Introduction to Binary Decision Diagram

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Outlines

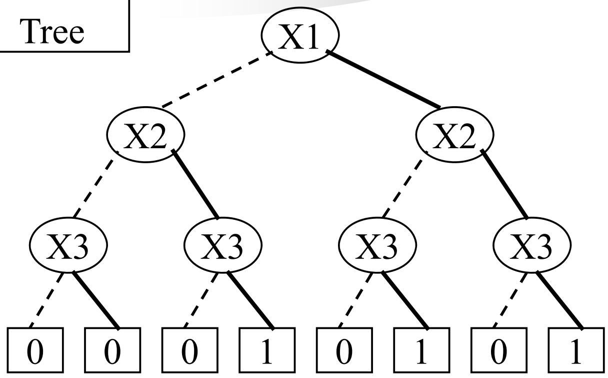
- Representing Boolean Functions
  - Decision graph structure
  - Reduction to canonical form
  - Effect of variable ordering
  - Variants to reduce storage
- Algorithms
  - General framework
  - Basic operations
    - » Restriction (Cofactor)
    - » If-Then-Else
  - Derived operations
  - Computing functional properties

Decision Structures

Truth Table

X1	X2	X3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

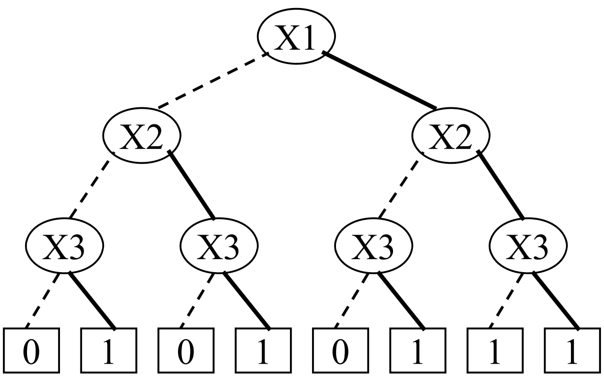
Decision Tree



- Vertex represents decision
- Follow dashed line for value 0
- Follow solid line for value 1
- Function value determined by leaf value

Binary Decision Diagram (BDD)

$f = x_1 x_2 + x_3$



- - - - : 0

—— : 1

terminal node :

- attribute
  - value(v) = 0
  - value(v) = 1

nonterminal node :

- index(v) = i
- two children
  - low(v)
  - high(v)

BDD

A BDD graph which has a vertex v as root corresponds to the function F_v :

(1) If v is a terminal node :

a) if value(v) is 1, then $F_v = 1$

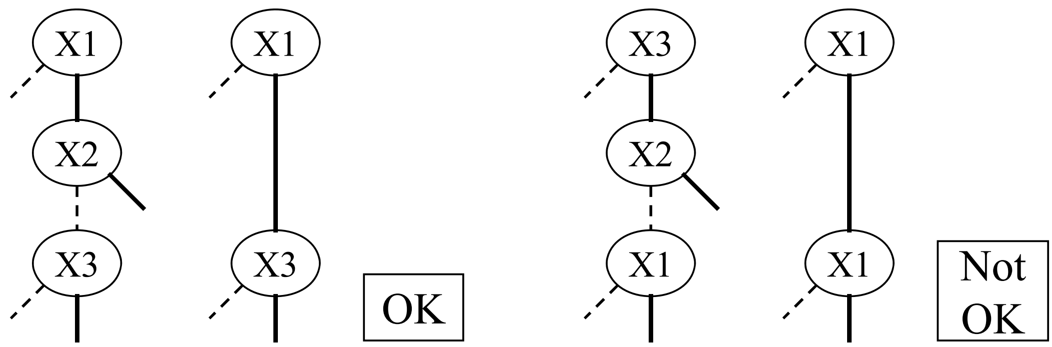
b) if value(v) is 0, then $F_v = 0$

(2) If F is a nonterminal node (with index(v) = i)

$$F_v(x_1, \ldots, x_n) = x_i' F_{low(v)}(x_{i+1}, \ldots, x_n) + x_i F_{high(v)}(x_{i+1}, \ldots, x_n)$$

Variable Ordering

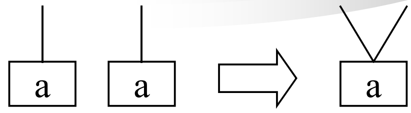
- Assign arbitrary total ordering to variable
  e.g. X1 < X2 < X3
- Variable must appear in ascending order along all paths

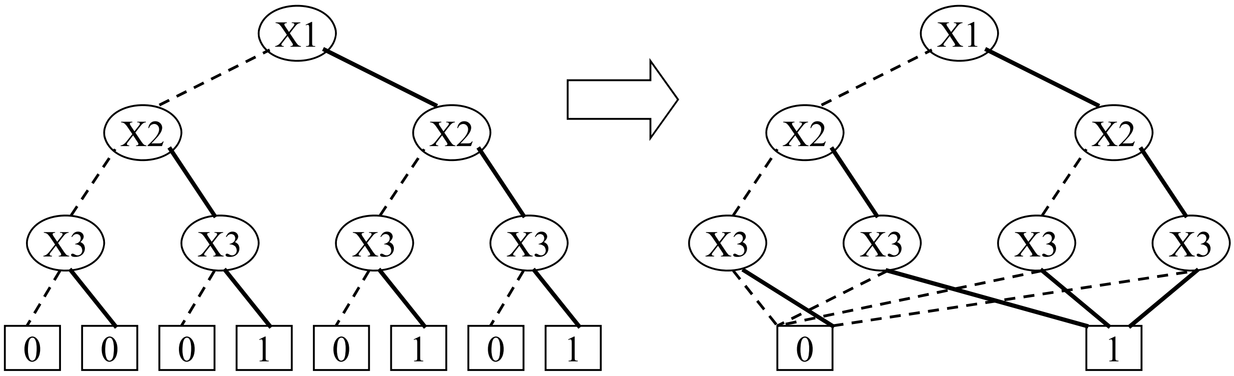


- Properties
  - No conflicting variable assignments along path
  - Simplifies manipulation

