# Lecture 1: AX and the column space of A

How to look at matrix times a vector $AX$:

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 5 & 7 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1. standard way: dot product of row x, get a component at a time——a low level way
2. vector wise——right way

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 5 & 7 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}$$

think of a matrix as a whole thing, not just a bunch of m times n numbers, but a thing——a matrix multiplies a vector to give another vector. $AX$ is a combination of the columns of $A$.

## Column Space:

A more general situation: take a matrix $A$, take all $X$s, and imagine all the outputs, what's that look like? It's the column space of $A$, denoted as $C(A)$, which is a space depends on $A$.

As to $A$ = $\begin{bmatrix} 1 & 3 & 8 \\ 1 & 3 & 8 \\ 1 & 3 & 8 \end{bmatrix}$, $C(A) = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$ for all X, is a line, and we can see rank(A) = 1.

> The Rank of a matrix A is the dimension of its column space.

Simply put, the columns of A (a set of vectors) form a space, which is the column space of A. The columns are the bases of that space.

The first A's rank is two, because the third column is a combination (multiply and add) of the others. The independent columns would be bases for the column space.

Rank-1 matrixs are the building blocks of linear algebra, which would be elaborated more detailed after. A special way to write those rank one matrixes is a column times a row:

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 3 & 8 \\ 1 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 8 \end{bmatrix}$$

## Independent Columns

### Find a basis for the column space of $A$ and Factor $A$ into $C$ times $R$

Basis of a space:

- a set of independent columns (No column is a combination of others)
- their combinations have to fill the space

To find a basis of $A$, the goal is to create a matrix $C$ whose columns directly from $A$, but not include any column that is a combination of previous columns. A natural construction of $C$:

- If column 1 of $A$ is not all zero, put it into the matrix $C$.
- If column 2 of $A$ is not a multiple of column 1, put it into $C$.
- If column 3 of $A$ is not a combination of column 1 and 2, put it into $C$. $Continue$

The final $C$ will be a "basis" for the column space of $A$.

Example: If $A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$, then $C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$. The column 3 of $A$ is the combination of 1 and 2, so it is dropped. It's obvious that the number of $C$'s columns is the rank of $A$, also the rank of $C$. **It counts the independent columns**, which is mentioned above.

Again:

> The column rank = The number of independent columns.
> **The rank of a matrix is the dimension of its column space.**

The matrix $C$ connects to $A$ by a third matrix $R$: $A = CR$. It's an important **factorization**(矩阵分解) of $A$.

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = CR$$

We can see it in the right way of matrix times vector in the beginning:

- column $i$ of $A$ is $C$ times column $i$ of $R$, take column 1 for example: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 * \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 * \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

## First great theorem: The column rank = The row rank

Next, We are going to prove a Great Theorem in Linear Algebra:

> The number of *independent columns* equals the number of *independent rows*.
>
> The column rank = The row rank

Obviously, $R$'s rank is equal to $C$'s rank, which is $A$'s column rank. Next, we need to prove $R$ is a basis for $A$'s row space:

- check1: check the vectors are independent
- check2: check their combinations produce all three of $A$'s rows

Matrix multiplication in another way: taking combinations of the rows of $R$, and we can get $A$ from the rows of $R$. Same to the columns, row $i$ of $A$ if row i of $C$ times $R$, take row 1 for example: $\begin{bmatrix} 1 & 3 & 8 \end{bmatrix} = 1 * \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} + 3 * \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$.
——The wonderful thing about matrix multiplication is that you can do it a lot of ways, it comes out the same every way, but each way tells you some different things.

So $R$ is a basis of the row space of $A$. The column space and row space of $A$ both have dimension 2, with 2 basis vectors——columns of $C$ and rows of $R$. Proof down.

In conclusion, the proof is exactly to look at the multiplication $CR$ in two ways. First, look at it as combinations of columns of $C$ to give $A$'s columns; Second, look as combinations of rows of $R$ to produce $A$'s rows. So the factorization $A = CR$ is the key idea.

$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ is a famous matrix in linear algebra, called the **row reduced-echelon form** of $A$ , with an identity there and other columns.

A big factorization for data science is "**SVD**" of A, when the first factor $C$ has $r$ orthogonal columns and the second factor $R$ has $r$ orthogonal rows. ——CUR, to be introduced in follow-up lectures.

How to deal with a matrix of size $10^5$, it is hard to be put into the fast memory.——sample a matrix

ABCx is also in the C(A), cause it's A times things.

# matrix multiplication:

A: m by n matrix, B: n by p matrix.

how to see $A$ times $B$:

- dot products of rows and columns——low level for beginners
- columns of $A$ times rows of $B$ is a high level way:

(m, n) (n, p) = (m, p) = sum of (m, 1) (1, p) ——Rank 1's matrixs are building blocks.

col1 times row1 + colk times row k + coln times rown

more in next lecture.

# Conclusion

column space of $A$.

a factorization: $A = CR$

The theorem: column rank = row rank