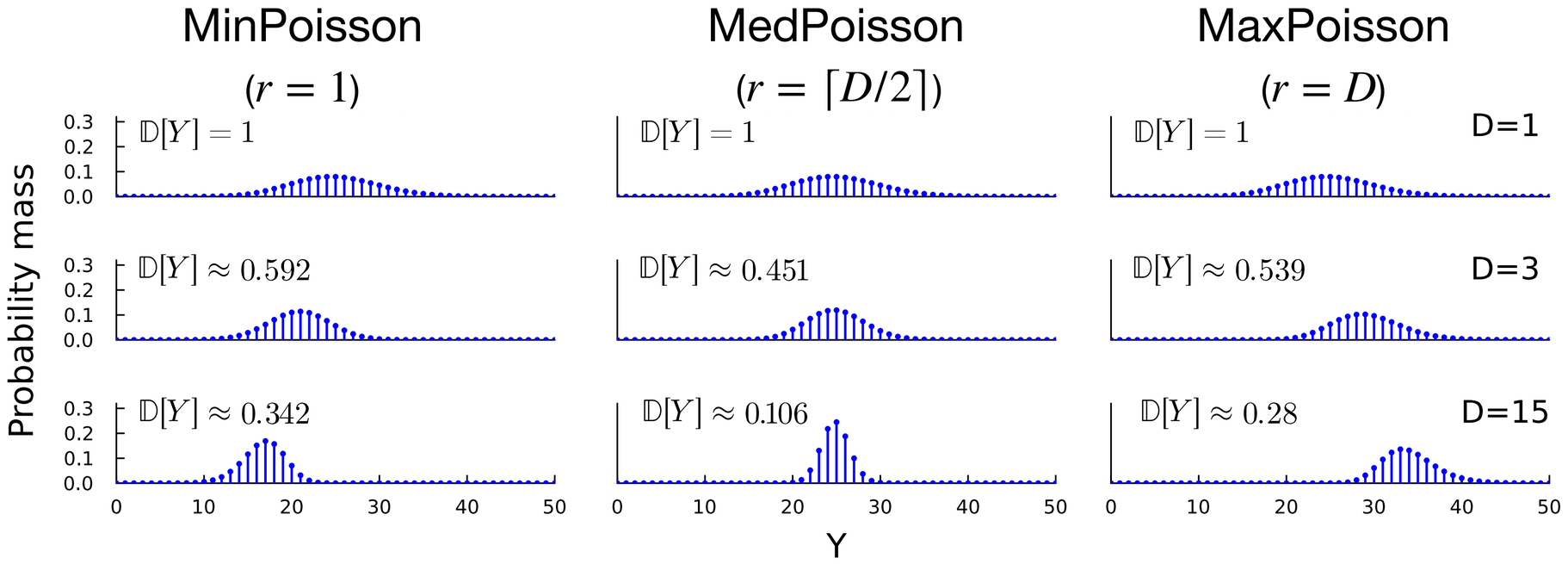

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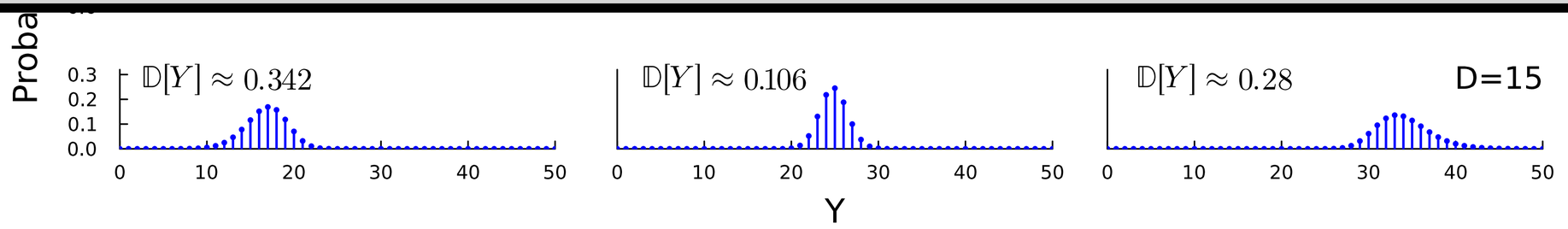
MinPoisson MedPoisson MaxPoisson

Key fact [Badiella, 2023]: For any μ, r, and $D > 1$, the Poisson order statistic $Y \sim \text{Pois}_{\mu}^{(r,D)}$ is underdispersed:

$$\mathbb{D}[Y] = \frac{\mathbb{V}[Y]}{\mathbb{E}[Y]} < 1$$

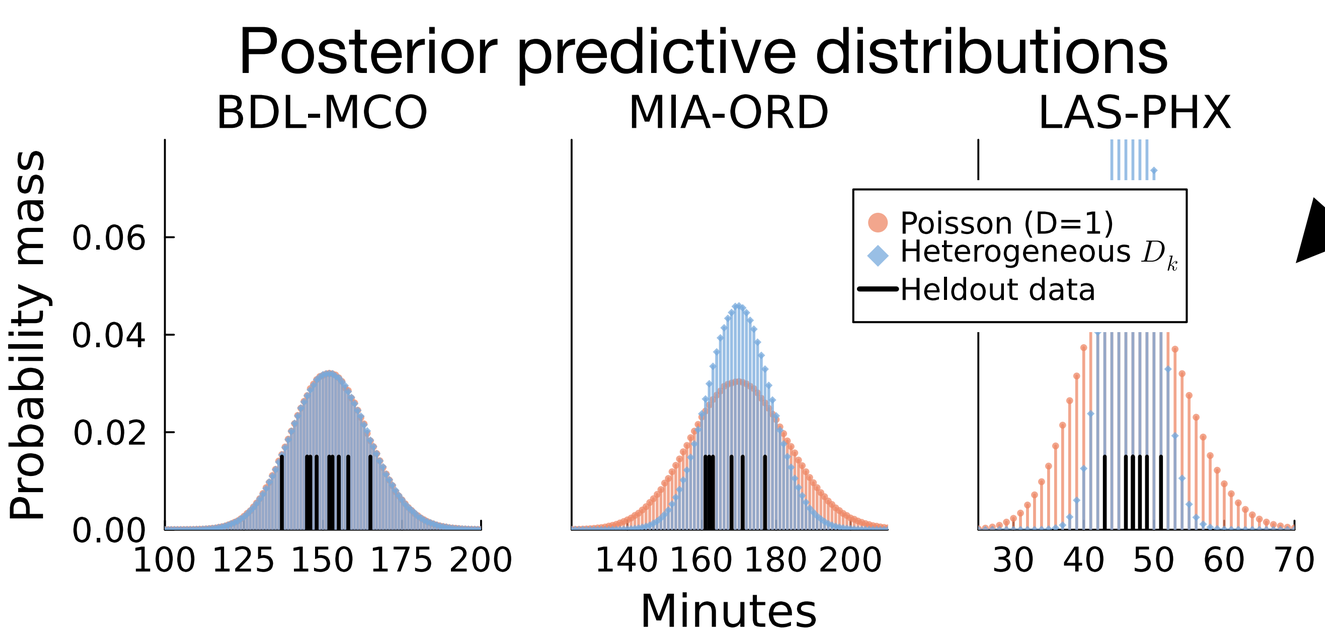


Talk outline

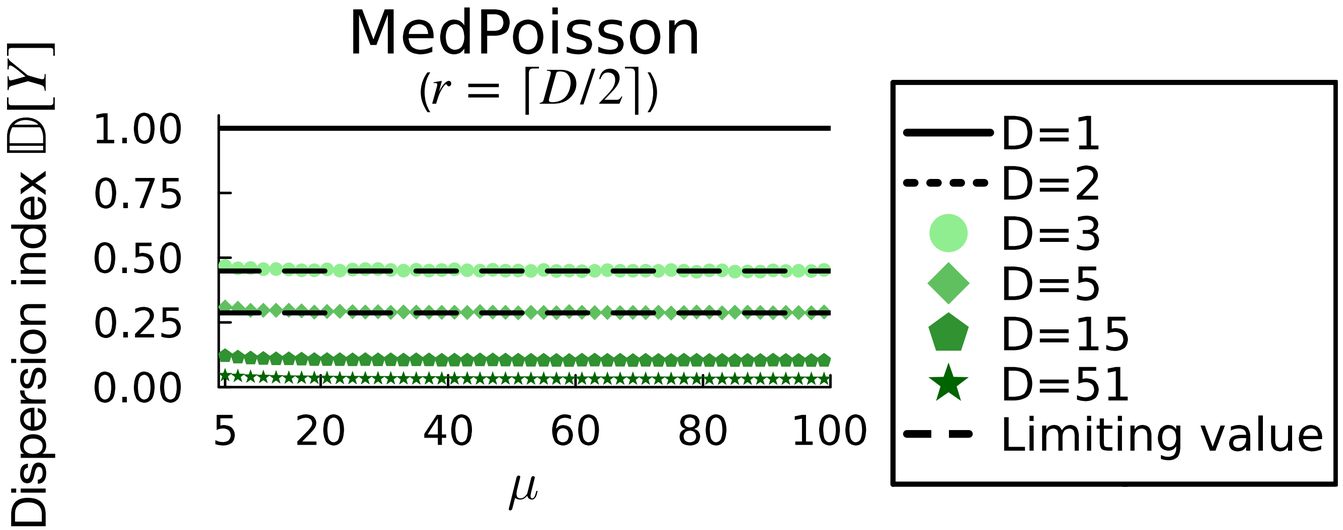
1. Dispersion properties of Poisson order statistics

