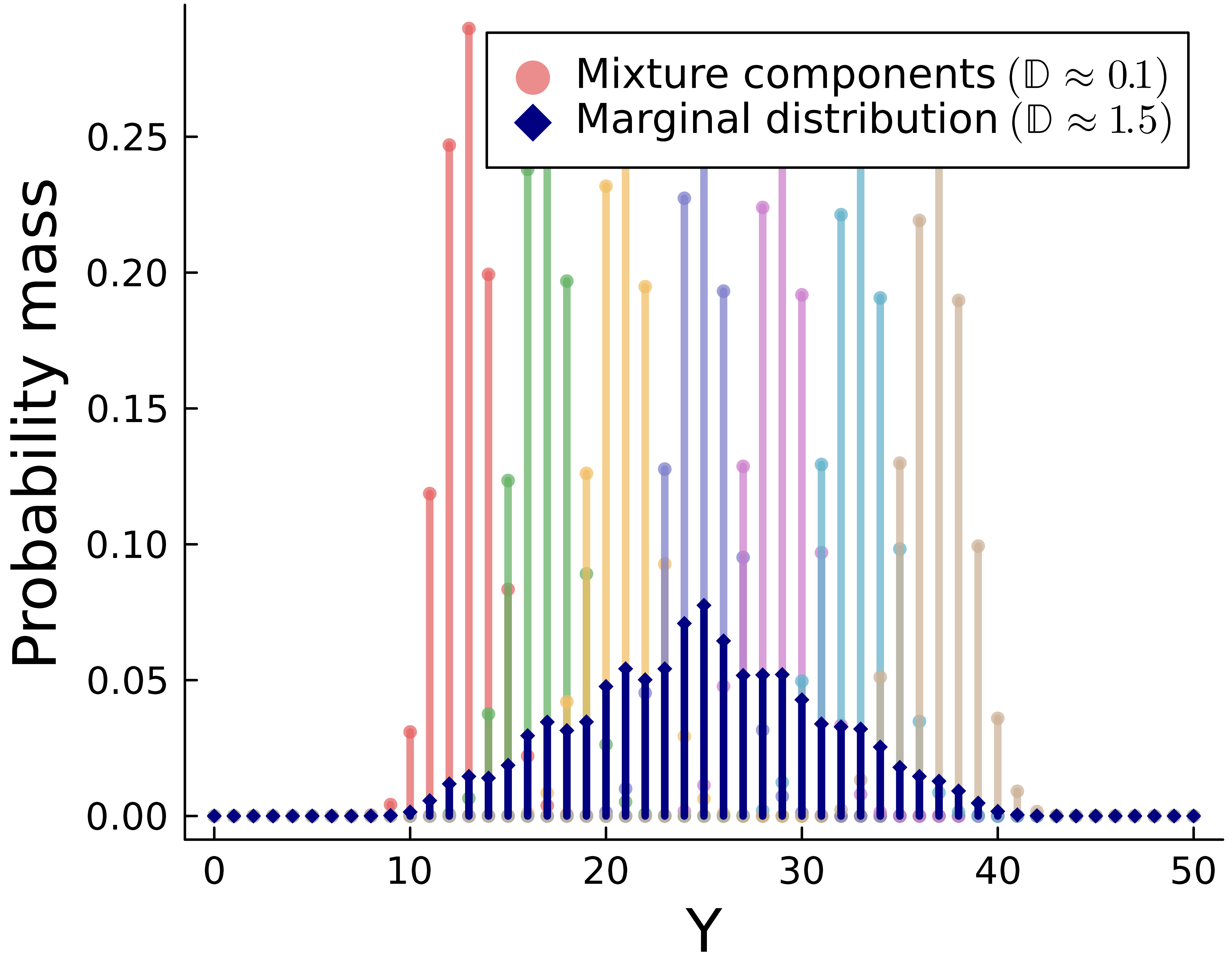
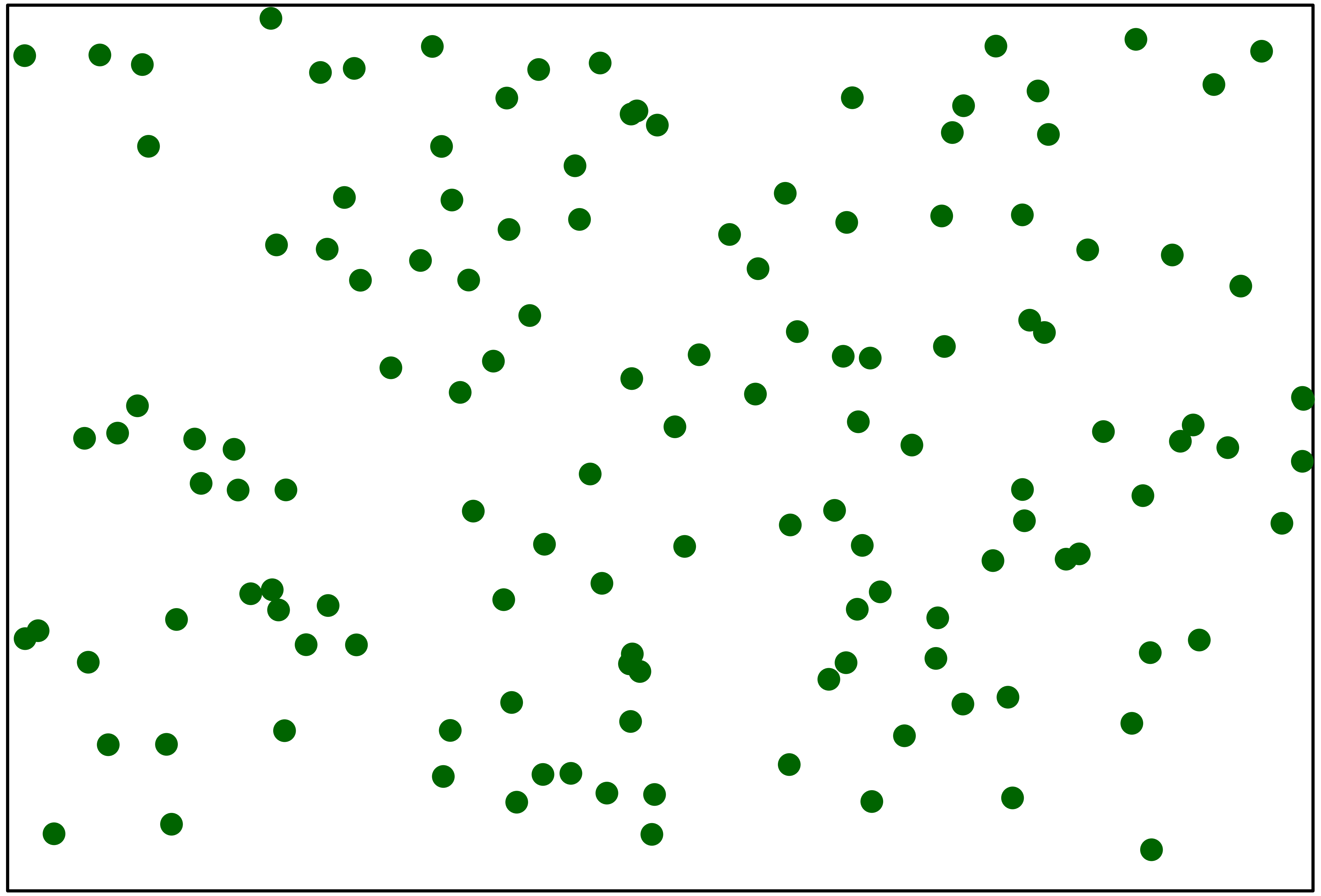
Modeling Latent Underdispersion with Discrete Order Statistics



Jimmy Lederman, University of Chicago

Work with Aaron Schein

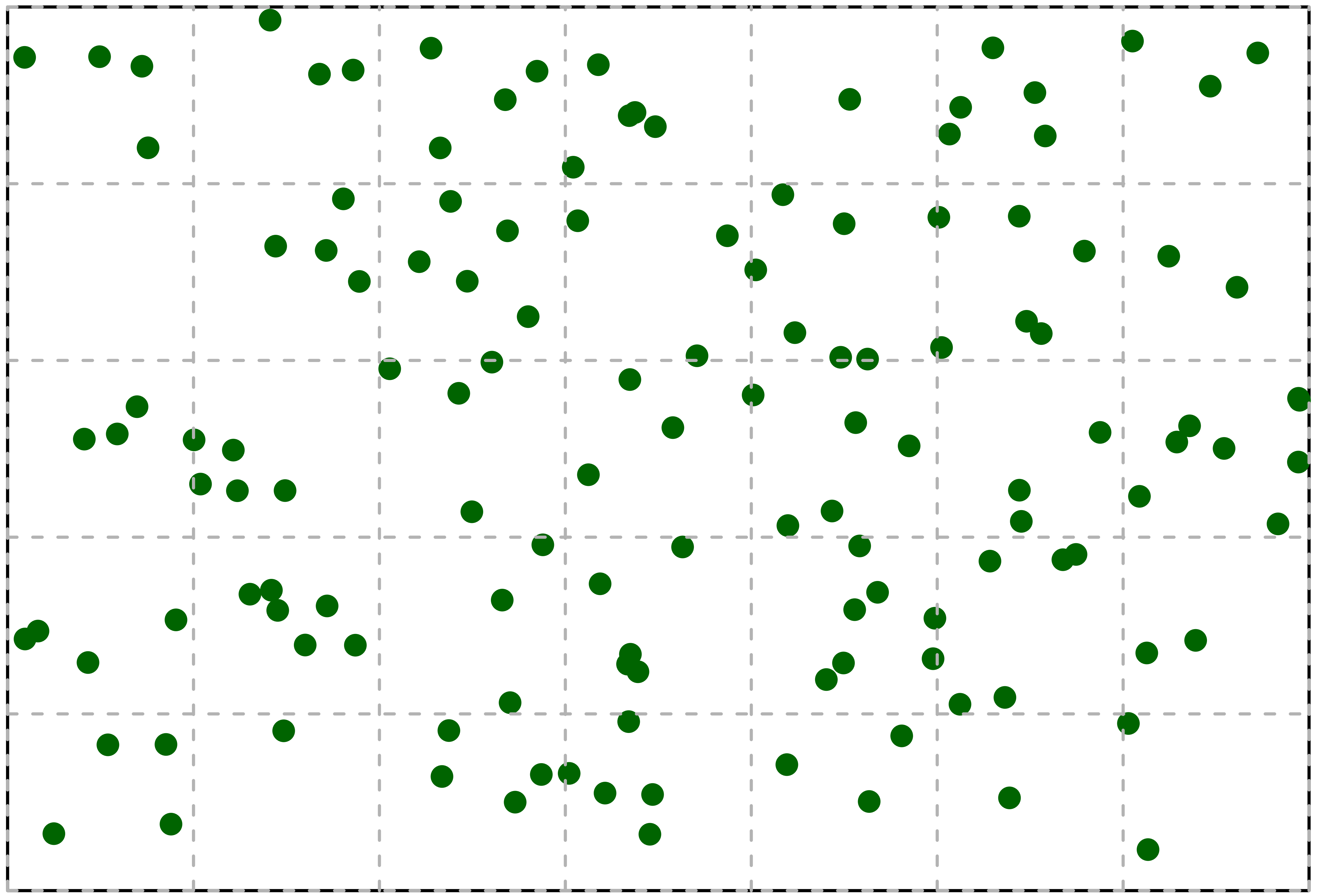
The Poisson process

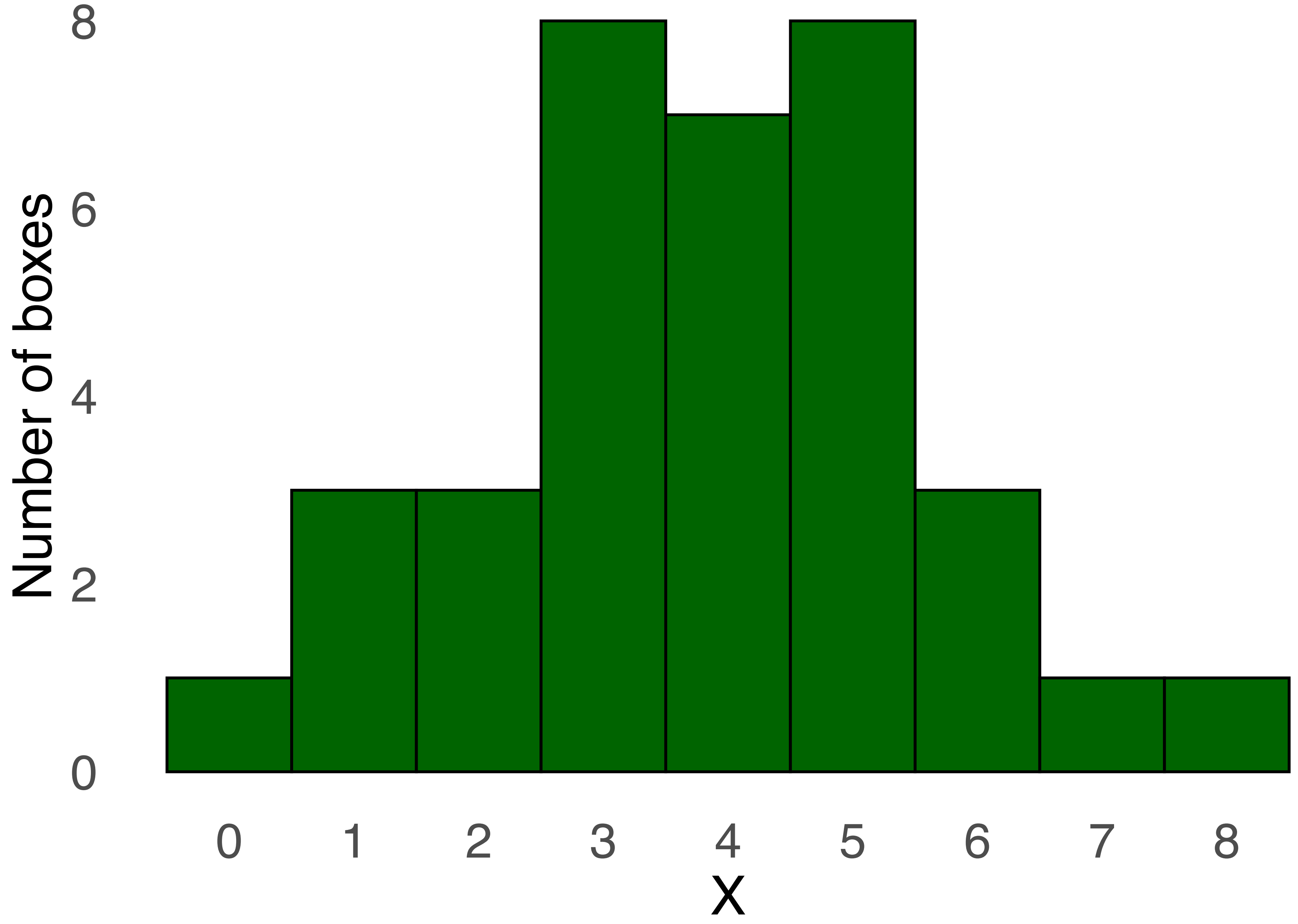


The Poisson process

X = number of points in each box

$X \sim \text{Poisson}(\mu)$





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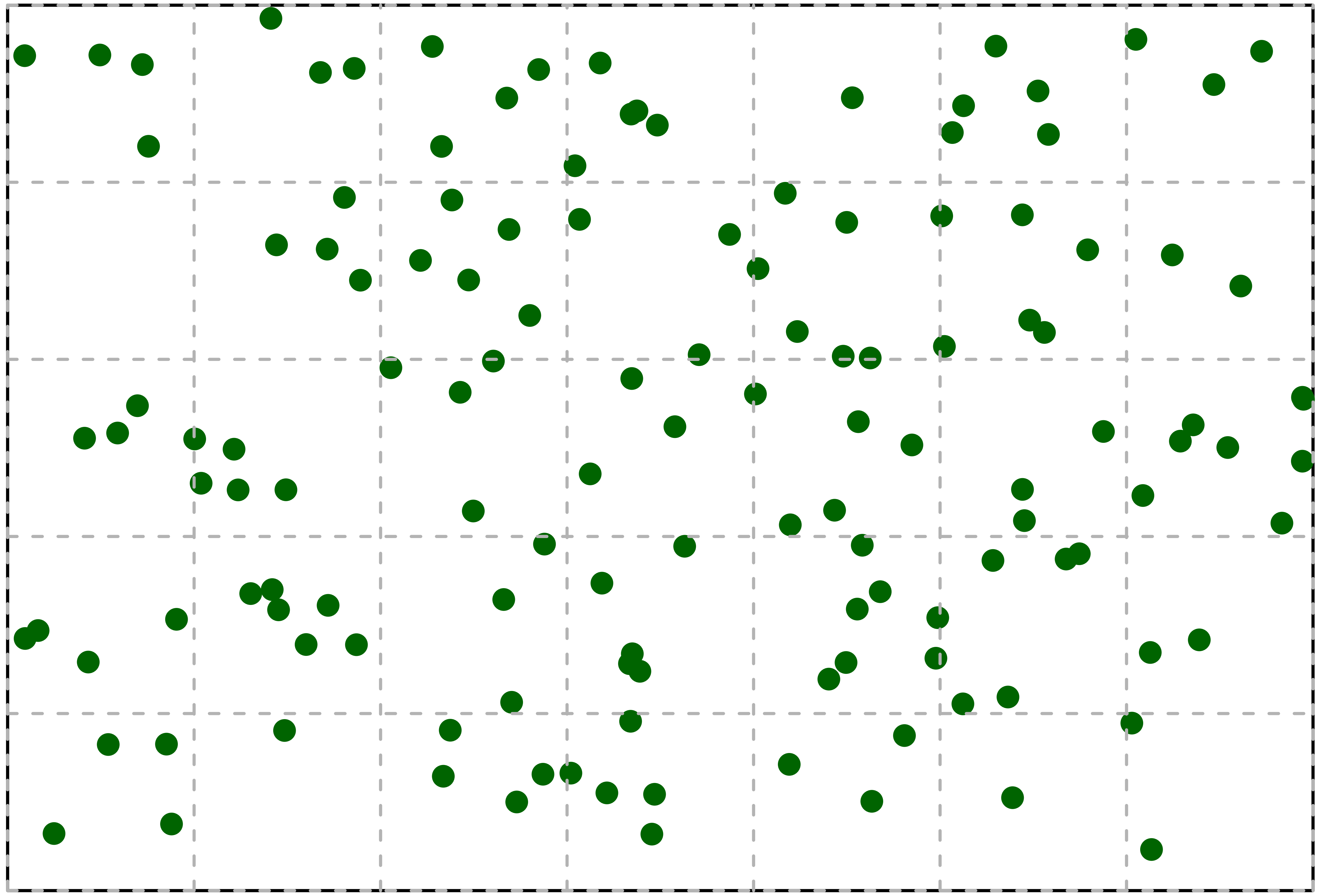
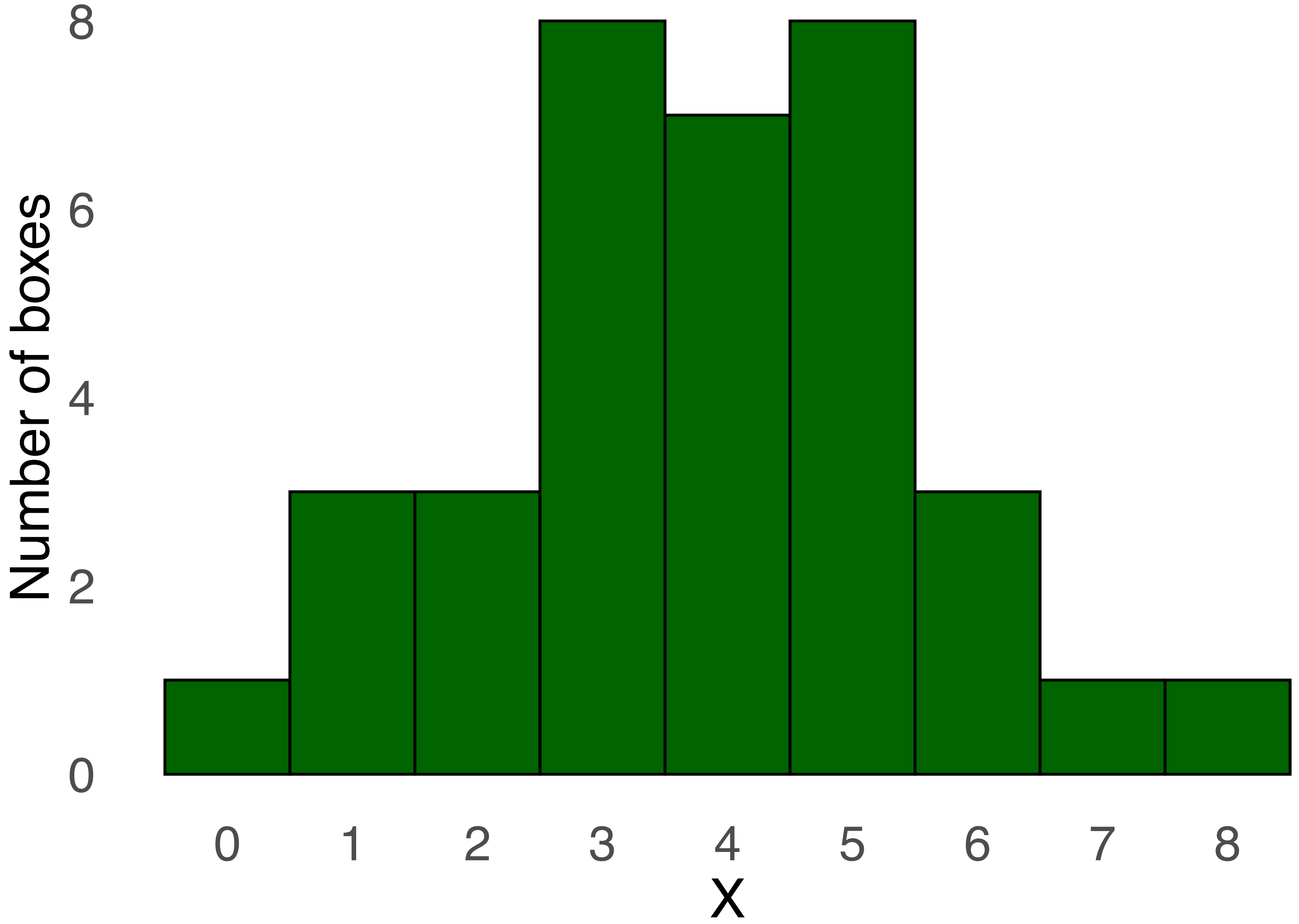
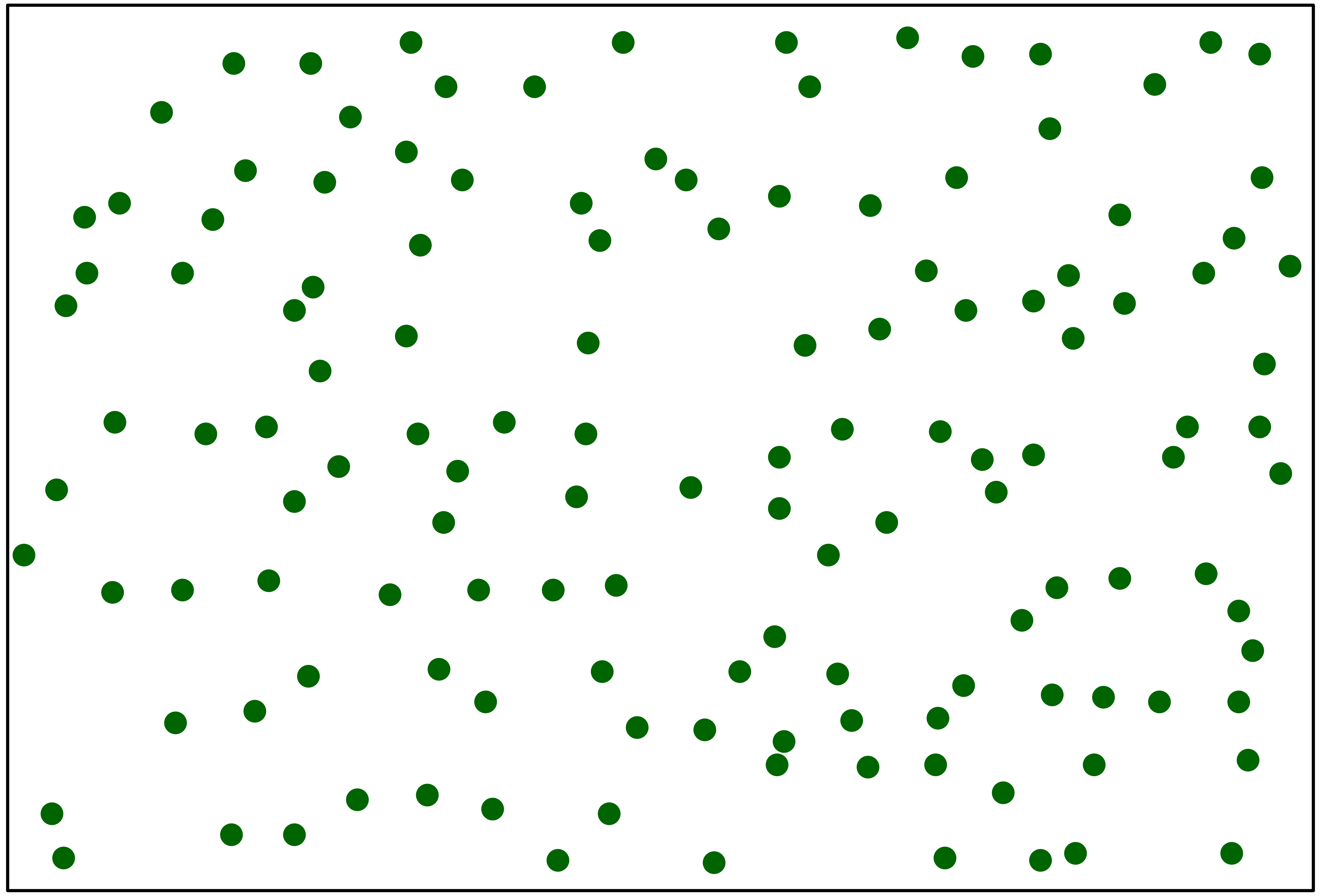