Reduction Rule #1

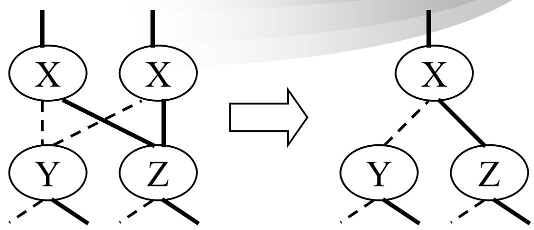
- Merge equivalent leaves

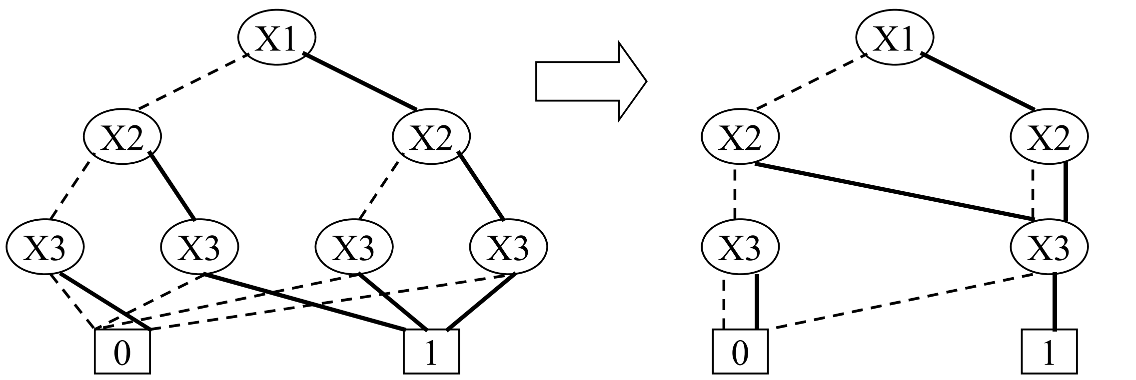




Reduction Rule #2

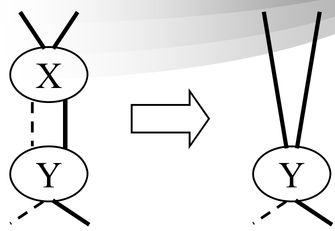
- Merge isomorphic nodes

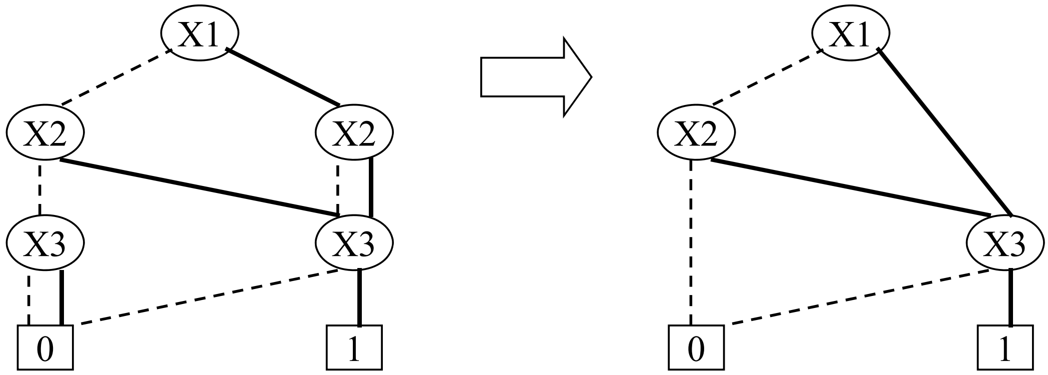




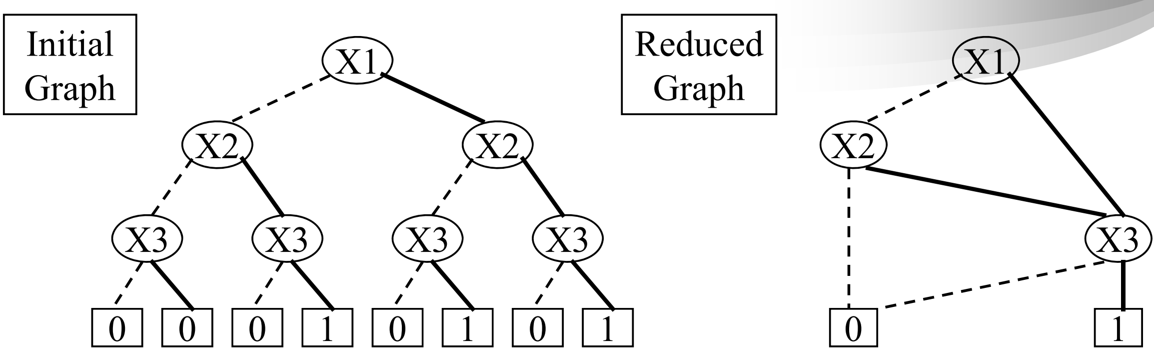
Reduction Rule #3

- Eliminate Redundant Tests





Example ROBDD

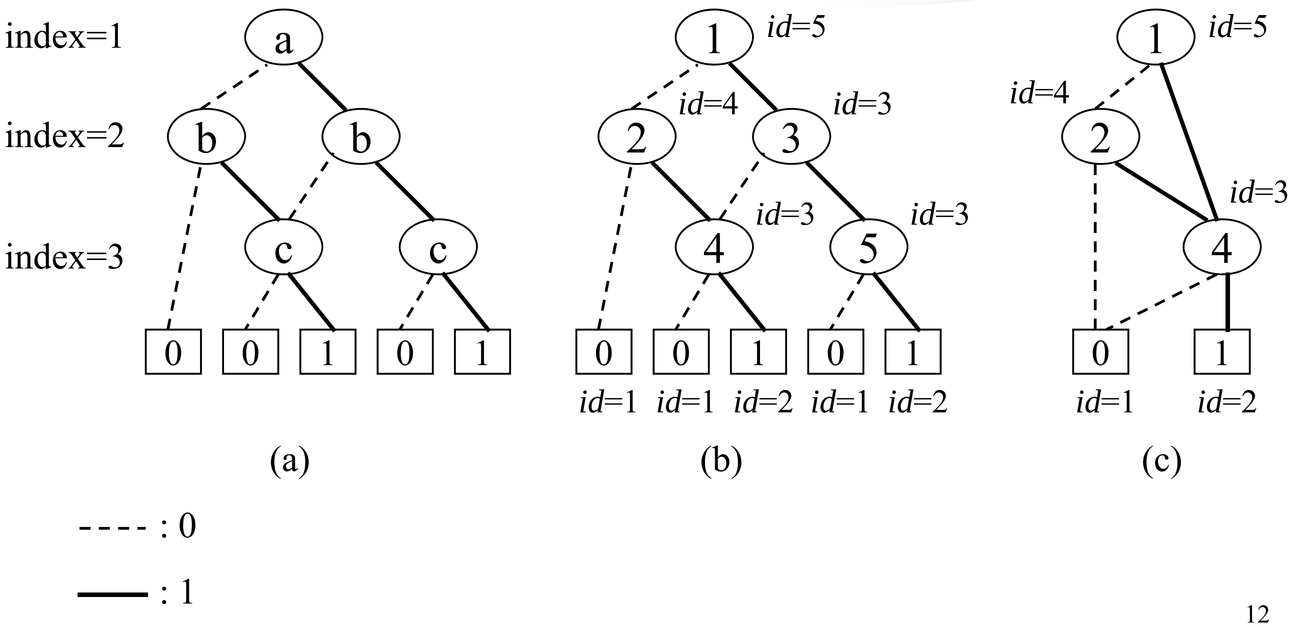


- Canonical representation of Boolean function for given variable ordering
  - Two functions equivalent iff graphs isomorphic
    - » can be tested in linear time
  - Desirable property : The simplest form is canonical

Reduce

- Visit OBDD bottom up and label each vertex with an identifier
- Redundancy
  - if id(low(v)) = id(high(v)), then vertex v is redundant
    $\Rightarrow$ set id(v) = id(low(v))
  - if id(low(v)) = id(low(u)) and id(high(v)) = id(high(u)), then set id(v) = id(u)
