AA/EE/ME 548: Linear Multivariable Control

Lecture 16

5/21/2025

Announcements

- Homework 3 is out (1 week already)
  - Kalman filter problem, MPC problem (HW2P4 adaptation), project update
- Project details on website
  - **Week 10 Monday Lightning Pitch June 2**: 40 students, 80 minutes… (~1.5 minutes pp, +1 min extra for each person in group)
    - Add slide to google slide deck
  - **Week 10 Wednesday Poster Session June 4**: Printing instructions on website. Assigned 2 posters to peer review.
    - Bring poster to session
  - **Finals Week Wednesday June 11**: Report/website submission
    - Submit to Canvas

Model Predictive Control

MPC Core Idea

Repeatedly solve an optimization problem over a finite horizon, execute first control, replan at next time step with updated state (and environment) information

Why?

- Intractable over long horizon
- Model mismatch/uncertain dynamics. Prediction far in the future is not reliable
- Solving Bellman/HJB is intractable (nonlinear dynamics, non-convex cost, high dimensionality)
- …

Receding horizon planning / MPC

1. Plan a trajectory over a **finite horizon** (i.e., a sequence of control inputs)
2. Execute the **first control input** in the sequence
3. Go to step 1.

- Solve long horizon problem with a sequence of shorter horizon problem.
  - Just make T smaller…right?

- **Potential issue?**

Potential challenges

- Terminal cost (e.g., reach goal) is not useful if horizon isn't long enough to reach the goal.
- **Persistent feasibility**. Will future MPC iterations be feasible?
- **Closed-loop stability**. Will the system eventually converge to the desired equilibrium point?

Need some “foresight” to account for future iterations and future state

- Let N be the shorter planning horizon
- Let t be the current time.
- We are planning from t to $t + N$

Cost-to-go

$$\begin{aligned} \min_{\boldsymbol{u}=(u_{t|t},\ldots,u_{t+N|t})} \sum_{k=0}^{N-1} g(x_{t+k|t}, u_{t+k|t}) &+ p(x_{t+N|t}) \\ \text{s.t.} \quad x_{t+k+1|t} &= f(x_{t+k|t}, u_{t+k|t}) \\ x_{t+k|t} \in X, u_{t+k|t} &\in U \quad \forall k \\ x(t) &= x_{t|t} \\ x_{t+N|t} &\in X_f \end{aligned}$$

Terminal set constraint

Something to ensure that the system will end up in a “reasonable” state

Persistent feasibility theorem

- If X_f is a *control invariant set* for the system $x_{k+1} = f(x_k, u_k)$ then the MPC law is persistently feasible. i.e., the next planning step will have a feasible solution

$$
\begin{aligned}
\min_{\boldsymbol{u}=(u_{t|t},\ldots,u_{t+N|t})} \quad & \sum_{k=0}^{N-1} g(x_{t+k|t}, u_{t+k|t}) + p(x_{t+N|t}) \\
\text{s.t.} \quad & x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k|t}) \\
& x_{t+k|t} \in X, u_{t+k|t} \in U \quad \forall k \\
& x(t) = x_{t|t} \\
& x_{t+N|t} \in X_f \ \text{Control invariant set}
\end{aligned}
$$

Proof (sketch)

- Suppose at time t, you have a feasible solution to the current planning problem. $\boldsymbol{u}_{\boldsymbol{0}}^{*} = (u_0, u_1, \dots, u_{N-1})$ and $x_N \in X_f$

- At the next planning step (shift time step forward by one), note that $\boldsymbol{u}_{\boldsymbol{1}} = (u_1, u_2, \dots, u_{N-1}, \tilde{u})$ is a feasible solution to the next planning problem where $\tilde{x}_{N+1} = f(x_N, \tilde{u}) \in X_f$ ($\tilde{u}$ is a feasible control corresponding to the control invariant set definition)

- We have just found a feasible solution to the next planning problem. So we will do at least as good, if not better at the next time step.