Dispersion

$$\mathbb{D}[X] = \frac{\mathbb{V}[X]}{\mathbb{E}[X]} = 1$$

Underdispersion: Spruce trees



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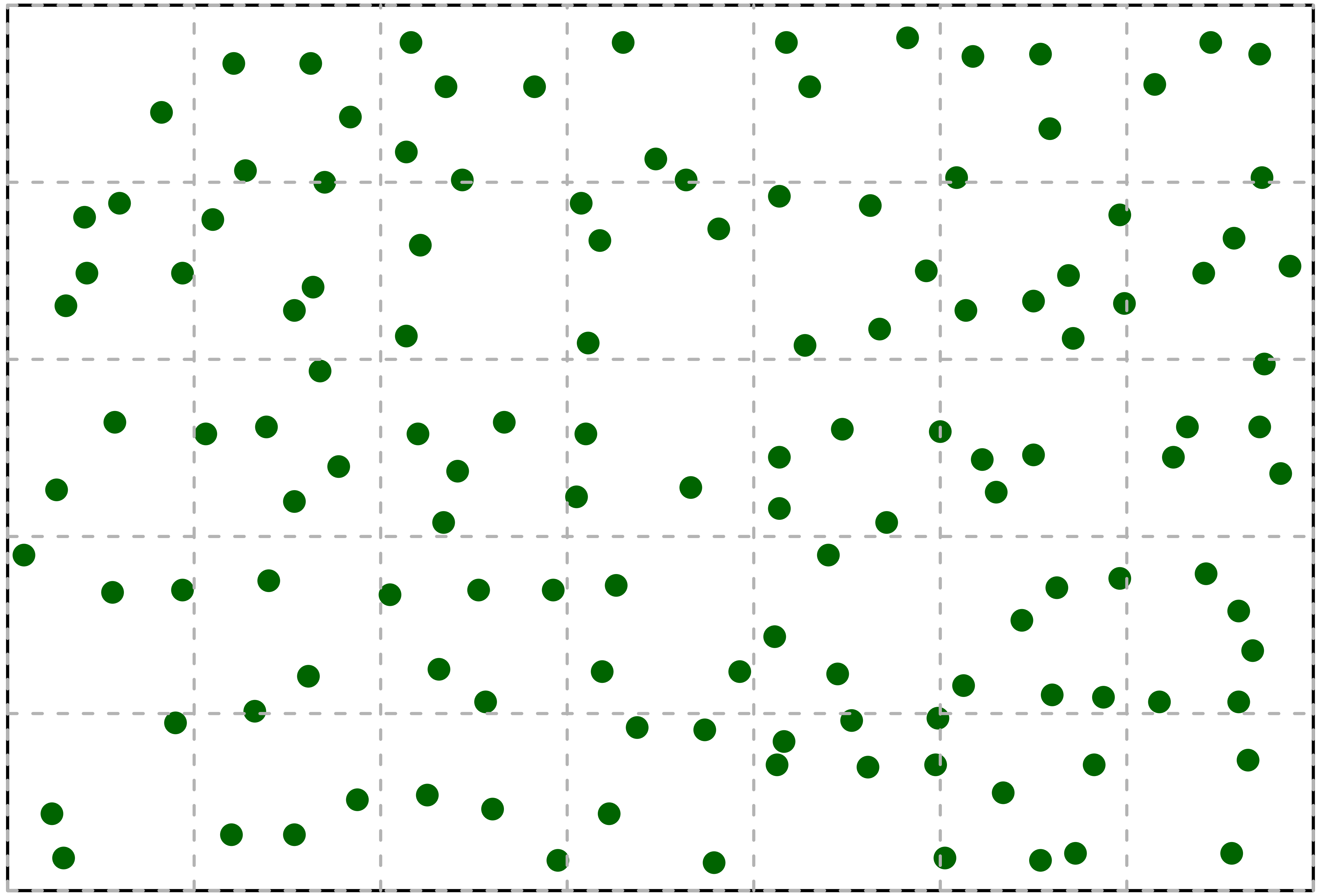

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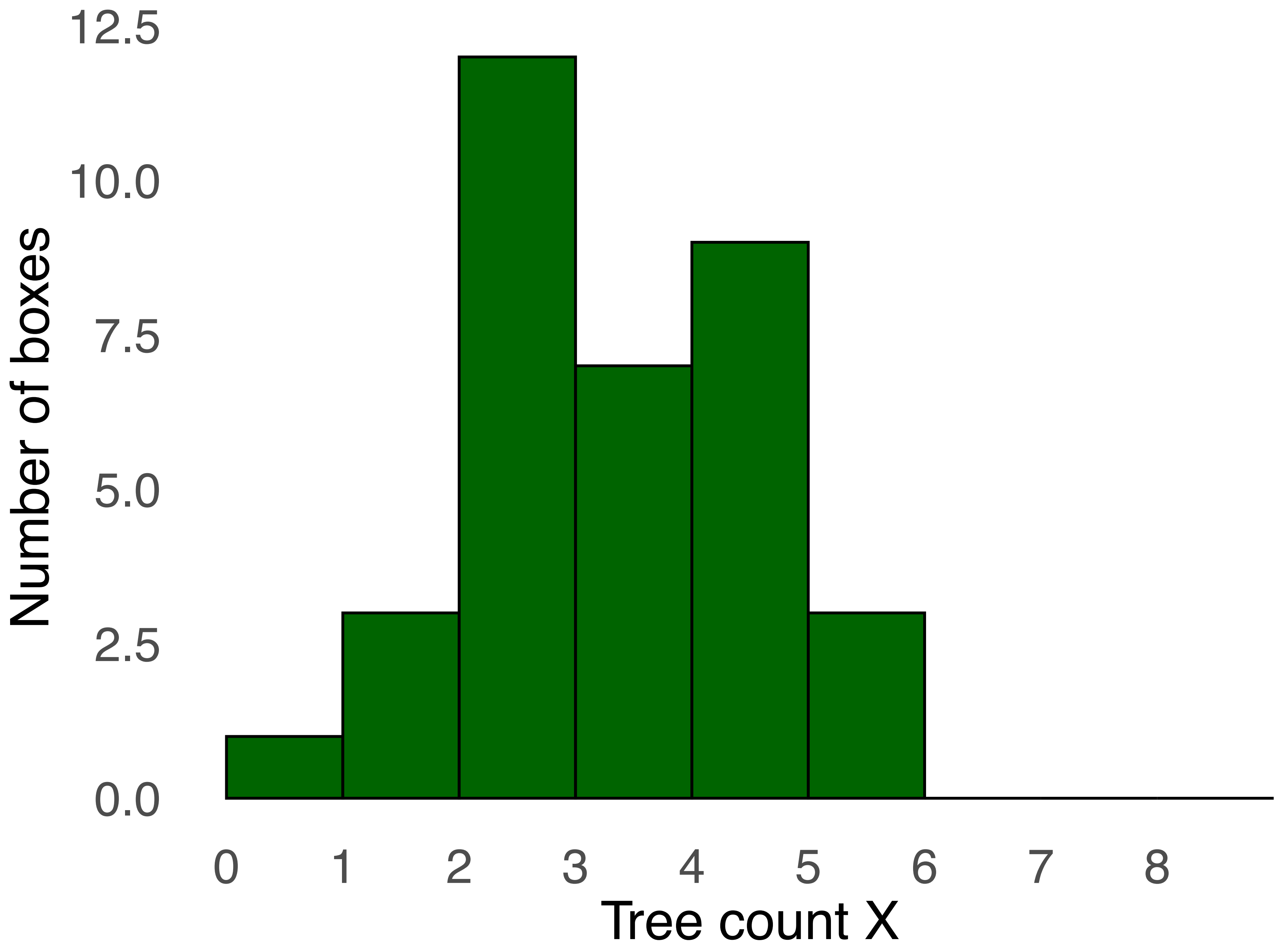

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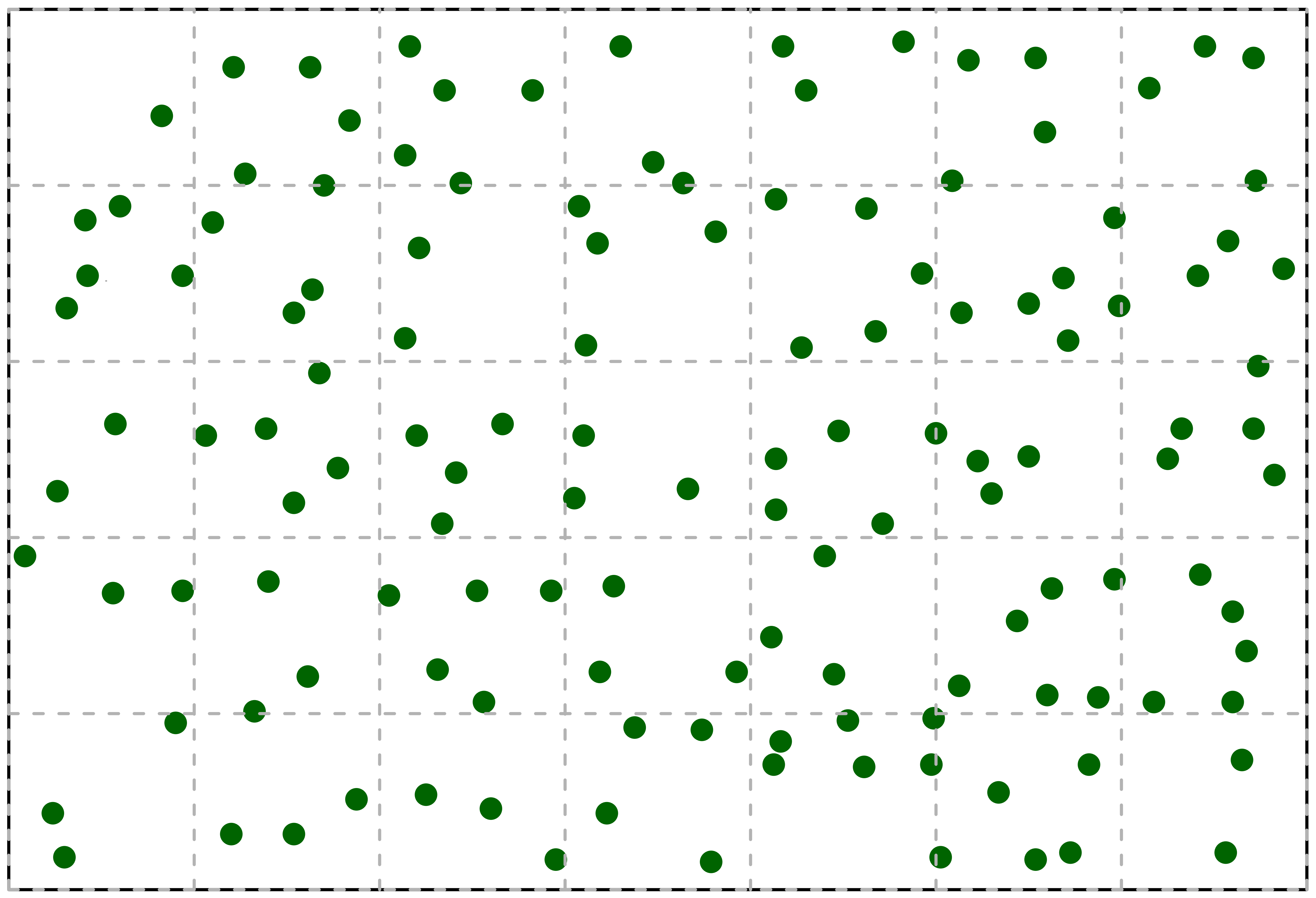

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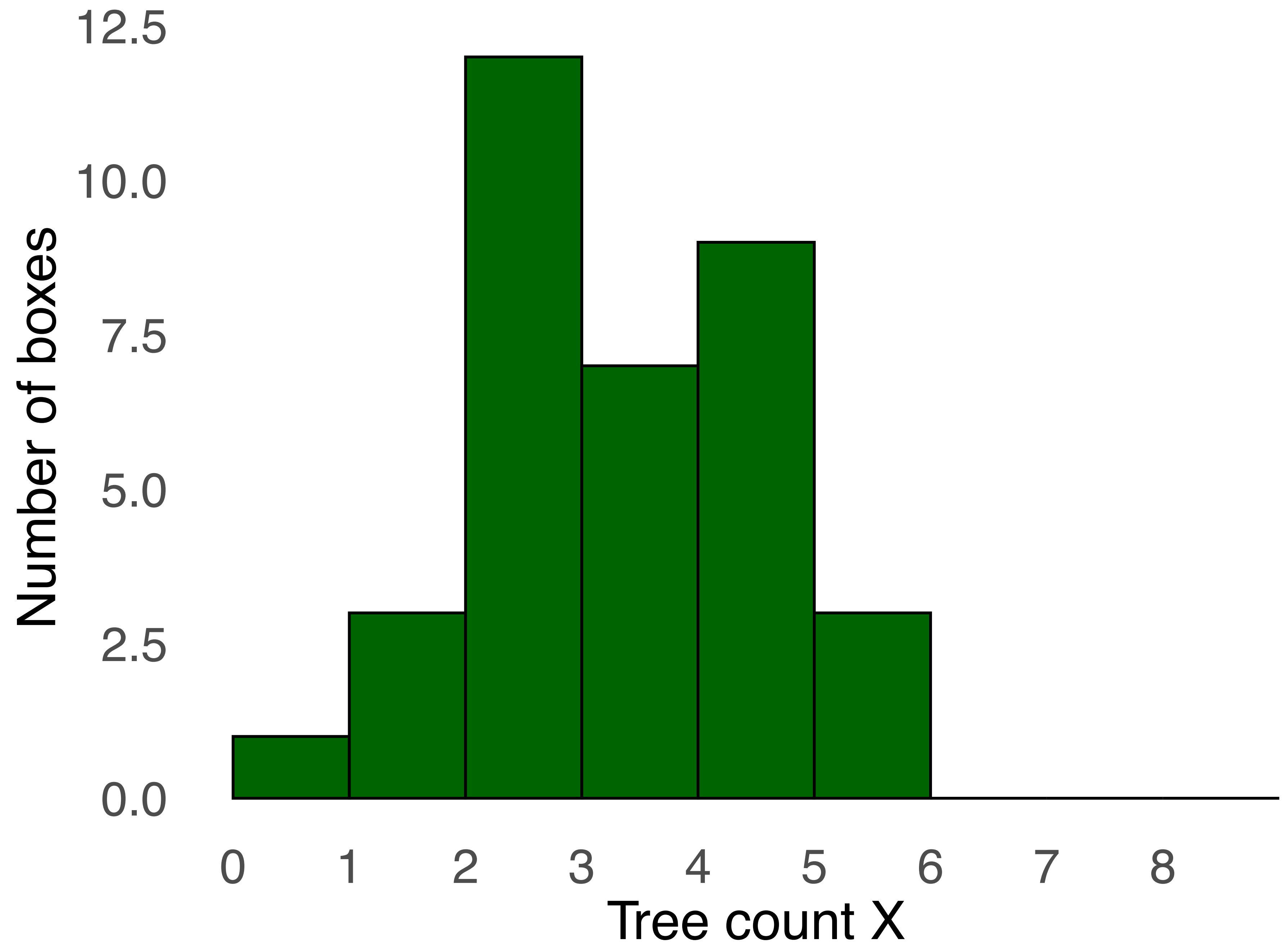
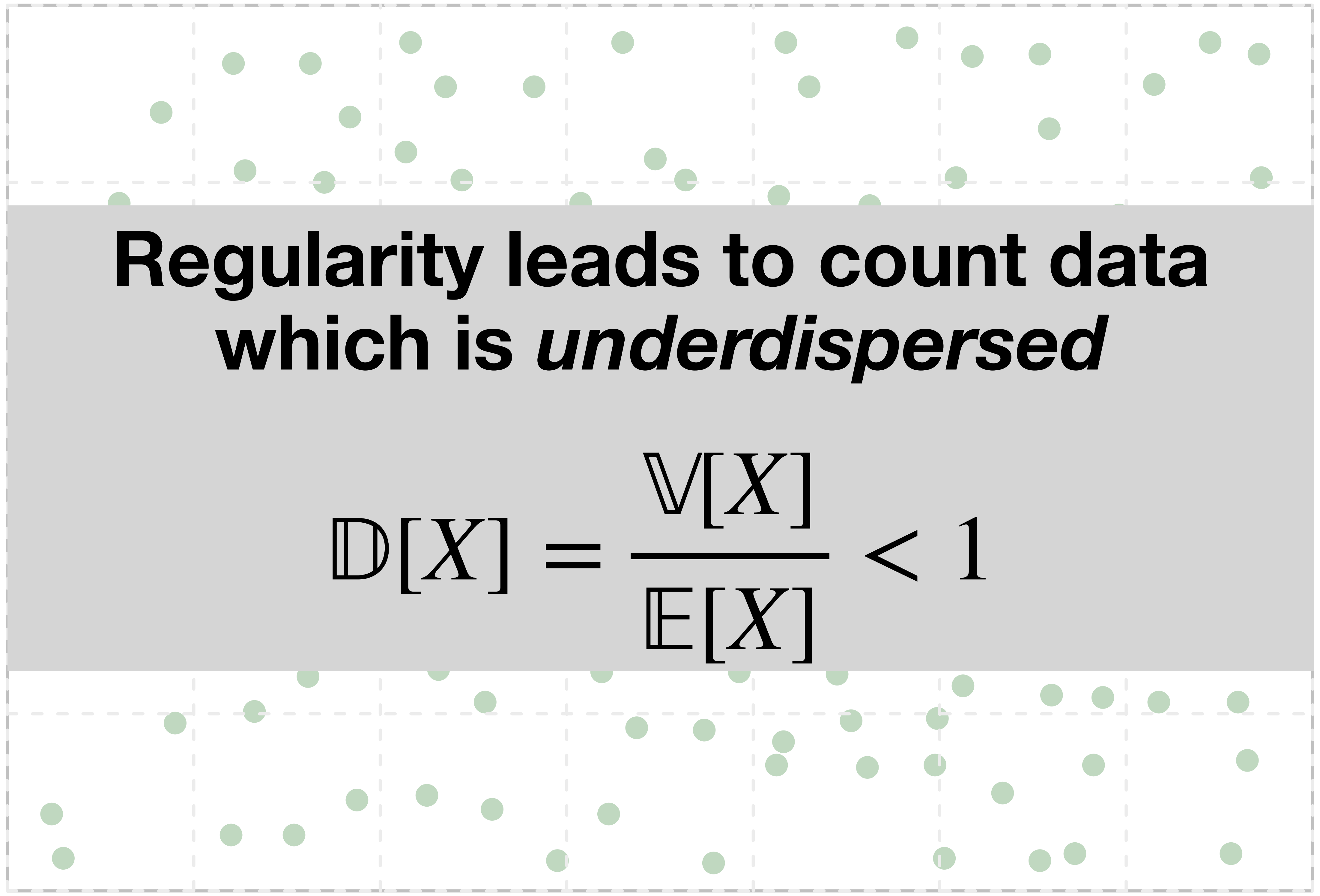