- A different identifier is given to each vertex at level i
- Terminated when root is reached
- An ROBDD is identified by a subset of vertices with different identifiers

Reduce



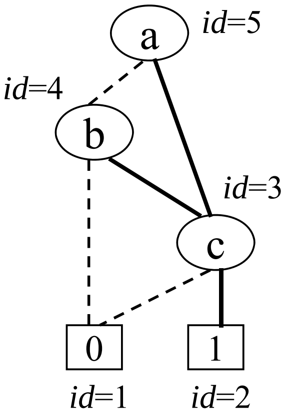
Construct ROBDD Directly

- Using a hash table called unique table
  - Contain a key for each vertex of an OBDD
  - Key : (variable, right children, left children)
  - Constructed bottom up
  - Each key uniquely identify the specific function
  - Look up the table can determine if another vertex in the table implements the same function

The Unique Table

- Represent an ROBDD
- A strong canonical form
- Check equivalence of two Boolean functions by comparing the corresponding identifiers
- Can represent multiple-output functions

Multi-Rooted ROBDD



Unique table

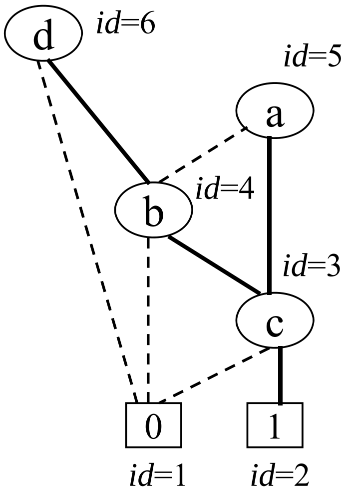
Key

Identifier	Variable	Right child	Left child
5	a	3	4
4	b	3	1
3	c	2	1

f = (a+b) c

variable order (a, b, c)