2. A data augmentation strategy for inference of the parameters of any discrete order statistic **(not only for the Poisson)**

3. Applications where we build and fit hierarchical models for conditionally underdispersed count data, yielding **more precise probabilistic predictions**



Dispersion of Poisson order statistics



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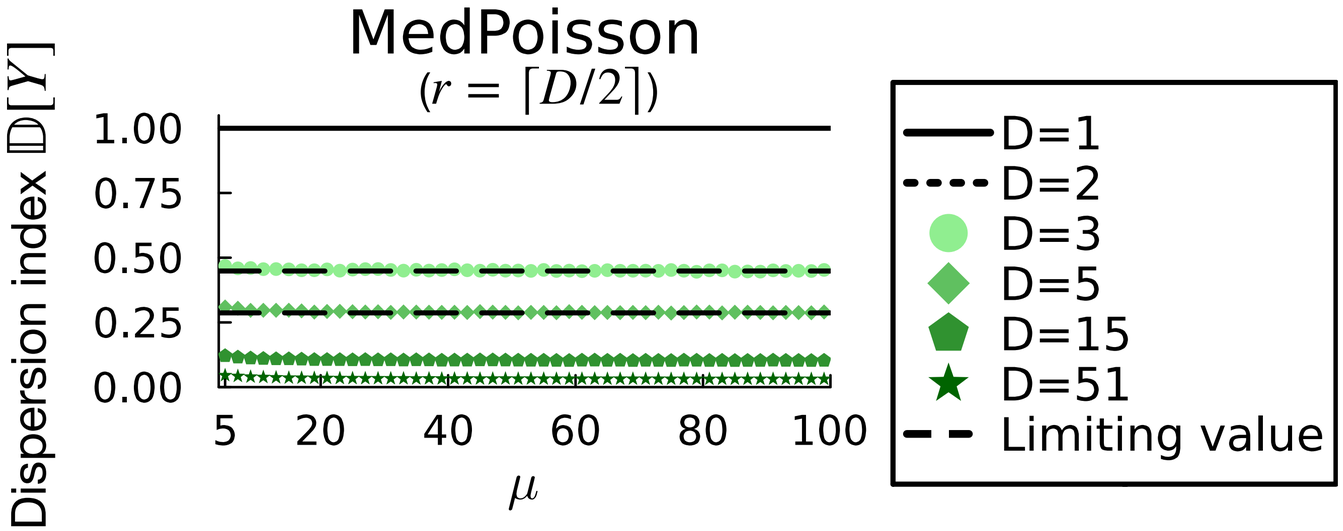
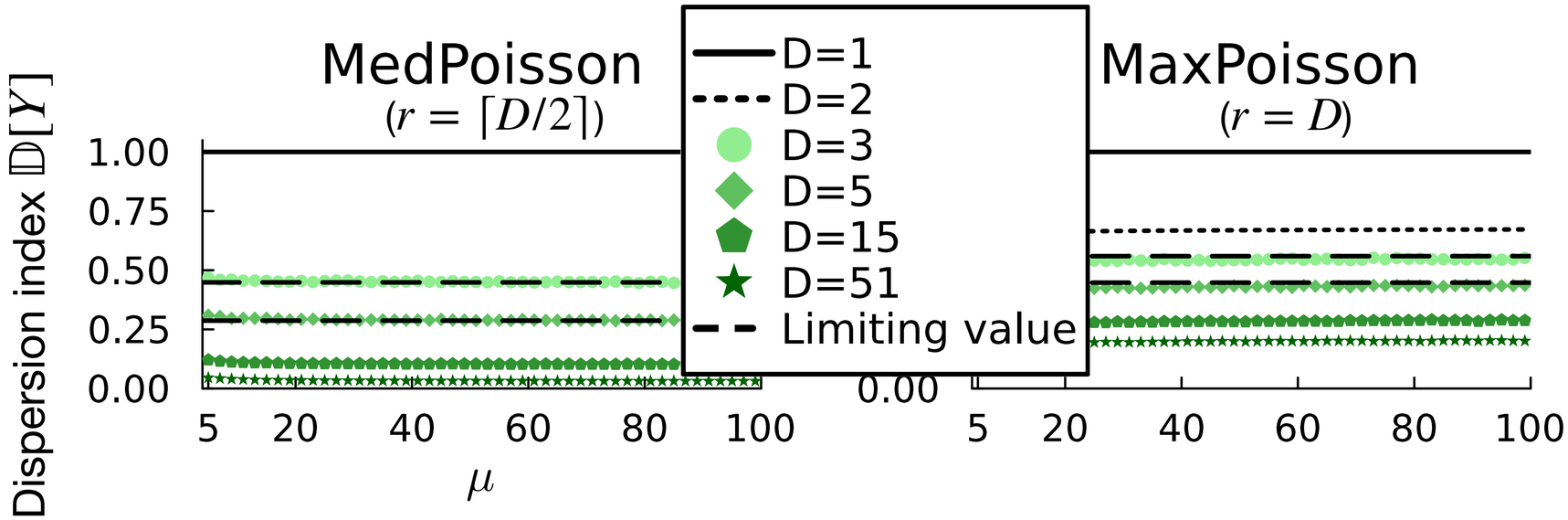


The dispersion $\mathbb{D}_{Y \sim \text{Pois}_{\mu}^{(r,D)}}[Y]$:

1. is stable across large values of μ for each rank r and order D
2. decreases as the order D increases

Dispersion of Poisson order statistics



The dispersion $\mathbb{D}_{Y \sim \text{Pois}_\mu^{(r,D)}}[Y]$:

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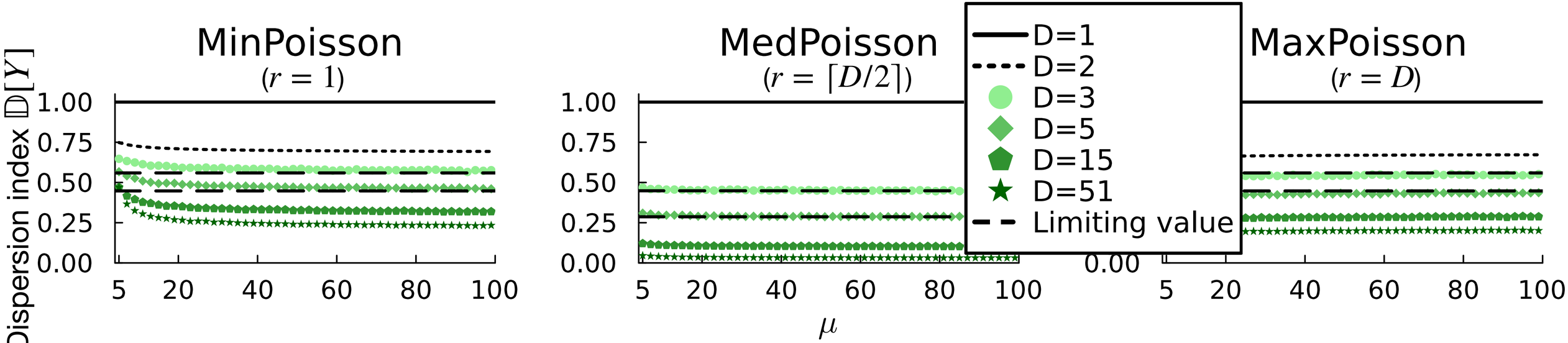

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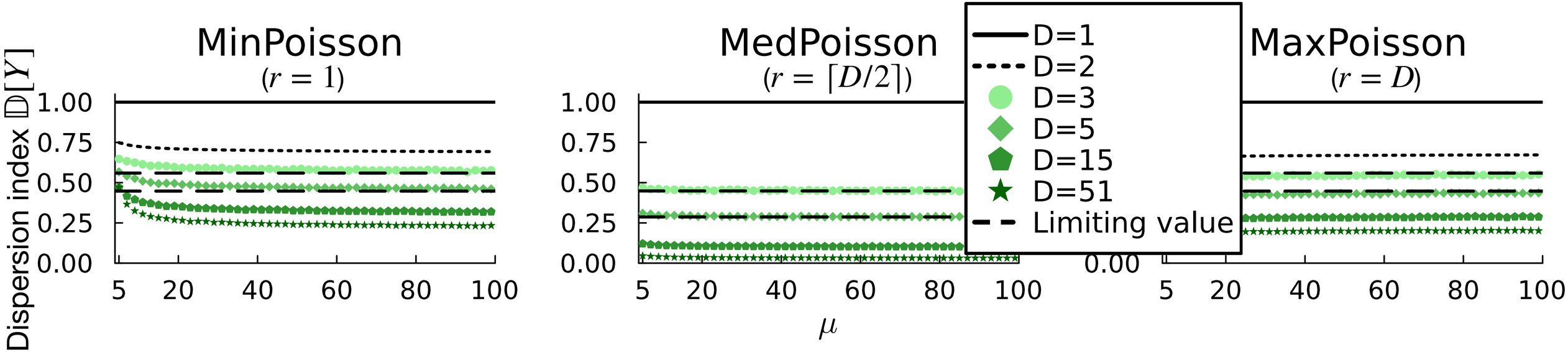

