

The second fundamental form and the Wilczynski equations for smooth projective surfaces

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Abstract

The classical Wilczynski equations [Wil07] play a role for smooth real projective surfaces $M \subset \mathbb{RP}^3$ up to projective transformation that is roughly analogous to the role played by the Gauss-Codazzi equations for surfaces in Euclidean space up to isometry. However, the Wilczynski equations only apply in the hyperbolic region of M , near “saddle points”.

Tabachnikov and Ovsienko [TO04] reformulated these equations in terms of a flat connection on a rank 4 vector bundle over M . By considering the conformal class of the second fundamental form \mathbf{II} of M , we generalize their formulation to arbitrary non-degenerate surfaces, including the elliptic case of locally-convex surfaces.

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1 Introduction

By giving several geometric constructions, we observe in section 2 that the conformal class of the second fundamental form of a smooth surface M in Euclidean 3-space, which we denote by Π_M , is invariant by projective transformations. That is, it is well-defined on M when M is regarded as a surface in the 3-dimensional real projective space. This is a slight departure from the linear system of quadrics $|\Pi|$ appearing in [GH79] or the differential invariants of [AG93], both of which generalize the Riemannian second fundamental form to projective geometry via general exterior differential calculus rather than low-dimensional synthetic geometry.

Throughout we shall use the notation:

- V , a 4-dimensional real vector space
- $P^3 := \mathbb{P}V$
- $M \rightarrow P^3$, a smooth immersed surface

There is a simple equivalence between the data of an immersion $M \rightarrow P^3$ up to projective transformations and a triple $(E, \nabla, [p])$ up to isomorphism, where:

1. E is a trivial rank 4 real vector bundle over M ,
2. ∇ is the trivial flat connection on E , and
3. $[p] \in \Gamma(\mathbb{P}E^*, M)$ is a smooth section of the fiber bundle $\mathbb{P}E^*$ over M , locally represented by smooth sections $p \in \Gamma(E^*, U)$ for appropriate neighborhoods $U \subset M$.

Namely, given M , we would set $E := M \times V^*$, and select a p whose value at $m \in M$ is proportional to the elements of $(V^*)^* \cong V$ represented by the image of m in $\mathbb{P}V$. The use of the $\mathbb{P}E^*$ rather than $\mathbb{P}E$ and $M \times V^*$ rather than $M \times V$ is somewhat gratuitous, but convenient in our context.

We shall also use the jet notions and notation appearing in [Vin01], [Gro86], and [BCG⁺91]. For example, a second-order partial differential equation for

functions on M is regarded as a hypersurfaces in the manifold of 2-jets of smooth functions on M , $J^2(M, \mathbb{R})$.

The main result, proved in section 3, is a non-trivial realization of $(E, \nabla, [p])$, by a second order differential system over M . We reiterate that only the locally-convex case is original, while the hyperbolic case is due to Wilczynski-Tabachnikov-Ovsienko. Moreover our formulation is not identical with theirs, and any errors are due to your present author. In summary:

Theorem. *Suppose that M is a non-degenerate surface in $\mathbb{P}V$, with a chosen smooth lift $X : M \rightarrow V$.*

1. *The restrictions of the linear functions $V \rightarrow \mathbb{R}$ to M along X all satisfy a certain codimension 2 integrable system of linear partial differential equations $E \subset J^2(M, \mathbb{R})$.*
2. *The characteristic variety of E , or kernel of the symbol system of E , is independent of the lift X and equals to $\Pi_M \subset S^2T^*M$.*
3. *There is a flat connection ∇ on the vector bundle E whose horizontal sections are the integral manifolds of the restriction of the Pfaffian contact system of $J^2(M, \mathbb{R})$ to E . (E, ∇) is isomorphic to the trivial bundle with the trivial connection.*
4. *Let $p : E \rightarrow J^0(M, \mathbb{R}) = M \times \mathbb{R}$ be the projection. The triple $(E, \nabla, [p])$ up to isomorphism determines M as a projective surface.*

2 Projective fundamental quadratic form

2.1 The hyperbolic tradition

Tabachnikov and Ovsienko [TO04] mention on page 110 a long-standing tradition of restricting attention to the hyperbolic points of surfaces in 3-space, meaning the points m at which the tangent plane to the surface locally meets the surface in two curves with distinct tangent directions at m . These tangent directions are by definition the *asymptotic directions at m* .¹

This tradition is evinced in the written record² in papers like [Wil07], [Spe18], [Ken23], and textbooks like [Eis09], [For20], and [Lan32]. The authors of [TO04] explain that the classical general theory of surfaces was usually implicitly real-analytic, admitting continuations into a complex domain. That

¹Although this is the quickest definition of the asymptotic directions, we shall give a few different ones in section 2.5, Definition 9.