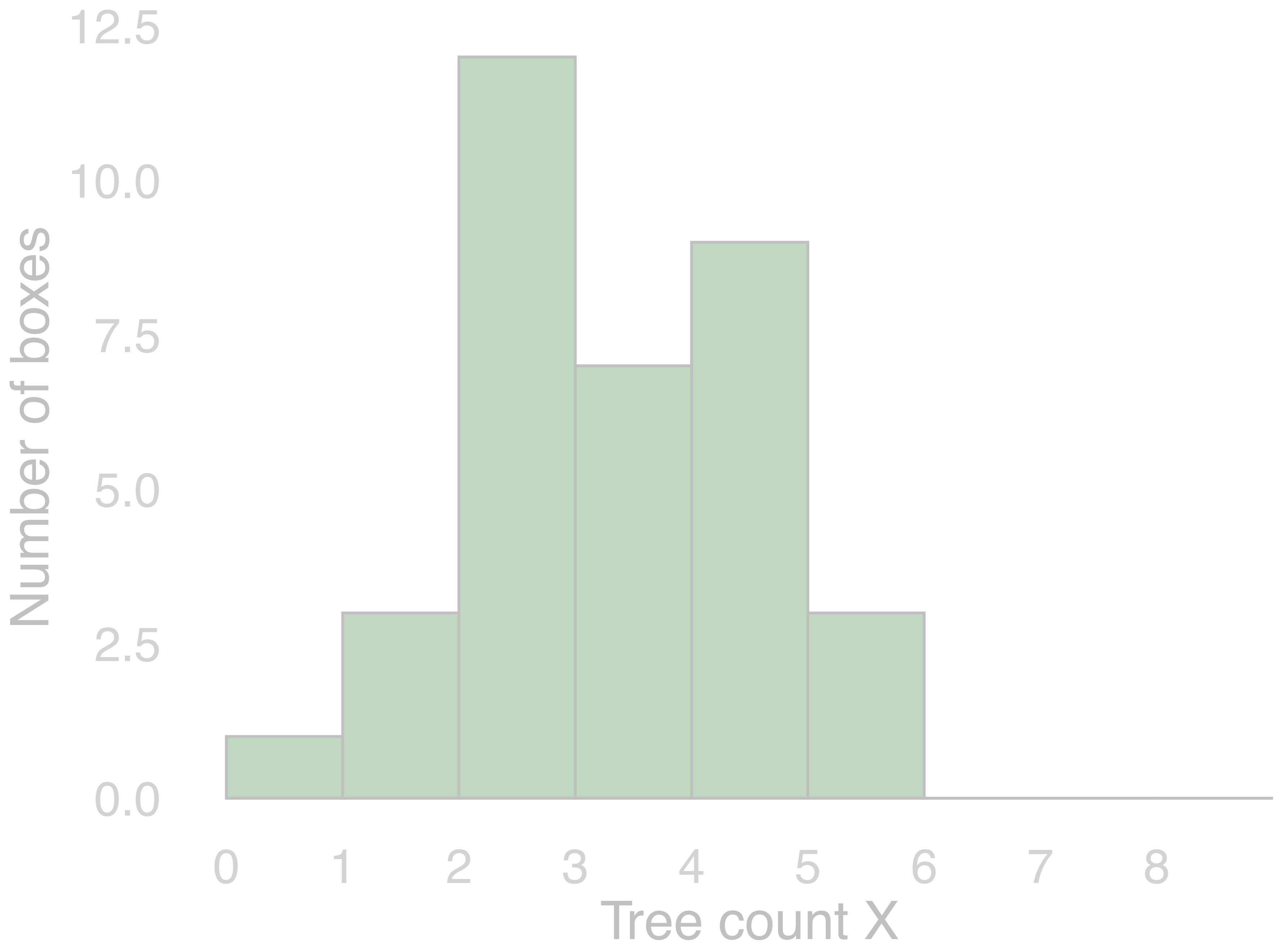
Dispersion → $\mathbb{D}[X] = \dfrac{\mathbb{V}[X]}{\mathbb{E}[X]} \approx .41$

Underdispersion: Spruce trees

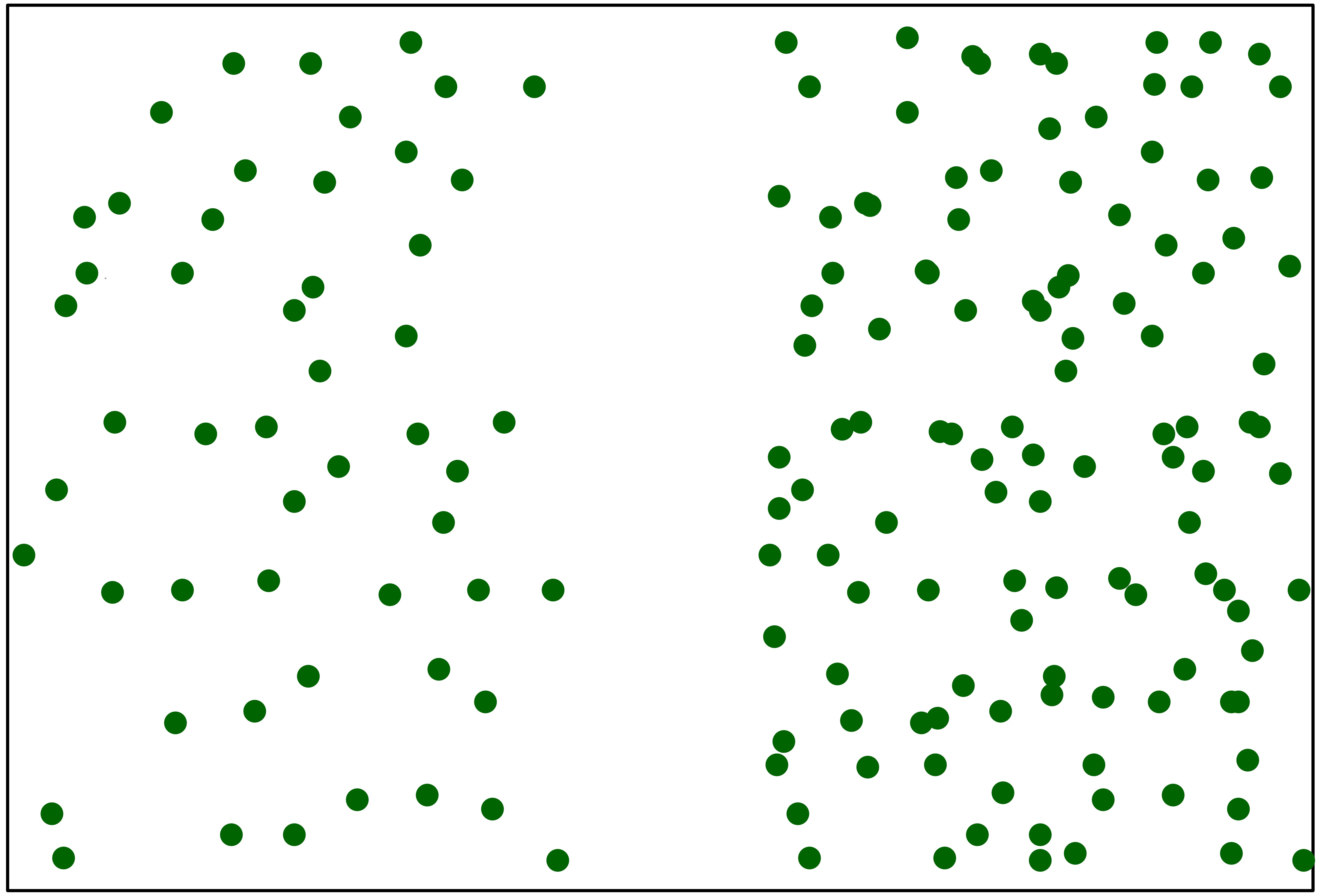


Regularity leads to count data which is *underdispersed*

$$\mathbb{D}[X] = \frac{\mathbb{V}[X]}{\mathbb{E}[X]} < 1$$

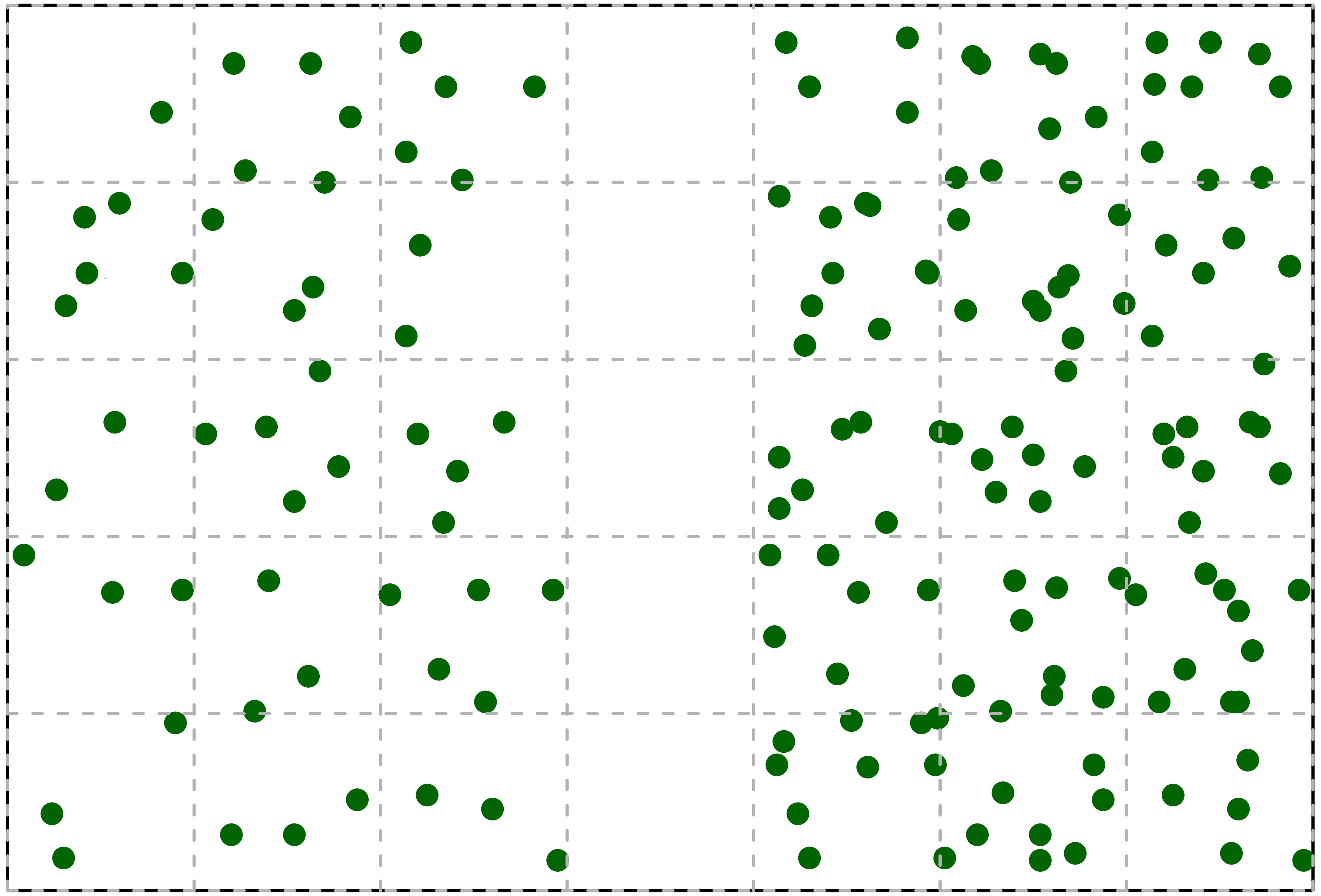


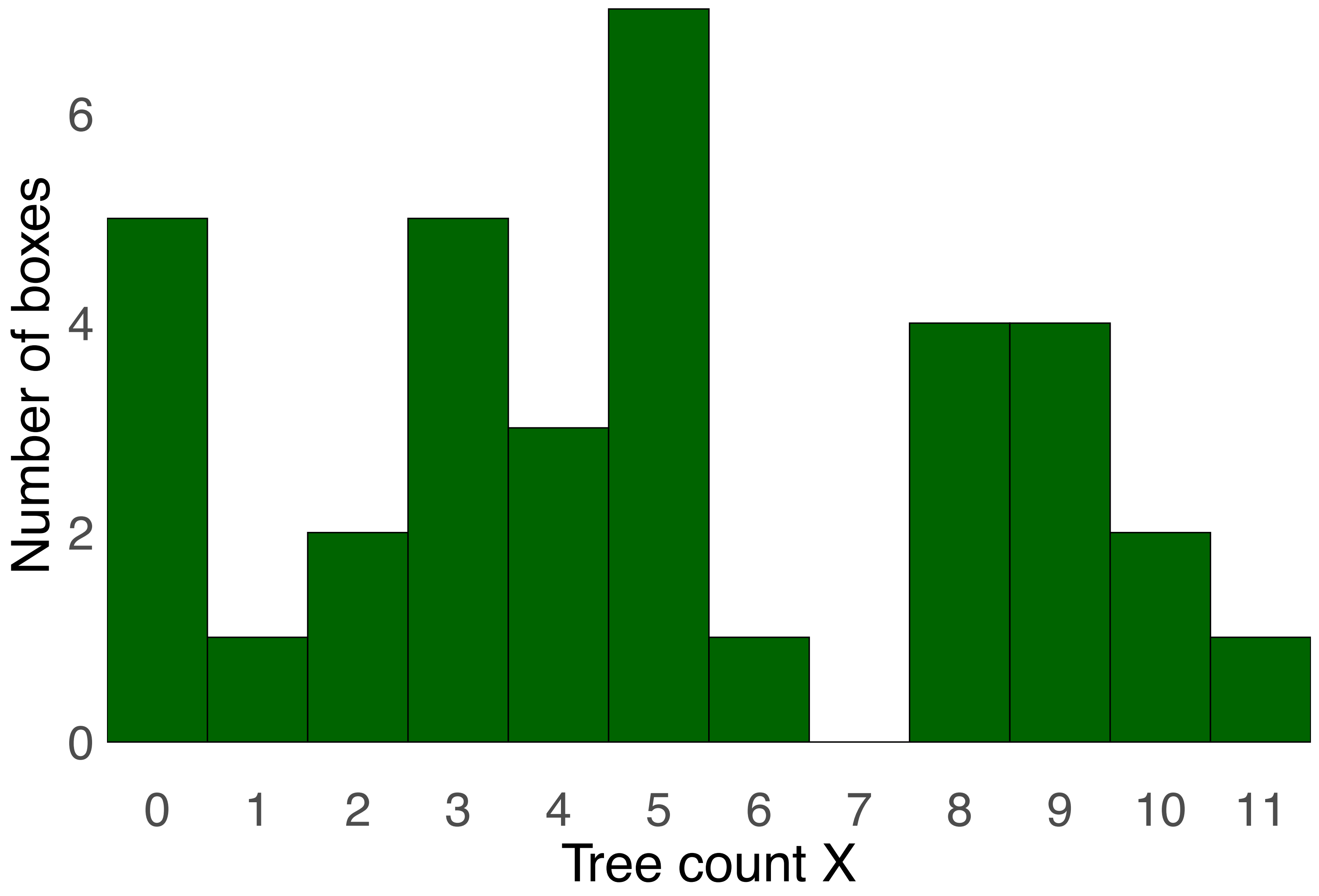
Conditional underdispersion



Conditional underdispersion

$$\mathbb{D}[X] = \frac{\mathbb{V}[X]}{\mathbb{E}[X]} \approx 2.20$$





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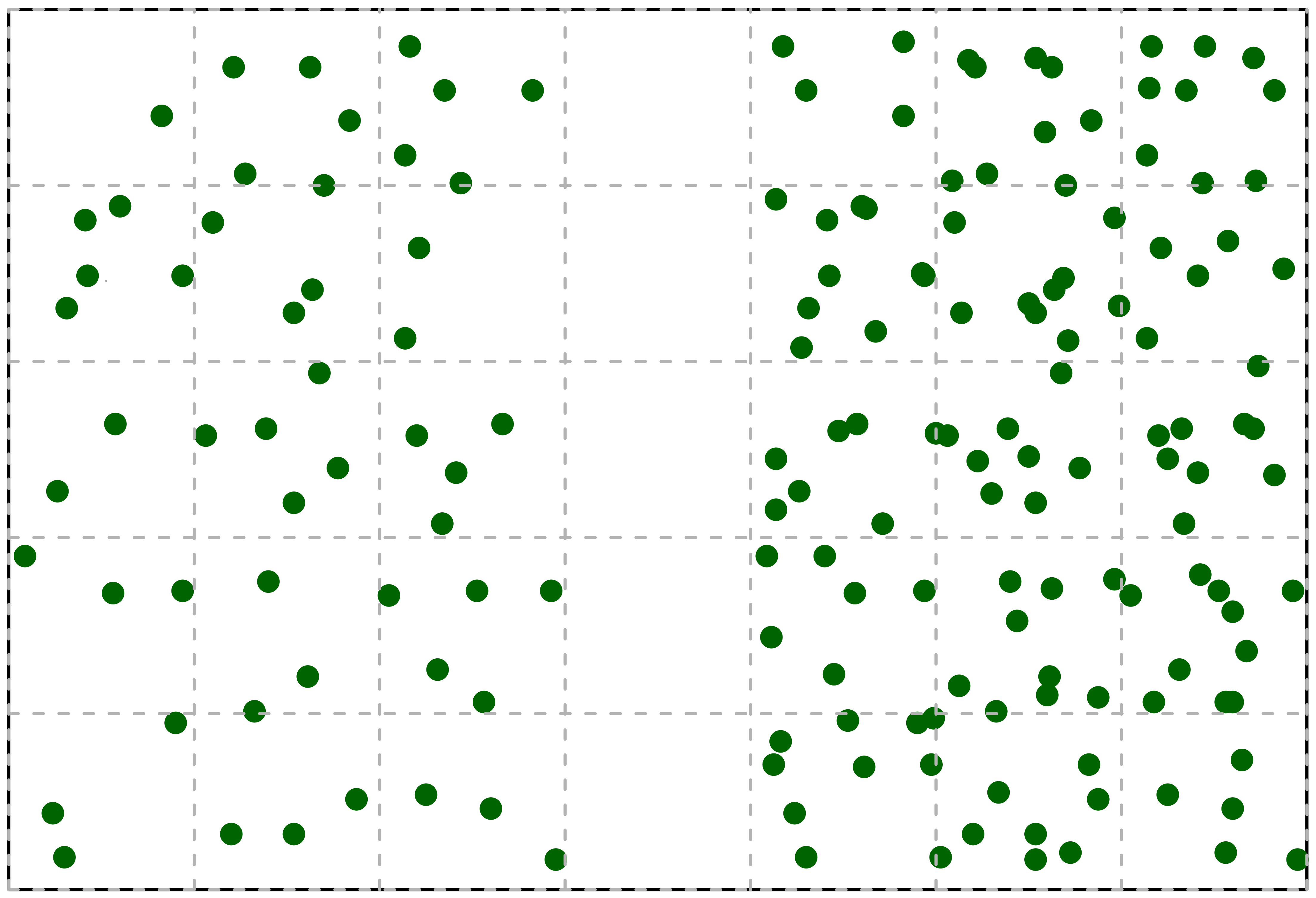
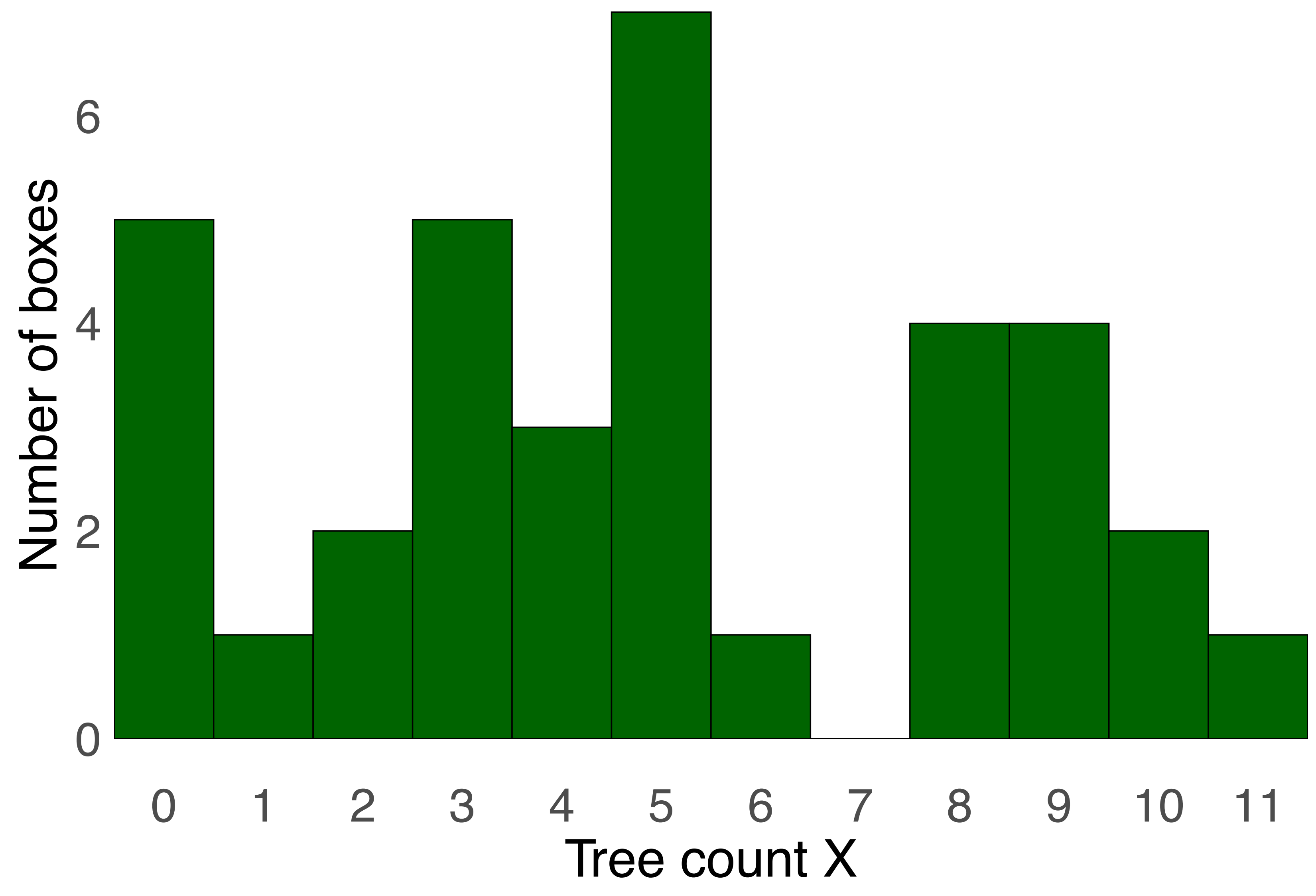
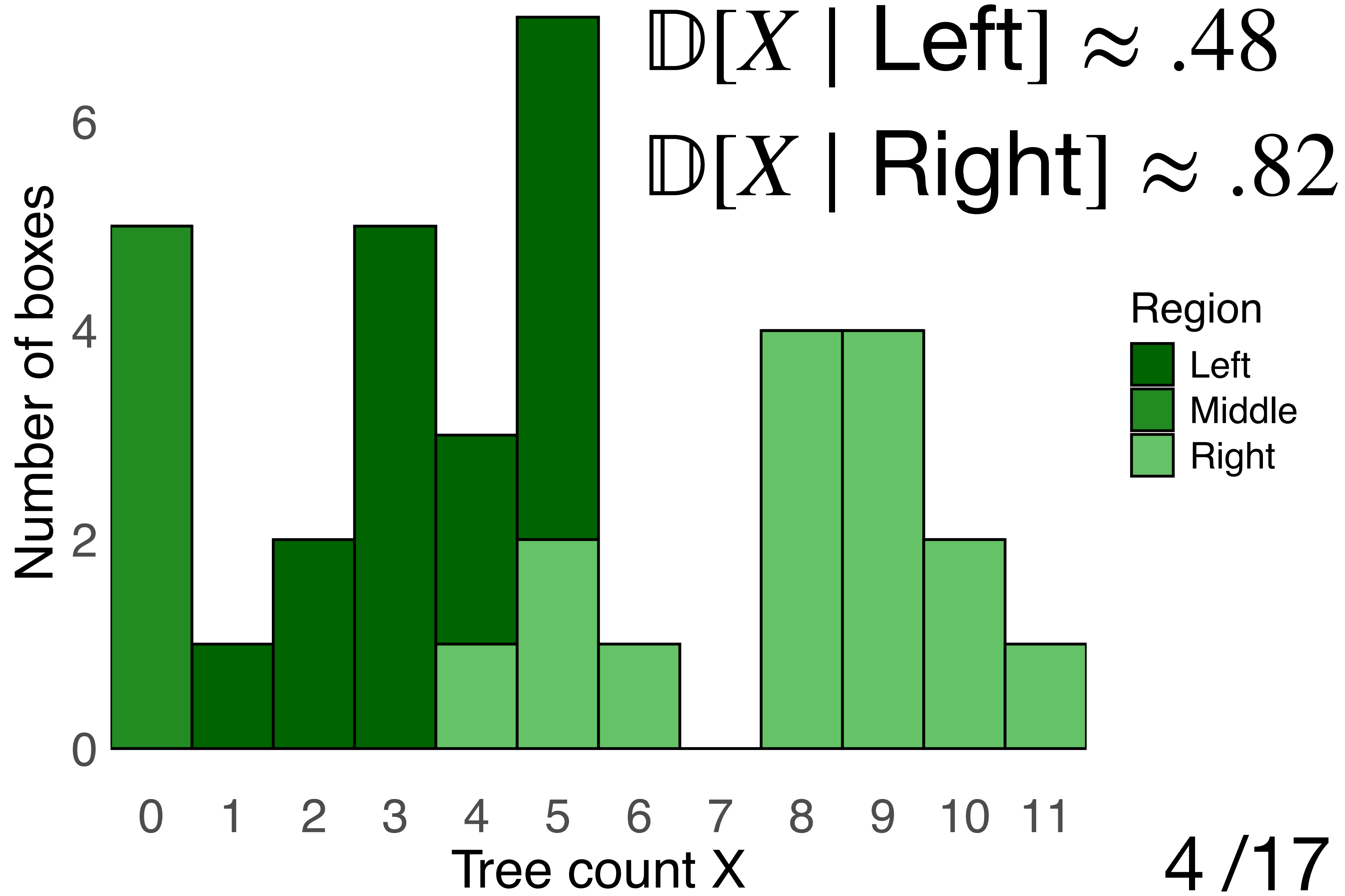