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We can use the order D as a "pseudo-index" to control the level of dispersion when building models

Inference for discrete order statistics via data augmentation

Our strategy revolves around inferring the latent $\mathbf{Z}_{1:D}$

$$Y \sim \text{Pois}_{\mu}^{(r,D)} \iff Y = Z^{(r,D)} \text{ where } Z_1, \ldots, Z_D \sim \text{Pois}(\mu)$$

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Data Augmentation: Sampling from $P_\mu(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y)$

$$Z^{(3,5)} = Y$$

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If $\mathbf{Z}_{1:D}$ were continuous

Data Augmentation: Sampling from $P_\mu(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y)$

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$\mathbf{Z}_{1:D}$ is **discrete**

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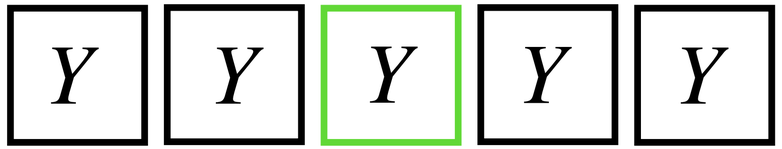
Example arrangements:



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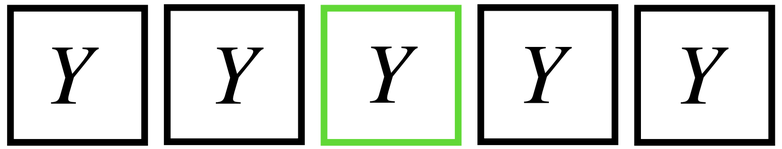

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2. Given C_d, sample each Z_d as a truncated Poisson, which can be made efficient

Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

Strategy: We sample the support $\mathbf{C}_{1:D}$ sequentially from $P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y)$

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Main idea: The past counts for each group are **sufficient statistics**

$$P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}) = P_\mu(Z^{(r,D)} = Y \mid n_d^{(<Y)}, n_d^{(=Y)}, n_d^{(>Y)})$$

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By Bayes rule, for $c \in \{<Y, =Y, >Y\}$:

$$p_d^{(c)} := P_\mu(C_d = c \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y) = \frac{P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}, C_d = c)}{P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1})} P_\mu(C_d = c)$$

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Takeaway: we can sample the support $\mathbf{C}_{1:D}$ in closed-form,

Therefore, we can sample the latent $\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y$ as desired

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Each observation $Y_i \in \mathbb{N}_0$ is the number of minutes for flight i between departure and arrival.

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$D_k \sim \text{OddBinomial}(D_{\text{max}}, \rho)$ such that $D_k \in \{1,3,5,\ldots,D_{\text{max}}\}$

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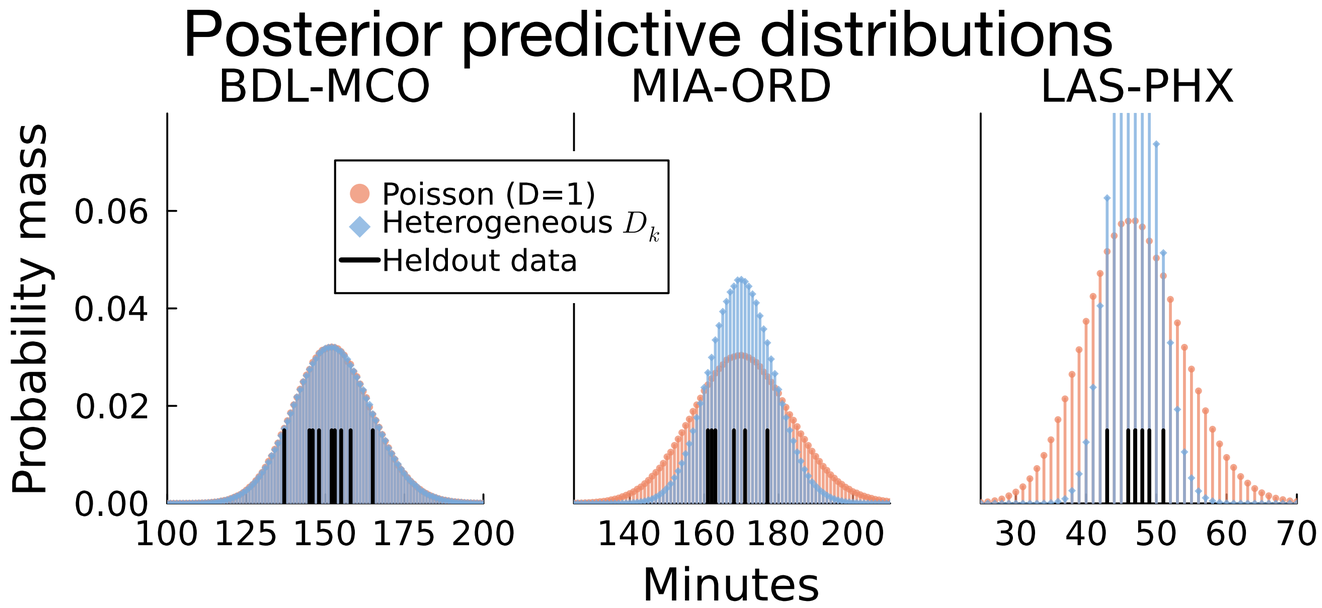

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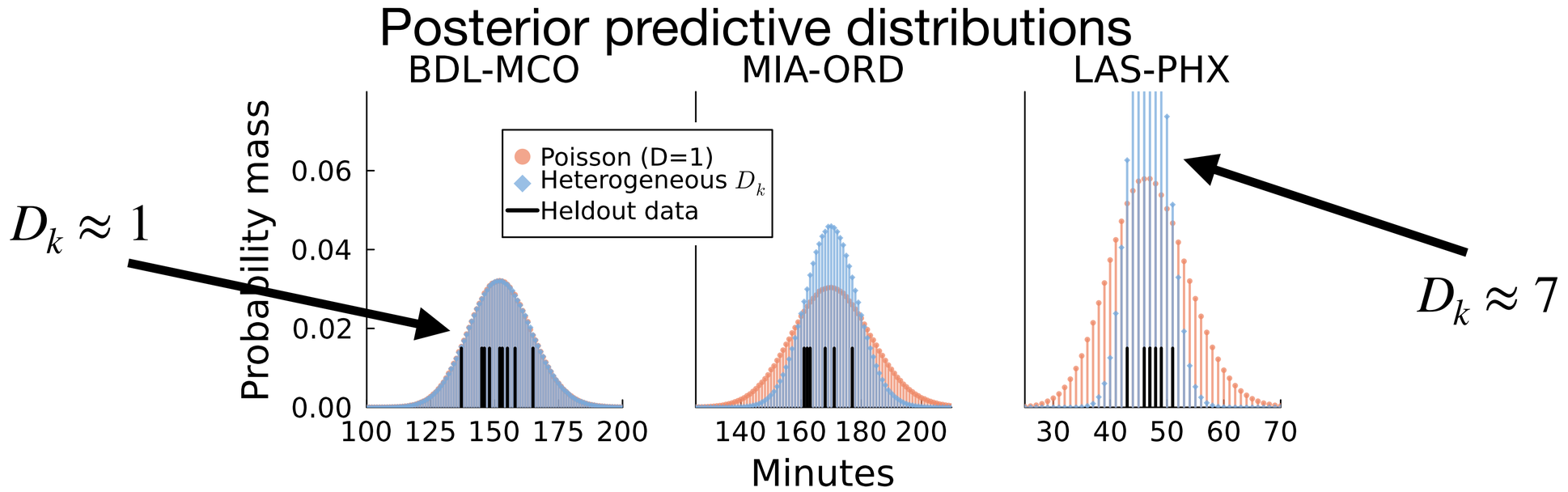

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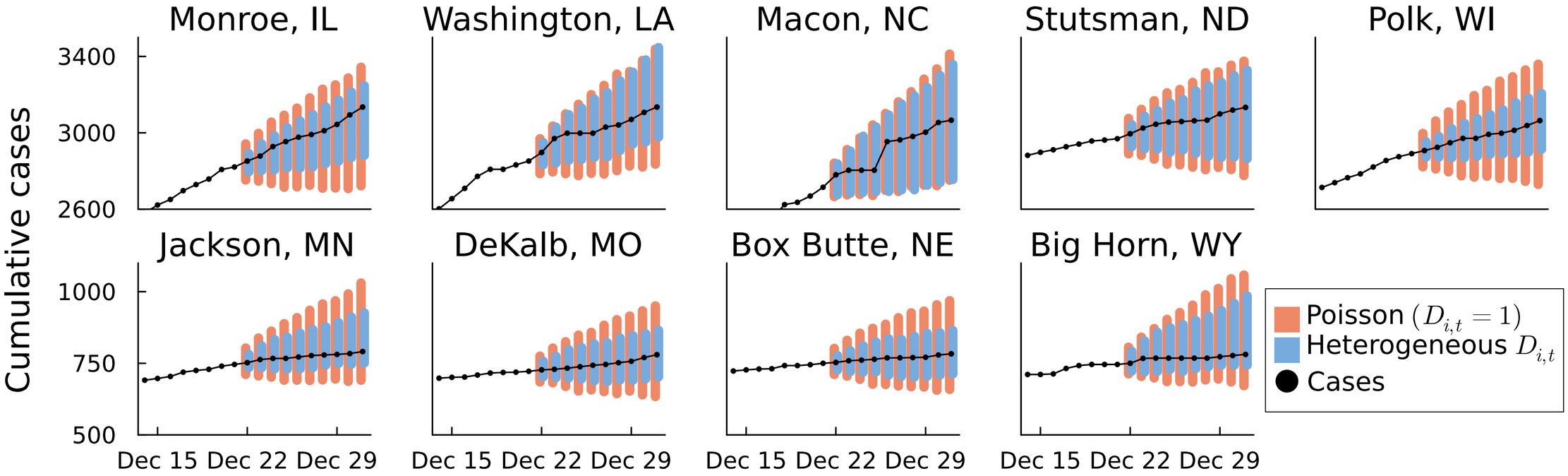