Multi-Rooted ROBDD



f is constructed first and is associated with *id*=5

g : *id*=6

Unique table

Key

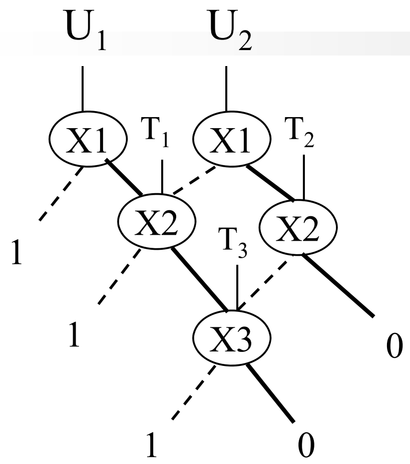
Identifier	Variable	Right child	Left child
6	d	4	1
5	a	3	4
4	b	3	1
3	c	2	1

f = (a+b) c

g = b c d

variable order (d, a, b, c)

The Unique Table

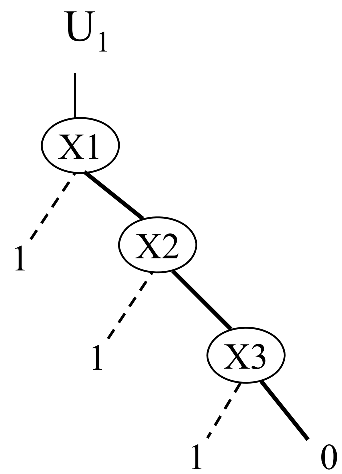


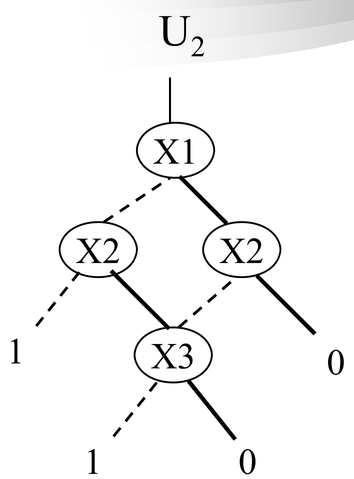
Hash Table Mapping
(X1, T1, 1) --> U1
(X1, T2, T1) --> U2
(X2, T3, 1) --> T1
(X2, 0 , T3) --> T2
(X3, 0 , 1) --> T3

true ↑ ↑ false

- Unique table : hash table mapping (Xi, G, H) into a node in the DAG
  - before adding a node to the DAG, check to see if it already exists
  - avoids creating two nodes with the same function
  - strong canonical form : pointer equality determines function equality

Non-Shared ROBDD

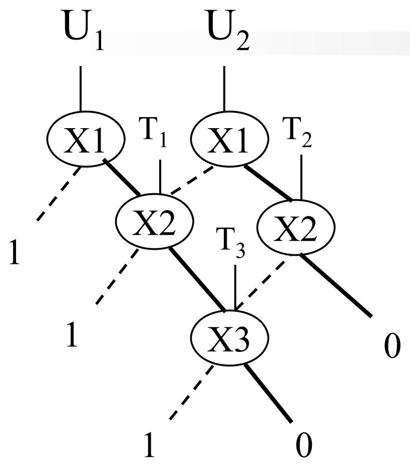




U1 = X1’ + X2’ + X3’

U2 = X1’X2’ + X1’X3’

Multi-Rooted (Shared) ROBDD



$U_1 = X_1'+X_2'+X_3' = (X_1, T_1, 1)$
$U_2 = X_1'X_2'+X_1'X_3' = (X_1, T_2, T_1)$

External functions
User functions

$T_1 = X_2'+X_3' = (X_2, T_3, 1)$
$T_2 = X_2'X_3' = (X_2, 0, T_3)$
$T_3 = X_3' = (X_3, 0, 1)$
$0 = (X_\infty, 0, 0)$
$1 = (X_\infty, 1, 1)$

Internal functions

- A DAG node F is represented by a tuple (Xi, G, H)
  - Xi is called the top variable of F
  - node (Xi, G, H) represents the function ite(Xi, G, H) = XiG + Xi'H
- DAG contains both external and internal functions

Separated vs. Shared

- Separated
  - 51 nodes for 4-bit adder
  - 12481 nodes for 64-bit adder
  - Quadratic growth
- Shared
  - 31 nodes for 4-bit adder
  - 571 nodes for 64-bit adder
  - Linear growth

Maintaining Shared ROBDD

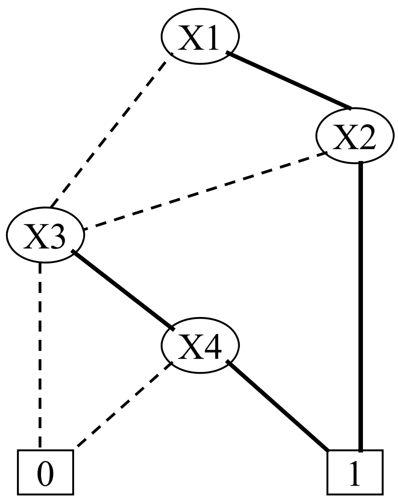
- Storage Model
  - Single, multiple-rooted DAG
  - Function represented by pointer to node in DAG
  - Maintain Unique (hash) table to keep canonical
- Storage Management
  - User cannot know when storage for node can be freed
  - Must implement automatic garbage collection
- Algorithmic Efficiency
  - Functions equivalent iff pointer equal
    - » if (p1 == p2) ...
  - Can test in constant time

