Particle MCMC for Inference in State-Space Models

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State-Space Models

In State-Space Modelling, we consider two random variables: $\{X_t\}_{t\geq 0}$, the latent variable, this is a Markov process which we do not observe. $\{Y_t\}_{t\geq 1}$ the observed variable, this is dependent on the Markov process. This is formalised as follows,

$$\begin{aligned} X_0 &\sim \mu(x_0), \\ X_t|(X_{t-1}=x_{t-1}) &\sim f(x_t|x_{t-1},\theta), \\ Y_t|(X_t=x_t) &\sim g(y_t|x_t,\theta). \end{aligned} \tag{1}$$

This leads to two major questions,

- **Filtering**: What is the latent variable at current time given as new observations are made, $p(x_T|y_{1:T},\theta)$?
- **Parameter Inference**: Can we make inference on the parameter values θ, $\pi(\theta)$?

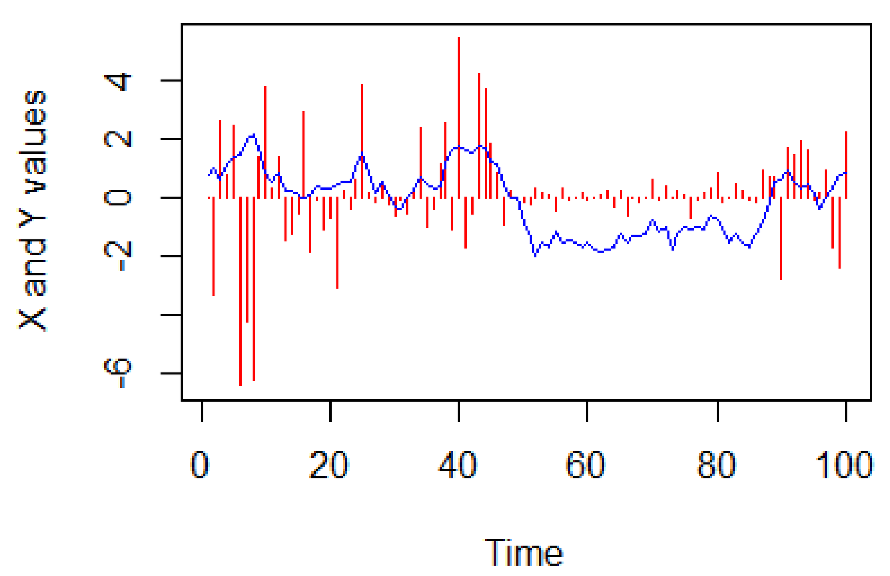
These problems are mostly intractable, thus we use **Monte Carlo** methods to make inferences on these distributions.

Stochastic Volatility

One area which we can apply this is within finance, where we model the log-volatility as a latent Markov process X and log-returns as Y such that,

$$\begin{aligned} X_t|(X_{t-1}=x_{t-1}) &\sim \mathcal{N}(\gamma x_t,\sigma^2), \\ Y_t|(X_t=x_t) &\sim \mathcal{N}(0,\exp(x_t)). \end{aligned} \tag{2}$$

Below is a simulation over 100 time periods. Red represents log-returns and blue represents log-volatility.



Bootstrap Filter

One way to approach **filtering** is with an algorithm known as bootstrap filter. This is described as follows,

1. **Initialise** particles by sampling $x_0^{(i)} \sim \mu(x_0)$,
2. Set $t \leftarrow 1$,
3. **Propagate** particles by sampling $\tilde{x}_t^{(i)} \sim f(\tilde{x}_t|x_{t-1}^{(i)})$,
4. **Assign weights** by calculating $\tilde{w}_t^{(i)} = g(y_t|x_t^{(i)})$ and normalise to $w_t^{(i)} \propto \tilde{w}_t^{(i)}$ where $\sum_i w_t^{(i)} = 1$,
5. **Resample** the particles with probability $w_t^{(i)}$ corresponding to particle $\tilde{x}_t^{(i)}$ to obtain $x_t^{(i)}$,
6. Set $t \leftarrow t+1$ and repeat until $t > T$ after this step.

Dangers of Resampling

In the bootstrap filter, (multinomial) resampling can be wasteful when weights are nearly uniform. For example, if $\{w_t^{(i)}\}_{i=1}^n = 1/n$, approximately **37% of particles are lost**. A simple solution is to monitor the effective sample size (ESS) and resample only when it falls below a threshold.

Particle MCMC

To approach the **parameter inference** problem, we use method known as the particle marginal Metropolis-Hastings (PMMH) algorithm, which we can use to obtain $p(x_{1:T},\theta|y_{1:T})$.

