STAT 621 HW 4

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Question 1

Consider the following AR(4) process: $y_t = 4 + 0.5y_{t-4} + w_t$ where $w_t \sim WN(0, 2)$. PAY ATTENTION to the "lag" and notice that the mean is not zero. For several of these questions, you have done similar problems before. This will serve as a good review.

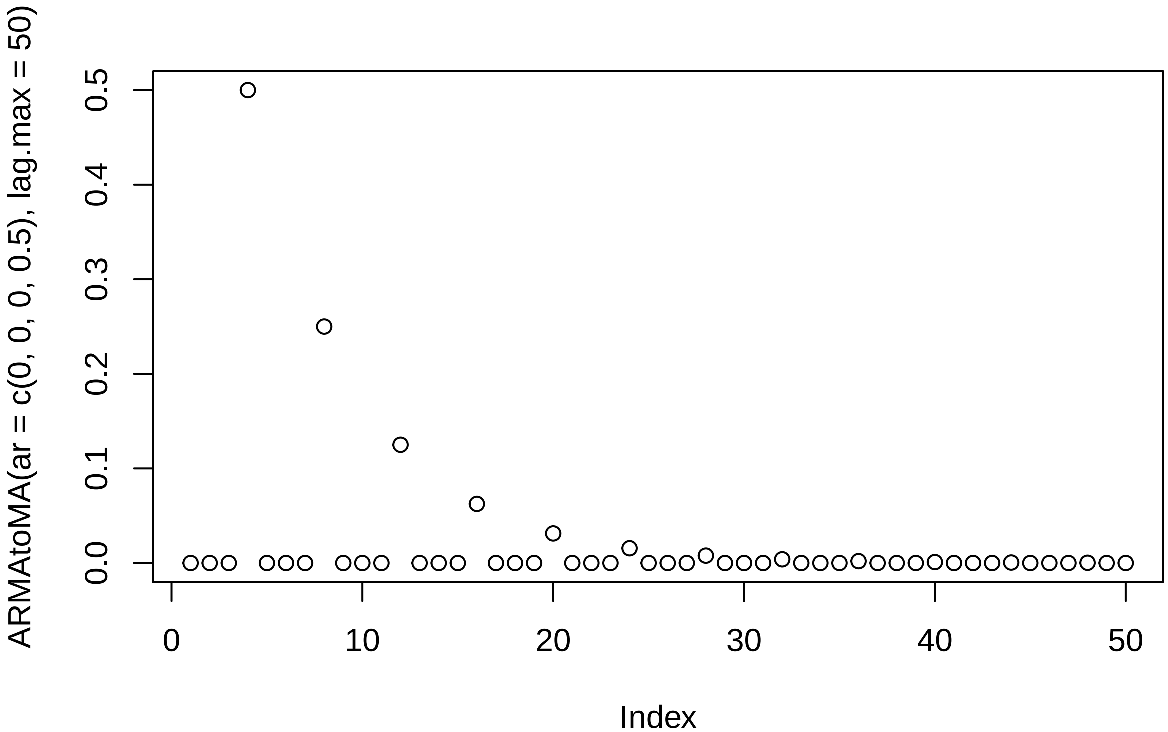
$$y_t = 4 + 0.5y_{t-4} + w_t$$

a. (10) Derive the MA(infinity) form of the model by "hand".

$$\begin{aligned} y_t &= 4 + 0.5y_{t-4} + w_t = 4 + 0.5(4 + 0.5y_{t-8} + w_{t-4}) + w_t \\ &= 4(1 + 0.5 + 0.5^2 + ...) + w_t + 0.5w_{t-4} + 0.5^2 w_{t-8} + ... \\ &= 4\sum_{j=0}^{\infty} 0.5^j + \sum_{j=0}^{\infty} 0.5^j w_{t-4j} \end{aligned}$$

b. (5) Verify that you obtain the same coefficients by using the ARMA to MA command in R. Plot these coefficients using R.

```
plot(ARMAtoMA(ar=c(0,0,0,0.5), lag.max = 50))
```



c. (5) Derive the mean of the process.