²Note, however, that the word “tradition” typically connotes the oral or cultural record, not the written publication record.

way a pair of asymptotic directions could always be taken for granted, whether real or complex conjugate.³

In the merely smooth real theory, the restriction to hyperbolic points ensures the familiar classical setting in which the 2-web of asymptotic directions, or asymptotic net, is present.

In section 2.6.1 we explain why the conformal class of the second fundamental quadratic form of a smooth surface M in Euclidean space, denoted Π_M , is invariant by projective transformations, and hence well-defined in projective geometry. Moreover: Near hyperbolic points this form encodes precisely the set of asymptotic directions, as the null directions of Π_M . Near *elliptic* points m , where the tangent plane of M locally meets M only at the point of tangency, Π_M is a conformal class of Riemannian metrics, or unoriented 1-dimensional complex structure. (The points m where Π_M is degenerate but non-zero are called *parabolic*. At parabolic points there is only one asymptotic direction, the kernel of Π_M .)

This suggests a break from the hyperbolic tradition, to incorporate locally-convex regions of surfaces and even parabolic points, by replacing the asymptotic net with the more general object, the conformal second fundamental form. In this setting, unlike in the Euclidean setting, there is no Riemannian metric arising by restriction from the ambient space, so there is no *first* fundamental form I. Therefore we will prefer the inordinal terms *fundamental quadratic form* or *fundamental form*, the conformal class being implicit. For referential convenience we still notate Π .

We shall see that this is actually the continuation of a different tradition, the study of *conjugate directions* (e.g. [Lan22], [Ste17]).

2.2 Other versions of Π

One of the consequences of [Cal82] is a generalization of the form Π to hypersurfaces in unimodular affine geometry. Further, in 1979 Griffiths and Harris exhibited linear systems ($|\Pi|$, $|\text{III}|$, $|\text{IV}|$...) of hypersurfaces of degrees (2, 3, 4...) associated with each point of a complex analytic submanifold of a complex linear or complex projective space, the linear system $|\Pi|$ generalizing the usual Π in a certain sense [GH79]. Although they emphasize the complex analytic case, their construction, which is based on the method of

³Indeed, one finds many old references to “nul-lines” on surfaces (e.g. [Lie78] p331, [For20] p81), meaning curves in the complex extension tangent to the null directions for the *first* fundamental form. Since the first fundamental form is always positive definite, null directions for this form are always complex. This is sensible only in the case that the original surface is real-analytic.

moving frames, still applies in the context of smooth submanifolds of a real linear or real projective space. Consult the book of Akivis and Goldberg [AG93] for an exposition along these lines.

The restrictions of these much more general constructions to the case of surfaces in real 3-space are computationally equivalent to the notion exposted here. However, the point of view of [GH79] would be that the linear system of quadrics at elliptic points is a system of empty varieties (since they do not have real points).

2.3 Metrics in 2 dimensions

In this section we explain how to translate between:

1. a quadratic form on a 2-dimensional real or complex vector space, up to multiplication by a scalar,
2. the corresponding isometric involution of a certain hyperbolic plane (in the real case) or a hyperbolic 3-space (in the complex case), and
3. the orthogonality operator on the (real or complex) projective space.

The reader already familiar with this translation may wish to skip to section 2.5.

2.3.1 Hyperbolic involutions

Note 1. In this section we will make use of some constructions in hyperbolic geometry. There is an unfortunate but not too serious terminological collision with the hyperbolic points of surfaces.

Proposition 2.

1. *There is a functor that associates:*
 - *To each 2-dimensional real vector space U a hyperbolic plane K with ideal boundary equal to $\mathbb{P}U = \mathbb{P}_{\mathbb{R}}U$.*
 - *To each isomorphism of such vector spaces an isometry of hyperbolic planes.*
2. *There is a functor that associates:*
 - *To each 2-dimensional complex vector space W an oriented hyperbolic 3-ball H with ideal boundary equal to $\mathbb{P}W = \mathbb{P}_{\mathbb{C}}W$*
 - *To each complex-linear isomorphism of such vector spaces an orientation-respecting isometry of hyperbolic spaces.*

Proof. (1) Given a real vector space U of dimension 2, consider the projective plane $\mathbb{P}(S^2U)$. Here S^2 denotes the second symmetric power functor. The square map $U \rightarrow S^2U$ embeds $\mathbb{P}U$ as a smooth conic C in this plane. We set K equal to the closure of its interior. Note that this is well-defined by C : In any affine chart containing C , K is the convex hull of C .