More precise probabilistic forecasts

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past case count — latent growth rate

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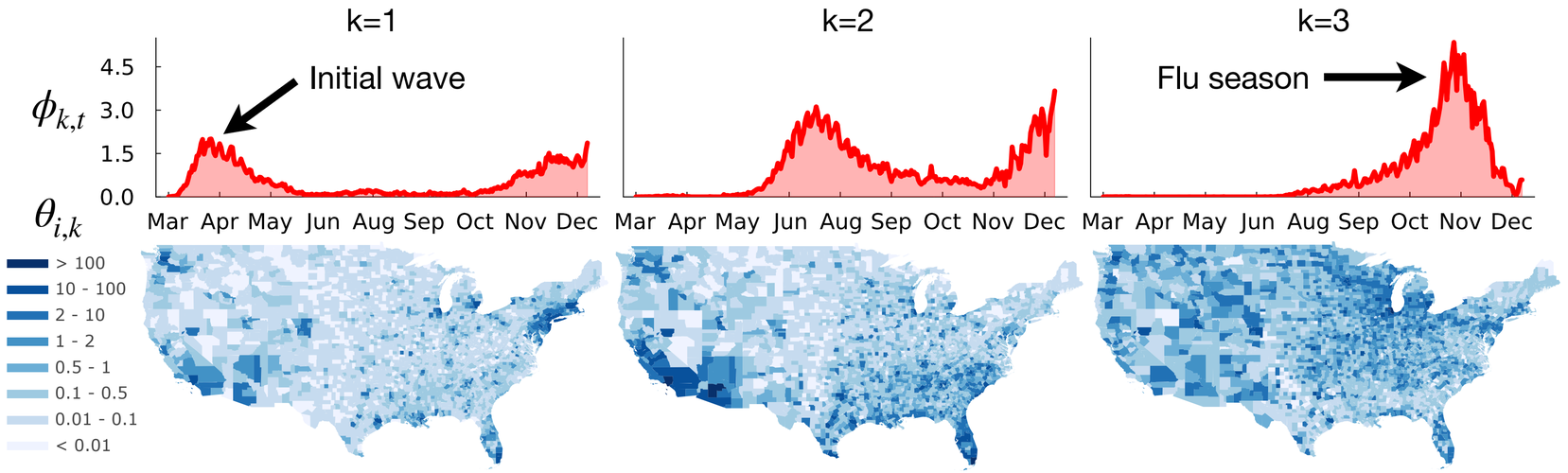

Smooth, interpretable structure

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To forecast, $\phi_{k,t}$ evolves over time: $\phi_{k,t} \sim \Gamma(a^{(\phi)} + b^{(\phi)}\phi_{k,t-1},\ b^{(\phi)})$

Data augmentation from [Acharya et al. 2015]

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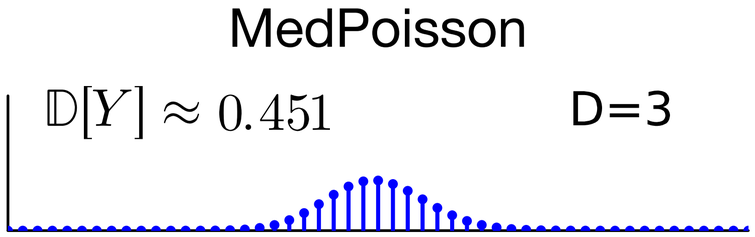
General tools enable modularity and increasing complexity

Summary



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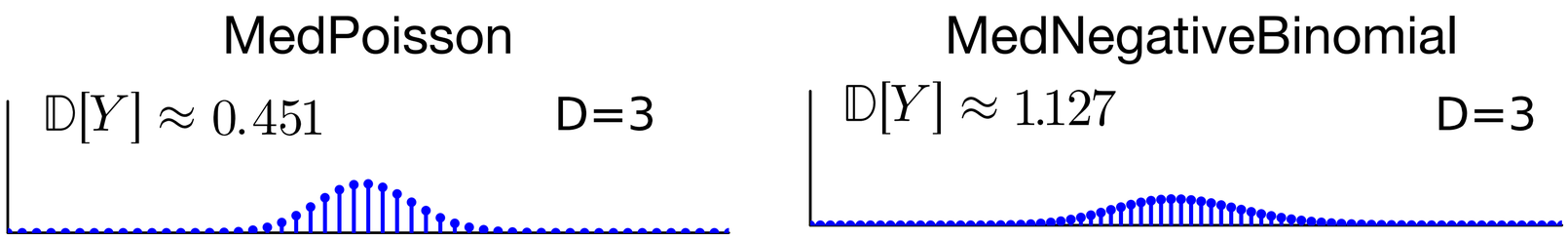

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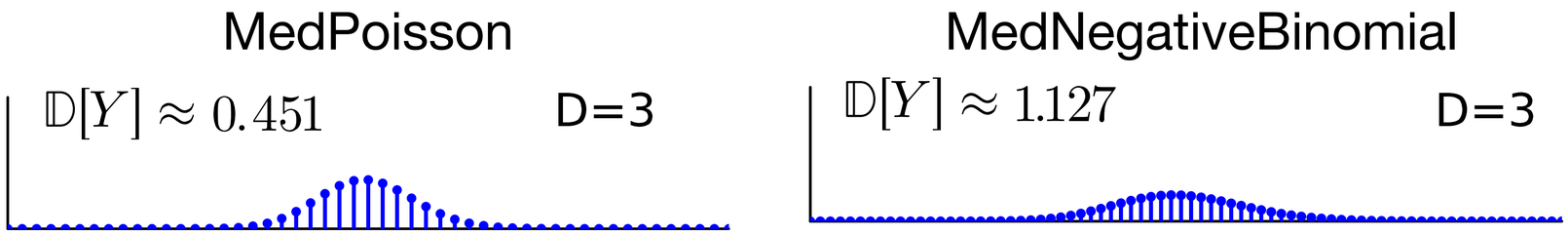

- We study **Poisson order statistics** as a tool for **modeling underdispersion** in probabilistic models of count data
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  - These tools might be useful for building Bayesian methods to model other phenomena, like extreme values and constrained parameter spaces

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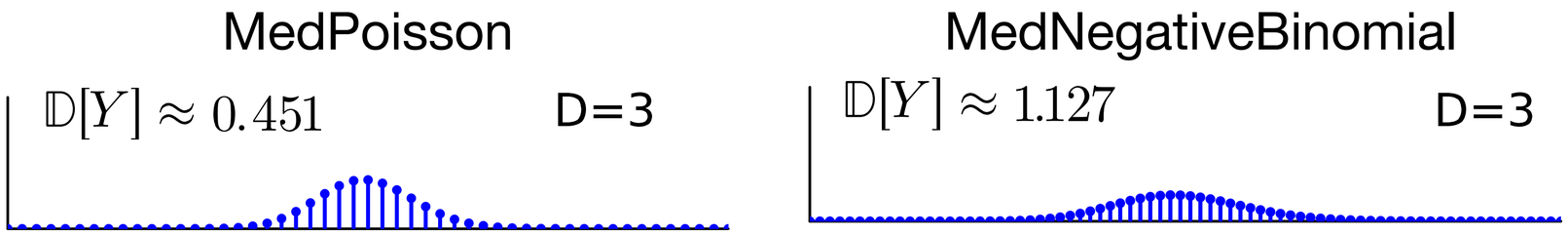

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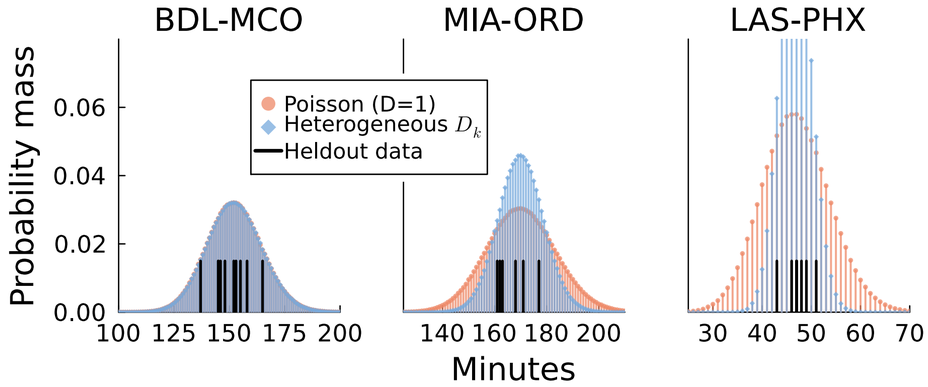