Ordering Effects

- The size of ROBDD depends on the ordering of variables

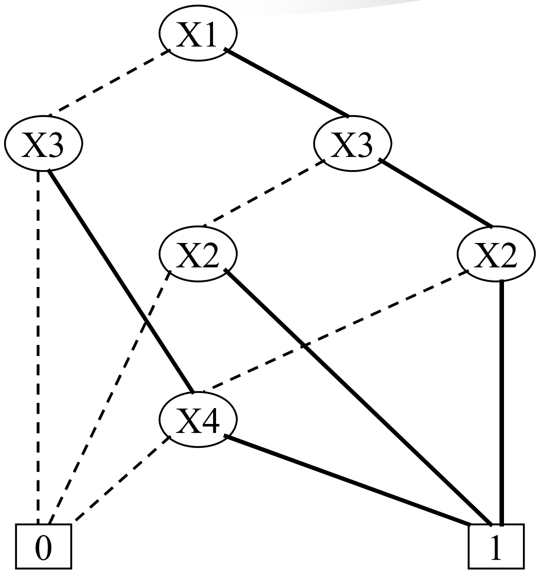
ex : $x_1x_2 + x_3x_4$

$x_1 < x_2 < x_3 < x_4$



Ordering Effects (cont'd)

$x_1 < x_3 < x_2 < x_4$



Sample Function Classes

Function Class	Best	Worst	Ordering Sensitivity
ALU (Add/Sub)	Linear	Exponential	High
Symmetric	Linear	Quadratic	None
Multipication	Exponential	Exponential	Low

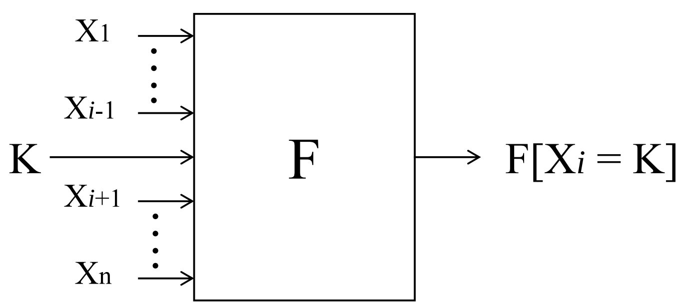
- General Experience
  - Many tasks have reasonable ROBDD representations
  - Algorithms remain practical for up to 100,000 vertex ROBDD
  - Heuristic ordering methods generally satisfactory

Symbolic Manipulation

- Strategy
  - Represent data as set of ROBDDs
    - » with identical variable orderings
  - Express solution method as sequence of symbolic operations
  - Implement each operation by ROBDD manipulation
- Algorithmic Properties
  - Arguments are ROBDDs with identical variable orderings
  - Result is ROBDD with same ordering
  - “Closure Property”
- Two Basic Operations
  - Restriction
  - If-Then-Else

Restriction Operation

- Concept
  - Effect of setting function argument Xi to constant K(0,1)
  - Also called Cofactor operation

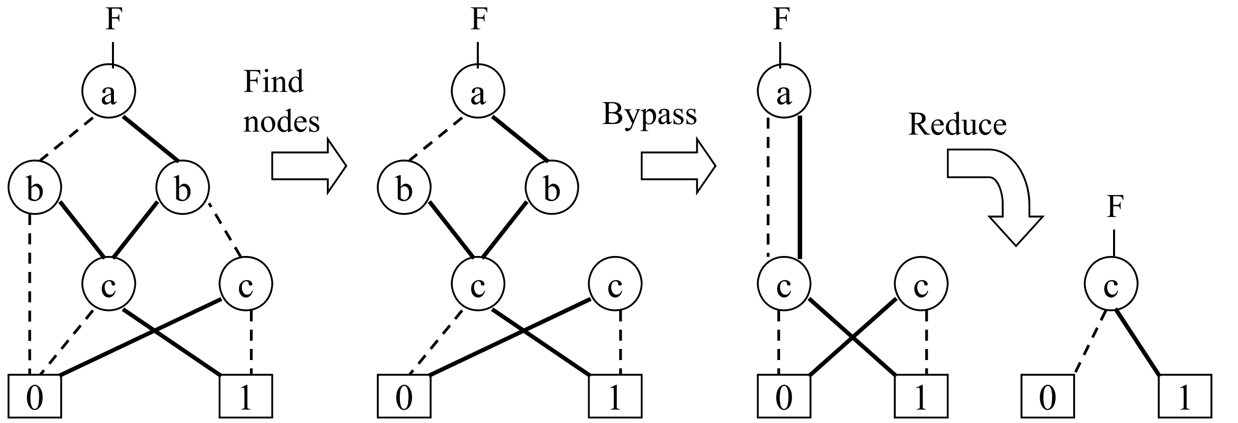


- Implementation
  - Depth-first traversal
  - Complexity near-linear in argument graph size

