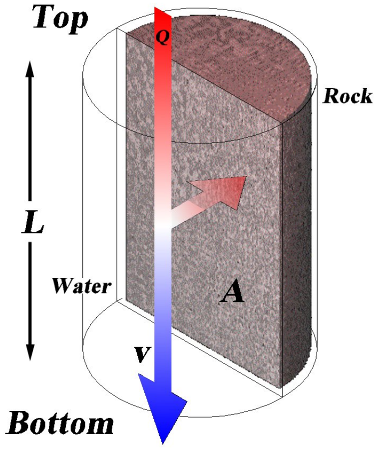
This theory is an attempt to model the heat exchange phenomena between a thermally charged fluid and a non-consolidated aggregate material in a non-pressurized, gravitationally charged hydrodynamic environment.



$$\frac{\partial Q(y,t)}{\partial t} = -k\left(\frac{A}{R}\right) T_{WR}(y,t)$$

$$\int_{Q(y,0)}^{Q(y,t_f)} d\,Q(y,t) \;=\; -k\left(\frac{A}{R}\right)\int_0^{t_f} T_{WR}\,(y,t)\,dt$$

$\ni$

Which yields the heat lost from 1 vertical "slab" over time, $\mathbf{0 < t < t_f}$ . . .

$$Q(y,t_f) - Q(y,0) = -k\left(\frac{A}{R}\right)\int_0^{t_f} T_{WR}(y,t)\,dt$$

Now, get the total heat lost over entire length, $\mathbf{0 < y < L}$

$$\int_0^L \left[Q(y,t_f) - Q(y,0)\right] dy = -k\left(\frac{A}{R}\right)\int_0^{t_f}\left[\int_0^L T_{WR}(y,t_l)dy\right] dt$$

or. . .

$$L\,\left[Q_{tot} - Q_0\right] = -k\left[\frac{A}{R}\right]\int_0^{t_f}\left[\int_0^L T_{WR}(y,t)\,dy\right] dt$$

Finally,

$$\boxed{Q_{tot} - Q_0 = -k\left(\frac{A}{R}\right)\int_0^{b_f}\left[\frac{1}{L}\int_0^L T_{WR}(y,t)dy\right] dt} \tag{1}$$

Now, consider for a vertical slot over a small time step...

$$Q(y+dt) - Q(y,t) = C_N\left[T_W(y,t+dt) - T_W(y,t)\right] \;\Rightarrow C_W = m_W c_W$$

and...

$$\frac{Q(y+dt) - Q(y,t)}{dt} \;=\; C_W \frac{[T_W(y,t+dt) - T_W(y,t)]}{dt}$$

or...

$$\dot{Q}(y,z) = C_W \dot{T}_W(y,t)$$

or...

$$dQ(y,t) = C_W \dot{T}(y,t)dt$$

Integrating...

$$\int_{Q(y,0)}^{Q(y,t_F} dQ(y,t) = C_W \int_0^{t_f} \dot{T}(y,t)dt$$
$$Q(y,t_f) - Q(y,0) = C_W \int_0^{t_f} \dot{T}_W(y,t)dt$$

Get total heat lost from vessel...

$$\boxed{Q_{tot} - Q_0 = C_W \int_0^{t_f} \left[\frac{1}{L}\int_0^L \dot{T}_W(y,t)dy\right] dt} \tag{2}$$

Equate the right hand side of (1) and (2)...

$$-k\left(\frac{A}{R}\right)\int_0^{t_f}\left[\frac{1}{L}\int_0^L T_{CW}(y,t)dy\right]\, dt \;=\; C_W \int_0^{t_f}\left[\frac{1}{L}\int_0^L \dot{T}_W(y,t)dy\right]\, dt$$

To be true, the integrands must be equal...

$$-k\left(\frac{A}{R}\right)T_{WR}(y,t) = C_W \dot{T}_W(y,t)$$

$$\boxed{\dot{T}_{WR}(y,t) = -\frac{kA}{C_{WR}R}T_{WR}(y,t)} \tag{3}$$

Now, to solve this, realize that the rate of heat loss by the water equals the rate of heat gained by the rock...