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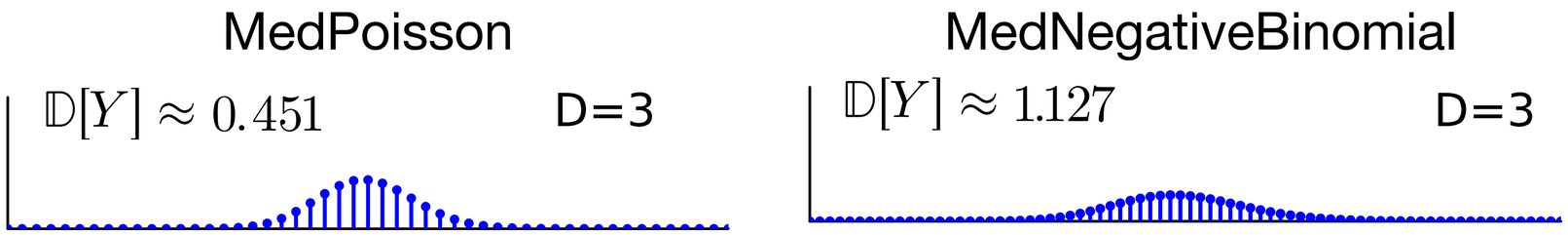

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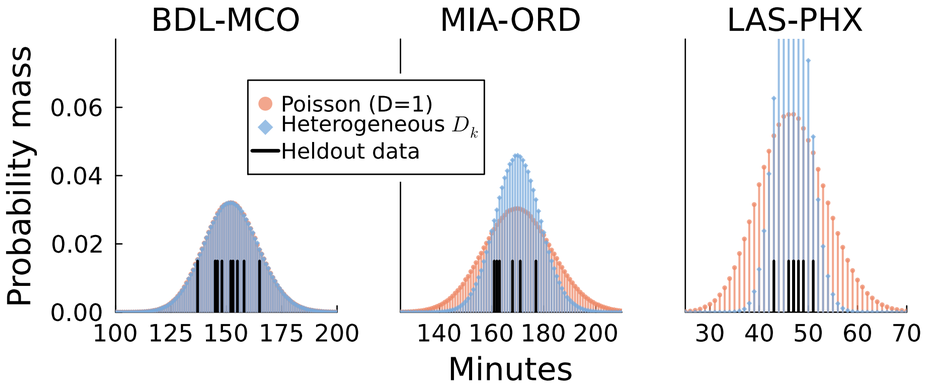
Thank you!



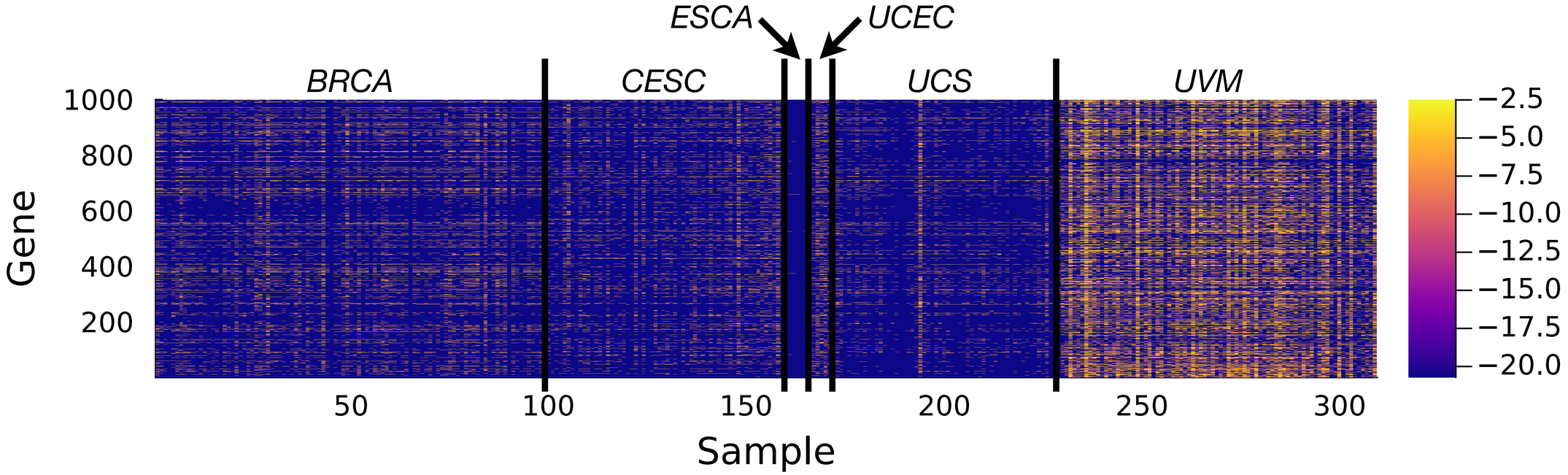
Case study: RNA-sequencing data

Is there evidence of conditional underdispersion in RNA-sequencing?

$Y_{i,j}$ is the read-count of gene j for subject i

$$Y_{i,j} \sim \text{MedNB}^{(D_{i,j})}_{\alpha_{i,j},\, p_j} \quad \text{where } \alpha_{i,j} := \sum_{k=1}^{K} \theta_{i,k}\phi_{k,j}$$

Posterior log-probability of underdispersion for each sample-gene pair



Exact form for support probabilities

For $c \in \{<Y, =Y, >Y\}$:

$$p_d^{(c)} = \frac{P_\theta(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}, C_d = c)}{P_\theta(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1})} P_\theta(C_d = c)$$

$$n_1 = \sum_{s=1}^{d-1} 1\{C_s = <Y\}$$

$$n_2 = \sum_{s=1}^{d-1} 1\{C_s = =Y\},$$

$$n_3 = \sum_{s=1}^{d-1} 1\{C_s = >Y\}$$

$$P_\theta(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}) = \begin{cases} 0 & \text{if } n_1 \geq r \text{ or } n_3 \geq D - r + 1 \\ F_\theta^{(r-n_1-n_2, D-d+1)}(Y) - F_\theta^{(r-n_1, D-d+1)}(Y-1) & \text{else if } n_2 < \min(r - n_1, D - n_3 - r + 1) \\ 1 - F_\theta^{(r-n_1, D-d+1)}(Y-1) & \text{else if } r - n_1 \leq n_2 < D - n_3 - r + 1 \\ F_\theta^{(r-n_1-n_2, D-d+1)}(Y) & \text{else if } D - n_3 - r + 1 \leq n_2 < r - n_1 \\ 1 & \text{otherwise} \end{cases}$$

Data augmentation

For a random variable $Y \sim f_\theta$, we can update θ with Poisson data augmentation if

1. We can represent Y as being generated from a latent Poisson:

$$Y \mid Z \sim h(Y \mid Z) \text{ where } Z \sim \text{Poisson}(\mu)$$

2. We can sample from the conditional distribution in closed-form

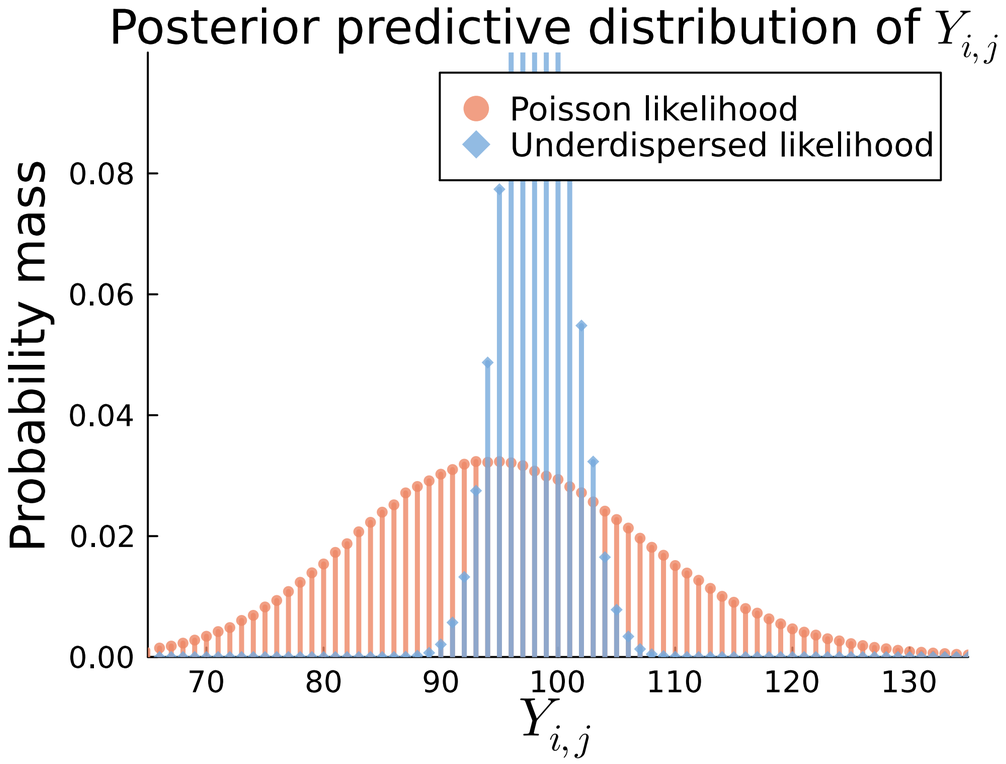
$$Z \mid Y \sim g(Z \mid Y, \theta)$$

Once we sample $Z \sim g(Z \mid Y, \theta)$, inference can proceed via Z, which is Poisson

