1st order Digital Low Pass Filter

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1 Calculation

A first order low pass filter in the Laplace domain can be defined as:

$$\frac{Y(S)}{X(S)}=H(S)=\frac{1}{S\tau+1} \qquad p@-\frac{1}{\tau} \tag{1}$$

Using the pole shown above, the digital system in the Z domain can be found using “zero pole mapping”.

$$H(Z)=\frac{1}{Z-e^{\frac{-T}{\tau}}} \tag{2}$$

Though the frequency response is approximated, the zero gain needs yet to be compensated. At zero Hertz, Z equals:

$$Z=e^{ST}=e^{(\sigma+\omega i)T}\Rightarrow Z_{\omega\rightarrow 0}=1 \tag{3}$$

So the zero gain for now equals:

$$H(1)=\frac{1}{1-e^{\frac{-T}{\tau}}} \tag{4}$$

If the digital system is being compensated for the zero gain, then the system becomes:

$$H(Z)=\frac{1-e^{\frac{-T}{\tau}}}{Z-e^{\frac{-T}{\tau}}}=\frac{Z^{-1}-Z^{-1}e^{\frac{-T}{\tau}}}{1-Z^{-1}e^{\frac{-T}{\tau}}} \tag{5}$$

Now calculating the output Y:

$$\begin{aligned} Y\left(1-Z^{-1}e^{\frac{-T}{\tau}}\right)&=X\left(Z^{-1}-Z^{-1}e^{\frac{-T}{\tau}}\right)\Rightarrow \\ Y&=YZ^{-1}e^{\frac{-T}{\tau}}+XZ^{-1}-XZ^{-1}e^{\frac{-T}{\tau}}\Rightarrow \\ Y&=YZ^{-1}e^{\frac{-T}{\tau}}+XZ^{-1}\left(1-e^{\frac{-T}{\tau}}\right) \end{aligned} \tag{6}$$

Now it seems that the system can be speed up one sample since all inputs are delayed.

$$Y=YZ^{-1}e^{\frac{-T}{\tau}}+X\left(1-e^{\frac{-T}{\tau}}\right) \tag{7}$$

A simplification can be made such that the exponent needs to be calculated once:

$$\begin{aligned} Y&=YZ^{-1}e^{\frac{-T}{\tau}}+X\left(1-e^{\frac{-T}{\tau}}\right) \\ Y&=YZ^{-1}-YZ^{-1}\left(1-e^{\frac{-T}{\tau}}\right)+X\left(1-e^{\frac{-T}{\tau}}\right) \\ Y&=YZ^{-1}+\left(X-YZ^{-1}\right)\left(1-e^{\frac{-T}{\tau}}\right) \end{aligned} \tag{8}$$

Now one constant is used. When transforming this system to a discrete time domain system you get:

$$y[n]=y[n-1]+\left(x[n]-y[n-1]\right)\left(1-e^{\frac{-T}{\tau}}\right) \tag{9}$$

Where y[n] is the current output, x[n] is the current input and y[n-1] is the previous input. Note:

$$e^{\frac{-T}{\tau}}=e^{-T\omega_c} \tag{10}$$

2 High Pass Filter

The system function of a high pass filter in the Laplace domain equals:

$$H=\frac{S\tau}{S\tau+1}=\frac{S}{S+\frac{1}{\tau}}=\frac{S}{S+\omega_c} \tag{11}$$

This has a pole and zero of:

$$\begin{gathered} p@-\omega_c \\ z@0 \end{gathered} \tag{12}$$

So in the Z domain the equivalent system function using pole zero mapping equals:

$$H=\frac{Z-e^{0}}{Z-e^{-T\cdot\omega_c}}=\frac{Z-1}{Z-e^{-T\cdot\omega_c}} \tag{13}$$

The gain needs to be compensated for $\lim_{\omega\rightarrow\infty} Z=0$.

$$\lim_{\omega_c\rightarrow\infty} H=\frac{0-1}{0-e^{-T\cdot\omega_c}}=\frac{1}{e^{-T\cdot\omega_c}} \tag{14}$$

Therefore the overall system function will be:

$$H=\frac{e^{-T\cdot\omega_c}\cdot(Z-1)}{Z-e^{-T\cdot\omega_c}} \tag{15}$$

Converting this to implementable Y = ... notation:

$$\begin{aligned} Y\cdot\left(1-Z^{-1}\cdot e^{-T\cdot\omega_c}\right)&=X\cdot e^{-T\cdot\omega_c}\cdot\left(1-Z^{-1}\right) \\ Y&=Y\cdot Z^{-1}\cdot e^{-T\cdot\omega_c}+X\cdot e^{-T\cdot\omega_c}-X\cdot e^{-T\cdot\omega_c}\cdot Z^{-1} \\ Y&=e^{-T\cdot\omega_c}\cdot\left(\left(Y-X\right)\cdot Z^{-1}+X\right) \end{aligned} \tag{16}$$