Restriction Algorithm

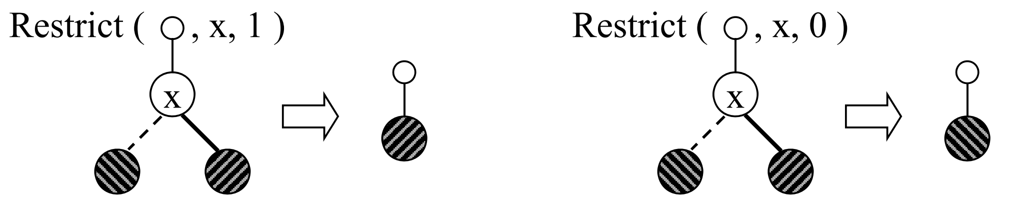
Restrict (F, x, k)
- Bypass any nodes for variable x
  - Choose Hi child for k = 1
  - Choose Lo child for k = 0
- Reduce result

e.g. Restrict variable b to 1

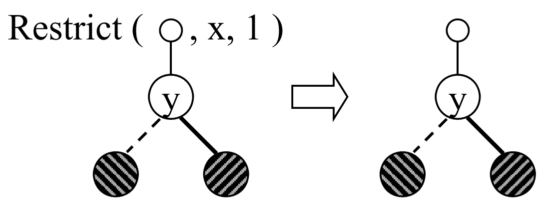


Special cases of Restriction

- Case 1 : Restrict on root node variable



- Case 2 : Restrict on variable less than root node
  - e.g. x < y



If-Then-Else Operation

- Concept
  - Basic technique for building ROBDD from network or formula
- Argument **I** (if), **T** (then), **E** (else)
  - Functions over variables X
  - Represented as ROBDDs
- Result
  - ROBDD representing composite function
  - IT + I'E
- Implementation
  - combination of depth-first traversal and dynamic programming
  - Worst case complexity : product of argument graph sizes

If-Then-Else Algorithm

- Recursive Formulation

  ITE (I, T, E) = x ITE(I[x=1], T[x=1], E[x=1]) +
  x'ITE(I[x=0], T[x=0], E[x=0])
- General Algorithm
  - Select top root variable x of I, T and E
  - Compute restrictions
    - » Guaranteed to be one of special cases
  - Apply recursively to get results Lo and Hi
  - Still remain canonical form
- Termination Conditions
  - I = 1 ==> Return T
  - I = 0 ==> Return E
  - T = 1, E = 0 ==> Return I
  - T = E ==> Return T

An ITE Example

- Given f = ab + bc + ac, g = c under the order a < b < c

ITE (f, g, 0)

= ITE[***a***, ITE(f(a=1), g(a=1), 0), ITE(f(a=0), g(a=0), 0)]

= ITE[***a***, ITE(b+bc+c, c, 0), ITE(bc, c, 0)]

b=1 *b=0*

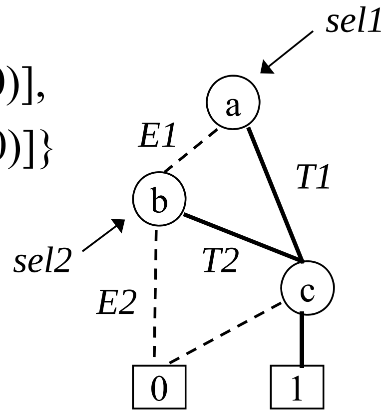
= ITE{ ***a***, ITE[***b***, ITE(1, c, 0), ITE(c, c, 0)],

ITE[***b***, ITE(c, c, 0), ITE(0, c, 0)]}

= ITE[***a***, ITE(b, c, c), ITE(b, c, 0)]

= ITE[***a***, c, ITE(b, c, 0)]

sel1 *T1* *E1=sel2* *T2*



Algorithmic Issues & Derived Operations

- Efficiency
  - Maintain computed table and unique table to increase efficiency
  - Worst case complexity product of graph sizes for I, T, E

- Derived operations
  - Express as combination of If-Then-Else and Restrict
  - Preserve closure property
    - » Result is a ROBDD with the same variable ordering

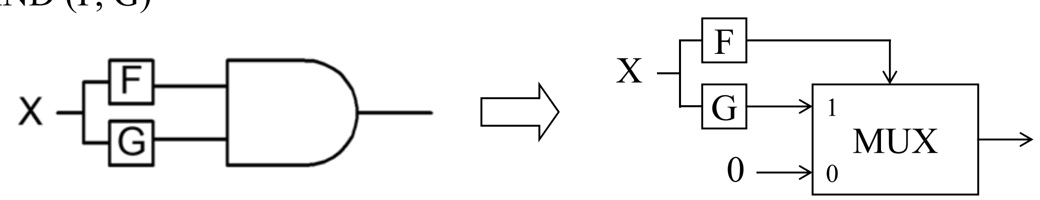
Detailed ITE Algorithm

```
ITE(f, g, h) {
    if (terminal case)
        return (r = trivial result) ;
    else {                                              /* exploit previous information */
        if (computed table has entry {(f, g, h), r} )
            return (r from computed table) ;
        else {
            x = top variable of f, g, h ;
            t = ITE(f_x, g_x, h_x) ;
            e = ITE(f_x', g_x', h_x') ;
            if (t == e)                                 /* children with isomorphic OBDDs */
                return (t) ;
            r = find_or_add_unique_table(x, t, e) ;     /* add r to unique table if not present */
            Update computed table with {(f, g, h), r} ;
            return (r) ;
        }
    }
}
```

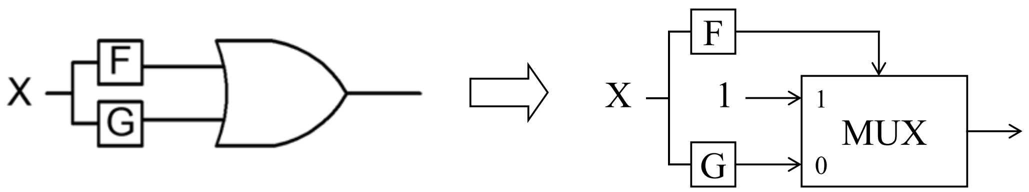

Derived Algebraic Operations

- Other common operations can be expressed in terms of If-Then-Else

AND (F, G) | If-Then-Else (F, G, 0)



