MAT 310 LINEAR ALGEBRA HOMEWORK 5

Section 2.3: 2a 3 9 11 12 13

(2a. omitted)

3. Let $g(x) = 3 + x$, and define linear transformations $T : P_2(R) \to P_2(R)$ and $U : P_2(R) \to R^3$ by

$$T(f(x)) = f'(x)g(x) + 2f(x)$$
$$U(a + bx + cx^2) = (a + b, c, a - b)$$

Denote by β and γ the standard (ordered) bases of $P_2(R)$ and R^3.

(a) Compute directly the matrices of U, T, and UT with respect to these bases. Then verify the last one with the theorem on the composition of linear transformations (2.11 in your textbook).

(b) Let $h(x) = 3 - 2x + x^2$. Compute the expression of $h(x)$ with respect to β and the expression of $U(h(x))$ with respect to γ. Then verify these expressions using the matrix of U from the previous part and the theorem 2.14 in your textbook.

(a) By a direct calculation,

$$\begin{aligned} T(\beta_1) &= 0(3 + x) + 2(1) &&= 2(1) + 0(x) + 0(x^2) \\ T(\beta_2) &= 1(3 + x) + 2(x) &&= 3(1) + 3(x) + 0(x^2) \\ T(\beta_3) &= 2x(3 + x) + 2(x^2) &&= 0(1) + 6(x) + 4(x^2) \end{aligned}$$

The matrix $[T]_\beta$ of T with respect to the basis β is by definition the matrix whose columns are the expressions of the $T(\beta_i)$ with respect to the basis β. Therefore this matrix is

$$[T]_\beta := \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

By another direct calculation:

$$\begin{aligned} U(\beta_1) &= (1 + 0, 0, 1 - 0) &&= (1, 0, 1) = \gamma_1 + \gamma_3 \\ U(\beta_2) &= (0 + 1, 0, 0 - 1) &&= (1, 0, -1) = \gamma_1 - \gamma_3 \\ U(\beta_3) &= (0, 1, 0) &&= (0, 1, 0) = \gamma_2 \end{aligned}$$

The matrix $[U]_\beta^\gamma$ of U with respect to the bases β and γ is by definition the matrix whose columns are the expression of the $U(\beta_i)$ with respect to the basis γ. Therefore this matrix is

$$[U]_\beta^\gamma := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

The last slew of direct calculations is:

$$U(T(\beta_1)) = U(2,0,0) = (2+0, 0, 2-0) = (2,0,2)$$
$$U(T(\beta_2)) = U(3,3,0) = (3+3, 0, 3-3) = (6,0,0)$$
$$U(T(\beta_3)) = U(0,6,4) = (0+6, 4, 0-6) = (6,4,-6)$$

Therefore the matrix of UT is

$$[UT]_\beta^\gamma := \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}$$

According to the theorem 2.11 from the textbook, this last matrix should equal to the following matrix product:

$$[U]_\beta^\gamma [T]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

This is true. It can be verified by direct calculation of each entry according to the definition of the matrix product.

(b) The expression of $h(x) = 3 - 2x + x^2$ with respect to the standard basis β of $P_2(R)$ is

$$[h(x)]_\beta = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

According to a direct calculation, $U(h(x)) = (3 + (-2), 1, 3 - (-2)) = (1,1,5)$. So the matrix of $U(h(x))$ with respect to the standard basis γ of R^3 is

$$[U(h(x)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

According to the theorem in your textbook, these expressions should satisfy the matrix equation

$$[U]_\beta^\gamma [h(x)]_\beta = [U(h(x))]_\gamma$$

This can be verified by direct calculation of the entries of the matrix product shown below:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

9. Find linear transformations $U, T : F^2 \to F^2$ such that $UT = 0$ (the zero linear transformation), but $TU \neq 0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

Notice that the condition $UT = 0$ is equivalent to the statement that the image of T is contained in the null space of U. So let's start with a transformation having null-space equal to the span of $(1, 0)$ and another having image equal to this span.

For $(a, b) \in F^2$, set $U(a, b) = (b, b)$ and $T(a, b) = (a + b, 0)$. For each such (a, b),

$$U(T(a, b)) = U(a + b, 0) = (0, 0)$$

Therefore $UT = 0$. Let's see if we also succeeded in arranging for TU to be non-zero:

$$T(U(a, b)) = T(b, b) = (b + b, 0) = (2b, 0)$$

We did.

Now, according to the theorem on composition of linear transformations, if A denotes the matrix of U and B denotes the matrix of T (all with respect to the standard basis of F^2, say), then the matrix product AB should equal to the zero matrix O:

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is verified by direct calculation of the matrix product.

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11. Let T be a linear transformation. Prove that $T^2 = 0$ if and only if $R(T) \subset N(T)$.

Suppose that the image of T is contained in the null space of T. By definition of the image, each $T(v)$ is in the image of T. By the assumption, $T(v)$ is also in the null space of T. Then by definition of the null space, $T(T(v)) = 0$. Since this is true for all v in the domain of T, it follows that $T^2 = 0$.

Now suppose that $T^2 = 0$. Each element of the image of T is of the form $T(v)$ for some v in the domain of T. By the assumption, $T(T(v)) = 0$. So $T(v)$ also belongs to the null space of T. Therefore the image of T is contained in the null space of T.

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12. Let T and U be linear transformations which are composable, i.e. so that the composition UT is defined.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one in this case?
(b) Prove that if UT is onto, then U is onto. Must T also be onto in this case?
(c) Prove that if U and T are one-to-one and onto, then UT is also.

(a) Suppose that UT is one-to-one. According to a theorem in your textbook, this condition is equivalent to the statement that for each non-zero x in the domain of T, $UT(x)$ does not equal to 0. For each such x, $T(x)$ is also not equal to zero (otherwise $U(T(x))$ would equal $U(0) = 0$!). Therefore T is also one-to-one in this case.

However, in this case U may not be one-to-one. It is not hard to think of examples, but the idea is this: What can happen is that U is one-to-one when restricted to the image of T, but has non-trivial null space elsewhere in its domain.

(b) Suppose that UT is onto. This means that, given a vector x in the target space of U, there is a vector y in the domain of T such that $U(T(y)) = x$. In this case, there is also a vector z such that $U(z) = x$, namely $z = T(y)$. Therefore U is also onto.

However, in this case T may not be onto. Again, it is not hard to think of examples, but the idea is: What can happen is that, even if the image of T is not the entire domain of U, the restriction of U to this image is already onto.

(c) Suppose that U and T are both one-to-one and onto. Then for each non-zero x in the domain of T, $T(x)$ is not equal to 0, and further $U(T(x))$ is not equal to 0. Therefore UT is one-to-one. Also, for each y in the target space of U, there is a vector z such that $U(z) = y$. Moreover, there is a vector w such that $T(w) = z$. For this w, $U(T(w)) = U(z) = y$. Therefore UT is also onto.

(13 omitted)