An application of the Poincare dodecahedral space to inertial orientation

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The task of an inertial orientation system is to determine the orientation, or rotation, of an object using only measurements of acceleration and, sometimes, angular velocity. This is identified as the geometric problem of integrating a certain $\mathfrak{so}(3)$-valued 1-form on I or $\mathbf{R}$ to a path in the configuration space SO(3). We describe two standard solutions as well as one new one derived from basic geometry. The latter has potentially superior numerical stability on account of a coordinate system well adapted to the symmetry of SO(3).

Contents

Introduction

The configuration space is SO(3), the 3-dimensional Lie group of orientation-preserving orthogonal transformations of $\mathbf{R}^3$, the symmetry group of the 2-sphere. Assume you are equipped with a 3-axis gyroscope, so that your input data consists of 3 real-valued functions of time v_x, v_y, v_z, the angular velocities. Having fixed the basis r_x, r_y, r_z for the Lie algebra $\mathfrak{so}(3)$ consisting of the infinitesimal generators of rotations about the 3 respective axes of the gyroscope when in some initial state, these functions together form an $\mathfrak{so}(3)$-valued function of time $v(t)$.

Our goal is to use $v(t)$ to compute the path $\gamma : \mathbf{R} \to$SO(3) through the configuration space over time. The fact that mere integration is unhelpful is due to the non-trivial topology of the configuration space, in particular that it is a non-abelian group. Less abstractly, it is because $v(t)$ only measures your angular velocity as expressed in the current frame.

The elegant solution of this problem is to interpret $v(t)$ as the expression of the tangent vector $\gamma'(t)$, located at $\gamma(t)$ in SO(3), with respect to an appropriate framing in SO(3). If left multiplication is used to define the action of SO(3) on itself as the configuration space, this framing is obtained by left translating the vectors r_x, r_y, r_z in $\mathfrak{so}(3) = T_e SO(3)$ using the differential of left multiplication. Roughly what is going on is that v tells us how to steer on the road, and the framing tells us how the road curves to fit SO(3).

Technically, the framing is a smooth function from the space of tangent vectors in SO(3) to $\mathfrak{so}(3)$. Smooth functions of the tangent bundle of a manifold to $\mathbf{R}$ or a vector space are called differential 1-forms if they are made of linear maps on each tangent space. The form just described is called the left-invariant Maurer-Cartan form, or translation form, and I will denote it ω.

The non-abelian fundamental theorem of calculus says that $\mathfrak{g}$-valued 1-forms η on a simply-connected piece of k-dimensional space can be integrated to G-valued functions (in our case $k = 1$, paths γ in G) provided that they, like the Maurer-Cartan form, satisfy the integrability condition $d\eta = -\frac{1}{2}[\eta, \eta]$. For us this equation is automatic because 2-forms on a 1-dimensional space are all trivial. Here integrated means the framing of the resulting path by ω matches η; $\gamma^*(\omega) = \eta$.

We use this language to describe implementations based on three different parameterizations of the configuration space: global matrix coordinates, coordinates on the unit quaternions or 3-sphere, and Poincare dodecahedral coordinates.

Global coordinates

SO(3), like many finite-dimensional Lie groups, can be realized as a matrix group, in this case as the subgroup of GL(3,$\mathbf{R}$) $\subset \mathbf{R}^9$ consisting of orientation-preserving orthogonal matrices (the determinant is 1 and the columns are orthonormal). The 9 matrix entries are global coordinate functions. The ordinary trivialization $\tau : T\mathbf{R}^{n^2} \to \mathbf{R}^{n^2}$ of the tangent bundle does not descend to the translation form on GL(n) or SO(n), but their relationship is well-known: With tangent vectors regarded as matrices via τ, the derivative of (left) matrix multiplication by a matrix g in GL is just (left) matrix multiplication by g. That is, $\omega_g = g^{-1}\tau_g$, where the operation on the right is matrix multiplication (this expression is sometimes written $g^{-1}dg$).