$$E(y_t) = E[4(1 + 0.5 + 0.5^2 + ...) + w_t + 0.5w_{t-4} + 0.5^2 w_{t-8} + ...] = 4(1 + 0.5 + 0.5^2 + ...)$$

where $1 + 0.5 + 0.5^2 + ... = \frac{2^{n-1}-1}{2^{n-1}}$. For n goes to infinity, $\sum_{j=0}^{\infty} x^j = 1/(1-x)$ when $|x| < 1$. Therefore, $E(y_t) = 4 * 1/(1-0.5) = 8$

d. (5) Derive the variance of the process.

$$Var(y_t) = Var(w_t + 0.5w_{t-4} + 0.5^2 w_{t-8}...) = (1 + 0.25 + 0.25^2 + ...)\sigma_w^2 = 1/(1-0.25) * \sigma_w^2 = 8/3$$

e. (5) Derive the autocorrelation function of the process.

$$\rho(h) = \gamma(h)/\gamma(0) = \gamma(h)/Var(y_t)$$
$$\gamma(h) = 0 \quad \text{if} \quad h = 1, 2, 3, 5, 6, 7, 9...(\text{not divisible by 4})$$
$$\gamma(4) = 0.5\sigma_w^2 + 0.5^3\sigma_w^2 + 0.5^5\sigma_w^2 + ... = 0.5(1 + 0.25 + 0.25^2 + ...)\sigma_w^2$$
$$\gamma(8) = 0.5^2\sigma_w^2 + 0.5^4\sigma_w^2 + 0.5^6\sigma_w^2 + ... = 0.5^2(1 + 0.25 + 0.25^2 + ...)\sigma_w^2$$
$$\gamma(h) = 0.5^{h/4}(1 + 0.25 + 0.25^2 + ...)\sigma_w^2 \quad \text{if} \quad h = 0, 4, 8, 12...(\text{divisible by 4})$$

Therefore, $\rho(h) = \gamma(h)/\gamma(0) = 0.5^{h/4}$ if h is divisible by 4 and $\rho(h) = 0$ for else. Same as the plot in question b).

f. (5) State the PACF of the process (you can use R to discover the function if the function is not obvious to you).

The PACF is 0 after lag(4) and also be zero for lag(1) lag(2) and lag(3).

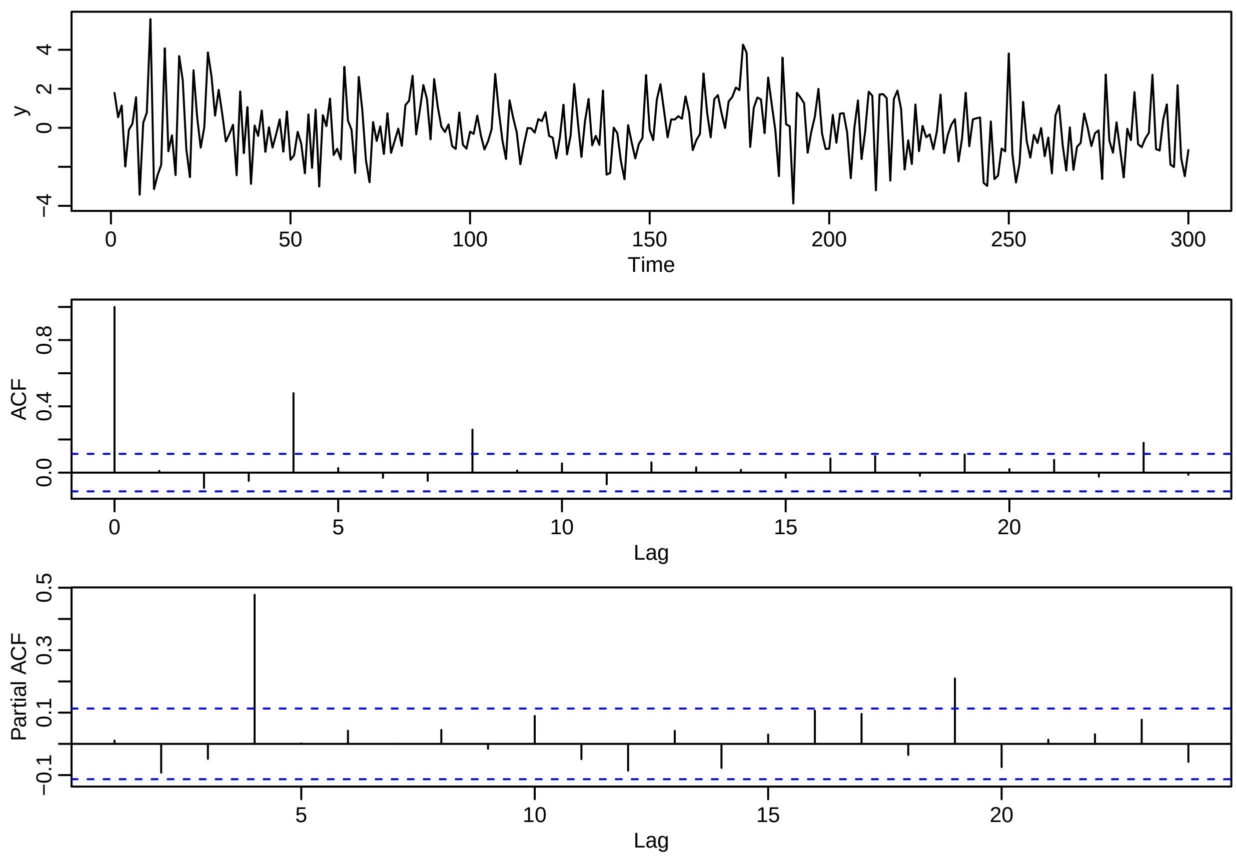
g. (5) Is the process a stationary causal autoregressive model? Why or why not?

