7. (a)

For m pigeons in n holes:

Let say $P_{ij} = \begin{cases} 1 & \text{if the } j\text{th pigeon is in } i\text{th hole} \\ 0 & \text{, otherwise} \end{cases}$

The 3 constraints in CNF clause:

① Every pigeon lives in some hole:

for pigeon j, it must exist in one of $1 \sim n$ holes

$\Rightarrow (P_{1j} \vee P_{2j} \vee P_{3j} \vee \cdots \vee P_{nj})$

for pigeons $1 \sim m \Rightarrow \bigwedge_{j=1}^{m} \left(\bigvee_{i=1}^{n} P_{ij} \right)$

② Every hole lives at most one pigeon:

for hole i, there should not be two different pigeons inside. So take 2 pigeons in m pigeons and check if these two both exist in the hole (C_2^m clauses)

so for $a, b \in \{1 \ldots m\}, a \neq b$, $\neg (P_{ia} \wedge P_{ib})$

$= (\neg P_{ia} \vee \neg P_{ib})$

$\Rightarrow \bigwedge_{i=1}^{n} \left\{ \bigwedge_{a=1}^{m-1} \bigwedge_{b=a}^{m} (\neg P_{ia} \vee \neg P_{ib}) \right\}$

③ There should be no pigeon in two holes

$\Rightarrow \bigwedge_{j=1}^{m} \left[\bigwedge_{a=1}^{n-1} \bigwedge_{b=a}^{n} (\neg P_{aj} \vee \neg P_{bj}) \right]$

$$① \wedge ② \wedge ③ \Rightarrow PHP_n^m = \bigwedge_{j=1}^{m} \left(\bigvee_{i=1}^{n} P_{ij} \right) \wedge \bigwedge_{i=1}^{n} \left[\bigwedge_{a=1}^{m-1} \bigwedge_{b=a}^{m} (\neg P_{ia} \vee \neg P_{ib}) \right]$$

$$\wedge \bigwedge_{j=1}^{m} \left[\bigwedge_{a=1}^{n-1} \bigwedge_{b=a}^{n} (\neg P_{aj} \vee \neg P_{bj}) \right]$$

(b)

$m = n = 4$

```
jimmyyang3125@Yang hw2 % python3 genPigeon.py 4 4
jimmyyang3125@Yang hw2 % minisat m4n4.cnf
============================[ Problem Statistics ]=============================
|                                                                             |
|  Number of variables:           16                                          |
|  Number of clauses:             52                                          |
|  Parse time:                  0.00 s                                        |
|  Eliminated clauses:          0.00 Mb                                       |
|  Simplification time:         0.00 s                                        |
|                                                                             |
============================[ Search Statistics ]==============================
| Conflicts |          ORIGINAL         |          LEARNT          | Progress |
|           |    Vars  Clauses Literals |    Limit  Clauses Lit/Cl |          |
===============================================================================
restarts              : 1
conflicts             : 0            (0 /sec)
decisions             : 1            (0.00 % random) (502 /sec)
propagations          : 0            (0 /sec)
conflict literals     : 0            ( nan % deleted)
Memory used           : 4.51 MB
CPU time              : 0.001992 s

SATISFIABLE
```

$m = n = 5$

```
jimmyyang3125@Yang hw2 % python3 genPigeon.py 5 5
jimmyyang3125@Yang hw2 % minisat m5n5.cnf
============================[ Problem Statistics ]=============================
|                                                                             |
|  Number of variables:           25                                          |
|  Number of clauses:            105                                          |
|  Parse time:                  0.00 s                                        |
|  Eliminated clauses:          0.00 Mb                                       |
|  Simplification time:         0.00 s                                        |
|                                                                             |
============================[ Search Statistics ]==============================
| Conflicts |          ORIGINAL         |          LEARNT          | Progress |
|           |    Vars  Clauses Literals |    Limit  Clauses Lit/Cl |          |
===============================================================================
restarts              : 1
conflicts             : 0            (0 /sec)
decisions             : 12           (0.00 % random) (6525 /sec)
propagations          : 20           (10875 /sec)
conflict literals     : 0            ( nan % deleted)
Memory used           : 4.51 MB
CPU time              : 0.001839 s

SATISFIABLE
```

$m = n = 6$

```
jimmyyang3125@Yang hw2 % python3 genPigeon.py 6 6
jimmyyang3125@Yang hw2 % minisat m6n6.cnf
============================[ Problem Statistics ]=============================
|                                                                             |
|  Number of variables:           36                                          |
|  Number of clauses:            186                                          |
|  Parse time:                  0.00 s                                        |
|  Eliminated clauses:          0.00 Mb                                       |
|  Simplification time:         0.00 s                                        |
|                                                                             |
============================[ Search Statistics ]==============================
| Conflicts |          ORIGINAL         |          LEARNT          | Progress |
|           |    Vars  Clauses Literals |    Limit  Clauses Lit/Cl |          |
===============================================================================
restarts              : 1
conflicts             : 0            (0 /sec)
decisions             : 17           (0.00 % random) (8915 /sec)
propagations          : 30           (15732 /sec)
conflict literals     : 0            ( nan % deleted)
Memory used           : 4.52 MB
CPU time              : 0.001907 s

SATISFIABLE
```

☆ satisfiable : there exist a solution

△ The solver is scalable on this problem because there is a trivial solution, meaning not much to try until it finds the sol.