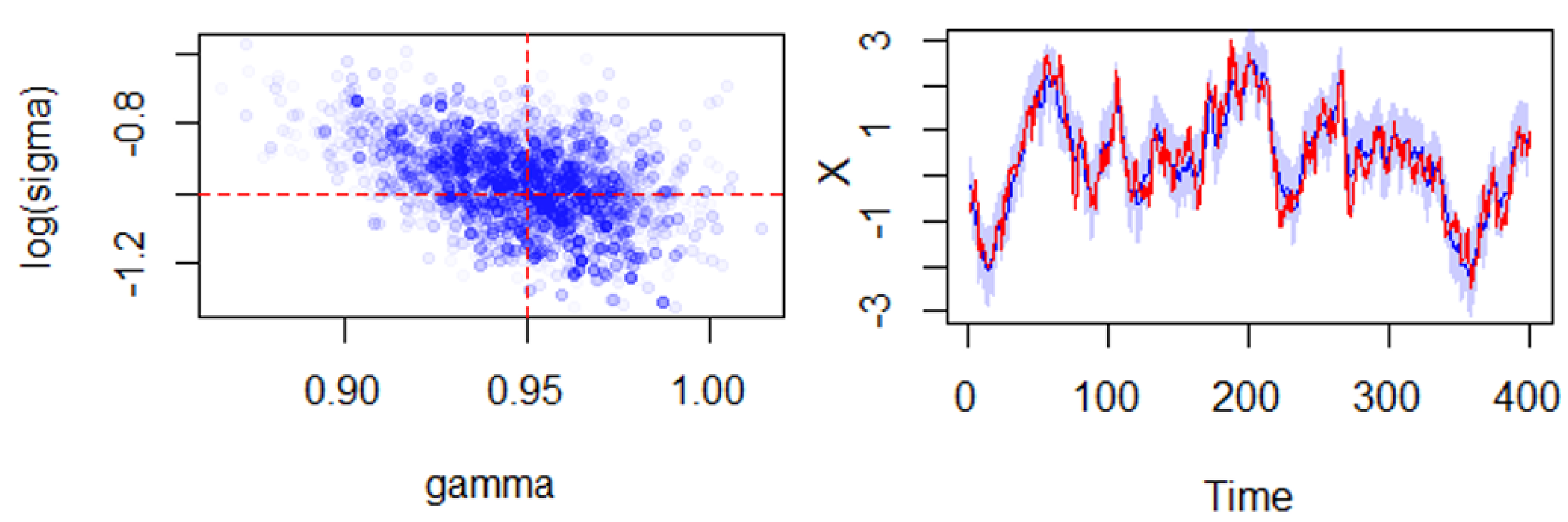
Define u as an auxiliary random variables generated by a **particle filter** $q_*(u|\theta)$ which contains particles $x_{1:T}^{1:n}$ and ancestor indices from resampling $a_{1:T}^{1:n}$. Each iteration of the PMMH algorithm can be described as follows,

1. Set parameters from previous iterations, $\theta \leftarrow \theta_{i-1}, u \leftarrow u_{i-1}, x_{1:T} \leftarrow x_{i-1,1:T}$
2. Propose a new parameter θ from proposal $q(\theta'|\theta)$,
3. Sample u' from $q(u'|\theta')$ and estimate $\hat{\pi}(\theta';u')$,
4. Sample k' from $q(k'|u',\theta')$, find the backwards path associated with $x_T^{(k')}$ denote the full path $x'_{1:T}$.
5. Calculate acceptance probability,
$$\alpha(\theta,u;\theta',u') = 1 \wedge \frac{\hat{\pi}(\theta';u')q(\theta|\theta')}{\hat{\pi}(\theta;u)q(\theta'|\theta)},$$
6. Generate $U \sim \text{Unif}(0,1)$. **If** $U \leq \alpha$, we **accept** the move: Set $\theta_i \leftarrow \theta', u_i \leftarrow u', x_{i,1:T} \leftarrow x'_{1:T}$. **Else**, we **reject** the move: Set $\theta_i \leftarrow \theta, u_i \leftarrow u, x_{i,1:T} \leftarrow x_{1:T}$.

Empirical Experiment

We test the PMMH algorithm on the stochastic volatility model with parameters $\theta = \{\gamma, \log\sigma\} = \{0.95, -1\}$. We set $\theta_0 = \{0.5, 0\}$, used **random walk proposal** $q(\theta'|\theta) \sim \mathcal{N}(\theta, h\boldsymbol{I})$ with $h = 0.005$. Each particle filter runs with $n = 500$ particles, and we ran this over 10,000 iterations.

After removing burn-in samples $b = 100$, we obtain the following plots. On the left is the joint likelihood for $p(\theta|y_{1:T})$. On the right plot is the latent variables posterior $p(x_{1:T}|y_{1:T})$ with red line denoting the true latent trajectory.



Further Work

- **Filtering**: Improve propagation by considering future observed value.
- **Parameter Inference**: Explore implementing the algorithm within a Gibbs sampling framework

References

[Fearnhead and Sherlock, 2025] Fearnhead, P. and Sherlock, C. (2025). MCMC for State Space Models. In *Handbook of Markov Chain Monte Carlo Volume 2*.