$1 - 0.5z^4 = 0$ then $z = \sqrt{\sqrt{2}} > 1$ it is causal.

h. (5) Simulate a series of length 300 from the AR model and plot the series, a histogram, and the ACF and PACF of the series.

```
set.seed(1001)
y=arima.sim(list(order=c(4,0,0),ar=c(0,0,0,0.5)),sd = sqrt(2),n=300)


par(mfrow=c(3,1),mar=c(2.5,2.5,0,0)+.5,mgp=c(1.6,.6,0))
plot(y)
acf(y)
pacf(y)
```



i. (5) Comment on what you have learned. Can you think of an example case-study for which this model might be appropriate?

The PACF of AR(p) is around 0 after lag p and the ACF decay exponentially. If there is gap between x_t and x_{t-p}, then ACF and PACF will be zero for whatever in between.

Question 2 # 3.1

$$x_t = w_t + \theta w_{t-1}$$

$E[x_t] = 0$. $Var(x_t) = (1+\theta^2)\sigma_w^2 = \gamma(0)$. The Autocovariance $\gamma(h) = \theta\sigma_w^2$ if $h = \pm 1$ and $\gamma(h) = 0$ if $|h| > 1$. The autocorrelation $\rho(h) = \theta/(1+\theta^2)$ if $h = 1$ and $\rho(h) = 0$ if $|h| > 1$.

$$\rho(1) = \theta/(1+\theta^2) = 1/(1/\theta + \theta) \quad \text{where} \quad |1/\theta + \theta| \geq 2$$

Therefore, $|\rho(1)| \leq 1/2$. The maximum $\rho(1) = 1/2$ when $\theta = 1$; the minimum $\rho(1) = -1/2$ when $\theta = -1$.

Question 3 # 3.4 Identify as ARMA(p,q) models (causal/invertible)

a)

$$x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1}$$

$(1-0.8B+0.15B^2)x_t = (1-0.3B)w_t \Rightarrow (1-0.5B)(1-0.3B)x_t = (1-0.3B)w_t \Rightarrow (1-0.5B)x_t = w_t \Rightarrow x_t = 0.5x_{t-1}+w_t$

Causal: $1 - 0.5z = 0 \Rightarrow z = 2 > 1$ it is causal.

No lag of w_t so it is not invertible.

b)

$$x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1} \Rightarrow (1 - B + 0.5B^2)x_t = (1-B)w_t$$

Causal:

$$1 - z + 0.5z^2 = 0 \Rightarrow z = 1 \pm i \Rightarrow |z| = \sqrt{1^2 + i^2} = 1.414 > 1$$

It is causal.

Not Invertible: $1 - z = 0 \Rightarrow z = 1$ is not outside the unit circle. It is not invertible.