Practical considerations

- The terminal set X_f is introduced artificially for the sole purpose of leading to a sufficient condition for persistent feasibility
- Want it to be large so it does not compromise on performance
- How do we pick X_f?
- These sets can be computed by using the MPT toolbox https://www.mpt3.org/

MPC stability theorem

Assume:

1. X, U contains the origin in their interior
2. $x = 0, u = 0$ is an equilibrium $f(0,0) = 0$
3. $\beta^-\|x\| \leq g(x,u) \leq \beta^+\|x\| \quad \forall x \in X, u \in U$
4. $X_f \subseteq X$ is control invariant
5. The terminal cost p is a local control Lyapunov function satisfying
   - $p(x) \leq \beta\|x\|$
   - $\exists\mu \quad p(x) - p(f(x,\mu(x))) \geq g(x,\mu(x)). \ \forall x \in X_f$

Then the origin of the closed-loop system under MPC control is exponentially stable with region of attraction X_0 (N-step BRS of X_f)

Proof (sketch)

- Show that the cost is Lyapunov stable for the closed-loop system
  - i.e., the cost is shrinking over time
- Suppose you have an optimal trajectory $\boldsymbol{u}_{\mathbf{0}}^{*} = (u_0, u_1, \dots, u_{N-1})$
- At the next time step $\boldsymbol{u}_{\mathbf{1}} = (u_1, u_2, \dots, u_{N-1}, \tilde{u})$ ($\tilde{u}$ keep $x_{N+1} \in X_f$)
- But $\boldsymbol{u}_{\mathbf{1}}$ is not necessarily optimal

Rough proof continued

$$J_1^*(x_1) \leq \sum_{k=1}^{N} g(x_k, u_k) + p(x_{N+1})$$

$$= \sum_{k=1}^{N-1} g(x_k, u_k) + g(x_N, \tilde{u}\ ) + p(x_{N+1})$$

$$= \sum_{k=1}^{N-1} g(x_k, u_k) + g(x_N, \tilde{u}) + p(x_{N+1}) + p(x_N) - \ p(x_N) + g(x_0, u_0) - g(x_0, u_0)$$

$$J_1^*(x_1) \leq J_0^*(x_0) + g(x_N, \tilde{u}) + p(x_{N+1}) - p(x_N) - g(x_0, u_0)$$

$$J_0^*(x_0)\ -\ \ J_1^*(x_1) \geq g(x_0, u_0) \geq \beta^- \|x_0\|$$

Cost is decreasing!

MPC stability theorem takeaway

- If we choose the cost-to-go to be a control Lyapunov function, then we can ensure stability (with some assumptions about the problem structure)

$$
\begin{aligned}
\min_{\boldsymbol{u}=(u_{t|t},\dots,u_{t+N|t})} \quad & \sum_{k=0}^{N-1} g(x_{t+k|t}, u_{t+k|t}) + p(x_{t+N|t}) \\
\text{s.t.} \quad & x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k|t}) \\
& x_{t+k|t} \in X, u_{t+k|t} \in U \quad \forall k \\
& x(t) = x_{t|t} \\
& x_{t+N|t} \in X_f \;\; \text{Control invariant set}
\end{aligned}
$$

Control Lyapunov function

What are reasonable choices of X_f and p?

Assuming linear dynamics & quadratic cost

Case 1: A is asymptotically stable

- $u = 0$ is a feasible control
- X_f: maximally positive invariant set for $x_{t+1} = Ax_t$
- p: Solution to the Lyapunov equation

$$-P + Q + A^T PA = 0$$

"Once you reach the end, just do nothing and you will asymptotically converge".

What are reasonable choices of X_f and p?

Assuming linear dynamics & quadratic cost

Case 2: A is open-loop unstable

- Let K_∞ be the optimal gain for infinite horizon LQR
- X_f: maximally positive invariant set for $x_{t+1} = (A + BK_\infty)x_t$
- p: Solution to the discrete Riccati equation

$$-P + Q + A^T PA - (A^T PB)(R + B^T PB)^{-1}(B^T PA) = 0$$

"Once you reach the end, apply the infinite horizon LQR controller".