For example, if $Y \sim$ NegativeBinomial(r,p), then if we sample $Z \mid Y \sim \text{CRT}(Y, r)$ then Z is marginally Poisson $Z \sim \text{Poisson}\left(r \frac{1}{1-p}\right)$

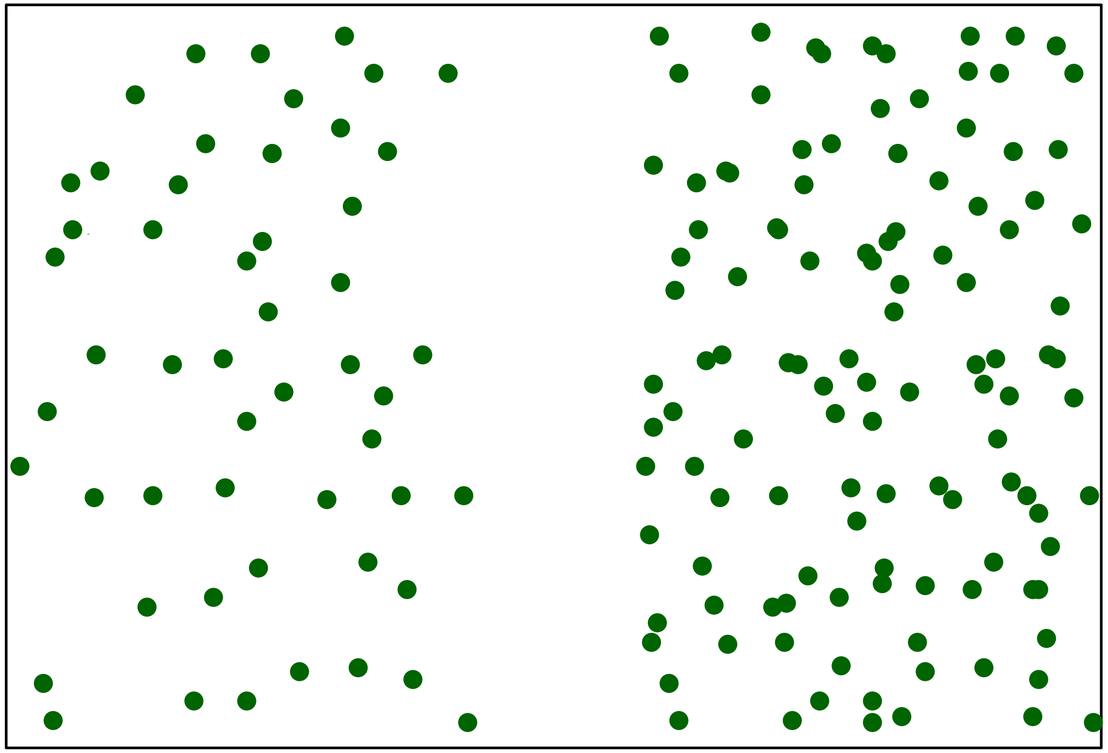
Why model underdispersion?

101	105	100	102	99
79	83	82	79	84
100	101	$Y_{i,j}$	102	98
80	81	78	82	84
99	103	98	100	100



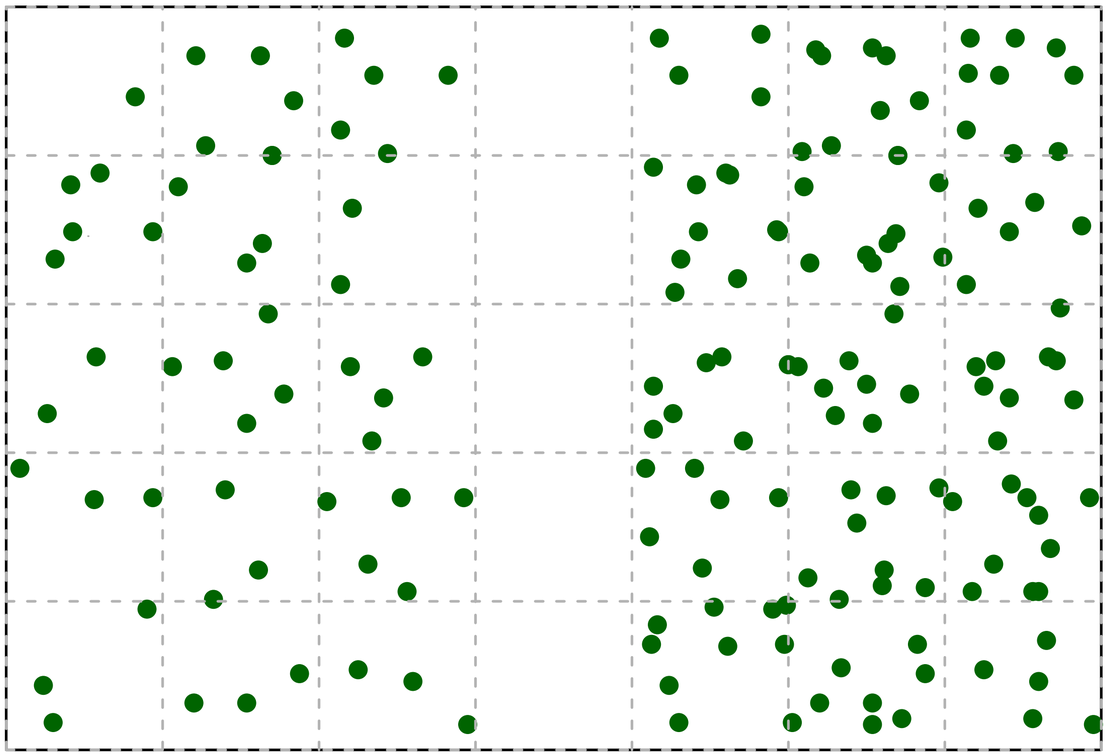
Capturing underdispersion allows us to make more precise probabilistic predictions than a Poisson likelihood would allow