K is the projective model of the hyperbolic plane. The hyperbolic metric of K can in fact be defined in terms of cross-ratios of points on projective lines (see e.g. [Tab05] page 43). Namely, if p and q are distinct points in the interior of K , let p', q' denote the points of intersection of $\partial K = C$ with the line spanned by p, q . Then the hyperbolic distance between p and q is defined by the formula

$$d(p, q) := \frac{1}{2} |\ln[p', p, q, q']| \quad (2.1)$$

When K is replaced with an arbitrary convex ball in a real projective space of arbitrary finite dimension, the metric defined by the formula (2.1) is also known as the *Hilbert metric* ([Guo14] page 2).

Since the cross-ratio is a projective invariant, any linear isomorphism $U \rightarrow U'$ induces an isomorphism of projective spaces $\mathbb{P}(S^2U) \rightarrow \mathbb{P}(S^2U')$ mapping $K(U)$ isometrically onto $K(U')$.

(2) Let W be a complex vector space of dimension 2. If V is any complex vector space, denote by \bar{V} the complex conjugate vector space, and denote an element $v \in V$ regarded as an element of \bar{V} by \bar{v} .

$W \otimes_{\mathbb{C}} \bar{W}$ has a canonical real structure σ equal to the \mathbb{C} -linear isomorphism

$$\begin{aligned} \sigma : W \otimes_{\mathbb{C}} \bar{W} &\rightarrow \overline{W \otimes_{\mathbb{C}} \bar{W}} \\ w_1 \otimes \bar{w}_2 &\mapsto w_2 \otimes \bar{w}_1 \end{aligned}$$

Denote the σ -real subspace of $W \otimes_{\mathbb{C}} \bar{W}$ by \bar{S}^2W . In fact, if a real structure is chosen on W with real subspace $U \subset W \cong U \otimes_{\mathbb{R}} \mathbb{C}$, \bar{S}^2W is the identified with the sum $S_{\mathbb{R}}^2U \oplus i\Lambda_{\mathbb{R}}^2U$ by means of the inclusions

$$\begin{aligned} S_{\mathbb{R}}^2U &\subset U \otimes_{\mathbb{R}} U \subset W \otimes_{\mathbb{C}} \bar{W} \\ \Lambda_{\mathbb{R}}^2U &\subset U \otimes_{\mathbb{R}} U \subset W \otimes_{\mathbb{C}} \bar{W} \end{aligned}$$

\bar{S}^2W is also equal to the Hermitian forms on W^* , meaning the forms which are \mathbb{R} -bilinear, complex-conjugate symmetric, and \mathbb{C} -sesquilinear (\mathbb{C} -linear and \mathbb{C} -antilinear in the respective arguments).

Denote by D the determinantal locus of $W \otimes_{\mathbb{C}} \overline{W}$, the complex quadric hypersurface consisting of tensors of rank less than 2. By definition the set of σ -real points of D is the image of the “Hermitian square”:

$$\begin{aligned} W &\rightarrow \overline{S}^2 W \\ w &\mapsto w \otimes \overline{w} \end{aligned}$$

Evidently the complex span of each w maps into the real span of its image $w \otimes \overline{w}$, so that there is a well-defined embedding

$$\mathbb{P}_{\mathbb{C}} W \rightarrow \mathbb{P}_{\mathbb{R}} \overline{S}^2 W \cong \mathbb{RP}^3$$

The restriction of the quadratic function defining D to $\overline{S}^2 W$ turns out to have signature $(1, 3)$; this is because this restriction is the determinant function with respect to a representation of the elements of $\overline{S}^2 W$ by complex-conjugate-symmetric 2×2 matrices. Therefore the image of this embedding is an ellipsoid.

We set H equal to the closure of the interior of this ellipsoid. H is the projective model of the hyperbolic 3-space. The hyperbolic metric of H is the Hilbert metric defined by the formula (2.1), just as in the case (1).