Some comments

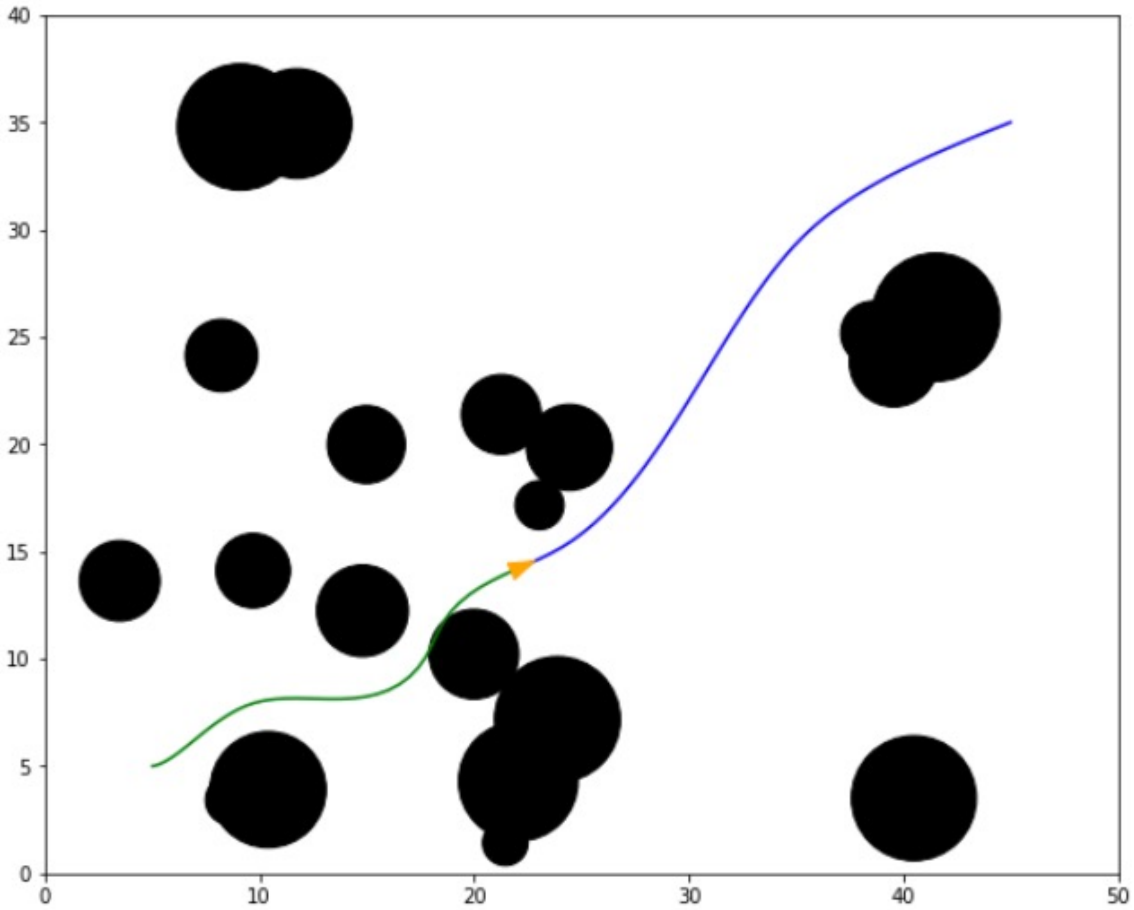
- Presented cases are suboptimal choices
- Need to make sure the control constraints are satisfied (increasing R)
- At present there is no other technique than MPC to design controllers for general large linear multivariable systems with input and output constraints with a stability guarantee
- Design approach (for squared 2-norm cost):
  - Choose horizon length N and the control invariant target set X_f
  - Control invariant target set X_f should be as large as possible for performance
  - Choose the parameters Q and R freely to affect the control performance
  - Adjust p as per the stability theorem
  - Useful toolbox (MATLAB): https://www.mpt3.org/
- In practice, sometimes choosing a good terminal cost is enough (i.e., don't need to enforce a terminal control invariant condition), though you may be sacrificing guarantees