$\mathbb{D}[X \mid \text{Left}] \approx .48$

$\mathbb{D}[X \mid \text{Right}] \approx .82$



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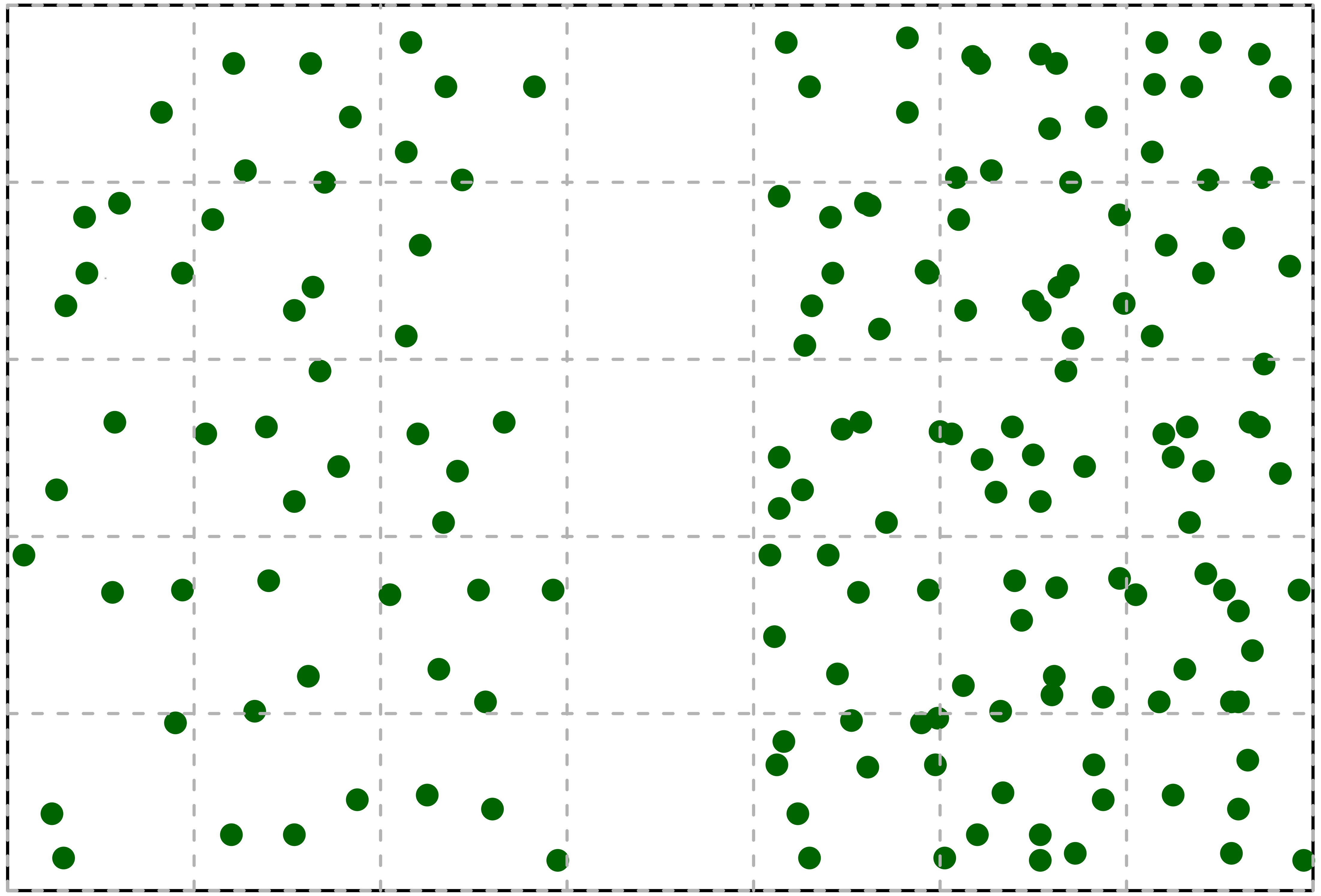

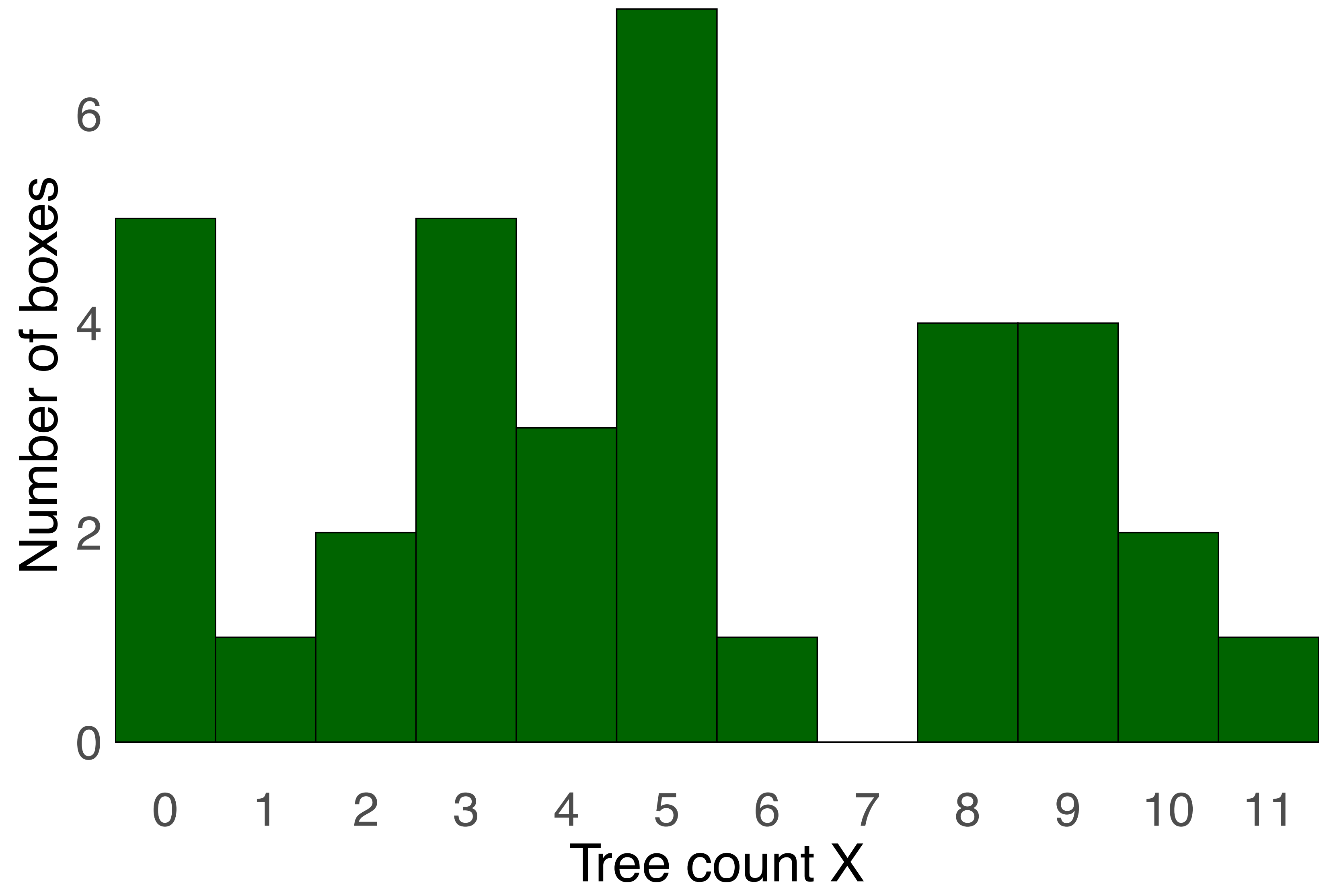

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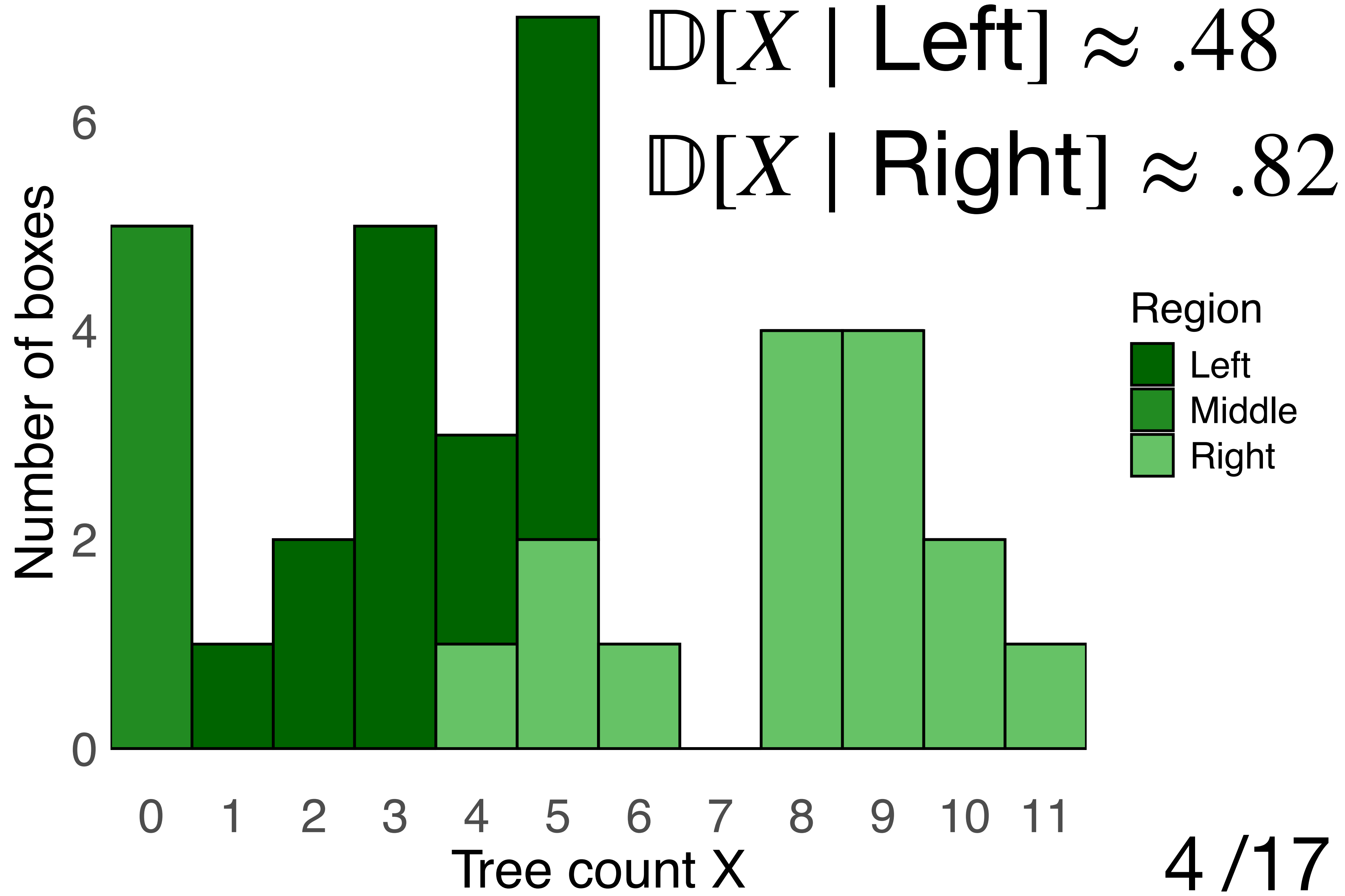
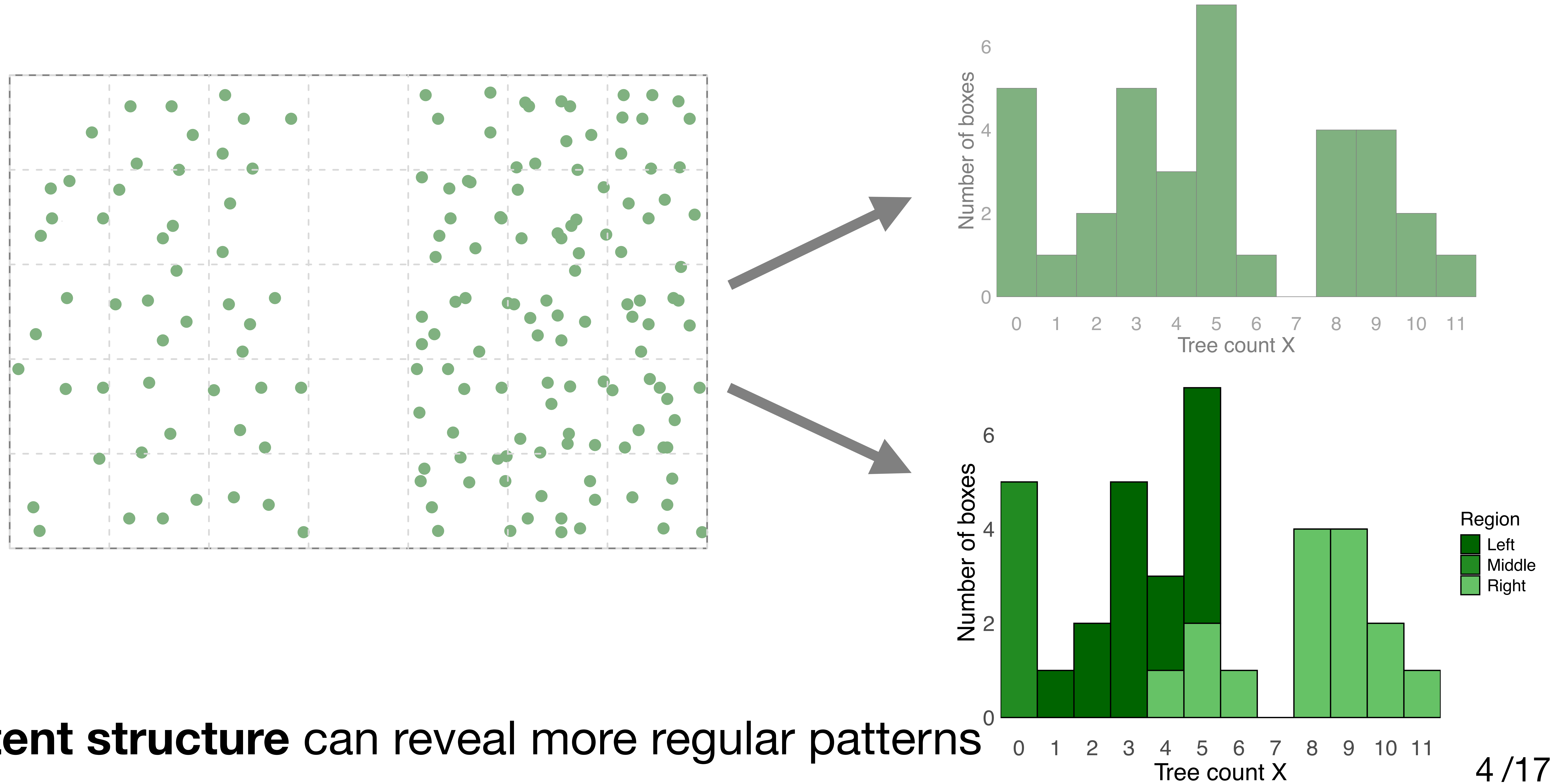


Marginal overdispersion can mask conditional underdispersion



Latent structure can reveal more regular patterns

Goal: we want to build **latent variable models** which allow for **conditional underdispersion**

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Count distributions which allow for underdispersion include:

- Conway-Maxwell Poisson [Conway, 1961]
- Double Poisson [Efron, 1986]
- Gamma count distribution [Winklemann, 1995]
- Generalized Poisson [Consul and Famoye, 2006]

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Problem: these distributions lack closed-form conjugate priors

Our solution: build models around a distribution which:

1. is capable of attaining underdispersion
2. can be written in terms of latent Poisson random variables

Poisson order statistics for underdispersed counts

Consider a count-valued datapoint $Y \in \mathbb{N}_0$ assumed to be a Poisson order statistic

$$Y \sim \text{Pois}_\mu^{(r,D)} \iff Y = Z^{(r,D)} \text{ where } Z_1, \ldots, Z_D \sim \text{Pois}(\mu)$$

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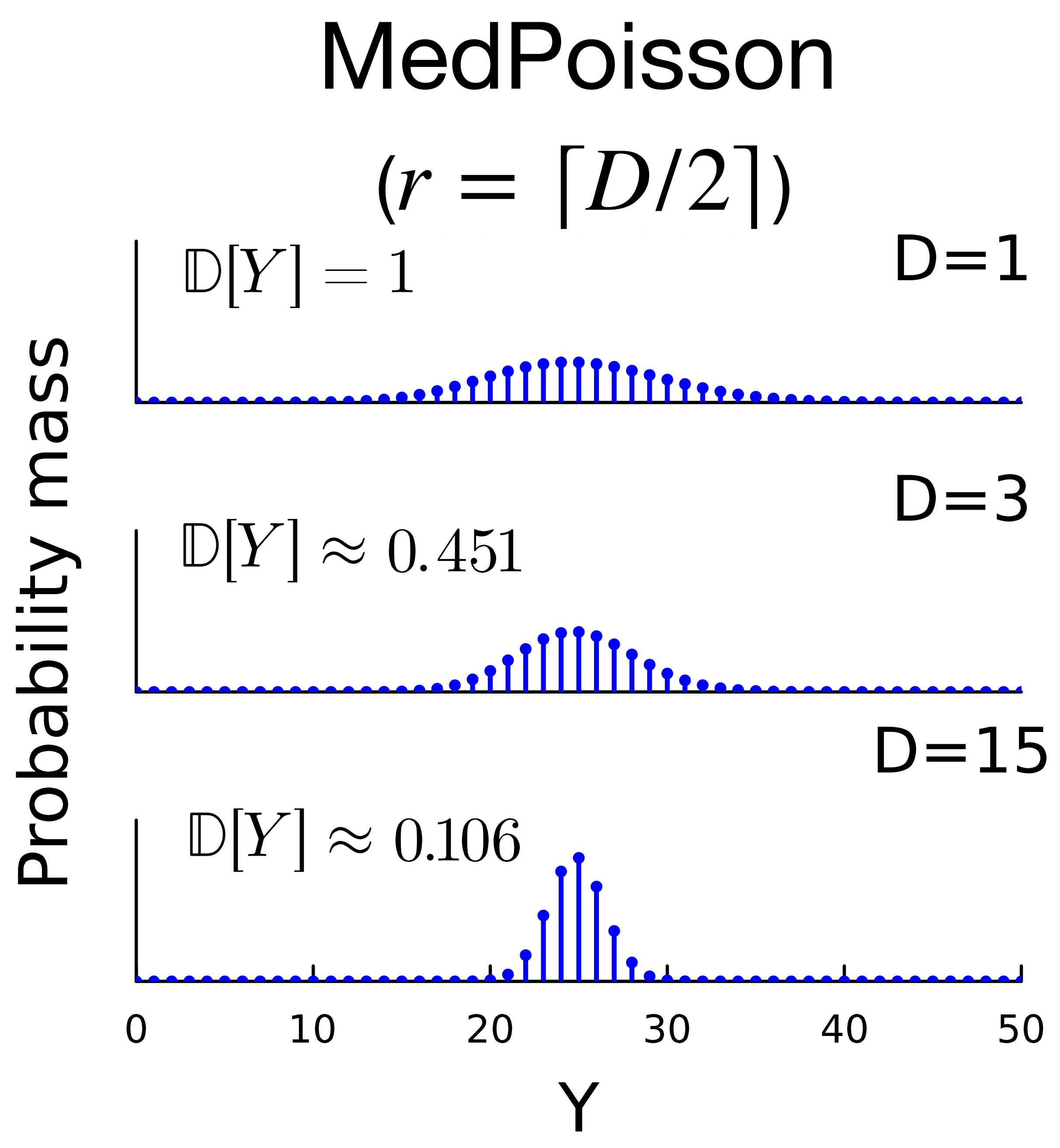