Question 4 # 3.6 AR(2) autoregressive polynomial and ACF

$$x_t = -0.9x_{t-2} + w_t = (-0.9)^2 x_{t-4} - 0.9w_{t-2} + w_t = \sum_{j=0}^{\infty}(-0.9)^j w_{t-2j}$$

$$\Phi(B)x_t = (1 + 0.9B^2)x_t = w_t$$

$\Phi(B)$ is the characteristic polynomial of the process. $1+0.9z^2 = 0$ we got complex roots. $z = \pm\frac{\sqrt{-0.9}}{0.9} = \pm\frac{i}{\sqrt{0.9}}$. It is causal.

$$Autocovariance \quad \gamma(0) = (1 + 0.81 + 0.81^2 + ...)\sigma_w^2 = 1/(1 - 0.81)\sigma_w^2$$

$$Autocovariance \quad \gamma(2) = (-0.9) * (1 + 0.81 + 0.81^2 + ...)\sigma_w^2 = -0.9/(1 - 0.81)\sigma_w^2$$

$$Autocovariance \quad \gamma(h) = (-0.9)^{h/2}/(1 - 0.81)\sigma_w^2 \quad \text{(h is divisible by 2)}$$

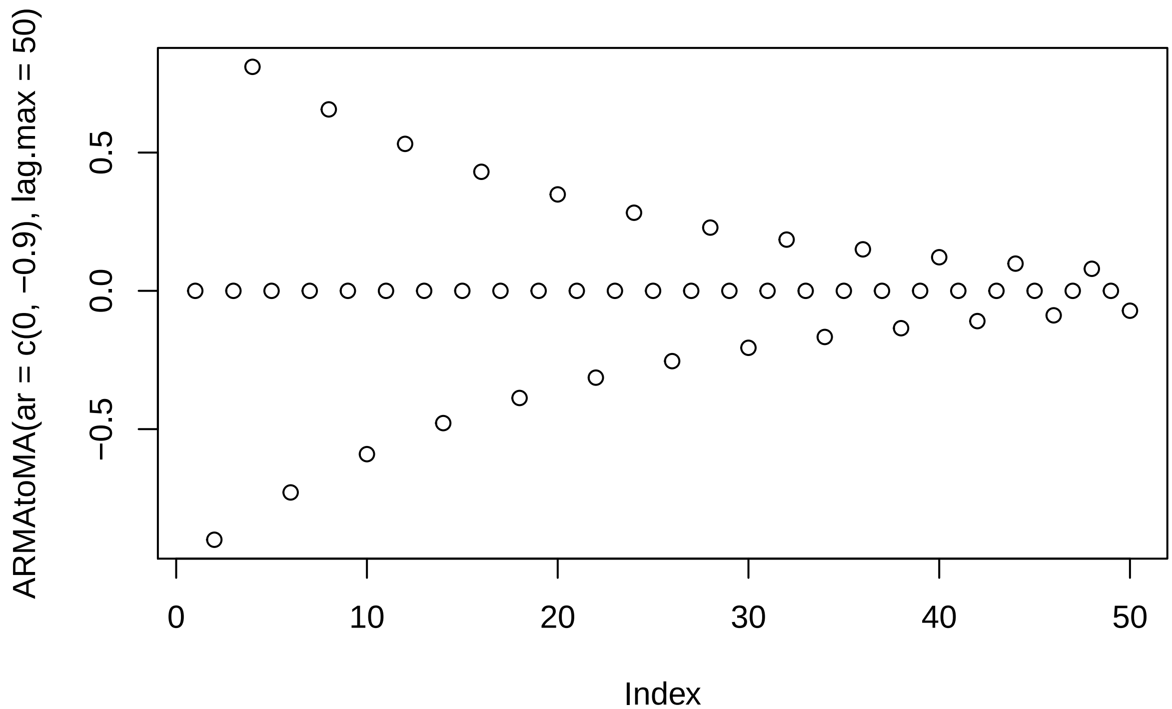
$$Autocorrelation \quad \rho(h) = (-0.9)^{h/2} \quad \text{(for h is divisible by 2)}$$

$\rho(h) = 0$ otherwise.

```
ARMAtoMA(ar=c(0,-0.9), lag.max = 10)
```

```
##  [1]  0.00000 -0.90000  0.00000  0.81000  0.00000 -0.72900  0.00000
##  [8]  0.65610  0.00000 -0.59049
```

```
plot(ARMAtoMA(ar=c(0,-0.9), lag.max = 50))
```