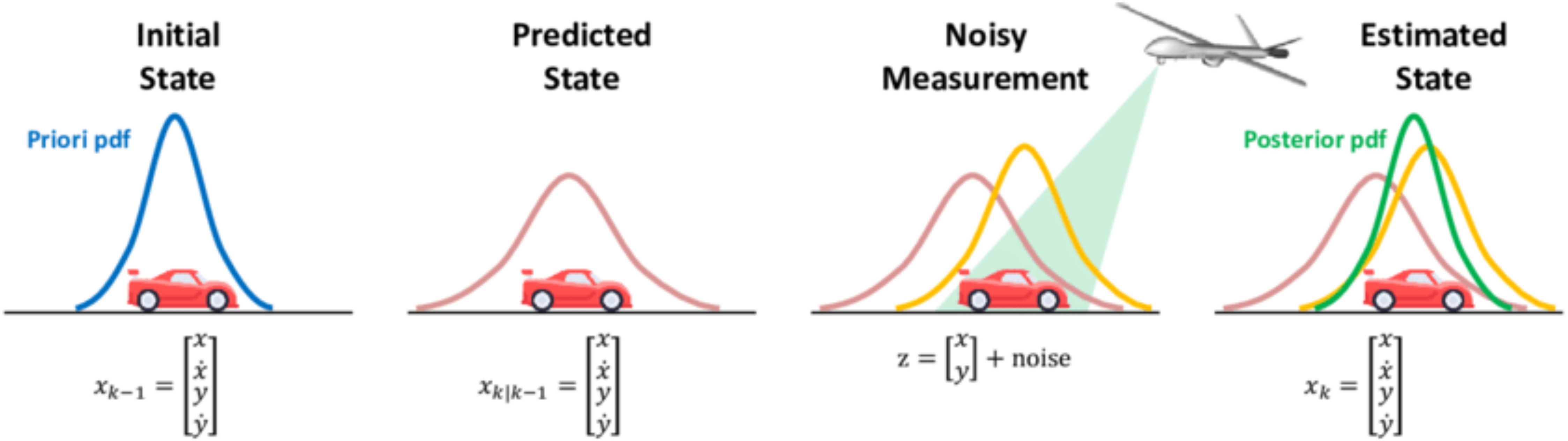
So far…

- Assumed perfect dynamics model
- Assumed no stochasticity
- What about moving obstacles? (Prediction models, human-robot interactions)
- Theorems hold in general but challenging for non-LQR-like structure (i.e., not linear & not quadratic)
- But provide conceptual understanding to what heuristics we would like to design for nonlinear systems
- There is a whole research area on *robust* MPC and *tube* MPC (beyond the scope of this course)

Code demo

Planning a trajectory to avoid moving asteroids

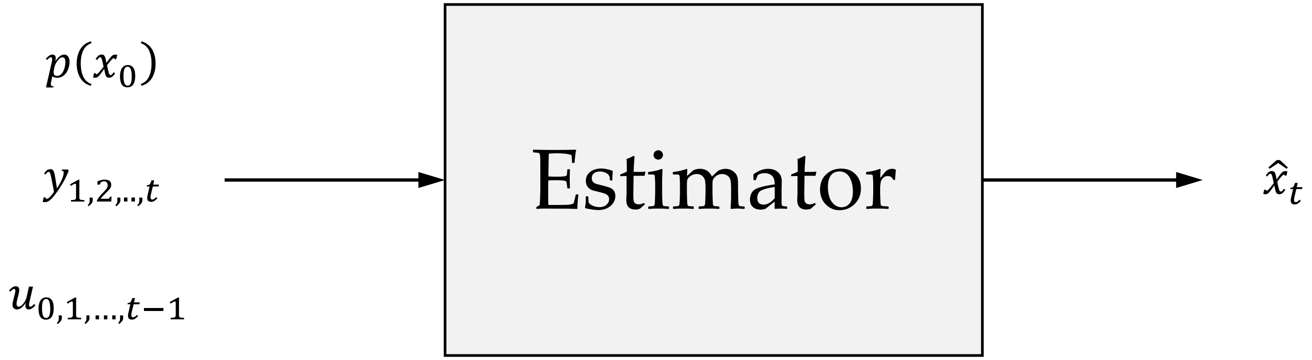




Kalman Filter (State estimation)

Problem set up

- Given initial state estimate, measurements, and control inputs, estimate the state.



Want to make sure $| x_t - \hat{x}_t|$ is small as possible.