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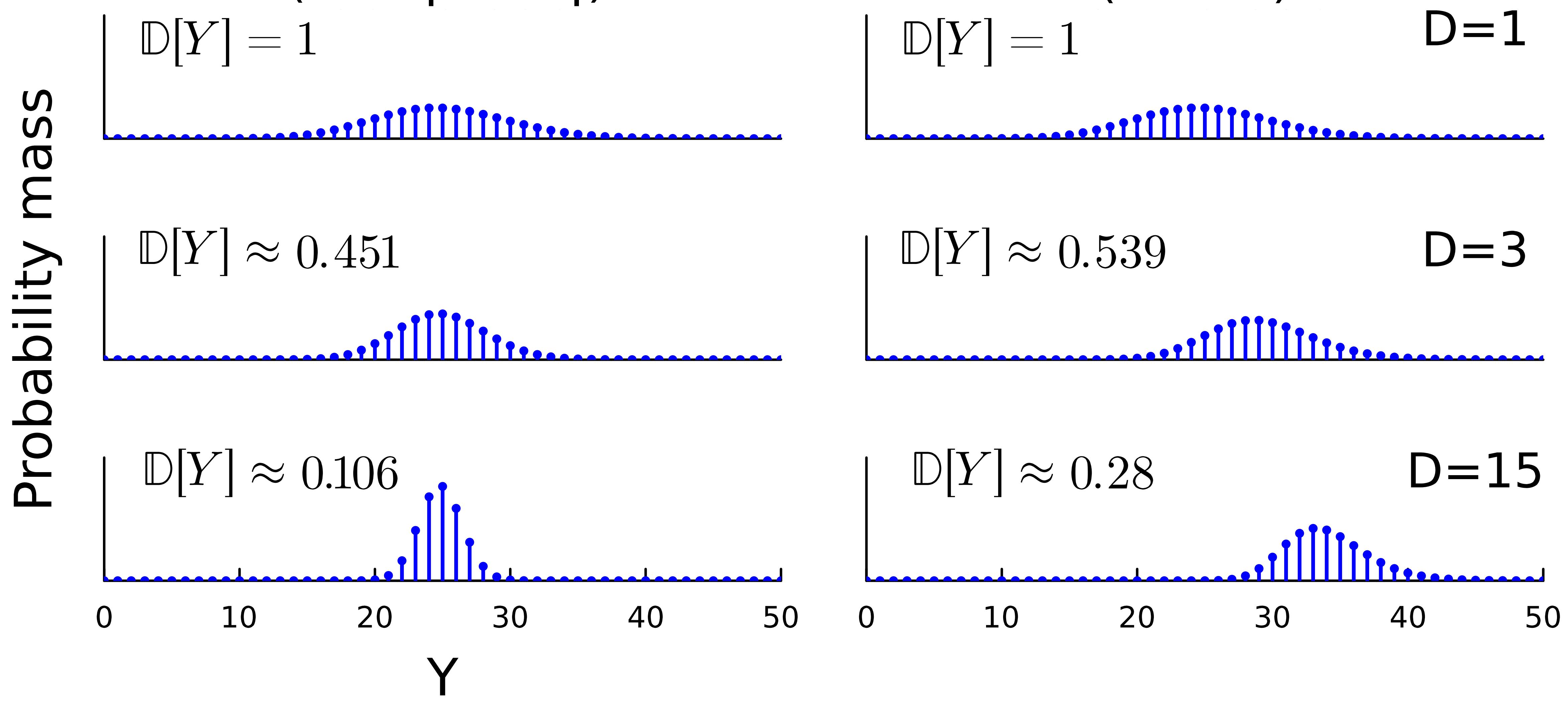
MedPoisson
($r = \lceil D/2 \rceil$)

MaxPoisson
($r = D$)



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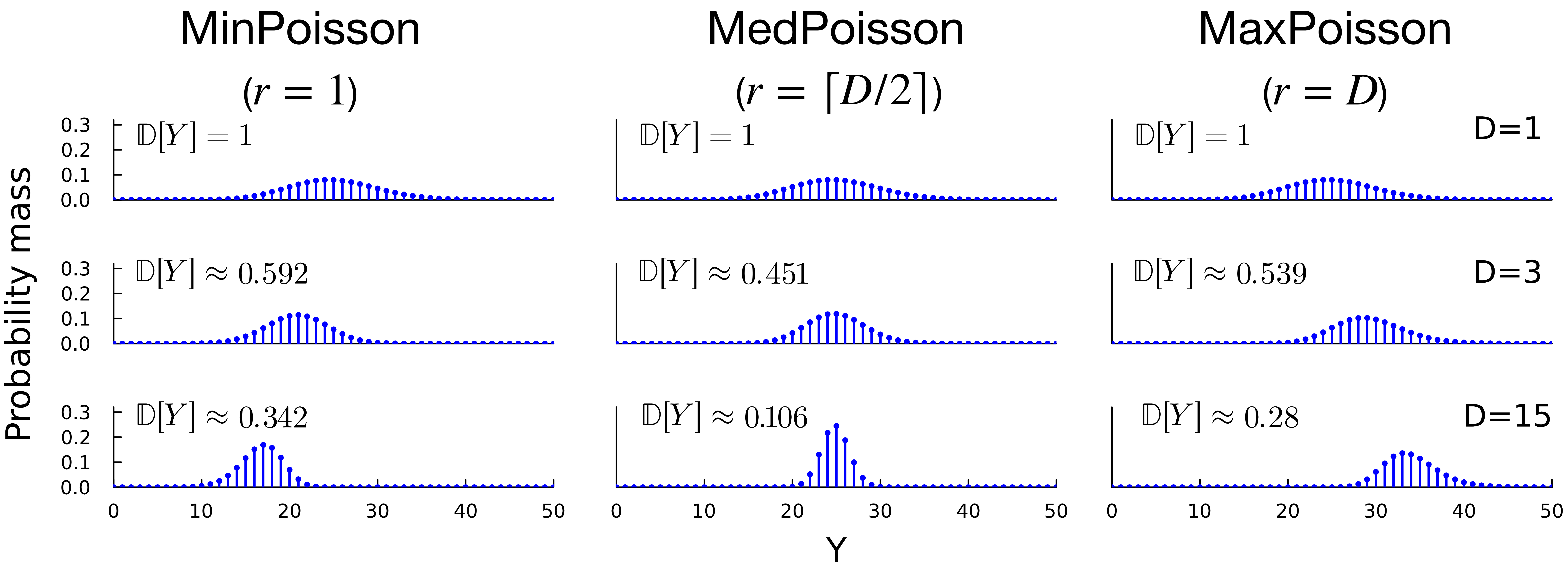

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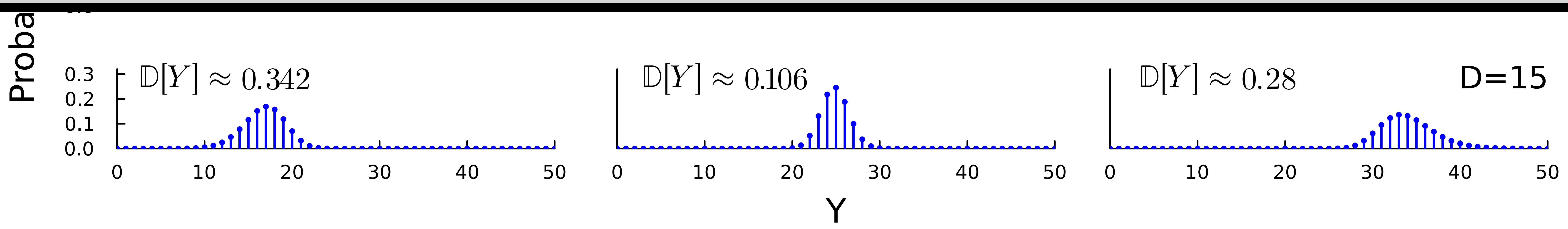
MinPoisson MedPoisson MaxPoisson

Key fact [Badiella, 2023]: For any μ, r, and $D > 1$, the Poisson order statistic $Y \sim \text{Pois}_{\mu}^{(r,D)}$ is underdispersed:

$$\mathbb{D}[Y] = \frac{\mathbb{V}[X]}{\mathbb{E}[X]} < 1$$

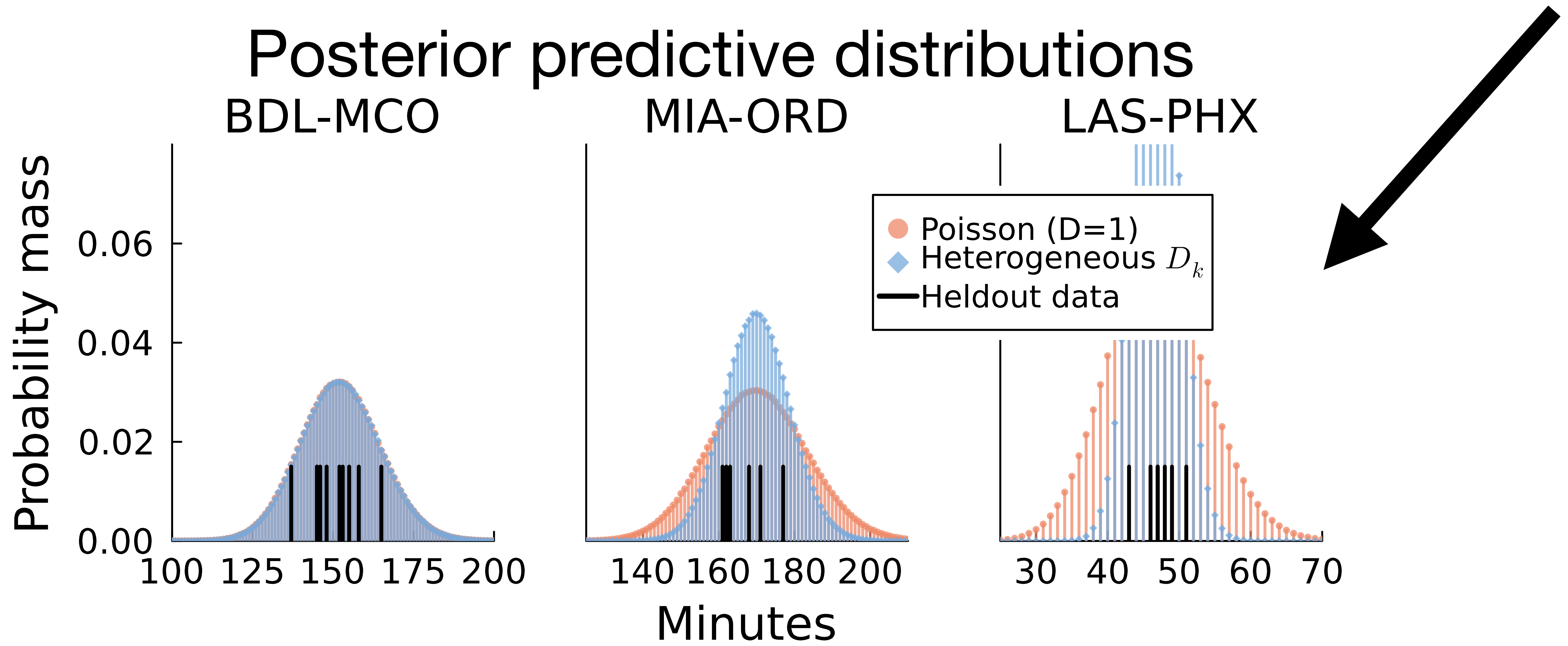


Talk outline

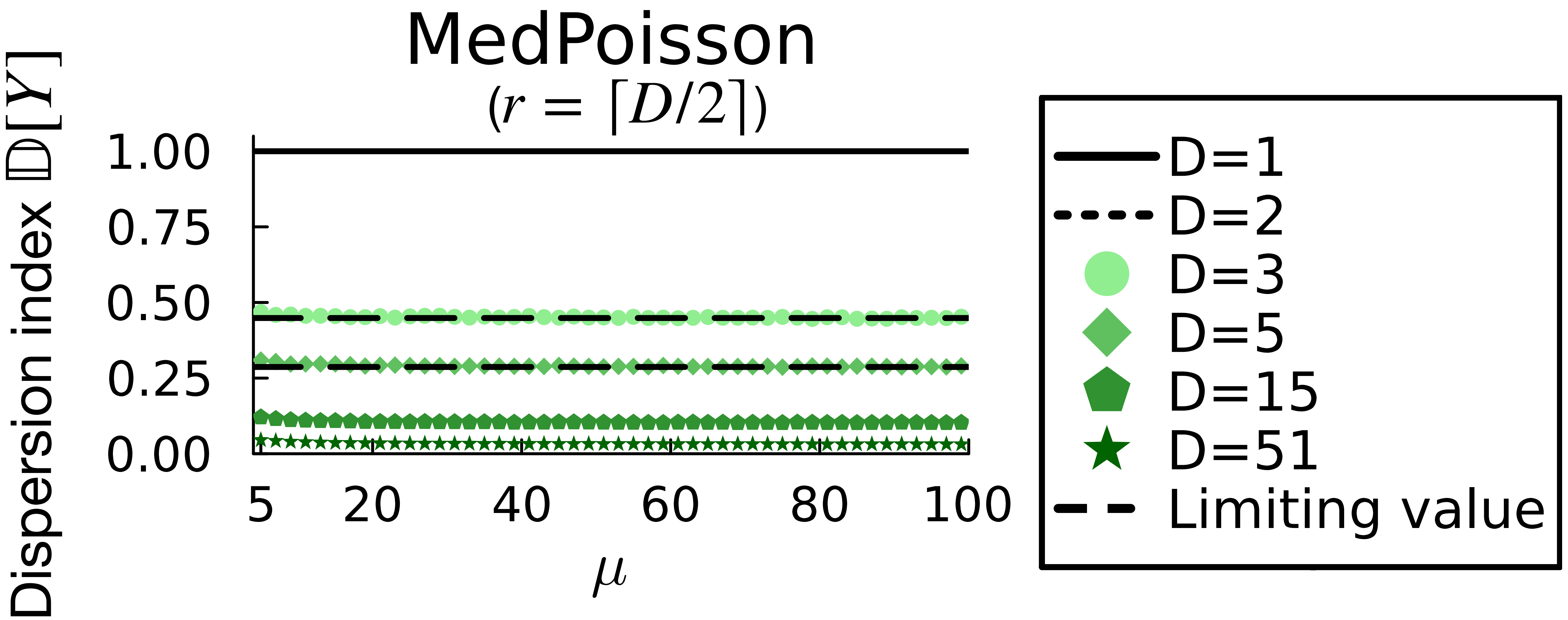
1. Dispersion properties of Poisson order statistics

2. A data augmentation strategy for inference of the parameters of any discrete order statistic **(not only for the Poisson)**

3. Applications where we build and fit hierarchical models for conditionally underdispersed count data, yielding **more precise probabilistic predictions**



Dispersion of Poisson order statistics



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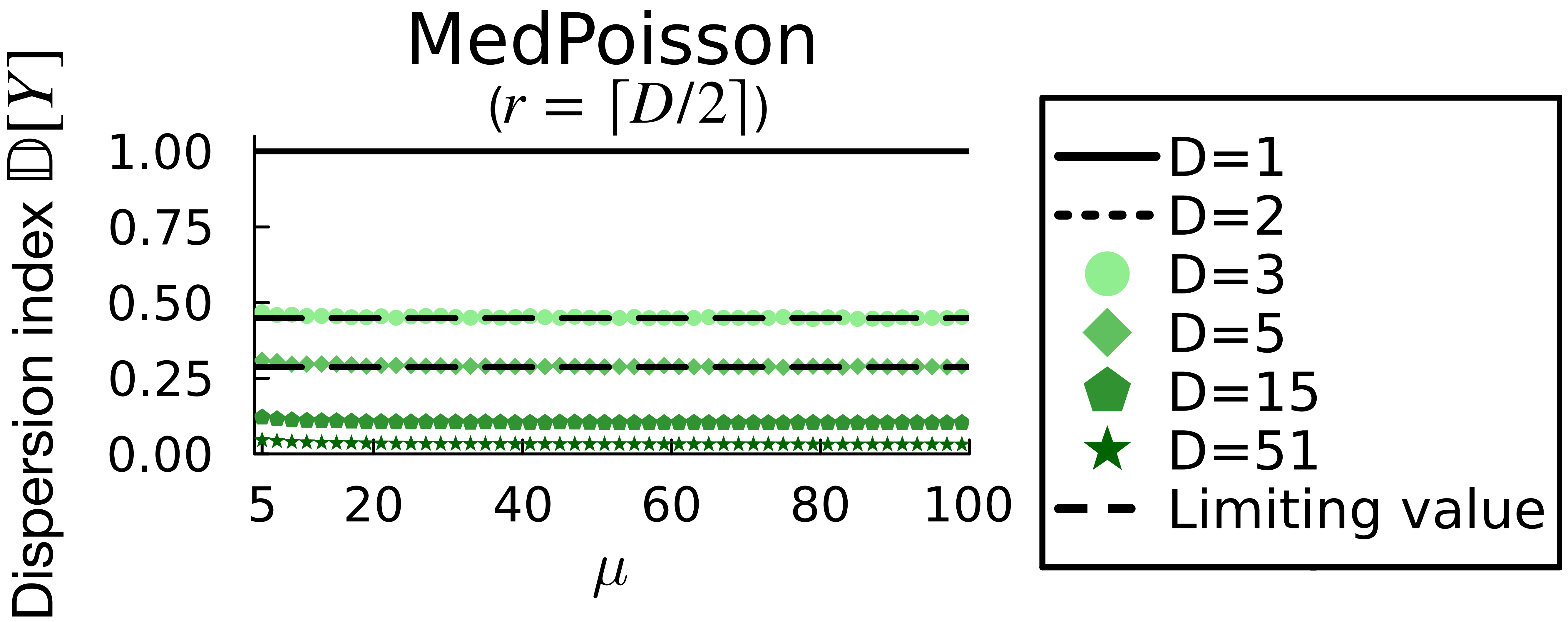
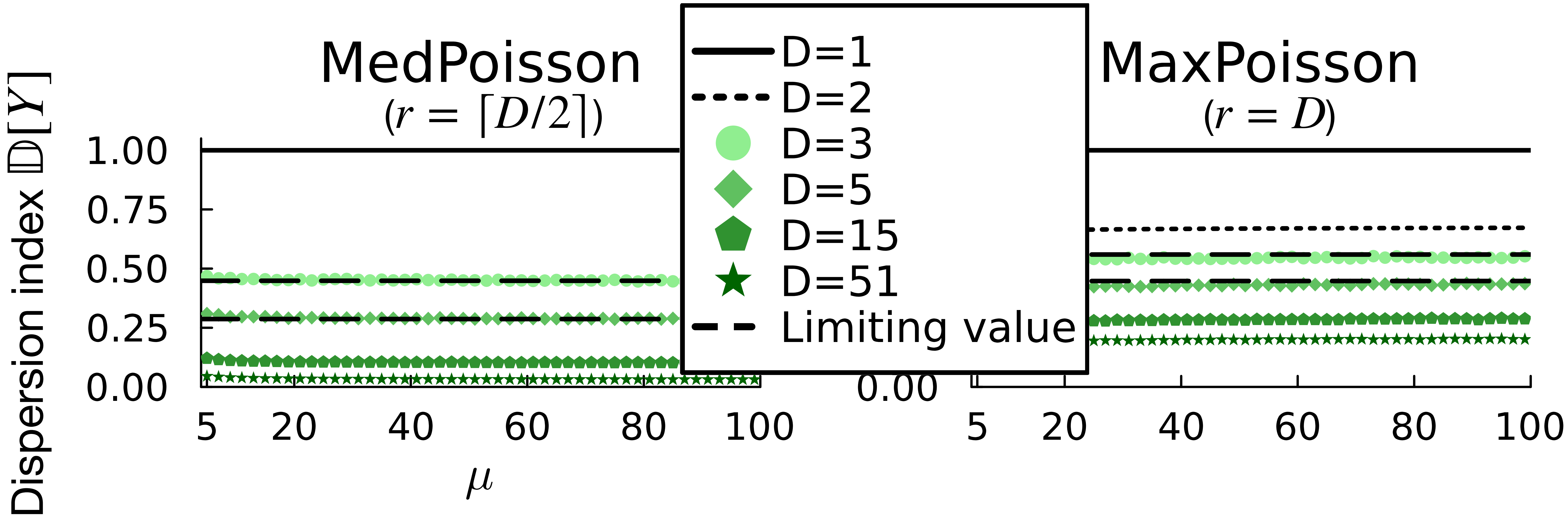


The dispersion $\mathbb{D}_{Y \sim \text{Pois}_\mu^{(r,D)}}[Y]$:

1. is stable across large values of μ for each rank r and order D
2. decreases as the order D increases

Dispersion of Poisson order statistics



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We can use the order D as a "pseudo-index" to control the level of dispersion when building models

Inference for discrete order statistics via data augmentation

Our strategy revolves around inferring the latent $\mathbf{Z}_{1:D}$

$$Y \sim \text{Pois}_{\mu}^{(r,D)} \iff Y = Z^{(r,D)} \text{ where } Z_1, \ldots, Z_D \sim \text{Pois}(\mu)$$

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First: $$P_{\mu}(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y) \propto 1(Y = Z^{(r,D)}) \prod_{d=1}^{D} \text{Pois}_{\mu}(Z_d)$$

Second: $$P(\mu \mid \mathbf{Z}_{1:D}) \propto g(\mu) \prod_{d=1}^{D} \text{Pois}_{\mu}(Z_d)$$