Question 5 # 3.9

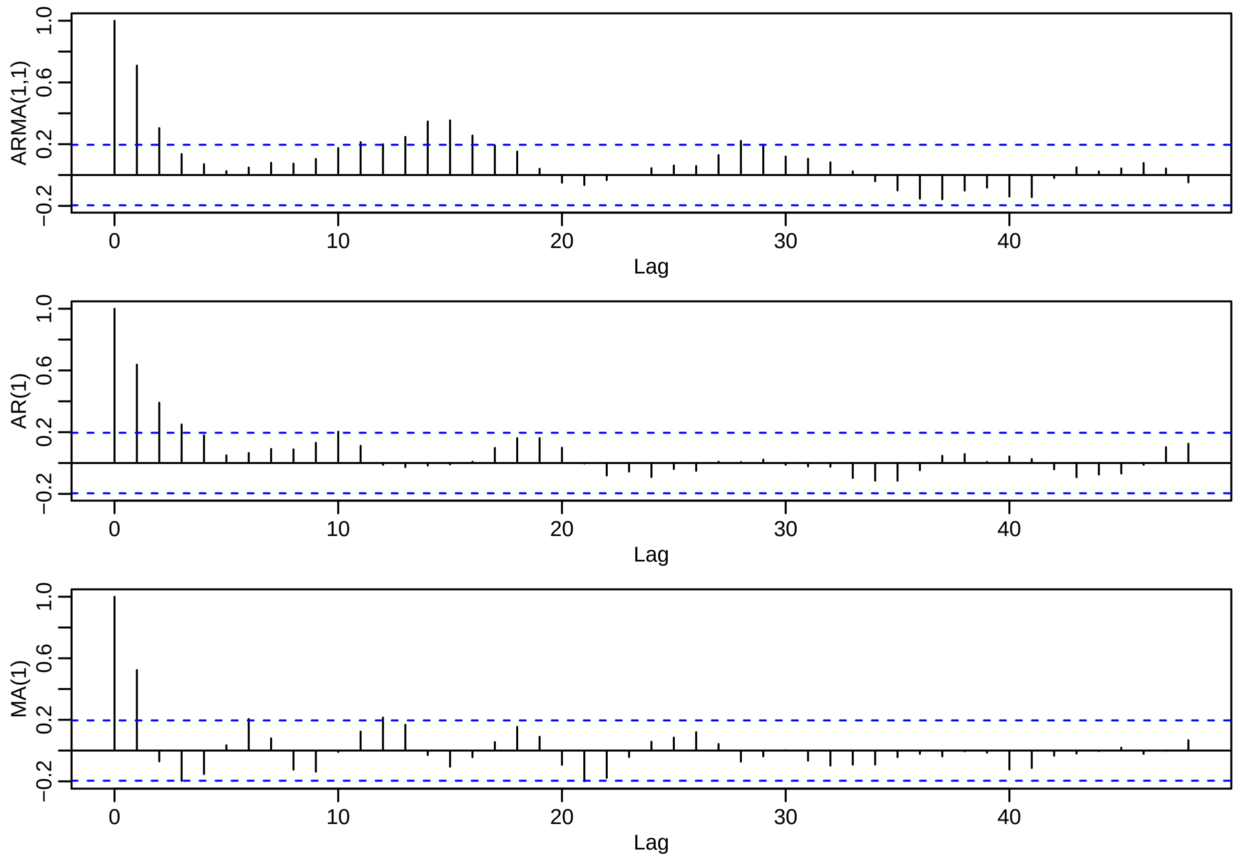
$$ARMA(1,1) \quad x_t = \phi x_{t-1} + \theta w_{t-1} + w_t = 0.6x_{t-1} + 0.9w_{t-1} + w_t$$