The boundary surface $\partial H \cong \mathbb{P}_{\mathbb{C}} W$ has a natural orientation, as a 1-dimensional complex manifold. The tangent spaces $TH|_{\partial H}$ have the induced orientation given by the equivalence class of any oriented basis for one of the tangent spaces $T(\partial H)$ followed by any inward-pointing normal to ∂H in H . Since H is orientable, the orientation of $TH|_{\partial H}$ so defined extends uniquely to an orientation of H .

The complex-linear isomorphisms $W \rightarrow W'$ induce real-linear isomorphisms of $\overline{S}^2 W$, and hence isometries $H(W) \rightarrow H(W')$. These isometries evidently extend biholomorphisms $\partial H \rightarrow \partial H'$, so they are orientation-respecting. \square

Proposition 3.

1. *The non-zero complex scalar equivalence classes of non-degenerate quadratic forms $q \in S^2 \mathbb{C}^{2*}$ are naturally identified with hyperbolic isometries of $H = H(\mathbb{C}^2)$ which are rotations by 180 degrees about some line.*
2. *The complex non-zero scalar equivalence classes of non-degenerate quadratic forms, elements in $S^2 \mathbb{C}^{2*}$, which have a real representative $q \in S^2 \mathbb{R}^{2*}$, belong to one of two types:*
 - (a) *The axis of rotation of the corresponding 3d isometry lies in the hyperbolic plane $K(\mathbb{R}^2)$ associated to the real part $\mathbb{R}^2 \subset \mathbb{C}^2$.*

- (b) *The axis of rotation of the corresponding 3d isometry is perpendicular to the hyperbolic plane $K \subset H$ with respect to the Hilbert metric.*
3. *The non-zero complex scalar equivalence classes of degenerate quadratic forms with a real representative $q \in S^2\mathbb{R}^{2*}$ are naturally identified with points of $\partial K(\mathbb{R}^2) \cong \mathbb{RP}^1$.*
4. *The restriction to the ideal boundary $\partial H \cong \mathbb{P}(\mathbb{C}^2)$ of one of the involutions (1), corresponding to non-degenerate q , is the q -orthogonality operator defined on the set of complex 1-dimensional subspaces of \mathbb{C}^2 .*

Proof. (1) Given $q \in S^2\mathbb{C}^{2*} \cong \Gamma(\mathbb{CP}^1, \mathcal{O}(2))$, non-degeneracy means that q has two distinct roots in \mathbb{CP}^1 . The unordered pair uniquely determines the complex non-zero scalar equivalence class of q . There is a unique line of $H = H(\mathbb{C}^2)$ joining these roots, regarded as ideal points. About each such line there is a unique rotation by 180 degrees.

(2) If, further, q is actually real, the set consisting of the two roots is preserved by complex conjugation. Thus the line they span in H is invariant by reflection in the plane K . Such lines either lie in the plane or are perpendicular to it with respect to the hyperbolic Hilbert metric of H .

(3) If q is degenerate but non-zero, it has 1 repeated root and this root determines the class of q . The root is real if and only if q is a complex multiple of a real form.

(4) By definition, the two roots of q are two transverse 1-dimensional complex subspaces of \mathbb{C}^2 which are individually equal to their own q -orthogonal complements, and they are the only two such subspaces. Assuming that the q -orthogonality is projective linear as a transformation of $\partial H(\mathbb{C}^2) \cong \mathbb{P}(\mathbb{C}^2)$, its extension to H must be the unique non-identity involutive isometry fixing the pair of ideal points given by the roots.

The assumption of projective linearity is in fact correct. To see this, use the matrix identity

$$v^t Q Q^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v = 0$$

where Q is the matrix of q with respect to a basis for \mathbb{C}^2 . The q -orthogonal complement of the span of $v \in \mathbb{C}^2$ is the span of $Q^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v$, which therefore depends projective-linearly on v .