(C)

$m = n+1 = 4$

```
jimmyyang3125@Yang hw2 % python3 genPigeon.py 4 3
jimmyyang3125@Yang hw2 % minisat m4n3.cnf
============================[ Problem Statistics ]=============================
|                                                                             |
|  Number of variables:            12                                         |
|  Number of clauses:              34                                         |
|  Parse time:                  0.00 s                                        |
|  Eliminated clauses:          0.00 Mb                                       |
|  Simplification time:         0.00 s                                        |
|                                                                             |
===============================================================================
Solved by simplification
restarts              : 0
conflicts             : 0            (0 /sec)
decisions             : 0            ( nan % random) (0 /sec)
propagations          : 4            (2335 /sec)
conflict literals     : 0            ( nan % deleted)
Memory used           : 4.46 MB
CPU time              : 0.001713 s

UNSATISFIABLE
```

$m = n+1 = 5$

```
jimmyyang3125@Yang hw2 % python3 genPigeon.py 5 4
jimmyyang3125@Yang hw2 % minisat m5n4.cnf
============================[ Problem Statistics ]=============================
|                                                                             |
|  Number of variables:            20                                         |
|  Number of clauses:              75                                         |
|  Parse time:                  0.00 s                                        |
|  Eliminated clauses:          0.00 Mb                                       |
|  Simplification time:         0.00 s                                        |
|                                                                             |
===============================================================================
Solved by simplification
restarts              : 0
conflicts             : 0            (0 /sec)
decisions             : 0            ( nan % random) (0 /sec)
propagations          : 5            (2752 /sec)
conflict literals     : 0            ( nan % deleted)
Memory used           : 4.46 MB
CPU time              : 0.001817 s

UNSATISFIABLE
```

$m = n+1 = 6$

```
jimmyyang3125@Yang hw2 % python3 genPigeon.py 6 5
jimmyyang3125@Yang hw2 % minisat m6n5.cnf
============================[ Problem Statistics ]=============================
|                                                                             |
|  Number of variables:            30                                         |
|  Number of clauses:             141                                         |
|  Parse time:                  0.00 s                                        |
|  Eliminated clauses:          0.00 Mb                                       |
|  Simplification time:         0.00 s                                        |
|                                                                             |
===============================================================================
Solved by simplification
restarts              : 0
conflicts             : 0            (0 /sec)
decisions             : 0            ( nan % random) (0 /sec)
propagations          : 7            (3659 /sec)
conflict literals     : 0            ( nan % deleted)
Memory used           : 4.47 MB
CPU time              : 0.001913 s

UNSATISFIABLE
```

☆ unsat: no solution

△ The solver is not scalable on this problem due to it may encounter many conflicts until it finds out there is no solution.

$m = n+1 = 100$ ⟶ 128042 conflicts!

```
jimmyyang3125@Yang hw2 % python3 genPigeon.py 101 100
jimmyyang3125@Yang hw2 % minisat m101n100.cnf
============================[ Problem Statistics ]=============================
|                                                                             |
|  Number of variables:         10100                                         |
|  Number of clauses:         1005051                                         |
|  Parse time:                  0.10 s                                        |
|  Simplification time:         0.45 s                                        |
|                                                                             |
============================[ Search Statistics ]==============================
| Conflicts |          ORIGINAL         |          LEARNT          | Progress |
|           |    Vars  Clauses Literals |    Limit  Clauses Lit/Cl |          |
===============================================================================
|       100 |   10100  1005051  2020000 |   368518      100    285 |  0.000 % |
|       250 |   10100  1005051  2020000 |   405370      250    179 |  0.000 % |
|       475 |   10096  1005051  2020000 |   445907      473    150 |  0.040 % |
|       812 |   10094  1005051  2020000 |   490498      808    170 |  0.059 % |
|      1318 |   10088  1005051  2020000 |   539548     1310    161 |  0.119 % |
|      2077 |   10086  1005051  2020000 |   593503     2067    159 |  0.139 % |
|      3216 |   10086  1005051  2020000 |   652853     3206    150 |  0.139 % |
|      4924 |   10070  1005051  2020000 |   718138     4902    137 |  0.297 % |
|      7486 |   10070  1005051  2020000 |   789952     7464    157 |  0.297 % |
|     11330 |   10056  1005051  2020000 |   868947    11295    162 |  0.436 % |
|     17096 |   10055  1005051  2020000 |   955842    17060    163 |  0.446 % |
|     25745 |   10050  1005051  2020000 |  1051426    25704    164 |  0.495 % |
|     38719 |   10035   993929  1997756 |  1156569    38147    170 |  0.644 % |
|     58180 |   10017   993929  1997756 |  1272226    57590    159 |  0.822 % |
|     87372 |    9945   993929  1997756 |  1399449    86737    155 |  1.535 % |
===============================================================================
restarts              : 349
conflicts             : 128642         (41820 /sec)
decisions             : 1560424        (0.00 % random) (507278 /sec)
propagations          : 6915956        (2248306 /sec)
conflict literals     : 18803119       (5.43 % deleted)
Memory used           : 220.08 MB
CPU time              : 3.07607 s

UNSATISFIABLE
```