Contents

1. Simulation of Random numbers

1.1. **Poisson Distribution.** We use inverse function method. First note that Poisson distribution satisfies

$$P(N = k+1) = P(N = k) \cdot \frac{\lambda}{k+1}$$

Then for $U \sim U(0,1)$,

$$F(k) = P_U(U \leq F(k)))$$

So we first generate U, then find the first K such that $U \leq F(k)$.

The pseudo code is the following

```
p = exp(−λ )
F = p
k = 0
//  generate u from U(0,1)
while U > F:
        k += 1
        p *=  λ/k
        F += p
return k
```

2. BSM model

We assume the stock price satisfies the following SDE,

$$dS = S\mu dt + S\sigma dw,$$

where w is standard brownian motion. By Ito's lemma, we have

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma w(t)}.$$

So to simulate the discounted stock price $e^{-rt}S(t)$ we need

- Current stock price $S(0)$: spot
- expiration date: T
- Interest rate r: r
- Average return μ: r. Because using risk neutral pricing, we can show $\mu = r$.
- Volatility σ: vol
- The Brownian motion $w(t)$

2.1. Estimation of $w(t)$.

To create Brownian motion, we begin with symmetric random walk. Let

$$X_k = \begin{cases} +1, & \text{if Head} \\ -1, & \text{if Tail} \end{cases}$$

and

$$M_t = \sum_{k=1}^{t} X_k$$

To approximate Brownian motion, we define the *scaled symmetric random walk* by

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}$$

- We move more frequently, from t times to nt times in the same period of time t.
- $E[W^{(n)}(t)] = 0$
- Var $W^{(n)}(t)$ = t
- By central limit theorem

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} = \frac{\sum X_k}{\sqrt{n}} = \sqrt{t}\frac{\sum X_k}{\sqrt{n}\sqrt{t}} \Rightarrow N(0, t)$$

3. Jump-Diffusion model

3.1. General Setup.

Assuming there are some jumps of the stock price, then the Jump-Diffusion model is

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW + dJ(t),$$

where

$$S(t-) = \lim_{\tau \to t^-} S(\tau)$$

and the jump process.

$$J(t) = \sum_{j=1} \tau_N \leq t(Y_j - 1)$$

We assume the jumps will happen at

$$0 < \tau_1 < \tau_2 < \cdots < \tau_N$$

and Y_j are random variables with positive values.

To justify the assumption of Y_j note that

$$\begin{aligned} dJ(t) &= 0 \text{ if } t \text{ is not equal to any of } \tau_j \\ &= Y_j - 1 if t = \tau_j \end{aligned}$$

Then according to the jump diffusion model

$$\begin{aligned} & S(\tau_j) - S(\tau_j -) = (Y_j - 1) \cdot S(\tau_j -) \\ \Rightarrow \quad & \frac{S(\tau_j)}{S(\tau_j -)} = Y_j \end{aligned}$$

That is, Y_j is the ratio of stock prices before and after the jump.

Proposition 3.1. *The solution of the jump diffusion model is*

$$S(t) = S(0) e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma w} \prod Y_j$$

3.2. **Specific Assumptions.** There are two contingent variables involved in the jump diffusion model. The number of jumps, $N(t)$ and the distributions of the jump ratio, Y_j. To simplify the model, we made the following assumptions,

- $N(t)$ is a Poisson process with parameter λ
- Y_j are independent to $W(t)$ and $N(t)$, and i.i.d. to log normal distribution $LN(a, b^2)$

For the Poisson distribution $N(t)$, the inter arrival times $\tau_{j+1} - \tau_j$ are i.i.d. with exponential distribution

$$P(\tau_{j+1} - \tau_j \leq t) = 1 - e^{-\lambda t}$$

Since product of log normal is still log normal, we have

$$\prod_{j=1}^{j=n} Y_j \sim LN(na, nb^2)$$

Condition on $N(t) = n$, the stock price

$$\begin{aligned} S(t) &= S(0) \cdot e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma w(t)} \cdot LN(na, nb^2) \\ &= S(0) \cdot LN((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t) \cdot LN(na, nb^2) \\ &= LN(log S(0) + (\mu - \frac{1}{2}\sigma^2)t + na, \sigma^2 t + nb^2) \end{aligned}$$