Now we are looking for a path γ with $\gamma^*(\omega) = \eta$, or equivalently $\omega(\gamma'(t)) = v(t)$. Using the formula, $\tau(\gamma'(t)) = \gamma(t)v(t)$. Since $\tau(\gamma'(t))$ is just the expression of $\gamma'(t)$ as a matrix whose entries are the derivatives of the entries of $\gamma(t)$, this is precisely the evolution equation for γ. Explicitly, if x_i, y_i, z_i are the components of $\gamma(t)$:

$$\begin{pmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \\ z_1' & z_2' & z_3' \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} (v_x r_x + v_y r_y + v_z r_z) \tag{1}$$

$$\begin{pmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \\ z_1' & z_2' & z_3' \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{pmatrix} \tag{2}$$

Coordinates on $S^3 \subset \mathbf{R}^4$

SO(3) is doubly covered by the 3-sphere, indeed by a group homomorphism. That means the local geometry and group structure of SO(3) and S^3 are the same. We get coordinates on SO(3) from the representation of S^3 as the unit sphere in $\mathbf{R}^4$. The multiplication is quaternion multiplication:

$$x \cdot y = (x_0, x_1, x_2, x_3) \cdot (y_0, y_1, y_2, y_3) = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = L(x)(y) \tag{3}$$

The form shown here represents left multiplication by x as a linear map $L(x)$ on the coordinate space $\mathbf{R}^4$. For the same reason as before, the derivative of this map, with respect to the usual global trivialization $T\mathbf{R}^4 = \mathbf{R}^4 \times \mathbf{R}^4$, is the same matrix. With $(1, 0, 0, 0)$ as the identity, and $\mathfrak{so}(3) = 0 \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R} \subset \mathbf{R}^4$, the translation form is $\omega_x(v_0, v_1, v_2, v_3) = L(x)^{-1}(v)$. With the identification of the r_x, r_y, r_z from before with $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$, from $\omega(\gamma'(t)) = v(t)$ we obtain the evolution equations:

$$L(x)^{-1}(x_0', x_1', x_2', x_3') = (0, v_x, v_y, v_z) \tag{4}$$

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} 0 \\ v_x \\ v_y \\ v_z \end{pmatrix} \tag{5}$$

We record for future use the fact that the left-invariant vector fields corresponding to $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ are $(-x_1, x_0, x_3, -x_2)$, $(-x_2, -x_3, x_0, x_1)$, and $(-x_3, x_2, -x_1, x_0)$.

These evolution equations are much more numerically stable than those associated to matrix coordinates, since the latter requires Gram-Schmidt orthogonalization at each step to get from a 9-dimensional space back to the true 3-dimensional configuration space, and the former only requires ordinary scaling from the 4-dimensional $\mathbf{R}^4$ to the unit 3-sphere. No normalization step is required in coordinate systems of dimension 3. The Euler angle coordinate system is 3-dimensional, but it is both maladapted to the geometry of SO(3) (not only must a frame be distinguished, but its axes must be ordered) and singular (see any discussion of gimbal lock). The following three-dimensional coordinates closely approximate the intrinsic geometry of the configuration space.

Poincare dodecahedral coordinates

Typical directions in a city first indicate a block or an intersection on a grid of roads, then microdirections within the neighborhood. In flat terrain, one can choose arbitrarily fine nested rectilinear grids to establish a discrete combinatorial domain for global navigation and a small geometric domain for local navigation. Such grids can be chosen to be globally homogeneous in the sense that the global symmetries act transitively on the set of regions, and isotropic in the sense that the regions are regular.

In positively-curved spaces like SO(3) and S^3, arbitrarily fine grids are not possible. An especially fine division of SO(3) into equal countries is obtained by translating a fundamental domain of the action of the subgroup $\mathbf{I} \subset$SO(3) consisting of symmetries of the icosahedron or dodecahedron.