$\dot{Q} = C_W \dot{T}_W$ and $\dot{Q}_R = C_R \dot{T}_R$

$$;:$$

$$;C_R = mRcR$$

and...

$$
\begin{aligned}
\dot{Q}_W = \dot{Q}_R &\rightarrow \\
\dot{T}_{WR} &= \dot{T}_W - \dot{T}_R \\
&= \dot{T}_W \left(1 - \frac{C_W}{C_R}\right) \\
&= \dot{T}_W \left(1 - \frac{C_R - C_W}{C_R}\right)
\end{aligned}
$$

$$\boxed{\dot{T}_W = \dot{T}_{WR} \left(\frac{C_R - C_W}{C_R}\right)^{-1}} \tag{4}$$

$$\boxed{\dot{T}_{WR}(y,t) = -\frac{kA}{m_W c_W R} \left(\frac{C_R - C_W}{C_R}\right) T_{WR}(y,t)} \tag{5}$$

This has the solution:

$$\boxed{T_{WR}(y,t) = T_{WR}(y,0)e^{-\alpha t}} \tag{6}$$

$$
\begin{aligned}
\text{Where } \alpha &= \frac{kA}{m_W c_W R} \left(\frac{C_R - C_W}{C_R}\right) \\
&\boxed{= \frac{kA}{R} \left(\frac{C_R - CW}{C_W CR}\right)}
\end{aligned} \tag{7}
$$

And finally,

$$\boxed{\dot{T}_{WR}(y,t) = (-\alpha)T - WR(y,0)e^{-\alpha t)}} \tag{8}$$

Let $T_{WR}(y,0) = T_{WR}(0,0)e(-\beta y)$ to represent the top temperature as a constant.

In this case, $\beta \equiv \left(\frac{C_R CW}{C_R - CW}\right)\frac{v}{kAL}$ where v is the flow velocity through the vessel.

$$\begin{aligned}
T_{WR}(y,t) =& T_{WR}(0,0)e^{-\beta y}e^{-\alpha t} \quad \ldots \text{from } (6) \\
\bar{T}_{WR}(t) =& \frac{1}{L}\int_0^L T_{WR}(0,0)e^{-\beta y}e^{-\alpha t}\,dy \\
=& \frac{1}{\beta L}T_{WR}(0,0)e^{-\alpha t}e^{-\beta y}|_0^L \\
=& \frac{T_{WR}(0,0)}{\beta L}(1 - e^{-\beta L}e^{-\alpha t}
\end{aligned}$$

So now (1) becomes ...

$$\begin{aligned}
Q_{tot} - Q0 =& -k\left(\frac{A}{R}\right)\left(\frac{T_{WR}(0,0)}{\beta L}\right)1 - e^{-\beta L}\int_0^{t_f} e^{-\alpha t}\,dt \\
\text{Let } \gamma =& -k\left(\frac{A}{R}\right)\left(\frac{T_{WR}(0,0)}{\beta L}\right)1 - e^{-\beta L}
\end{aligned}$$

$$\text{and so } \boxed{Q_{tot} = Q_0 - \frac{\gamma}{\alpha}(1 - e^{-\alpha t})} \; (11)$$

skipping the check part for now...

$$Q_{tot} - Q0 = -C_W \left(\frac{C_R}{C_R - C_W} \right) \frac{\alpha}{\beta L} T_{WR}(0,0)(1 - e{-}\beta L) \int_0^{t_f} e^{-\alpha t}\, dt$$

and...

$$\gamma = C_W \left(\frac{C_R}{C_R - C_W} \right) \frac{\alpha}{\beta L} T_{WR}(0,0)(1 - e{-}\beta L)$$
$$\boxed{Q_{tot} = Q_0 - \frac{\gamma}{\alpha}(1 - e^{-\alpha t})} \tag{13}$$

and finally, we can write everything together...

$$\boxed{\dot{Q} = -\gamma e^{-\alpha t} = -\left(\frac{k^2 A^2 (C_R - C_W)}{R v C_R C_W} \right) T_{top}(1 - e^{-\beta L}) e^{-\alpha t}}$$