AA/EE/ME 548: Linear Multivariable Control

Lecture 11

5/05/2025

Announcements

- Homework 2
  - PLEASE start early! Or at least look at it early. Some parts may be helpful for your project! Problem 4 may be helpful!
- Project proposal
  - Feedback released today
  - PLEASE start early!
  - Homework 3 will have a "project update"
- Homework solutions out
  - Please check your homework 1 and submit reflection
  - Thank your homework 1 folks
- Week 5 folks, worked examples due Wednesday!

Course so far…

Linear Multivariable Control

COURSE SYLLABUS

Course information

Week-by-week schedule

HOMEWORK

Homework solution and worked example guidelines

Homework 1

Homework 2

Homework 3

PROJECT

Project

LECTURES

Introduction

State-space representation

Introduction to optimization

Control certificate functions

Sequential Decision-Making

CODE EXAMPLES ← Code demo and worked examples

CLF controller

HJ reachability basics

Week-by-week schedule

Note: *Italicized* text indicates planned topics, but subject to change.

Date	Week	Topic	Milestones	Links
March 31 April 2	1	Introduction, state-space dynamics, linearization, continuous and discrete time dynamics	hw 1 out	lec01 pdf
April 7 April 9	2	Intro to optimization, Control Lyapunov Functions, control invariant sets, Control barrier functions, CLF-CBF-QP		lec03 pdf, lec04 pdf
April 14 April 16	3	Guest lecture (Dr. Max Cohen) CBFs in the real-world!, Guest lecture (Dr. Edward Schmerling) Introduction to sequential decision-making		lec05 pdf, Pre-reading for guest lecture: Control barrier functions via reduced-order models, lec06 pdf
April 21 April 23	4	Value function, Bellman equation, value iteration, stochastic DP	hw 1 due; hw 2 out	lec07 pdf
April 28 April 30	5	HJB, HJI, HJ reachability, linear quadratic regulator	Project proposal due	lec09 pdf, lec10 pdf, Pursuit-Evasion, HJ code basics
May 5 May 7	6	*Tracking LQR, iLQR, Trajectory optimization*		
May 12 May 14	7	*Model predictive control*	hw 2 due hw3 out	
May 19 May 21	8	*Guest lecture(?), Kalman filter*		
May 26 May 28	9	(No lecture; Memorial Day) *Technical communication*	hw 3 due	
June 2 June 4	10	Project spotlight presentation Project poster presentation	Due project pitch Due project poster	
	Finals	Due final report or website		

The key equations so far…

- Bellman equation

$$V^*(x,t) = \min_{u \in \mathcal{U}(x)} \Big(J(x,u,t) + V^*(f(x,u,t), t+1) \Big)$$

- Infinite horizon case

$$V^{(k+1)}(x) \leftarrow \min_{u \in \mathcal{U}(x)} \Big(J(x,u,t) + \gamma V^{(k)}(f(x,u,t)) \Big)$$

- Stochastic case

$$V^*(x_t,t) = \min_{u_t \in \mathcal{U}(x_t)} \mathbb{E}_{x_{t+1} \sim p(x_{t+1} | x_t, u_t, t)} \Big[J(x_t,u_t,t) + V^*(x_{t+1}, t+1) \Big]$$

- Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(x,t) + \min_{u \in \mathcal{U}(x)} \Big(J(x,u,t) + \nabla V(x,t)^T f(x,u,t) \Big) = 0$$

- Hamilton-Jacobi-Isaacs equation

$$\frac{\partial V}{\partial t}(x,t) + \min_{u \in \mathcal{U}(x)} \max_{d \in \mathcal{D}(x)} \Big(J(x,u,d,t) + \nabla V(x,t)^T f(x,u,d,t) \Big) = 0$$

Disturbance acts 2^{nd}!

Reach case

"regular"

$$\frac{\partial V}{\partial t}(x,t) + \min_{u \in \mathcal{U}(x)} \nabla V(x,t)^T f(x,u,t) = 0$$

"w/ disturbance"

$$\frac{\partial V}{\partial t}(x,t) + \min_{u \in \mathcal{U}(x)} \max_{d \in \mathcal{D}(x)} \nabla V(x,t)^T f(x,u,d,t) = 0$$

"tube"

$$\frac{\partial V}{\partial t}(x,t) + \min\left(0, \min_{u \in \mathcal{U}(x)} \max_{d \in \mathcal{D}(x)} \nabla V(x,t)^T f(x,u,d,t)\right) = 0$$

$$V(x,T) = J_T(x)$$

Avoid case

"regular"

$$\frac{\partial V}{\partial t}(x,t) + \max_{u \in \mathcal{U}(x)} \nabla V(x,t)^T f(x,u,t) = 0$$

"w/ disturbance"

$$\frac{\partial V}{\partial t}(x,t) + \max_{u \in \mathcal{U}(x)} \min_{d \in \mathcal{D}(x)} \nabla V(x,t)^T f(x,u,d,t) = 0$$

"tube"

$$\frac{\partial V}{\partial t}(x,t) + \min\left(0, \max_{u \in \mathcal{U}(x)} \min_{d \in \mathcal{D}(x)} \nabla V(x,t)^T f(x,u,d,t)\right) = 0$$

$$V(x,T) = J_T(x)$$

Solving for the value function

- Discrete & finite state, control, time
  - Doable by looping through x, u, t (homework 2)

- Continuous state, control, time
  - Solve PDE. Expensive
  - Include HJ reachability

- Continuous state, control, discrete time
  - Generally hard, no closed form

This week

- But possible for simple problem formulation
- Linear Quadratic Regulator
  - Discrete time, continuous state & controls
  - Time-invariant: Q, R, A, B are constant.

$$\begin{aligned} \min_{u_0,\ldots,u_{T-1}} \quad & \textstyle\sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t + x_T^T Q x_T \\ \text{subj. to} \quad & x_{t+1} = A x_t + B u_t \\ & x_0 = x_{\text{curr}} \end{aligned}$$

$$Q = Q^T, Q \succeq 0, Q_T = Q_T^T, Q_T \succeq 0, R = R^T, R \succ 0$$

We use Bellman equation!

$$V^*(x,t) = \min_{u \in \mathcal{U}(x)} \Big(J(x,u,t) + V^*(f(x,u,t), t+1) \Big)$$

- Just apply this recursive formula!
- Things to keep in mind:
  - Functional form of V.
    - Is $V(x, t+1)$ the same functional form as $V(x, t+1)$?
  - Is computing the minimum over u tractable and closed form?
    - What kinds of functions is finding the minimum easy?
- If yes to the above, then finding $V^*(x,t)$ at t requires the "same work" for all t

https://stanford.edu/class/ee363/lectures.html