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Data augmentation

difficult

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easy

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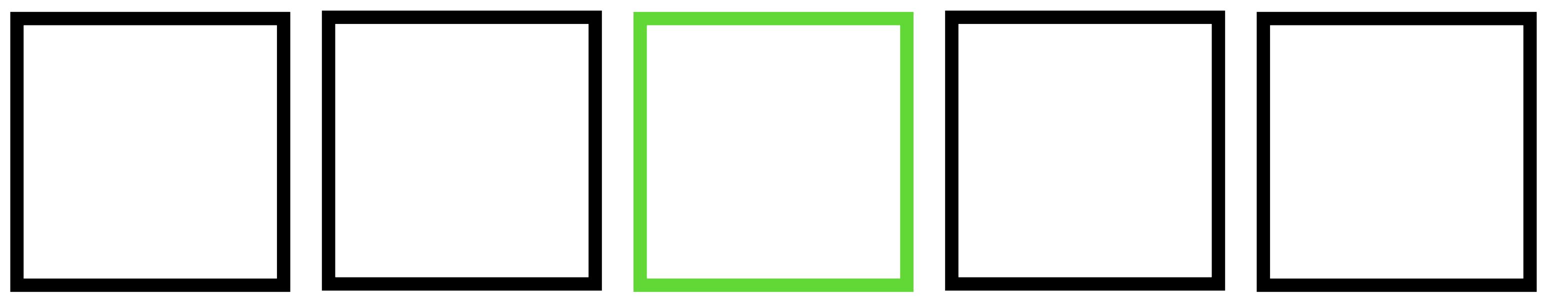
Data Augmentation: Sampling from $P_\mu(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y)$

$$Z^{(3,5)} = Y$$



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$\boxed{<Y}\ \boxed{<Y}\ \boxed{\ Y\ }\ \boxed{>Y}\ \boxed{>Y}$ If $\mathbf{Z}_{1:D}$ were continuous

Data Augmentation: Sampling from $P_\mu(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y)$

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All valid arrangements

that satisfy the constraint

$$\boxed{<Y}\ \boxed{<Y}\ \boxed{Y}\ \boxed{>Y}\ \boxed{>Y}$$

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Only the support of each Z_d matters to satisfy the constraint $Z^{(r,D)} = Y$

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Introduce a categorical random variable for each Z_d denoted $C_d \in \{<Y, =Y, >Y\}$ which determines the support of Z_d

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1. $\mathbf{Z}_{1:D}$ are conditionally independent of $Z^{(r,D)} = Y$ given $\mathbf{C}_{1:D}$

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2. Given C_d, sample each Z_d as a truncated Poisson, which can be made efficient

Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

Strategy: We sample the support $\mathbf{C}_{1:D}$ sequentially from $P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y)$

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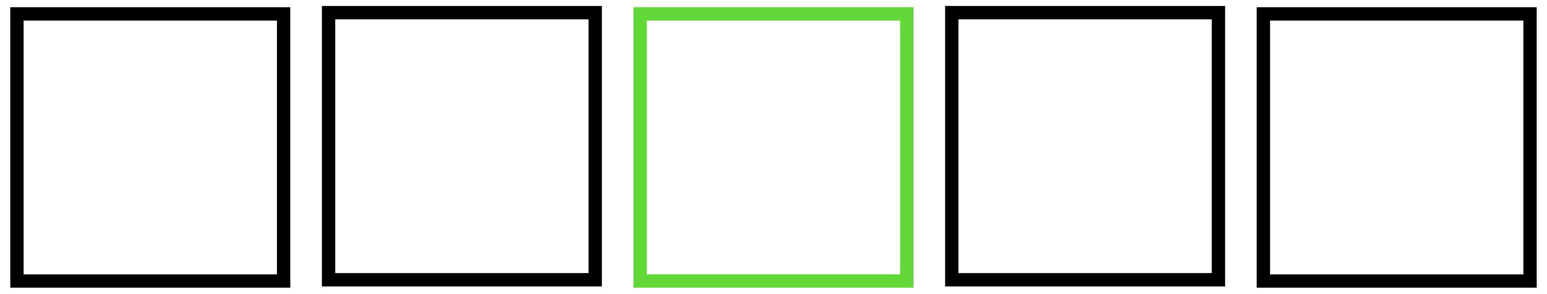
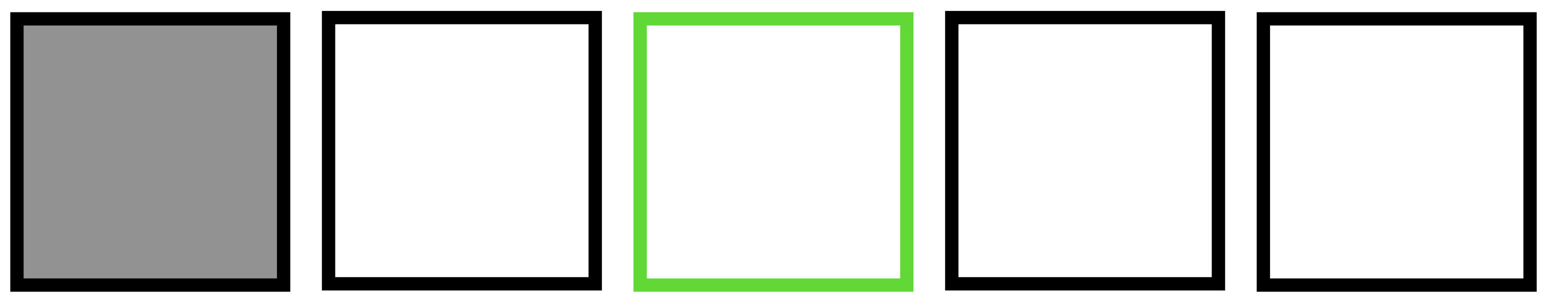


Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

Strategy: We sample the support $\mathbf{C}_{1:D}$ sequentially from $P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y)$

$$Z^{(3,5)} = Y$$



$$C_1 = {<}Y$$

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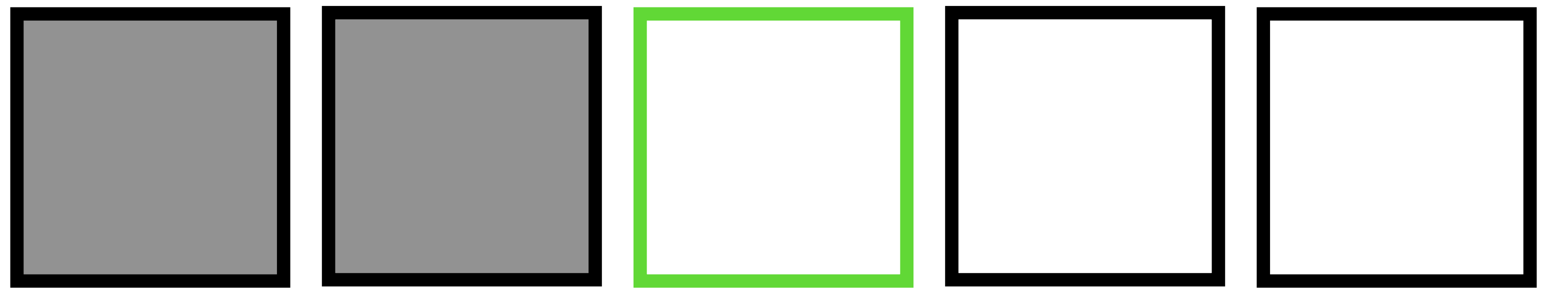

$$C_1 = <Y$$

$$C_2 = <Y$$

Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

Strategy: We sample the support $\mathbf{C}_{1:D}$ sequentially from $P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y)$

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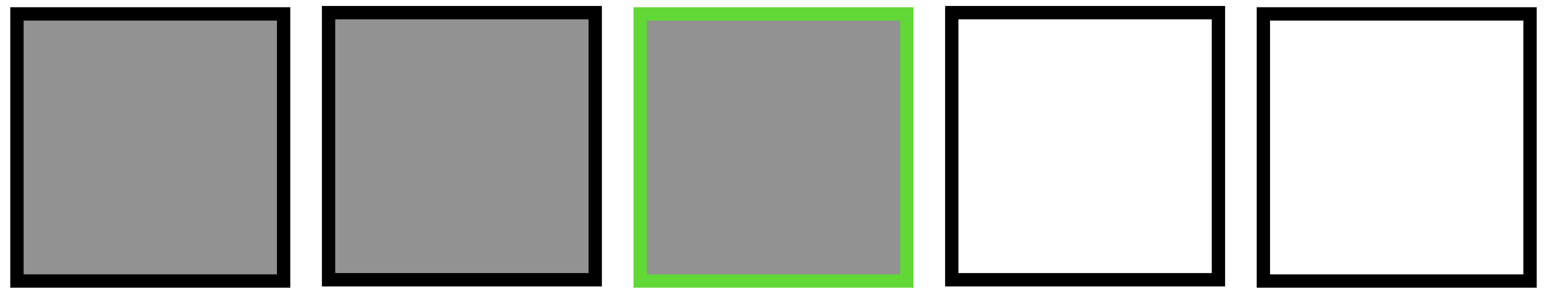

$C_1 = {<}Y \quad C_3 = {=}Y$

$C_2 = {<}Y$

Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

Strategy: We sample the support $\mathbf{C}_{1:D}$ sequentially from $P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y)$

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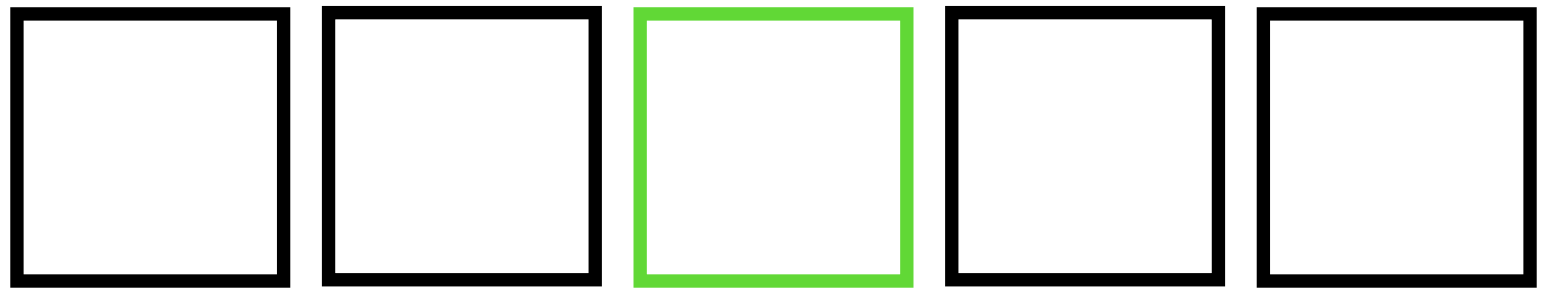

$$\mathbf{C}_d \,|\, \mathbf{C}_{1:d-1} \sim \text{Categorical}(p^{(<Y)}, p^{(=Y)}, p^{(>Y)})$$

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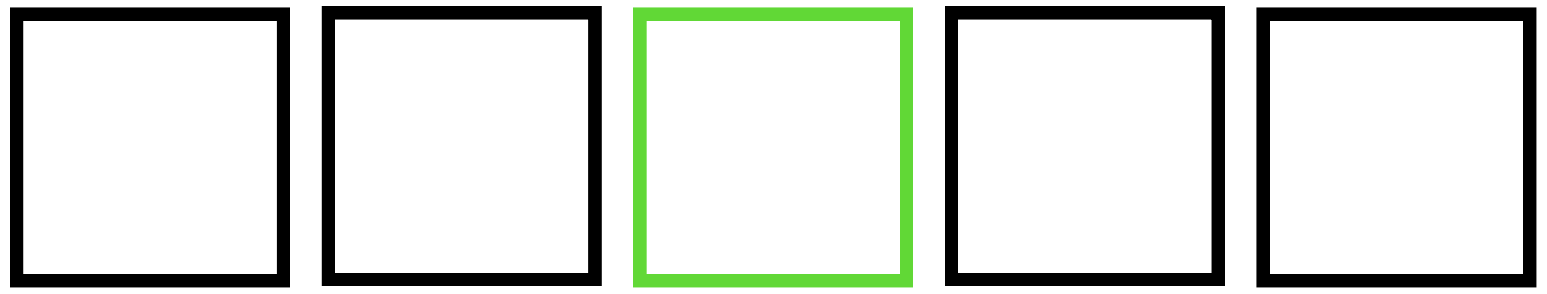


$$n^{(<Y)} = \sum_{s=1}^{d-1} 1\{C_s = <Y\}$$

$\mathbf{C}_d \,|\, \mathbf{C}_{1:d-1} \sim$ Categorical$(p^{(<Y)}, p^{(=Y)}, p^{(>Y)})$

$$n^{(=Y)} = \sum_{s=1}^{d-1} 1\{C_s = =Y\},$$

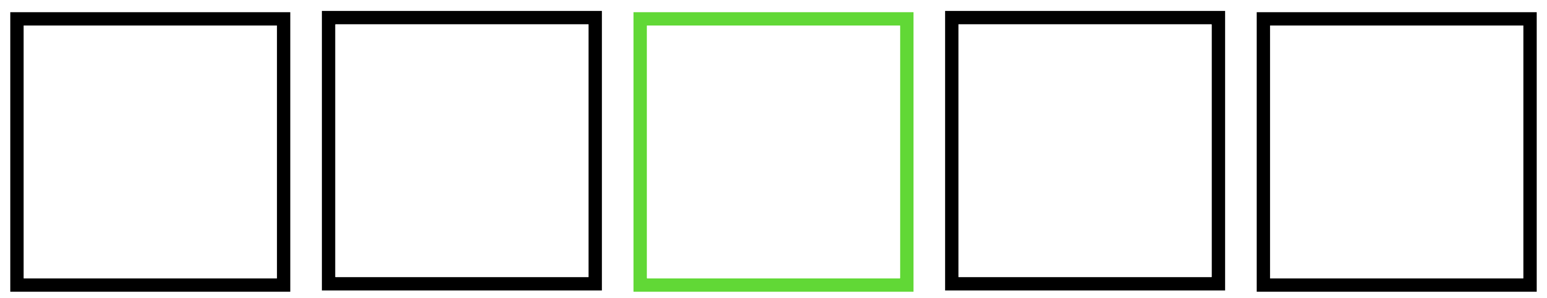
We can calculate these probabilities efficiently in closed-form

$$n^{(>Y)} = \sum_{s=1}^{d-1} 1\{C_s = >Y\}$$

Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

Strategy: We sample the support $\mathbf{C}_{1:D}$ sequentially from $P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y)$

$$Z^{(3,5)} = Y$$



$$\mathbf{C}_d \mid \mathbf{C}_{1:d-1} \sim \text{Categorical}(p^{(<Y)}, p^{(=Y)}, p^{(>Y)})$$

$$n^{(<Y)} = \sum_{s=1}^{d-1} 1\{C_s = <Y\}$$

$$n^{(=Y)} = \sum_{s=1}^{d-1} 1\{C_s = =Y\},$$

$$n^{(>Y)} = \sum_{s=1}^{d-1} 1\{C_s = >Y\}$$

We can calculate these probabilities efficiently in closed-form

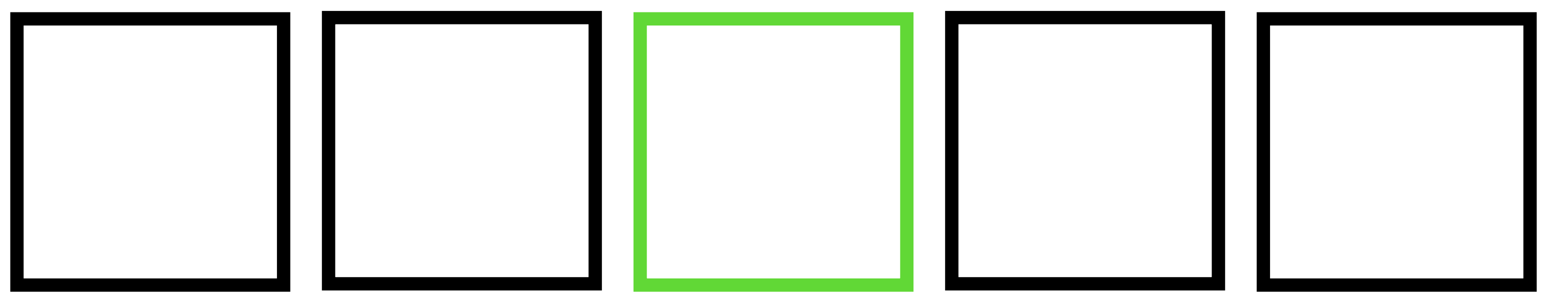
Main idea: These past counts are **sufficient statistics**

$$P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}) = P_\mu(Z^{(r,D)} = Y \mid n^{(<Y)}, n^{(=Y)}, n^{(>Y)})$$

Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

Strategy: We sample the support $\mathbf{C}_{1:D}$ sequentially from $P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y)$

$$Z^{(3,5)} = Y$$



$$n^{(<Y)} = \sum_{s=1}^{d-1} 1\{C_s = <Y\}$$

$$n^{(=Y)} = \sum_{s=1}^{d-1} 1\{C_s = =Y\},$$

$$n^{(>Y)} = \sum_{s=1}^{d-1} 1\{C_s = >Y\}$$

$$\mathbf{C}_d \,|\, \mathbf{C}_{1:d-1} \sim \text{Categorical}(p^{(<Y)}, p^{(=Y)}, p^{(>Y)})$$

We can calculate these probabilities efficiently in closed-form

Main idea: These past counts are **sufficient statistics**

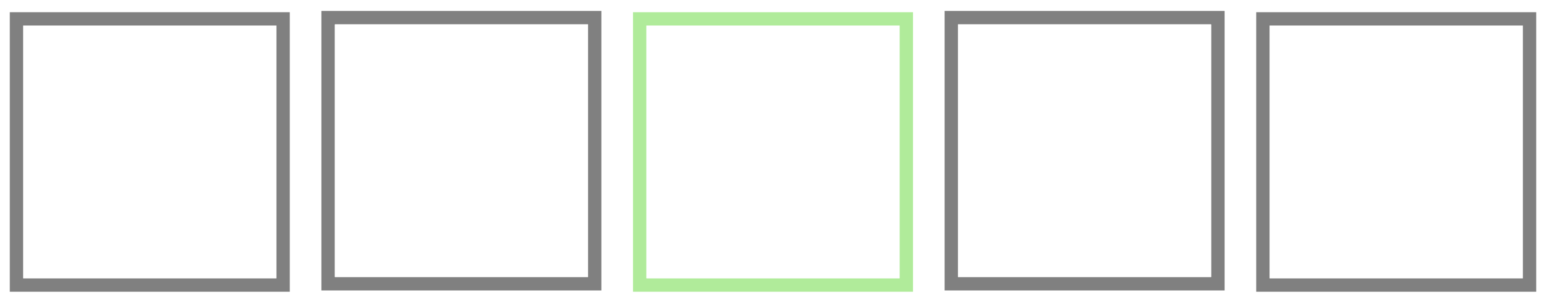
$$\boxed{P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1})} = P_\mu(Z^{(r,D)} = Y \mid n^{(<Y)}, n^{(=Y)}, n^{(>Y)})$$

By Bayes rule, for $c \in \{<Y, =Y, >Y\}$:

$$p_d^{(c)} := P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y) = \frac{\boxed{P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}, C_d = c)}}{\boxed{P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1})}} P_\mu(C_d = c)$$

Sampling the support $\mathbf{C}_{1:D}$ from $P_\mu(\mathbf{C}_{1:D} \mid Z^{(r,D)} = Y)$

$$Z^{(3,5)} = Y$$



Takeaway: we can sample the support $\mathbf{C}_{1:D}$ in closed-form,

Therefore, we can sample the latent $\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y$ as desired

$$P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}) = P_\mu(Z^{(r,D)} = Y \mid n^{(<Y)}, n^{(=Y)}, n^{(>Y)})$$

$$p_d^{(c)} := P_\mu(C_d \mid \mathbf{C}_{1:d-1}, Z^{(r,D)} = Y) = \frac{P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}, C_d = c)}{P_\mu(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1})} P_\mu(C_d = c)$$

Inference Summary

First:

$$\mathbf{Z}_{1:D} \sim P_\theta(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y)$$

Second:

$$P(\theta \mid \mathbf{Z}_{1:D}) \propto g(\theta) \prod_{d=1}^{D} f_\theta(Z_d)$$

Algorithm B Exact simulation of $\mathbf{Z}_{1:D} \sim P_\theta(\mathbf{Z}_{1:D} \mid Z^{(r,D)} = Y)$

1: **Input**: observation $Y \in \mathbb{Z}$, order $D \in \mathbb{N}$, rank $r \in [D]$, parent distribution f_θ
2: **Initialize**: $N_0^{(<Y)} = N_0^{(=Y)} = N_0^{(>Y)} = 0$
4: **for** $d = 1 \ldots D$ **do**
5: Compute $[p_d^{(<Y)}, p_d^{(=Y)}, p_d^{(>Y)}]$ `// as defined in Equation (6)`
6: Sample $C_d \sim \text{Categorical}\left(p_d^{(<Y)}, p_d^{(=Y)}, p_d^{(>Y)}\right)$ `// where` $C_d \in \{<Y, =Y, >Y\}$
7: $N_d^{(c)} \leftarrow N_{d-1}^{(c)} + \mathbb{1}\{C_d = c\}$ for $c \in \{<Y, =Y, >Y\}$ `// update sufficient stats`
8: Sample $Z_d \sim \underset{\mathbb{Z}_{C_d}}{\text{trunc}} f_\theta$ `// as given in Theorem 5.1`
`// (Optional) assess/execute the break conditions in Theorem 5.2`

Output: $\{Z_1, \ldots, Z_D\}$

Using this data augmentation scheme, we can build Bayesian models with Poisson order statistic distributions

Case study: predicting flight times

Each observation $Y_i \in \mathbb{N}_0$ is the number of minutes for flight i between departure and arrival.

$$Y_i \overset{\text{ind.}}{\sim} \mathrm{MedPois}_{\mu_i}^{(D_{\mathrm{route}[i]})} \text{ where } \mu_i \overset{\text{def}}{=} a_{\mathrm{orig}[i]} + b_{\mathrm{dest}[i]} + c_{\mathrm{route}[i]}\,\mathrm{dist}_{\mathrm{route}[i]}$$

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heterogeneous dispersion for each route k

$D_k \sim$ OddBinomial(D_{max}, ρ) such that $D_k \in \{1,3,5,\ldots,D_{\text{max}}\}$