OR (F, G) | If-Then-Else (F, 1, G)



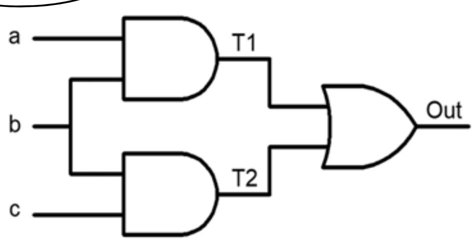
ITE Operators

Operator	**Equivalent *ite* form**
0	0
$f \bullet g$	*ite* ($f, g, 0$)
$f \bullet g'$	*ite* ($f, g', 0$)
f	f
$f' \bullet g$	*ite* ($f, 0, g$)
g	g
$f \oplus g$	*ite* (f, g', g)
$f + g$	*ite* ($f, 1, g$)
$(f + g)'$	*ite* ($f, 0, g'$)
$(f \oplus g)'$	*ite* (f, g, g')
g'	*ite* ($g, 0, 1$)
$f + g'$	*ite* ($f, 1, g'$)
f'	*ite* ($f, 0, 1$)
$f' + g$	*ite* ($f, g, 1$)
$(f \bullet g)'$	*ite* ($f, g', 1$)
1	1

Generating ROBDD from Network

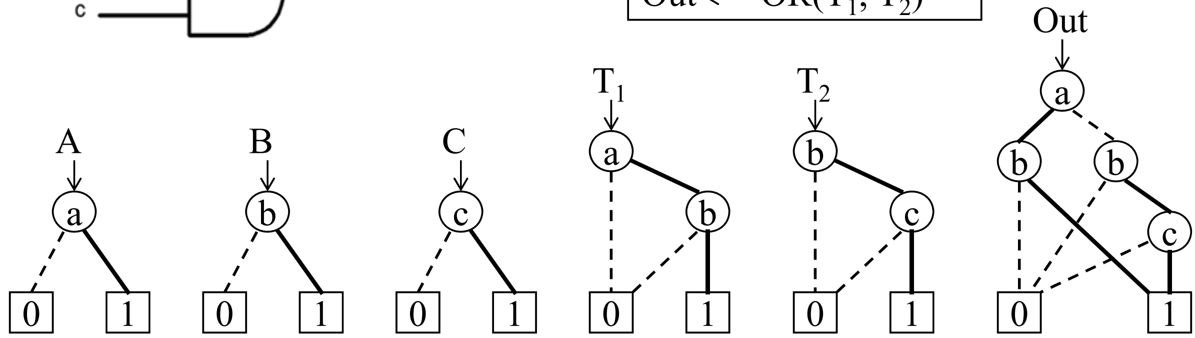
- Task : Represent output functions of gate network as ROBDDs

Network

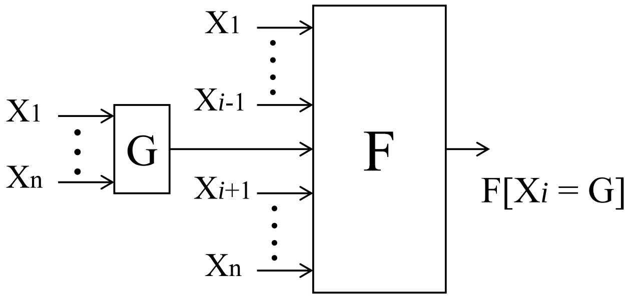


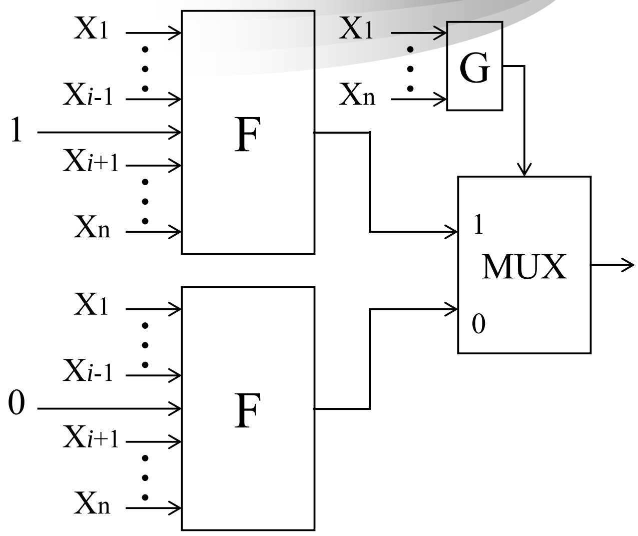
A $\leftarrow$ new_var("a")
B $\leftarrow$ new_var("b")
C $\leftarrow$ new_var("c")
T_1 $\leftarrow$ AND(A, B)
T_2 $\leftarrow$ AND(B, C)
Out $\leftarrow$ OR(T_1, T_2)

