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1984 Phys. Educ. 19 24

(http://iopscience.iop.org/0031-9120/19/1/313)

Phys. Educ., Vol 19, 1984. Printed in Northern Ireland.

Combination of thin lenses—a computer oriented method

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The treatment of geometric optics in elementary physics courses is mostly restricted to the thin lens approximation. Even with this restriction calculations of combinations of thin lenses are impeded by annoying sign conventions (Jenkins and White 1957, Sears 1958). In this article a modified approach to the calculation of image formation is described which eliminates the difficulty of sign conventions; the calculation of image formation by combinations of lenses is straightforward and the arithmetic is very suitable for computer treatment at an elementary level—in contrast to more advanced methods such as matrix methods (Gerrard and Burch 1975). With the increasing use of the microcomputer in undergraduate education such a method can be very useful. Our computer adapted method makes use of the connection between the composition of lenses and the mathematics of fractional linear transformations.

Considering only paraxial approximations, the image formation by a single thin lens is usually described by

$$\frac{1}{u}+\frac{1}{v}=\frac{1}{f} \tag{1}$$

where u, v and f represent the object, image and focal distances respectively. When using equation (1) appropriate sign conventions have to be observed. These sign conventions impede the calculation of image formation by optical instruments with more than one lens.

Taking the principal axis as the x axis, equation (1) can be rewritten as

$$\frac{1}{x_1-x_0}+\frac{1}{x_i-x_1}=\frac{1}{f} \tag{2}$$

where x_1, x_0 and x_i refer to the positions of the lens, object and image respectively. With equation (2) the calculation of image formation becomes straightforward and sign conventions are no longer needed. Equation (2) can be written as

$$x_i=\frac{A-Bx_0}{C-x_0} \tag{3a}$$

where

$$A=x_1^2 \tag{4}$$

$$B=x_1+f \tag{5}$$

$$C=x_1-f \tag{6}$$

When this is the first of a series of lenses equation (3a) becomes

$$x_{i_1}=\frac{A_1-B_1x_0}{C_1-x_0} \tag{3b}$$

With the condition

$$D_1=A_1-B_1C_1\neq 0 \tag{7}$$

equation (3b) is identical to the general expression for a fractional linear transformation (FLT) or homography (Duncan 1968). We see from equations (4), (5) and (6) that condition (7) is identical to $f\neq 0$. In this way we can relate the physical object 'thin lens' to the mathematical object 'fractional linear transformation'.

Now we can describe the image formation of a thin lens by the operator P_1 working on x_0

$$x_{i_1}=P_1x_0 \tag{8}$$

where P_1 is defined by equation (3b). If we use two lenses, we can write for the second lens

$$x_{i_2}=P_2x_{i_1} \tag{9}$$

Combining equations (8) and (9)

$$x_{i_2}=P_2\circ P_1x_0 \tag{10}$$

where we can regard

$$P_{12}=P_2\circ P_1 \tag{11}$$

as a multiplication of P_1 and P_2. Note that the set of FLT is closed under this multiplication. This means that two subsequent FLT can always be replaced by a single FLT. If the first two transformations are represented by A_1, B_1, C_1 and A_2, B_2, C_2 respectively, the product transformation A_{12}, B_{12}, C_{12} can by a simple calculation be shown to satisfy

$$A_{12}=(A_2C_1-A_1B_2)/(C_2-B_1) \tag{12}$$

$$B_{12}=(-B_1B_2+A_2)/(C_2-B_1) \tag{13}$$

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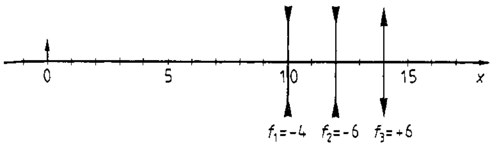

Figure 1 Arrangement of three thin lenses for example calculated

$$C_{12}=(C_1C_2-A_1)/(C_2-B_1) \qquad (14)$$

It is equally easily shown that from $D_1 \neq 0$ and $D_2 \neq 0$ it follows $D_{12} \neq 0$.

In contrast to this, two thin lenses cannot always be replaced by a single thin lens. To see this we eliminate x_1 and f from equations (4), (5) and (6) and get the condition

$$E_1 \equiv (B_1+C_1)^2-4A_1=0 \qquad (15)$$

Condition (15) characterises a subset of the set of FLT which corresponds to thin lenses. This subset (15) is not closed under multiplication (11). In other words two thin lenses can be related to a single FLT but this FLT does not necessarily correspond to a thin lens. This can be demonstrated by a simple example

$$A_1=1, \quad B_1=2, \quad C_1=0$$
$$A_2=0, \quad B_2=-1, \quad C_2=1$$

From equations (12)–(14) it follows that

$$A_{12}=-1, \quad B_{12}=-2, \quad C_{12}=1$$

and from equation (8) we get

$$E_1=E_2=0 \quad \text{and} \quad E_{12}=5$$

which shows that the resulting FLT does not belong to subset (15).

If we want to replace two lenses by a single one, we must get rid of the restriction imposed by condition (15). This condition originates from the fact that the three parameters A_i, B_i and C_i of the FLT are related to only two lens parameters f and x_1—see equations (4)–(6). We can avoid the restriction (15) by adding an extra parameter in equation (2) which then reads

$$\frac{1}{x_r-x_0}+\frac{1}{x_i-x_s}=\frac{1}{f} \qquad (16)$$

However, this equation does not correspond to a thin lens (except when $x_r=x_s$). Equation (16) introduces the concept of thick lens (Hecht and Zajac 1974). A detailed treatment shows that equation (16) defines a new subset of FLT characterised by

$$E_i \equiv (B_i+C_i)^2-4A_i \geqslant 0 \qquad (17)$$

It can be shown that this subset is now closed under multiplication (11). In fact we have replaced two thin lenses by a single thick lens. The parameters x_r and x_s represent the positions of the principal planes.

Example

In this example the image is calculated of the combination of three lenses shown in figure 1.

$$\begin{array}{lll} A_1=x_{1_1}^2=100 & B_1=x_{1_1}+f_1=6 & C_1=x_{1_1}-f_1=14 \\ A_2=x_{1_2}^2=144 & B_2=x_{1_2}+f_2=6 & C_2=x_{1_2}-f_2=18 \\ A_3=x_{1_3}^2=196 & B_3=x_{1_3}+f_3=20 & C_3=x_{1_3}-f_3=8 \\ A_{12}=118 & B_{12}=9 & C_{12}=12\tfrac{2}{3} \\ A_{123}=-122\tfrac{2}{3} & B_{123}=-16 & C_{123}=16\tfrac{2}{3} \end{array}$$

$$x_{i_3}=\frac{-122\frac{2}{3}+16\times 0}{16\frac{2}{3}-0}=-7.36$$

Conclusion

In this article a computer adapted method is described for calculating image formation by combinations of thin lenses. Apart from its easy implementation on a computer, this method has the additional advantage of eliminating the annoying sign conventions. It is sufficient to write a simple computer program which includes equations (3)–(6) and (12)–(14). With given values of x_0, x_{1_i} and f_i the computer can calculate in no time the position x_{i_n} of the image.

References

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Faraday lecture

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