Perhaps surprisingly, the fundamental domain is also a dodecahedron. (Think of blowing up balloons at each element of $\mathbf{I}$ until they cover. Studying the elements directly, there are 12 elements closest to the identity, the 6 minimal rotations in face-center axes and their inverses.) It lifts to S^3 to a dodecahedron whose faces are sectors of great spheres. Choosing it to be centered at the identity, stereographic projection from the negative of the identity sends it to a dodecahedron in $\mathbf{R}^3$ whose faces are sectors of ordinary 2-spheres. *A configuration is specified by a point in this dodecahedron and a group element g indicating which of the 60 blocks we are in.* Really, g is a vertex of the Cayley graph on these 12 generators, and when our point travels across a face, the new g is obtained by following the edge labelled by the generator corresponding to that face. The point emerges from the opposite face in a dodecahedron identified with the first by a minimal $2\pi/10$ clock-wise rotation, applied with respect to the usual translation of the ambient space one would use to compare ordinary dodecahedra sharing a face.

The stereographic projection sends (x_0, x_1, x_2, x_3) to $\frac{2}{1+x_0}(x_1, x_2, x_3)$. Applying the differential of this map to the left-invariant vector fields on S^3 corresponding to our chosen basis of $\mathfrak{so}(3)$, we actually get the usual coordinate vector fields of $\mathbf{R}^3$ (!) scaled by the function $1+|u|^2$, where $u=(u_1,u_2,u_3)\in\mathbf{R}^3$. The evolution equations are therefore as simple as one could hope:

$$u_1' = (1+|u|^2)v_x \tag{6}$$
$$u_2' = (1+|u|^2)v_y \tag{7}$$
$$u_3' = (1+|u|^2)v_z \tag{8}$$

In fact, as we are about to see, the largest that $1+|u|^2$ gets on the puffy dodecadedral domain of interest is about $(1+0.317^2)\approx 1.100$. That means the algorithm locally deviates from naive component-by-component integration only very little, as our coordinate system very closely approximates Euclidean space. The global information is encoded discretely, and hence optimally stably, by the group element g. In principle it is always possible to approximate the local geometry of, say, a Riemannian manifold better and better by choosing smaller and smaller coordinate systems–it's just that we are not always so lucky as to have a symmetric way of choosing small coordinate systems covering the whole space.

Let's be more precise about the shape of the fundamental domain. If the edge length of a Euclidean dodecahedron is 1, the radius of the inscribed sphere is

$$r_i = \frac{\phi^2}{2\sqrt{3-\phi}} = \frac{1}{20}\sqrt{250+110\sqrt{5}} \tag{9}$$

The midradius, the distance from the center to the center of an edge, is

$$r_m = \frac{\phi^2}{2} = (3+\sqrt{5})/4 \tag{10}$$

Here ϕ is the so-called golden ratio, $(1+\sqrt{5})/2$. A quick application of the Law of Cosines gives the distance a from the center of a face to the midpoint of one of its edges:

$$a = 1/\sqrt{2-2\cos(2\pi/5)} \tag{11}$$

Observe that if the angle between two neighboring face-center axes is θ, then

$$\cos(\theta/2) = r_i/r_m = 1/\sqrt{3-\phi} = 4/\sqrt{5-\sqrt{5}} \tag{12}$$
$$\sin(\theta/2) = a/r_m = \frac{2\sqrt{2}}{\sqrt{(1-\cos(2\pi/5))(3+\sqrt{5})}} \tag{13}$$

Now the quaternion corresponding to a clock-wise rotation by α about the axis determined by a unit vector (x_1,x_2,x_3) is $(\cos(\alpha/2),\sin(\alpha/2)(x_1,x_2,x_3))$. If the z-axis passes through the center of one of the faces, let $(\sin\theta(\cos(2\pi k/5),\sin(2\pi k/5)),\cos\theta)$ (for $k=0,1,2,3,4$)

be the axes of the neighboring 5 faces. Then the 12 closest group elements are the following 6 and their inverses:

$$R_z = (\cos(2\pi/10), 0, 0, \sin(2\pi/10)) \tag{14}$$

$$R_k = \begin{pmatrix} \cos(2\pi/10) \\ \sin(2\pi/10)\sin\theta\cos(2\pi k/5) \\ \sin(2\pi/10)\sin\theta\sin(2\pi k/5) \\ \sin(2\pi/10)\cos\theta \end{pmatrix} \tag{15}$$

$$= \begin{pmatrix} \cos(2\pi/10) \\ \frac{r_i^2-a^2}{r_m^2}\sin(2\pi/10)\cos(2\pi k/5) \\ \frac{r_i^2-a^2}{r_m^2}\sin(2\pi/10)\sin(2\pi k/5) \\ \frac{2r_i a}{r_m^2}\sin(2\pi/10) \end{pmatrix} \tag{16}$$