□

Corollary 4. *The real non-zero scalar equivalence classes of non-degenerate real quadratic forms $q \in S^2\mathbb{R}^{2*}$ are naturally identified with the non-identity projective linear involutions of $\partial K = \mathbb{P}(\mathbb{R}^2)$ and also with the non-identity involutive isometries of the hyperbolic plane $K(\mathbb{R}^2)$:*

- *reflections of the hyperbolic plane, with two fixed points on the ideal line $\mathbb{P}(\mathbb{R}^2)$, or*
- *180 degree rotations of the hyperbolic plane about some point, with no fixed points on $\mathbb{P}(\mathbb{R}^2)$.*

Note 5. From now on we will not distinguish explicitly between orientation-preserving isometric involutions of the hyperbolic 3-space, non-zero complex scalar equivalence classes of complex quadratic forms in 2 dimensions, and their associated orthogonality operators. Similarly we will not distinguish between isometric involutions of the hyperbolic plane, non-zero real scalar equivalence classes of real quadratic forms in 2 dimensions, and their associated orthogonality operators.

2.4 Characteristics and envelopes of families of surfaces

We shall recall the definitions of the envelope and characteristics of a differentiable family of smooth surfaces in 3-dimensional space, as they appear in a textbook like [Eis09], which will be needed in section 2.5. However, the reader may also wish to consult [Mat16], by the present author, for a conceptual generalization more in the spirit of this paper.

Suppose that $F(x, t)$ is a smooth real-valued function, where $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, and suppose that for each t , 0 is a regular value of $F(x) := F(x, t)$. Thus F defines a family of smooth surfaces M_t in \mathbb{R}^3 , parameterized by $t \in \mathbb{R}$. Namely, $M_t := \{x | F(x, t) = 0\}$.

Definition 6.

1. *The t^{th} characteristic of the family defined by F is the subset of M_t defined by the condition $\partial_t F = 0$.*
2. *The envelope of the family defined by F is the union of the t^{th} characteristics.*

Note 7. The generic characteristic of a 1-parameter family of planes is a line.

2.5 Asymptotic directions

We still denote by $M \rightarrow P^3$ a smooth immersed surface. We shall denote by M^* the dual surface in P^{3*} , consisting of the points dual to the tangent planes of M . M^* is not necessarily immersed, even though M is.

Recall that the *osculating plane* of a smooth curve γ in P^3 , at a point $p \in \gamma$, is the limit plane of a secant plane through 3 distinct points belonging to the curve as the 3 points approach simultaneously to p (provided that this limit exists).

Proposition 8. *Choose a tangent direction b to M at a point m . The following are equivalent:*

1. *The osculating plane at m of a curve belonging to M and tangent to b is equal to the tangent plane of M at m .*
2. *Let (u, v) be local coordinates on M , and let $X(u, v) \in V$ be a homogeneous parameterization of $M \subset P^3$, with respect to which $\frac{\partial}{\partial u}|_m$ corresponds to b .
 X_{uu} belongs to the span of X , X_u , and X_v evaluated at m .*
3. *The tangent line through b makes jet-contact with M of order greater than 1 at m .*

If M is also non-planar at m , meaning that the dual M^ is either a smooth surface or a curve in P^{3*} , then these conditions are equivalent to:*

4. *The envelope of the tangent planes to M along a curve belonging to M and tangent to b at m has for its characteristic at m the line of P^3 through b .*

Definition 9. *A tangent direction b to a surface M satisfying the conditions of the previous proposition is called asymptotic.*

The condition Proposition 8(4) is part of a slightly more general construction:

Proposition 10. *Let b and c be tangent directions to M in P^3 at m . The following are equivalent:*

1. *The tangent line through b is the m^{th} characteristic of the envelope of the tangent planes to M along any smooth curve belonging to M and tangent to c .*

2. The tangent line through c is the m^{th} characteristic of the envelope of the tangent planes to M along any smooth curve belonging to M and tangent to b .

For the proofs of Propositions 8 and 10, formulated slightly differently, see [Eis09] pages 128-129.

Definition 11. *Tangent directions b and c to a surface M satisfying the conditions of the Proposition 10 are called conjugate.*

So an asymptotic direction is precisely a self-conjugate direction.

2.6 The projective fundamental form II in the smooth category

2.6.1 Definition by conjugacy

The following proposition seems to have been well-known to classical authors. The proof can be found in [Eis09] on page 127. For our purposes, we can use the definition of the second fundamental form II for surfaces in Euclidean space appearing in ([dC76] page 154); an explicit formula is reproduced in section 3.2, in the proof of Proposition 26, equation (3.2).