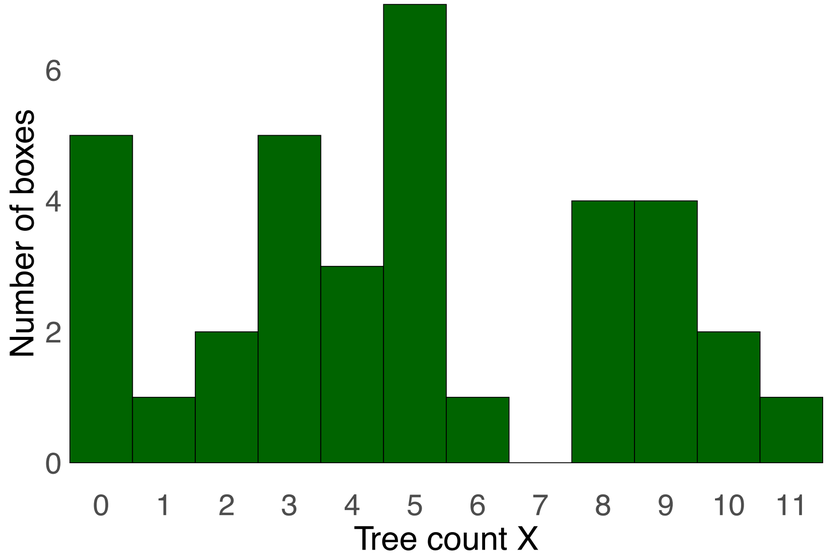
Conditional underdispersion



Conditional underdispersion

$$\mathbb{D}[X] = \frac{\mathbb{V}[X]}{\mathbb{E}[X]} \approx 2.20$$





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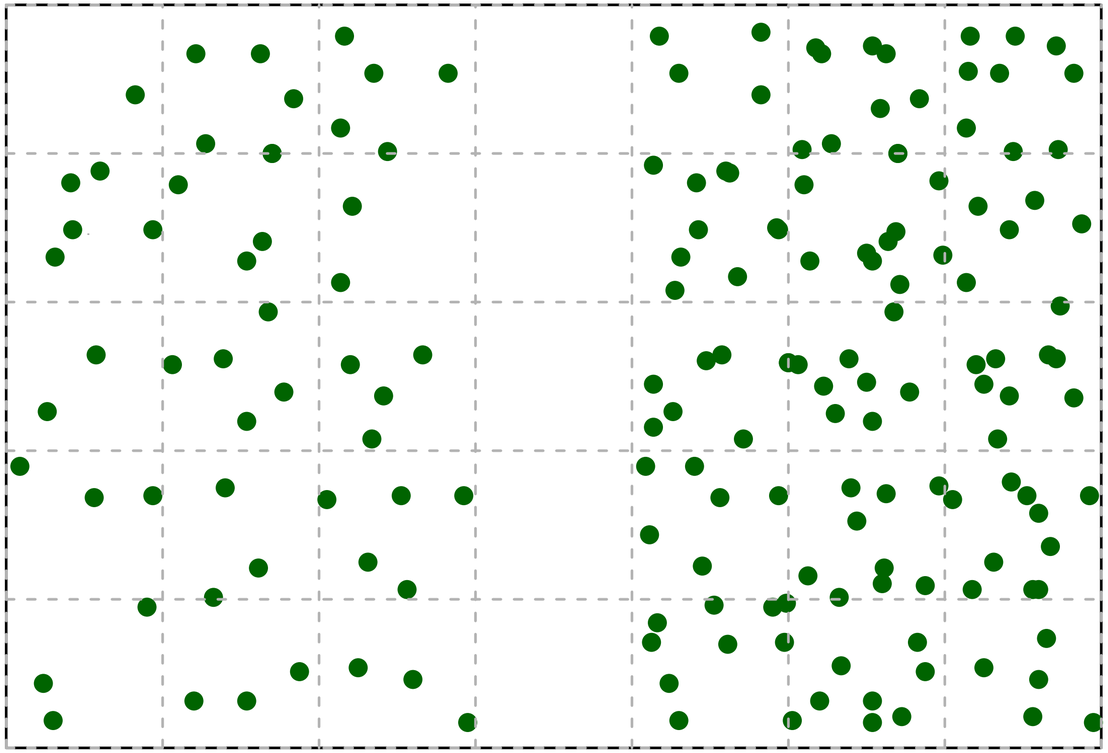
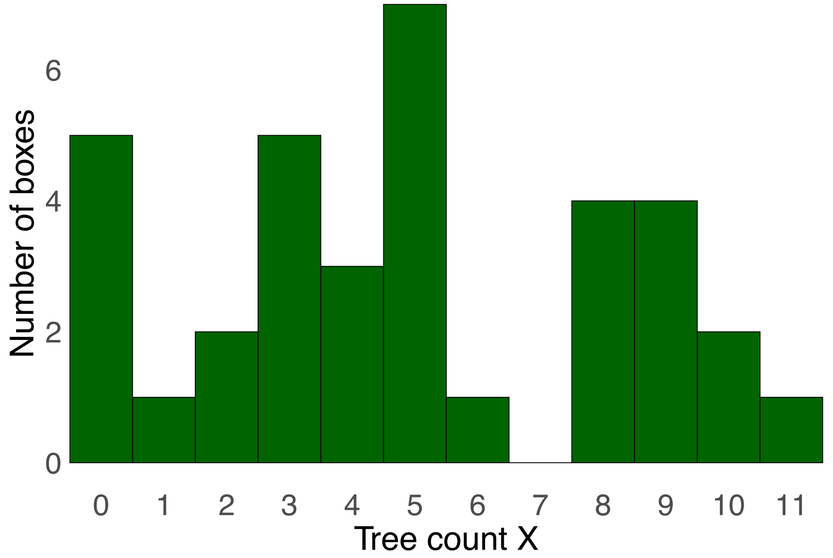
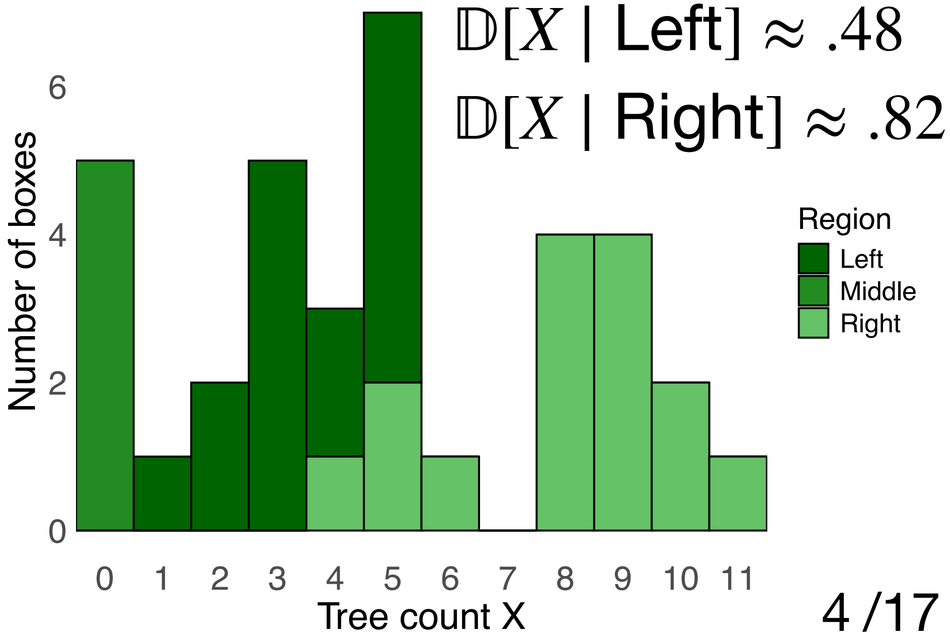




$$\mathbb{D}[X \mid \text{Left}] \approx .48$$

$$\mathbb{D}[X \mid \text{Right}] \approx .82$$



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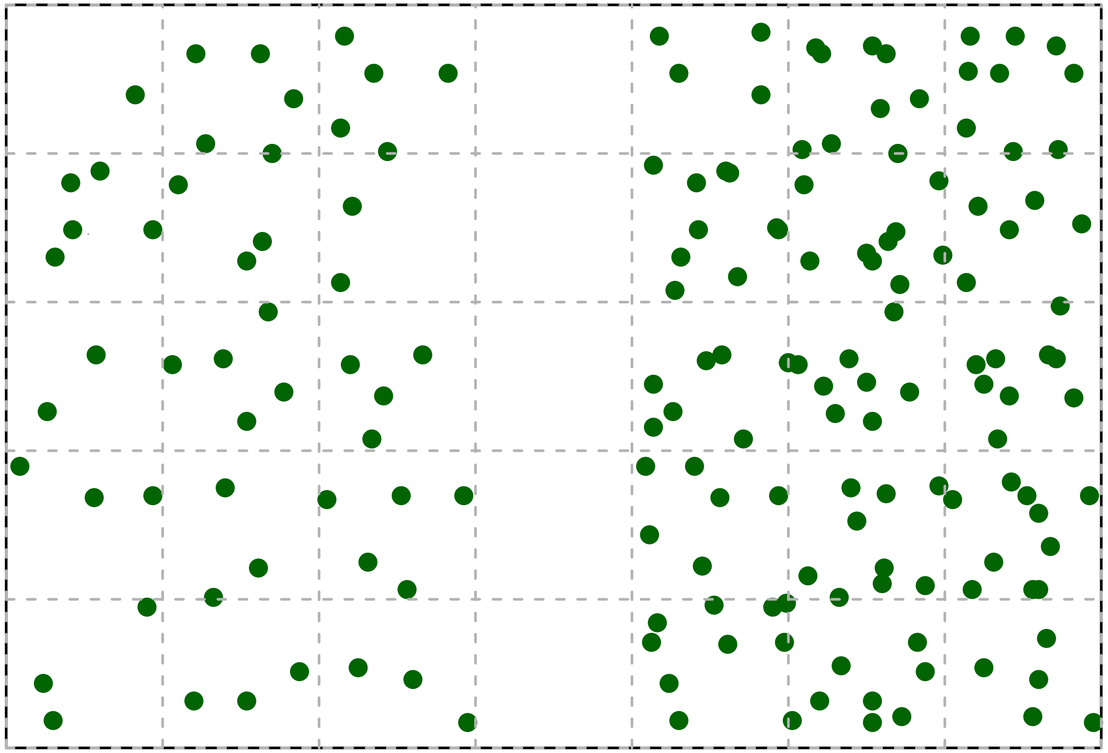