They are arranged symmetrically about e, so consider R_z. Its dot product in $\mathbf{R}^4$ with e is $\cos(2\pi/10)$, so the angle between them is $2\pi/10$. That means the great 2-sphere S of points equidistant from e and R_z (a portion of which bounds our fundamental domain) is deflected an angle $2\pi/20$ from one passing through e. Our stereographic projection $S^3 - \{-e\} \to \mathbf{R}^3$ restricts to ordinary stereographic projection on any great 2-sphere through e, in particular the 2-sphere spanned by the great circle L through e and R_z and any one of the S^1-many great circles lying in S through L. Then the ordinary formulae for the radius and distance of the center from the origin of the stereographic image of a great circle deflected an angle α apply: $r = \cot(\alpha/2) + \tan(\alpha/2)$ and $d_c = \cot(\alpha/2) - \tan(\alpha/2)$. In our case, this family of circles projects to a family sharing the same center and radius, $r \approx 6.472$ and $d_c \approx 6.155$, and sweeping out an ordinary 2-sphere with this radius and central distance. The center of a face in $\mathbf{R}^3$ is $r - d_c = 2\tan(2\pi/40) \approx 0.317$ away from the origin. Note that the faces are sectors of spheres and are puffed out rather than in.

The point of computing the nearby group elements explicitly is to get a formula for their action on the configuration space in the $\mathbf{R}^3$ coordinates. We'll need this to apply the "translations" (not Euclidean translations unfortunately) to get a configuration point that has crossed a face back into the domain. For example, if the configuration point has crossed the face corresponding to R_z, we project it back to S^3, quaternion left multiply by R_z^{-1}, then project back again to $\mathbf{R}^3$.

One can verify by direct computation or by constructing a model of the dodecahedral tiling that the choice of generators

$$R_z = (12345) \tag{17}$$
$$R_0 = (15243) \tag{18}$$
$$R_1 = (13254) \tag{19}$$
$$R_2 = (14352) \tag{20}$$
$$R_3 = (15324) \tag{21}$$
$$R_4 = (13542) \tag{22}$$

specifies an ismorphism $A_5 \to \mathbf{I}$, and so the discrete part of the coordinates of a configuration can be encoded easily as an alternating 5-cycle. This is not trivial; there are 20 elements of order 5 in A_5, belonging to two conjugacy classes, and not every choice of six work. One must ensure for example that the sets $\{g^{(-1)2}, g^{-1}, g, g^2\}$ for the generators g are disjoint and that the consecutive products and the products with (12345) are all distinct and conjugate.

Error correction using acceleration measurements

In the presence of gravity, a 3-axis accelerometer can be used to ascertain which direction is down in the frame of a device. This vector determines a circle's worth S^1 of potentially correct configurations in SO(3), those whose z-axis matches the reading (assuming the initial state, corresponding to 1 in SO(3), has the z-axis pointing down). If our algorithm for producing $\gamma(t)$ from $v(t)$ does not produce a configuration matching one on this circle, the error is likely the integral error associated with any discrete integration algorithm. Since the acceleration measurements are more accurate (in particular, they cannot have integral error), it makes sense to modify the algorithm so that at each step, instead of adding to $\gamma(t)$ the usual small displacement associated to the angular velocity $v(t)$, we average $v(t)$ with a velocity u that points along the most efficient path to the circle of valid configurations. Given the acceleration measurement, we want u to be a vector field on SO(3) that is zero on S^1, always points towards it otherwise, and gradually decreases in magnitude as we approach it. Moreover we certainly want it to be symmetric in the sense that it should be invariant under the isometries of SO(3) that preserve S^1. A natural choice of such a field is the field orthogonal to the fibers of the Hopf fibration $S^1 \to S^3 \to S^2$ obtained by lifting a longitudinal pole-to-pole vector field on S^2, descended from S^3 to SO(3).