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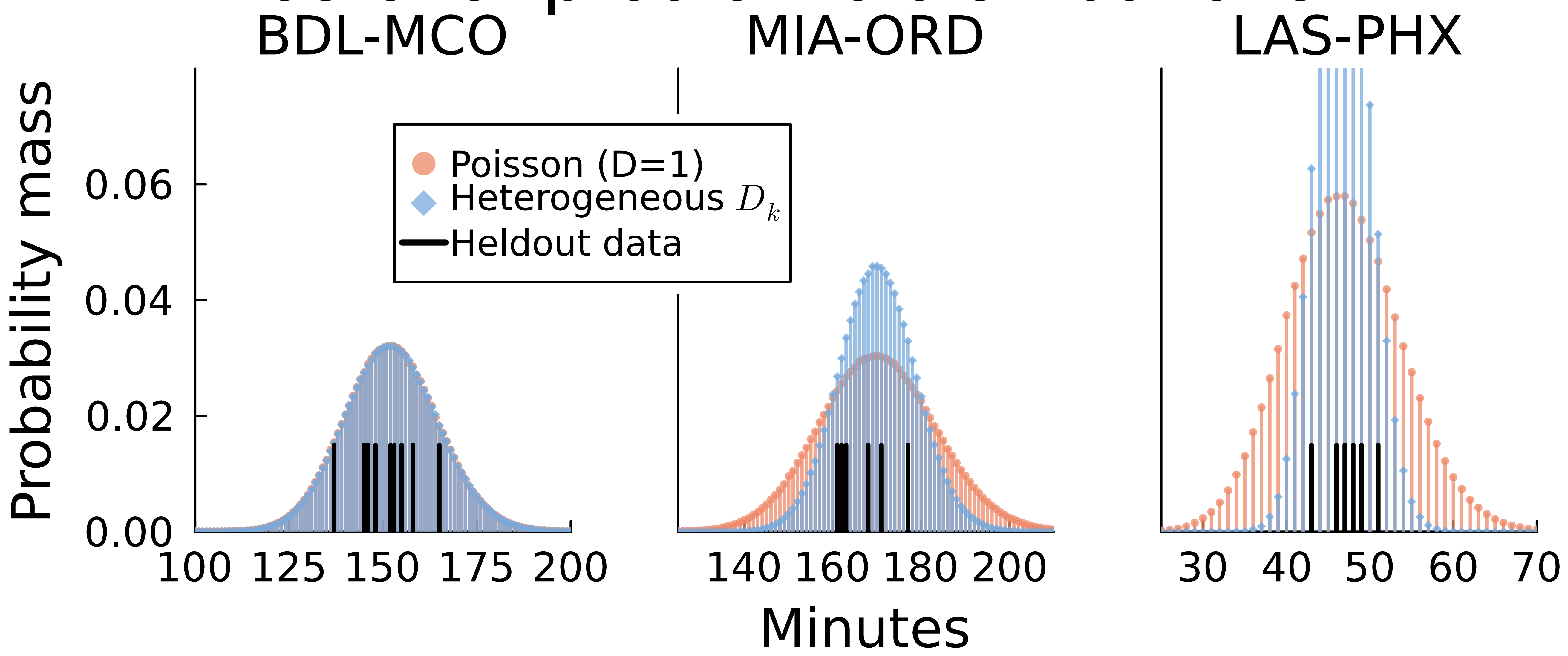
Posterior predictive distributions



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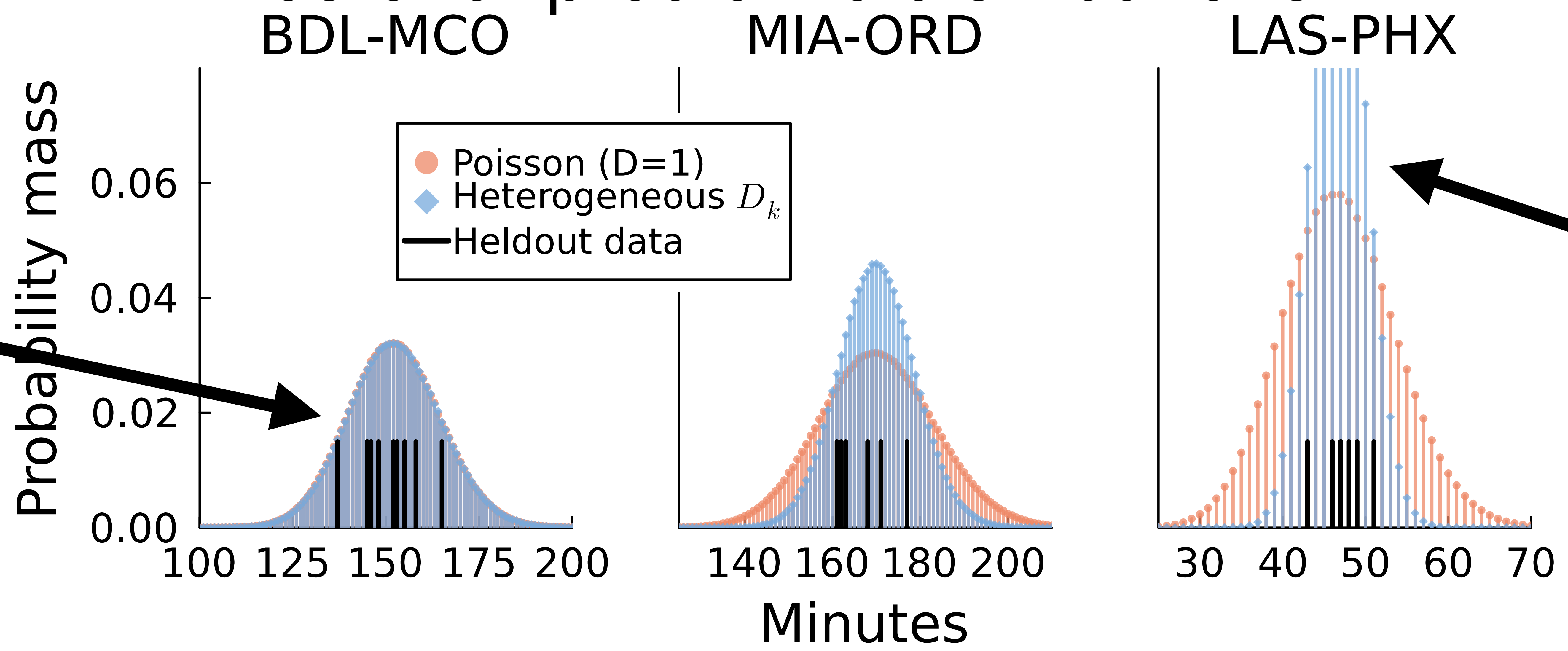
heterogeneous dispersion for each route k

$D_k \sim \text{OddBinomial}(D_{\text{max}}, \rho)$ such that $D_k \in \{1,3,5,\ldots, D_{\text{max}}\}$

Posterior predictive distributions

$D_k \approx 1$



$D_k \approx 7$

Case study: forecasting COVID-19 cases

Each $Y_{i,t}$ is the cumulative number of COVID-19 cases at time t in county i.

$$Y_{i,t} \sim \text{MedPois}_{\mu_{i,t}}^{(D_{i,t})} \quad \text{where} \quad \mu_{i,t} \;:=\; Y_{i,t-1} \;+\; \log(\text{pop}_i)\left(\varepsilon + \alpha \sum_{k=1}^{K} \theta_{i,k}\phi_{k,t}\right)$$

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heterogeneous
dispersion

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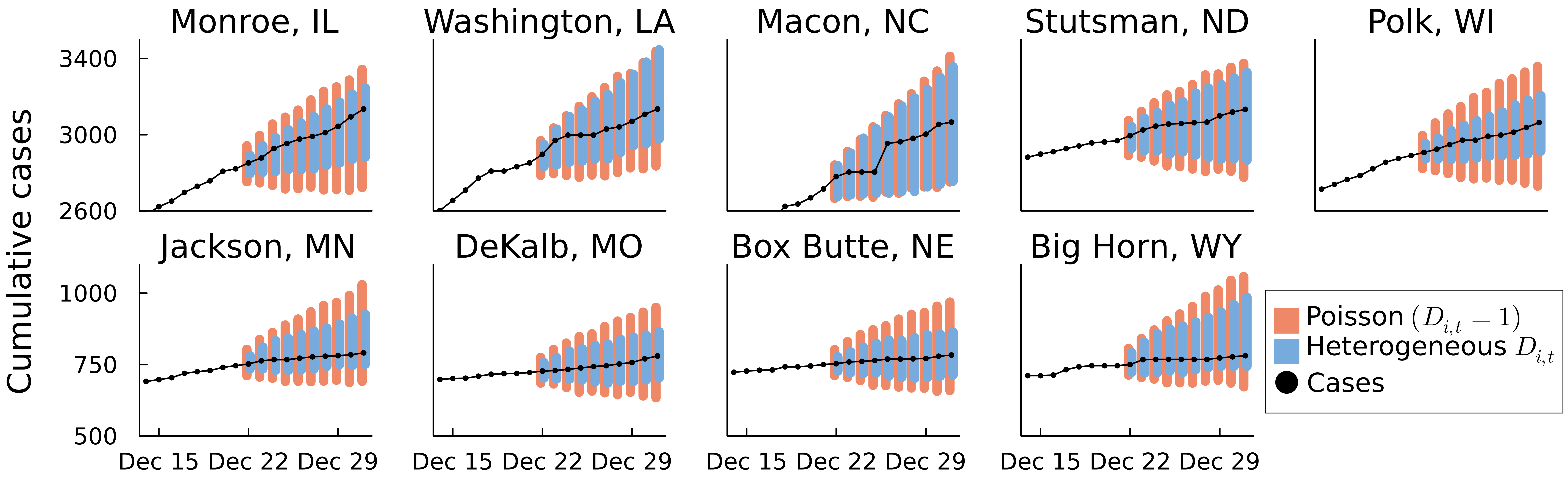

More precise probabilistic forecasts

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To forecast, $\phi_{k,t}$ evolves over time: $\phi_{k,t} \sim \Gamma(a^{(\phi)} + b^{(\phi)}\phi_{k,t-1},\, b^{(\phi)})$

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Data augmentation from [Acharya et al. 2015]

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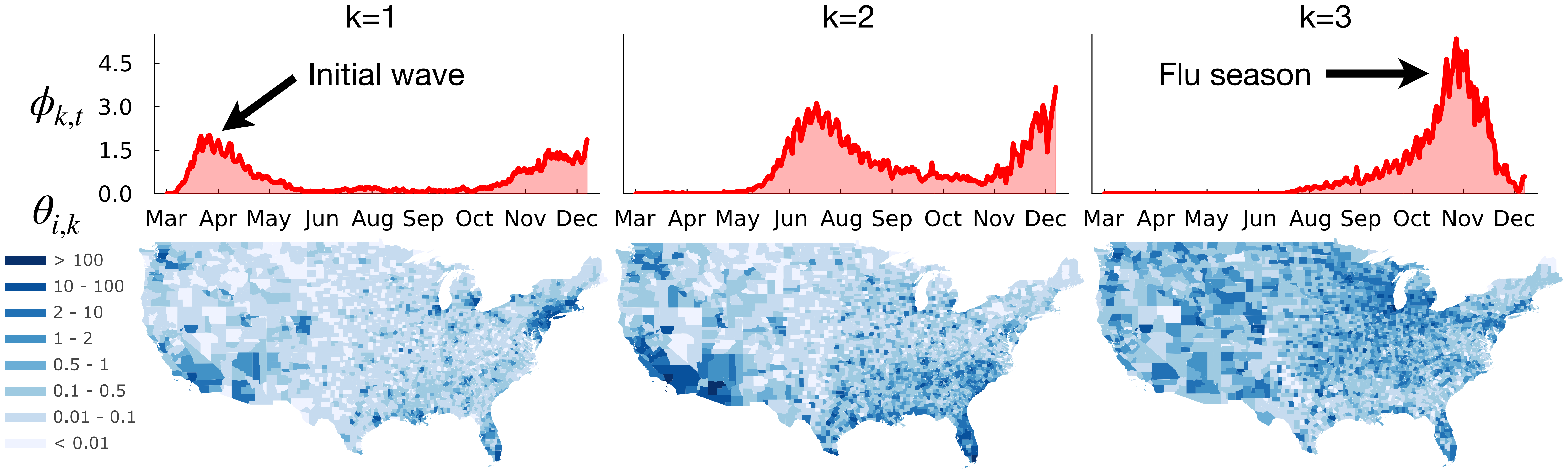

Smooth, interpretable structure

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For heterogeneous dispersion for each data point:

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where $\beta_{i,q} \sim \mathcal{N}(0,1)$ and $\tau_{q,t} \sim \mathcal{N}\left(a^{(\tau)} + b^{(\tau)}\tau_{q,t-1},\, 1/\lambda_t^{(\tau)}\right)$

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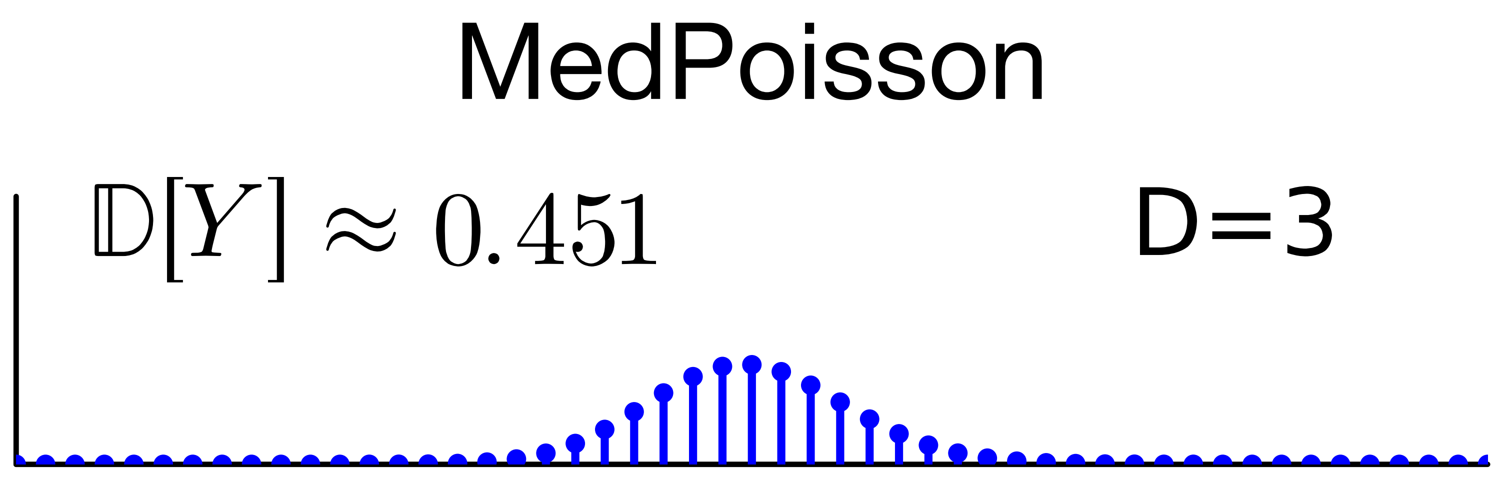
General tools enable modularity and increasing complexity

Summary



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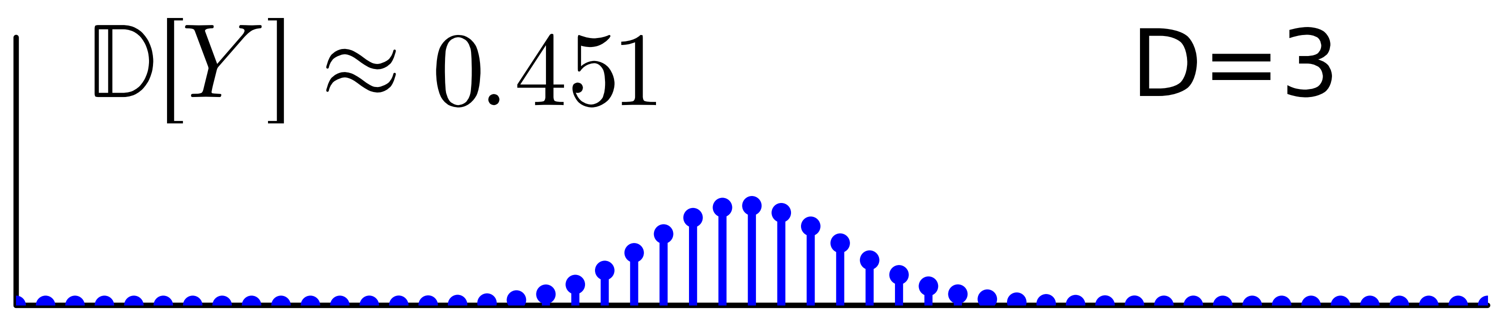
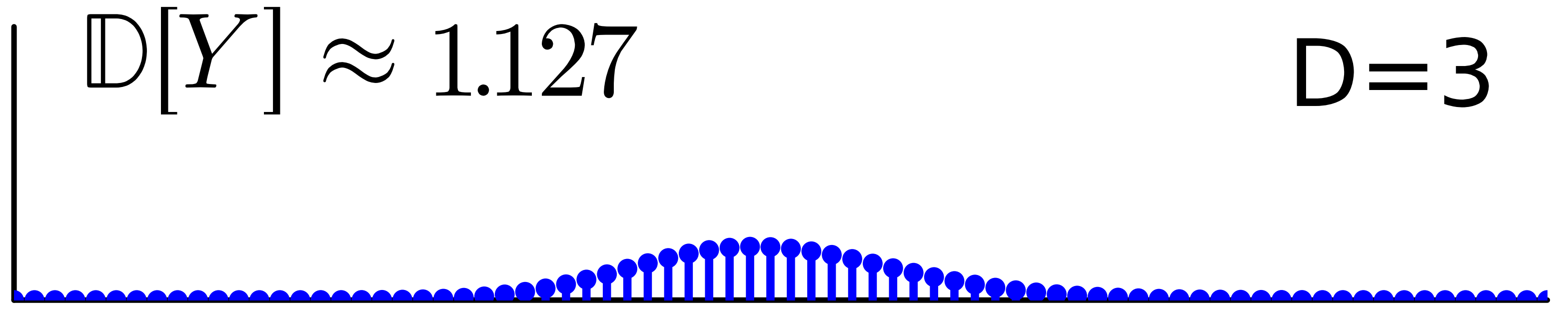
Summary

MedPoisson



MedNegativeBinomial



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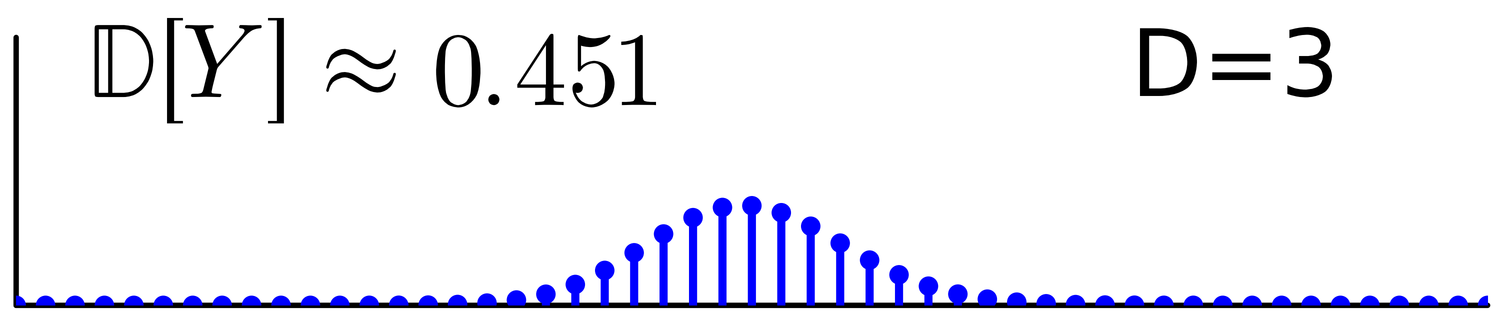

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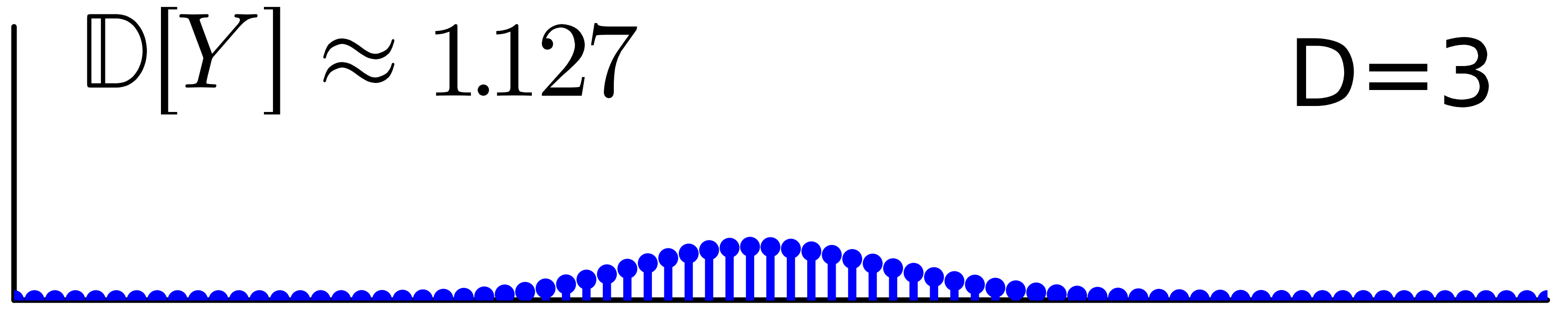

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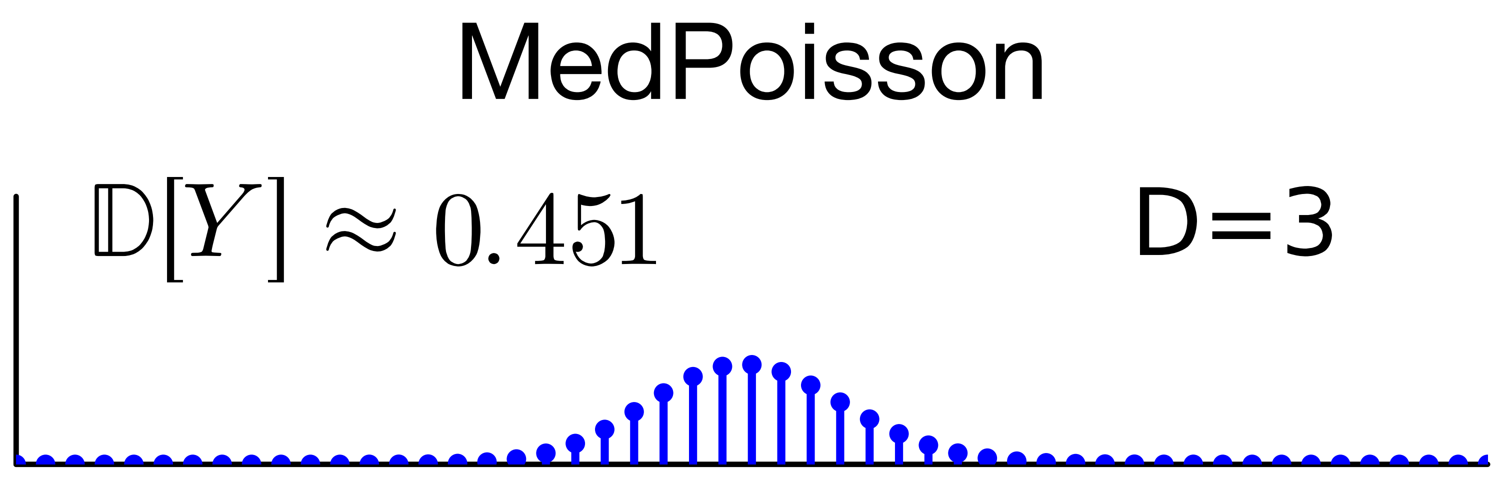