Evaluation



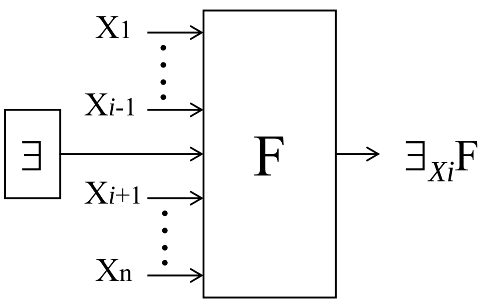
Functional Composition

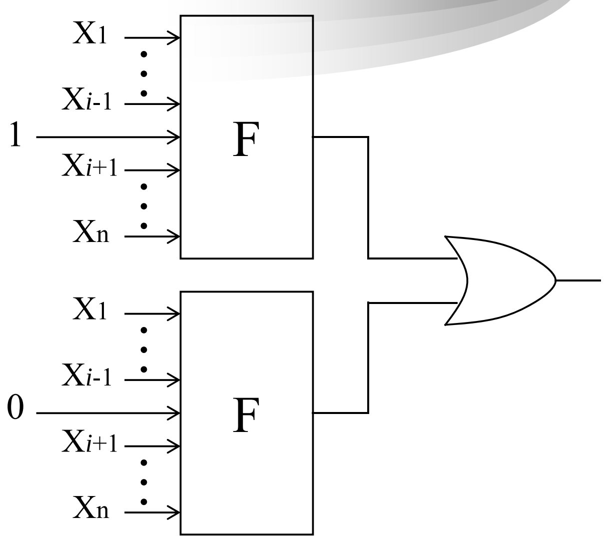




- Create new function by composing functions F and G
- Useful for composing hierarchical modules

Variable Qualification





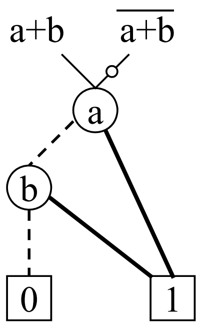
- Eliminate dependency on some argument through qualification

Variants & Optimizations

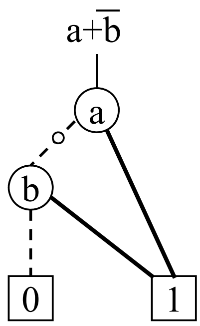
- Concept
  - Refinements to ROBDD representation
  - Do not change fundamental properties
- Objective
  - Reduce memory requirement
  - Improve algorithmic efficiency
  - Make commonly performed operations faster
- Common Optimizations
  - Share nodes among multiple functions
  - Negated arcs

Negation Arcs

- Concept
  - Dot on arc represents complement operator
    - » Invert function value
  - Can appear internal or external arc



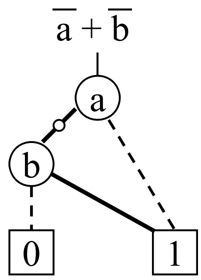
external

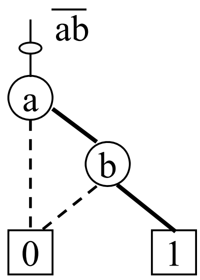


internal

Effect of Negation Arcs

- Storage Savings
  - At most 2X reduction in numbers of nodes
- Algorithmic Improvement
  - Can complement function in constant time
- Problem
  - Negation arc allow multiple representations of a function

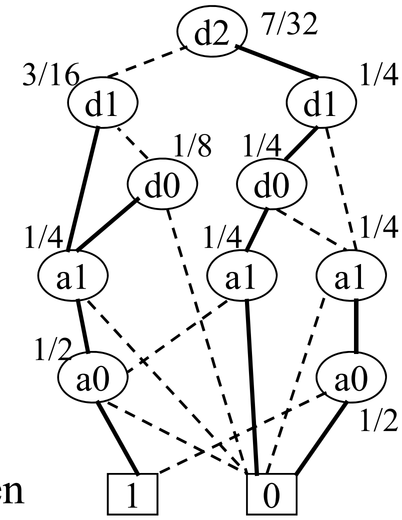




  - Modify algorithms with restricted conversions for use of negative arcs

Density Computation

- Definition
  - p(F) : fraction of variable assignments for which F = 1
- Applications
  - Testability measures
  - Probability computations
- Recursive Formulation
  - p(F) = [p(F[x=1]) + p(F[x=0])] / 2
- Computation
  - Compute bottom-up, starting at leaves
  - At each node, average density of children



Characteristic Function

Let E be a set and $A \subseteq E$

The characteristic function of A is the function

$X_A : E \to \{ 0, 1 \}$

$X_A(x) = 1$ if $x \in A$

$X_A(x) = 0$ if $x \notin A$

Ex :

$E = \{ 1, 2, 3, 4 \}$

$A = \{ 1, 2 \}$

$X_A(1) = 1$

$X_A(3) = 0$

Characteristic Function

Given a Boolean function

$f : B^n \to B^m$

the mapping relation denoted as $F \subseteq B^n \times B^m$ is defined as

$F(x, y) = \{ (x, y) \in B^n \times B^m \mid y = f(x) \}$

The characteristic function of a function f is defined for (x, y) s.t. $X_f(x, y) = 1$ iff $(x, y) \in F$

Characteristic Function

Ex : y = f(x1, x2) = x1 + x2

x1	x2	y
0	0	0
0	1	1
1	0	1
1	1	1

Fy(x1, x2, y) =

x1	x2	y	F
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Summary

- ROBDD
  - Reduced graph representation of Boolean Function
  - Canonical for given variable ordering
  - Size sensitive to variable ordering
- Algorithmic Principles
  - Operations maintain closure property
    - » Result ROBDD with same ordering as arguments
    - » Can perform further operations on results
  - Limited set of basic operations to implement
    - » Restrict, If-Then-Else
    - » Other operations defined in terms of basic operations