$$AR(1) \quad x_t = 0.6x_{t-1} + w_t = \sum_{j=0}^{\infty} 0.6^j w_{t-j}$$

$$MA(1) \quad x_t = 0.9w_{t-1} + w_t$$

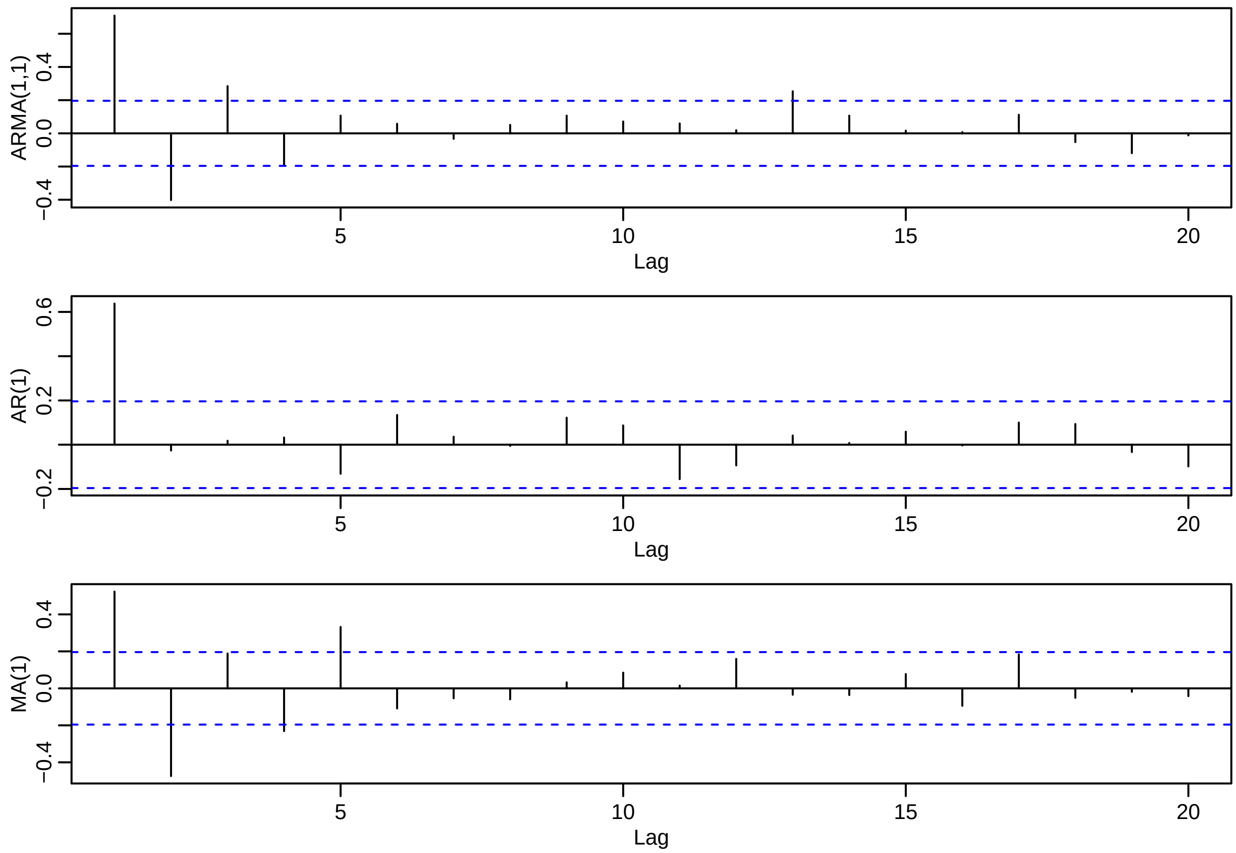
```r
x1=arima.sim(list(order=c(1,0,1),ar=0.6,ma=0.9),n=100)
x2=arima.sim(list(order=c(1,0,0),ar=0.6),n=100)
x3=arima.sim(list(order=c(0,0,1),ma=0.9),n=100)

par(mfrow=c(3,1),mar=c(2.5,2.5,0,0)+.5,mgp=c(1.6,.6,0))
acf(x1,48,ylab="ARMA(1,1)")
acf(x2,48,ylab="AR(1)")
acf(x3,48,ylab="MA(1)")
```



For ARMA(1,1), ACF exponentially decay. For AR(1), ACF decay exponentially after lag(1); Theoretically, $\rho(0) = 1; \rho(1) = 0.6; \rho(h) = 0.6^h$ For MA(1), ACF is zero after lag(1). Theoretically, $\rho(0) = 1; \rho(1) = 0.9/(1 + 0.81) \approx 0.5; rho(h) = 0, else$

```r
par(mfrow=c(3,1),mar=c(2.5,2.5,0,0)+.5,mgp=c(1.6,.6,0))
pacf(x1,ylab="ARMA(1,1)")
pacf(x2,ylab="AR(1)")
pacf(x3,ylab="MA(1)")
```



For ARMA(1,1), PACF exponentially decay; For AR(1), PACF is zero after lag(1); For MA(1), PACF decay exponentially after lag(1).