Let $F_{n,t}(x)$ denote the cdf of $S(t)$, then the unconditional distribution of $S(t) \leq x$ is

$$P(S(t) \leq x)) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \cdot F_{n,t}(x)$$

3.3. **Price of options.** We want the discounted value of the stock is a martingale. Look at the diffusion model

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dW + dJ(t),$$

Our first guess is that let $\mu = r$ will work because it will cancel the effect of r. Now we look at $dJ(t)$. Note that $J(t)$ is a **compound Poisson distribution**. Use total law of expectation we see that

$$\begin{aligned} E(J(t)) &= E(N(t))E(Y-1) \\ &= \lambda t E(Y-1) \end{aligned}$$

In other words, $J(t) - \lambda t E(Y-1)$ is a martingale. So to make the discounted price of the stock a martingale, we should have

$$\frac{dS(t)}{S(t-)} = r dt + \sigma dW + dJ(t) - \lambda E(Y-1)dt.$$

So

$$\mu = r - \lambda E(Y-1)$$

3.4. **Simulation Of Stock Price.** We consider the method assuming that $t_i's$ are fixed, the number of jumps in each subintervals follows poisson distribution with λ or λ/T. Here T is the number of subintervals.

Question 3.2. How do we estimate λ?

By the solution of the jump diffusion model, we have the following difference equation

$$S(t_{i+1}) = S(t_i) \cdot e^{(\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma(w(t_{i+1} - w(t_i))} \prod_{N(t_i)+1}^{N(t_{i+1})} Y_j$$

To make it easier for recursion, we let $x(t) = log(S(t))$ then

$$x(t_{i+1}) = x(t_i) + (\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sum_{N(t_i)+1}^{N(t_{i+1})} log(Y_j)$$

If we assume further that Y_j's are i.i.d. log normal(a, b^2), and let N be $N(t_{i+1}) - N(t_i)$ Then

$$\sum_{N(t_i)+1}^{N(t_{i+1})} log(Y_j) = aN + \sqrt{N} b \cdot \text{normal}(0,1)$$

In this equation, there are three random numbers to be generated,

```
Generate Z_1, Z_2 ~ N(0,1)
Generate N ~ Poisson(λ/T)
Generate logY_1+....+logY_N = aN+√NbZ_2
```

4. Interest Rate Model

We actually want to price the price of bonds like the following equation

$$P(t,T) = E_t[e^{-\int_t^T r(u)du}]$$

However there will be too many bonds to price, since different bonds may have different maturity date. So instead, we model the interest rate first. Once we have $r(u)$, then the price of the bonds $P(t,T)$ will automatically be no arbitrage. Because the discounted price of P is of the form

$$\begin{aligned} e^{-\int_0^t r(u)du} P(t,T) &= E_t[e^{-\int_0^t r(u)du} e^{-\int_t^T r(u)du}] \\ &= E_t[e^{-\int_0^T r(u)du}] \\ &= E_t[X] \end{aligned}$$

where X has nothing to do with t. So it will be a martingale. We note that if discounted price is martingale then there is no arbitrage.

4.1. **Market price of Risk.** Here are some facts about market price of risk.
It is a constant λ such that

$$\frac{\mu - r}{\sigma} = \lambda \tag{1}$$

Choosing a market price of risk is also referred to as choosing a probability measure. If we have two financial product f and g, such that we choose σ_g as λ, then $\frac{f}{g}$ is a martingale for all f. This g is called numeraire, means the method to discount. By equation (1), we have

$$df = (r + \sigma_g \sigma_f) f dt + \sigma_f dw.$$

4.2. **Vasicek Model.** We assume that

$$dr = a(b - r)dt + \sigma dz$$

Since interest rate is not directly tradable, we trade derivatives on bounds (?) Assume the price of the derivative

$$P_t(T) = V(t, T, r)$$

Differentiate it we have

$$dV = [V_t + V_r \cdot a(b - r) + \frac{1}{2}\sigma^2 V_r r]dt + V_r \sigma dz$$

Choosing market price of risk equaling 0 (money market?) by equation (1) we have

$$[V_t + V_r \cdot a(b - r) + \frac{1}{2}\sigma^2 V_r r] = rV$$