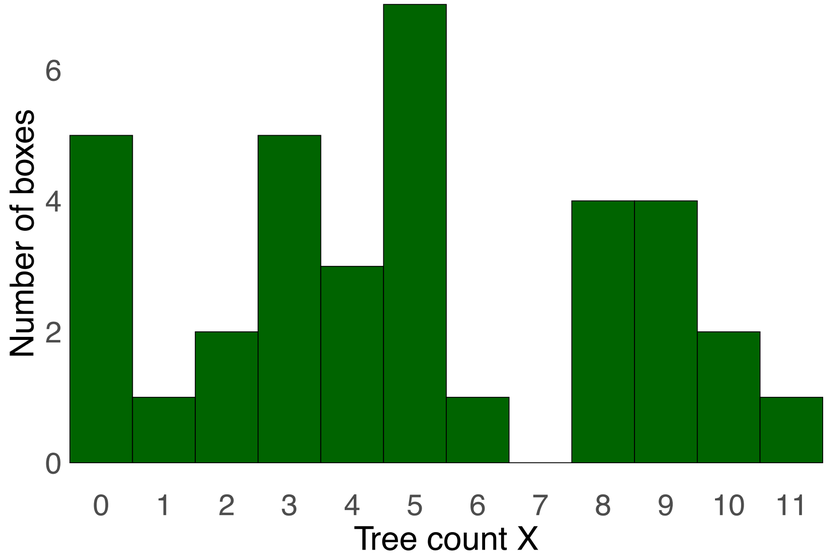

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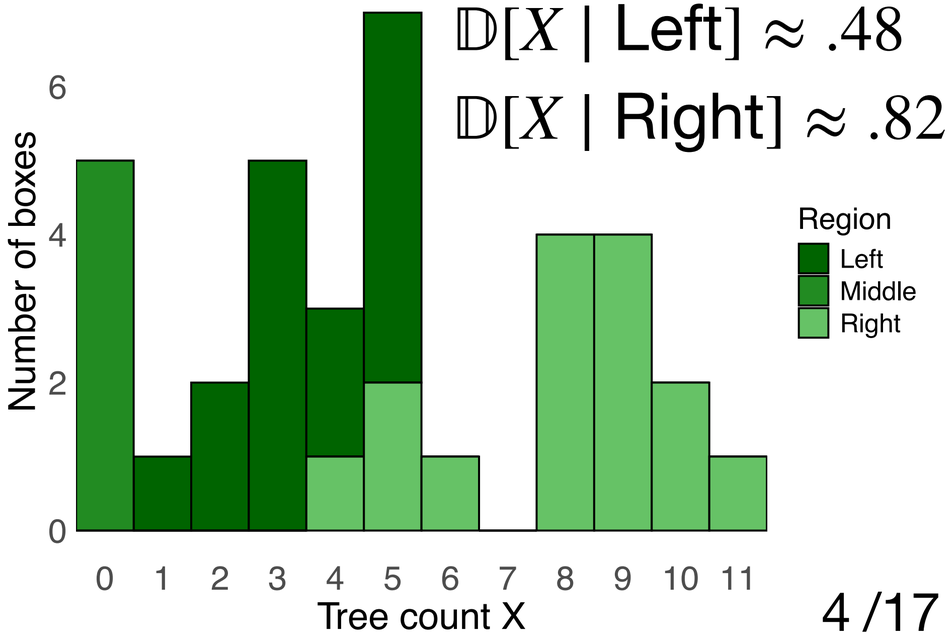
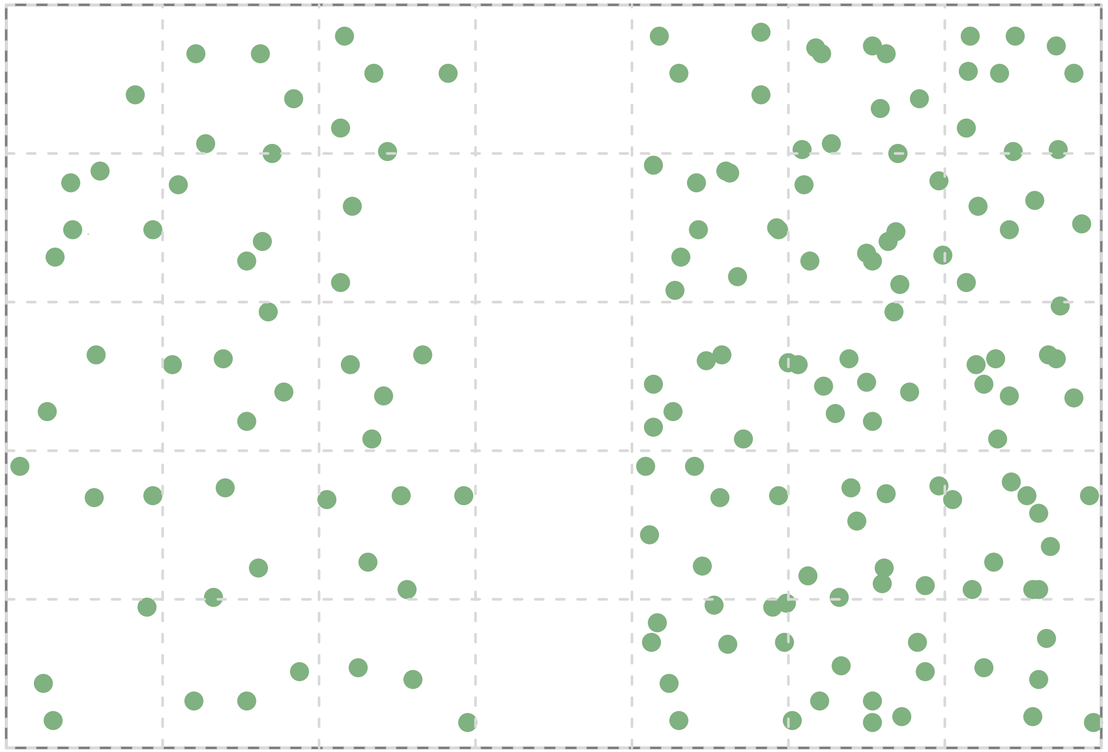
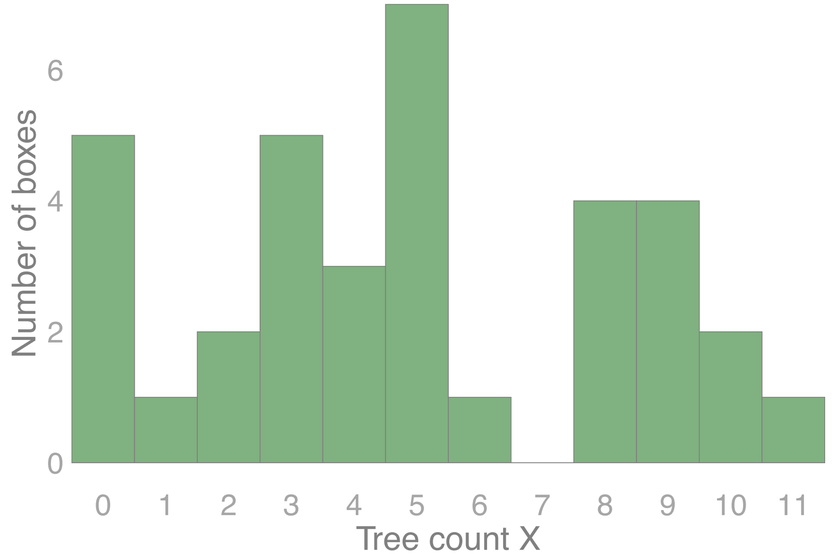
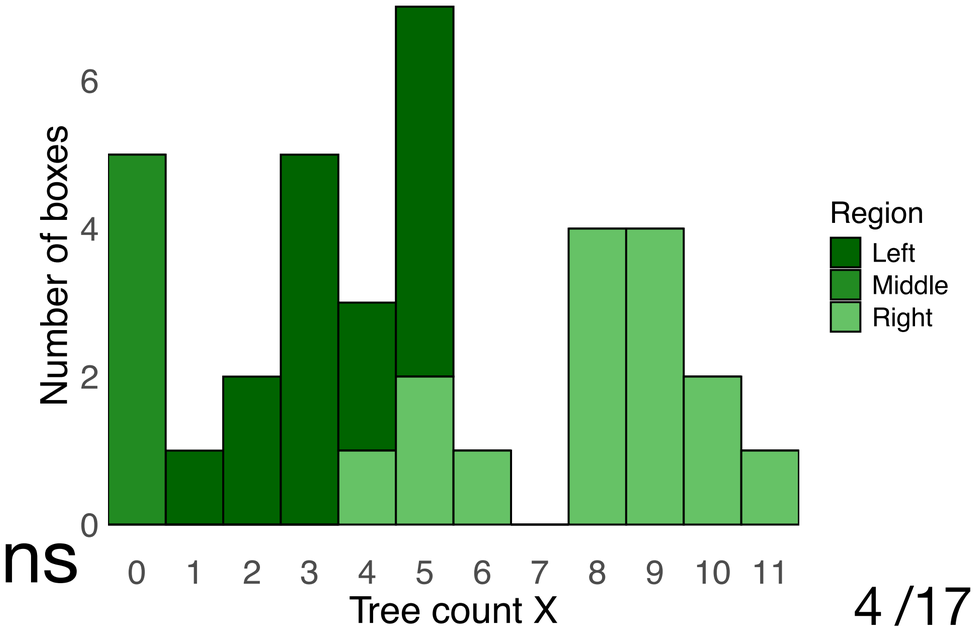


Marginal overdispersion can mask conditional underdispersion







Latent structure can reveal more regular patterns

Inference Summary

First:

$$\mathbf{Z}_{1:D} \sim P_\theta(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y)$$

Second:

$$P(\theta \mid \mathbf{Z}_{1:D}) \propto g(\theta) \prod_{d=1}^{D} f_\theta(Z_d)$$

Algorithm B Exact simulation of $\mathbf{Z}_{1:D} \sim P_\theta(\mathbf{Z}_{1:D} \mid Z^{(r,D)}=Y)$

Input: observation $Y \in \mathbb{Z}$, order $D \in \mathbb{N}$, rank $r \in [D]$, parent distribution f_θ

Initialize: $N_0^{(<Y)} = N_0^{(=Y)} = N_0^{(>Y)} = 0$

for $d = 1 \dots D$ **do**

  Compute $[p_d^{(<Y)}, p_d^{(=Y)}, p_d^{(>Y)}]$ // as defined in Equation (6)

  Sample $C_d \sim \text{Categorical}\left(p_d^{(<Y)}, p_d^{(=Y)}, p_d^{(>Y)}\right)$ // where $C_d \in \{<Y, =Y, >Y\}$

  $N_d^{(c)} \leftarrow N_{d-1}^{(c)} + \mathbb{1}\{C_d = c\}$ for $c \in \{<Y, =Y, >Y\}$ // update sufficient stats

  Sample $Z_d \sim \underset{\mathbb{Z}_{C_d}}{\text{trunc}}\, f_\theta$ // as given in Theorem 5.1

  // (Optional) assess/execute the break conditions in Theorem 5.2

Output: $\{Z_1, \dots, Z_D\}$

Using this data augmentation scheme, we can build Bayesian models with Poisson order statistic distributions