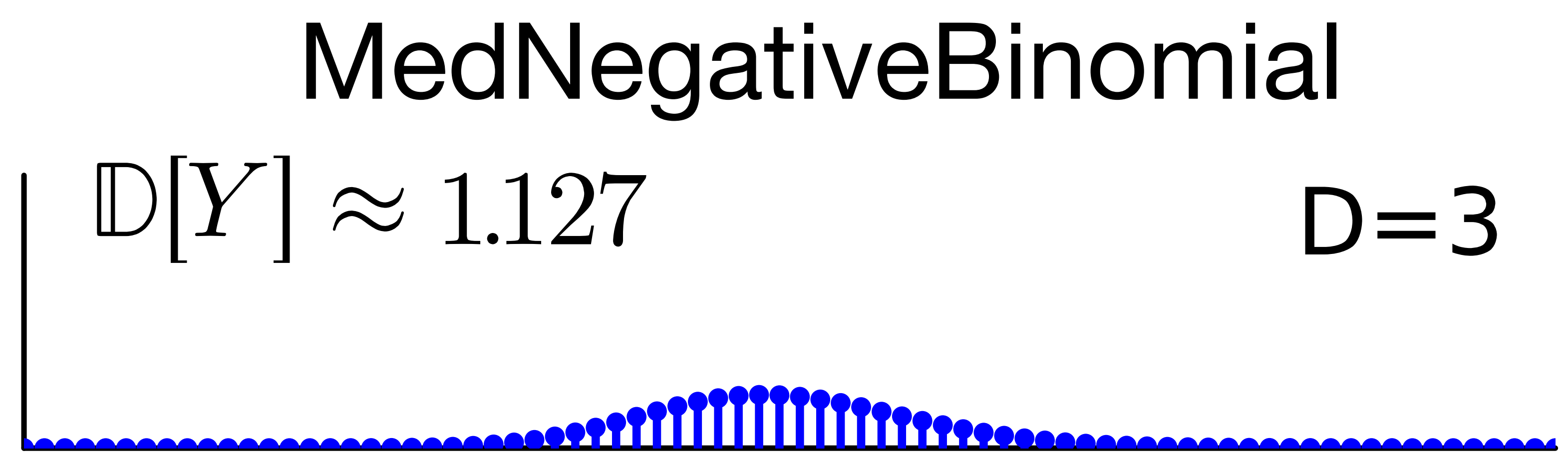

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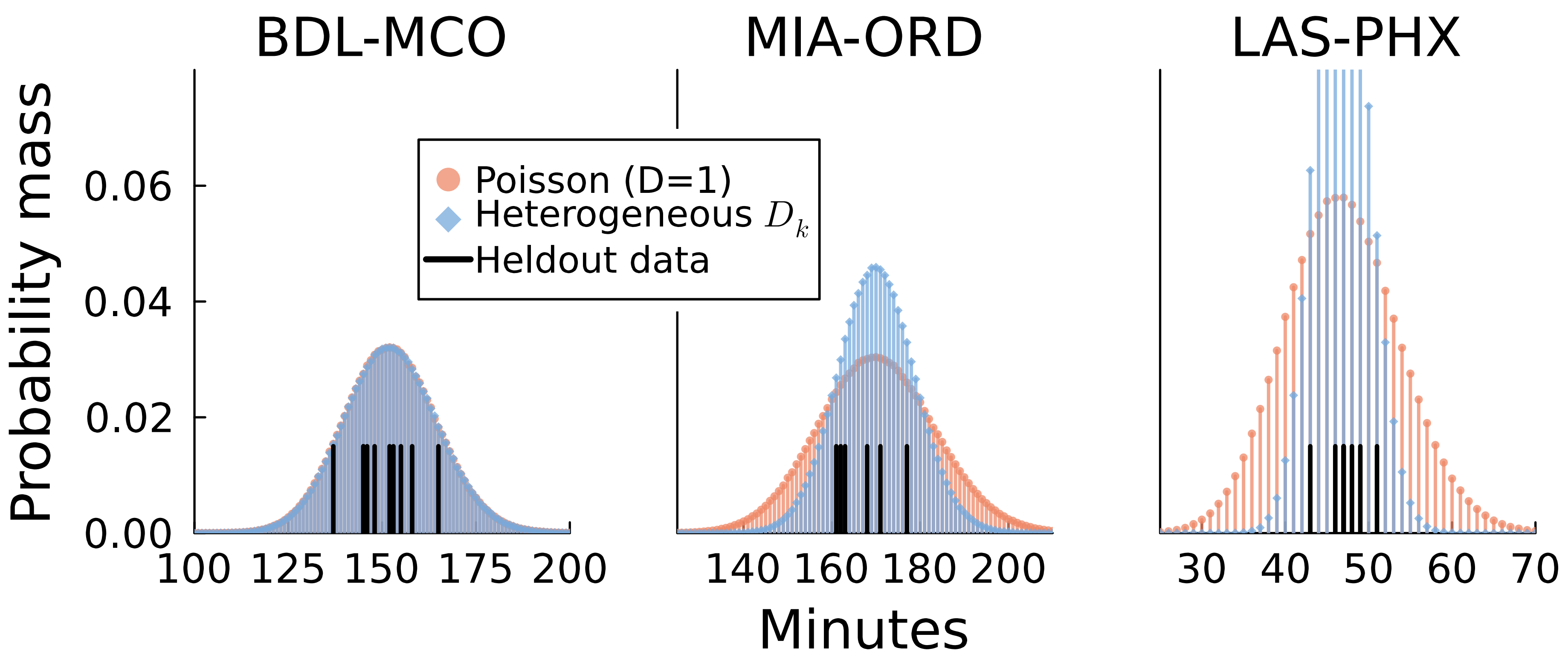

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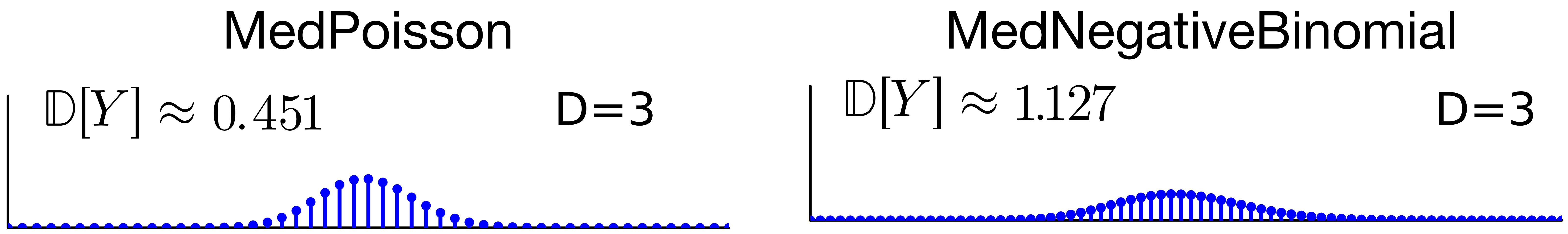

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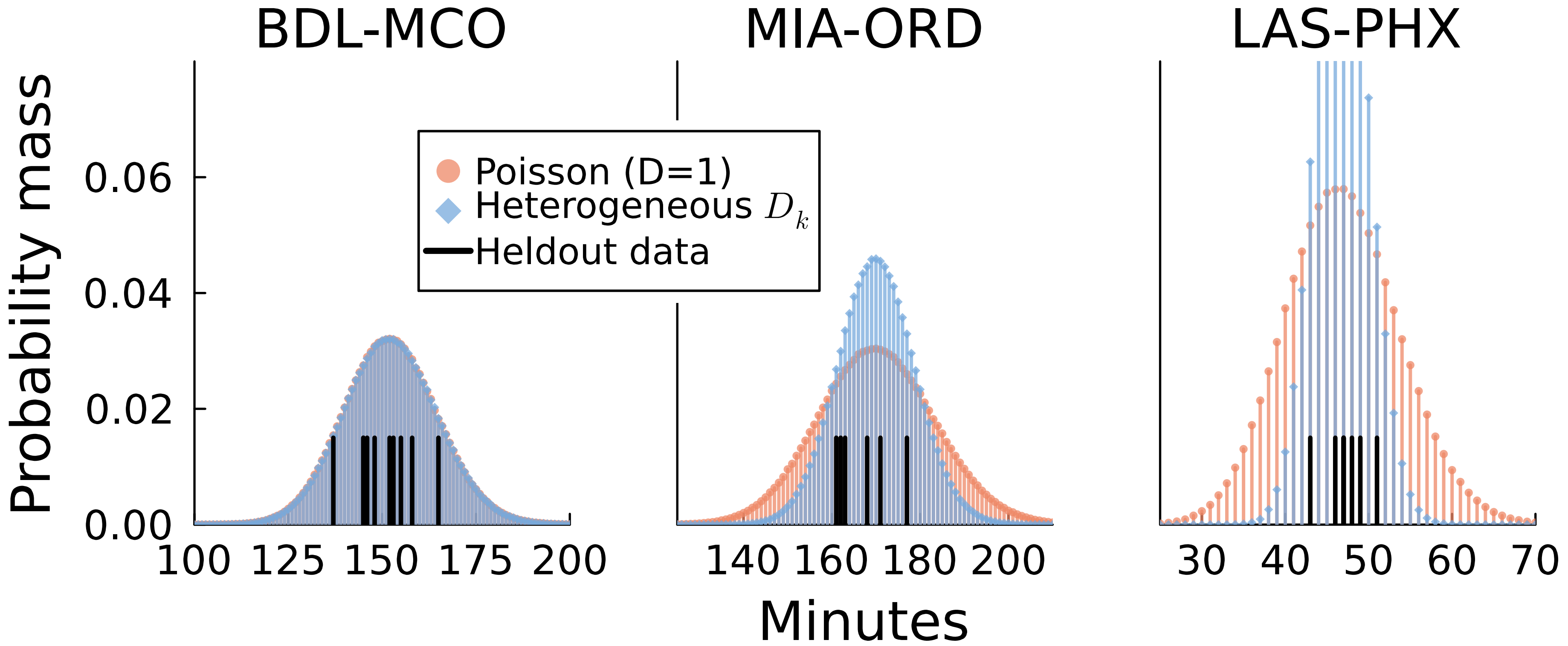
Thank you!



Exact form for support probabilities

$$n_1 = \sum_{s=1}^{d-1} 1\{C_s = <Y\}$$

$$n_2 = \sum_{s=1}^{d-1} 1\{C_s = =Y\},$$

$$n_3 = \sum_{s=1}^{d-1} 1\{C_s = >Y\}$$

For $c \in \{<Y, =Y, >Y\}$:

$$p_d^{(c)} = \frac{P_\theta(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}, C_d = c)}{P_\theta(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1})} P_\theta(C_d = c)$$

$$P_\theta(Z^{(r,D)} = Y \mid \mathbf{C}_{1:d-1}) = \begin{cases} 0 & \text{if } n_1 \geq r \text{ or } n_3 \geq D - r + 1 \\ F_\theta^{(r-n_1-n_2,\, D-d+1)}(Y) - F_\theta^{(r-n_1,\, D-d+1)}(Y-1) & \text{else if } n_2 < \min(r - n_1, D - n_3 - r + 1) \\ 1 - F_\theta^{(r-n_1,\, D-d+1)}(Y-1) & \text{else if } r - n_1 \leq n_2 < D - n_3 - r + 1 \\ F_\theta^{(r-n_1-n_2,\, D-d+1)}(Y) & \text{else if } D - n_3 - r + 1 \leq n_2 < r - n_1 \\ 1 & \text{otherwise} \end{cases}$$

Data augmentation

For a random variable $Y \sim f_\theta$, we can update θ with Poisson data augmentation if

1. We can represent Y as being generated from a latent Poisson:

$$Y \mid Z \sim h(Y \mid Z) \text{ where } Z \sim \text{Poisson}(\mu)$$

2. We can sample from the conditional distribution in closed-form

$$Z \mid Y \sim g(Z \mid Y, \theta)$$

Once we sample $Z \sim g(Z \mid Y, \theta)$, inference can proceed via Z, which is Poisson

For example, if $Y \sim$ NegativeBinomial(r,p), then if we sample $Z | Y \sim$ CRT(Y, r) then Z is marginally Poisson $Z \sim$ Poisson $\left(r\frac{1}{1-p}\right)$

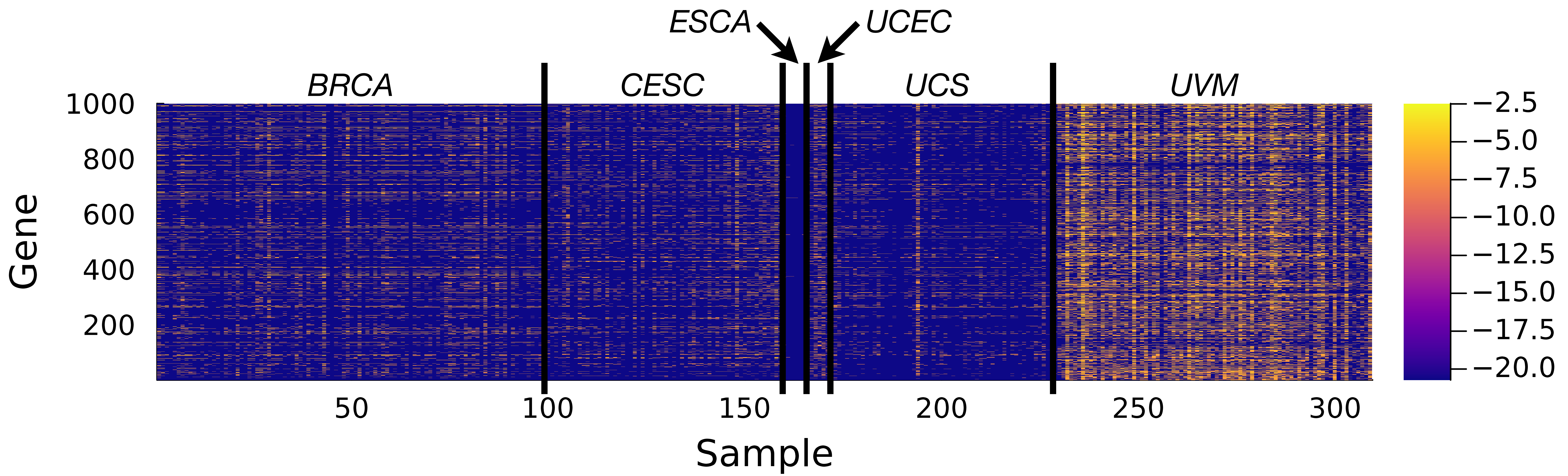
Case study: RNA-sequencing data

Is there evidence of conditional underdispersion in RNA-sequencing?

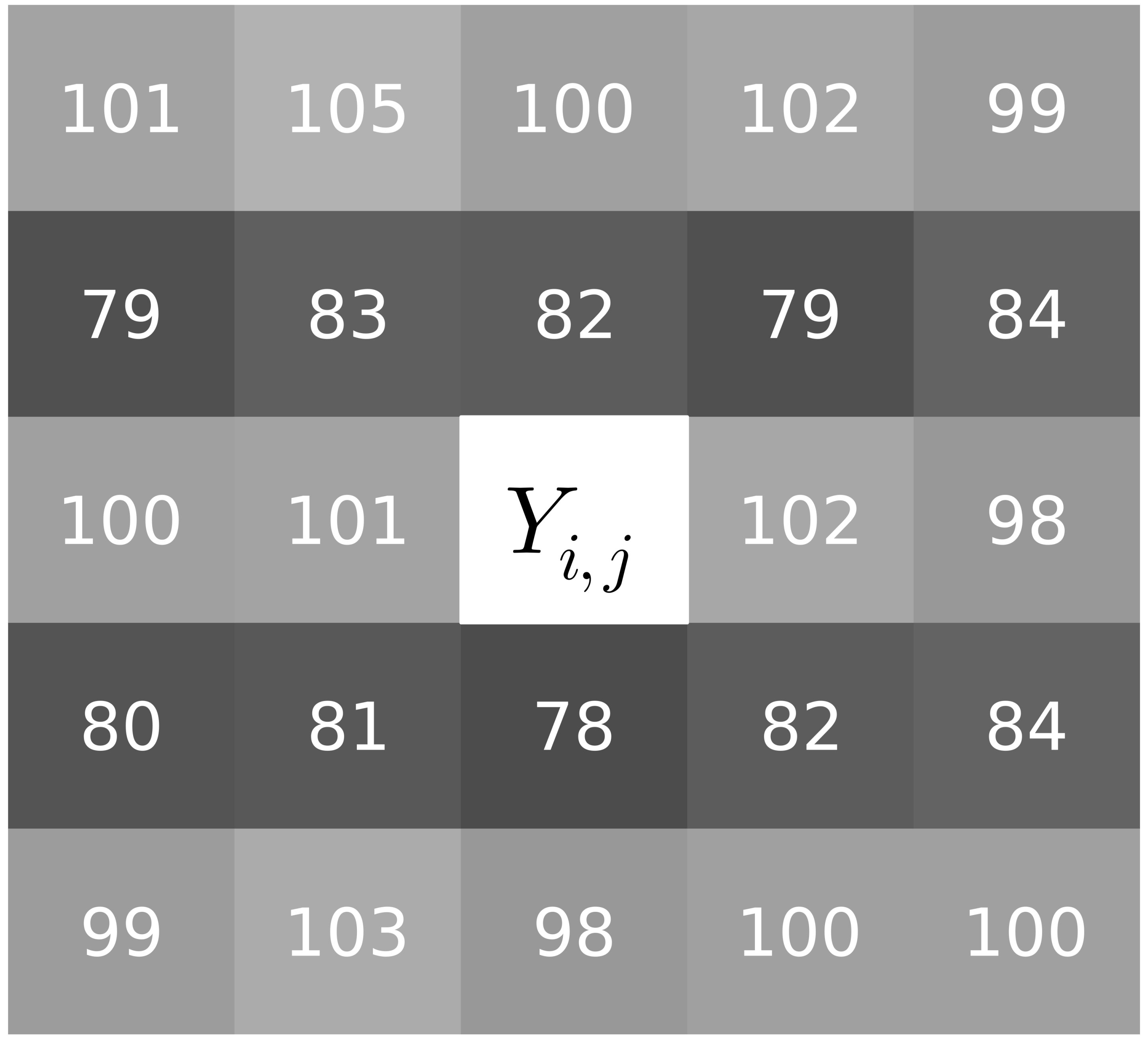
$Y_{i,j}$ is the read-count of gene j for subject i

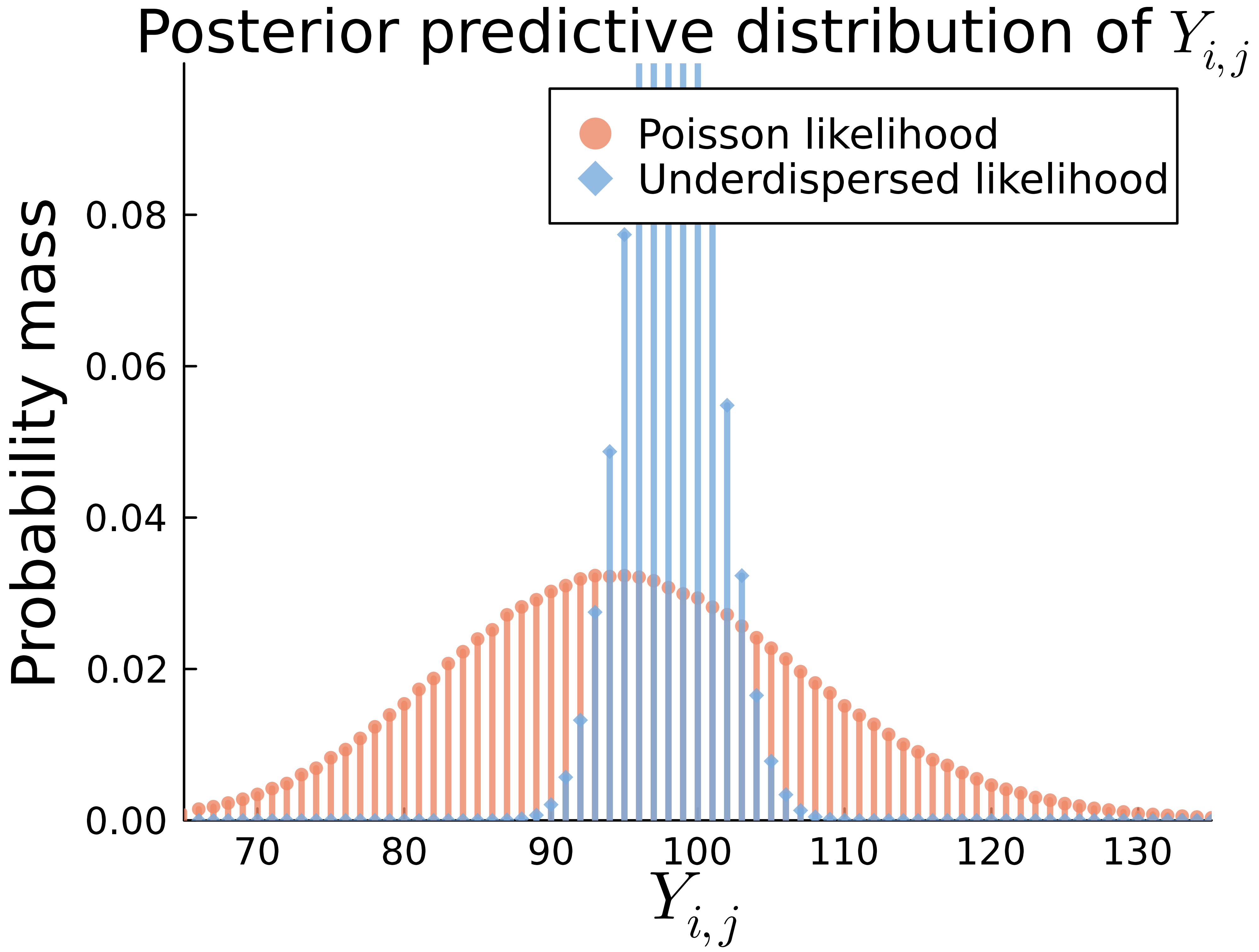
$$Y_{i,j} \sim \text{MedNB}^{(D_{i,j})}_{\alpha_{i,j}, p_j} \quad \text{where } \alpha_{i,j} := \sum_{k=1}^{K} \theta_{i,k}\phi_{k,j}$$

Posterior log-probability of underdispersion for each sample-gene pair



Why model underdispersion?





Capturing underdispersion allows us to make more precise probabilistic predictions than a Poisson likelihood would allow