Proposition 12. *Tangent directions b and c to a surface M are conjugate if and only if $\text{II}(b, c) = 0$, where II is the second fundamental form of the surface M , calculated in any affine chart for P^3 with respect to any Euclidean metric compatible with the affine structure.*

Thus, not only is the conformal class of the form II well-defined on any smooth projective surface, but the representation of it by means of its orthogonality operator is given by a projectively-manifest construction. This justifies the following definition:

Definition 13. *The fundamental form II_M of a smooth projective surface M in P^3 is the conformal class of metrics preserved by the automorphisms of the projectivized tangent spaces of M which preserve the conjugate relation.*

Note 14. It seems that classical authors did not observe that this conformal class is actually determined by its orthogonality operator (conjugacy), as described in section 2.3. Hence the hyperbolic tradition. Moreover, in some areas *conjugate directions* eventually came to mean the generalization: any basis for a tangent space of a hypersurface of a Riemannian manifold of arbitrary dimension which diagonalizes its second fundamental form there. This definition insinuates that the presence of the second fundamental form is a prerequisite for the notion of conjugacy, which would preclude a definition of II in terms of conjugacy.

Note 15. The construction defining Π can be extended to a natural mapping, from the space of 2-jets of curves in M , into P^3 :

$$\psi : J^2(1, M) \rightarrow P^3$$

We use the language of [Mat16]. Let γ be a smooth curve in M having 2-jet j and tangent direction t at m . We defined $\Pi_M(t)$ to be the direction of the characteristic line of the first-order envelope of the family of tangent planes to M along γ at m ; this only depended on the 1-jet t of γ . On the other hand, the second-order envelope of this family, which is the edge of regression of the developable surface comprised of the characteristic lines along γ , presumably depends only on the 2-jets of γ . If $p \in P^3$ denotes the point of the edge of regression lying on the characteristic through $m \in M$, we set $\psi(j) = p$.

In case M is hyperbolic, so that it admits asymptotic curves, the 2-jets j_m of an asymptotic curve through $m \in M$ satisfy $\psi(j_m) = m$; that is, the asymptotic curves are *equal* to the edges of regression of the associated tangent-plane family, and this characterizes them (as pointed out e.g. by [Eis09] page 129).

2.6.2 Definition by line duality

Let M^* denote the dual surface of M , in P^{3*} . Assume that M is non-degenerate in the sense that the map $M \rightarrow M^*$ is a diffeomorphism.

Denote the projective duality which maps lines of P^3 to lines of P^{3*} by $*$. Its formula with respect to Plücker line coordinates is given by the scalar equivalence class of the duality $\Lambda^2 V \rightarrow \Lambda^2 V^*$, which is furnished by a choice of volume form on V .

Choose local coordinates of M and regard M and M^* as simultaneously parameterized surfaces. Let ∂ be a tangent direction at a point of the coordinate space, with corresponding tangent lines ∂M and ∂M^* .

Proposition 16. *Use Corollary 4 to identify the form Π_M of Definition 13 with the corresponding involution on the set of (based) tangent lines to M . Then:*

$$\Pi_M(\partial M) = (\partial M^*)^* \tag{2.2}$$

This proposition is an immediate consequence of the following:

Lemma 17. *Suppose that $\gamma^*(t) \in P^{3*}$ is a non-degenerate smooth curve. γ^* can be regarded as a family of planes in P^3 . The t^{th} characteristic of this family is the line dual to the tangent line of γ^* at $\gamma^*(t)$.*

Proof. Let us use the definition of the t^{th} characteristic as the limit of the intersection of the t^{th} family member with those for nearby t .*

According to projective duality, the line dual to the line intersection of 2 planes is the one spanned by the 2 dual points. Thus the limit of the intersection of the family of planes is the limit of the secant lines of the dual family of points. \square

This justifies the following equivalent definition.

Definition 18. *The involution Π_M at non-degenerate points of M is*

$$\Pi_M(\partial M) = (\partial M^*)^*$$

2.6.3 Definition by intrinsic and extrinsic contact

Note 19. (Background terminology) In this section, we use freely the notions of developable surfaces and basic line geometry in the manner of [Mat16] sections 2.5 and 2.6.

Let $Q = \text{Gr}(2, 4)$ denote the 4-manifold of lines in P^3 , regarded as a quadric in $P^5 := \mathbb{P}(\Lambda^2 V)$. The lines passing through a given point comprise a plane in Q known as an α plane, while the lines lying in a given plane comprise a plane in Q known as a β plane. The tangent directions to α and β planes at a given point $l \in Q$ are known as *null directions* and they comprise a doubly-ruled quadric N_l in $\mathbb{P}T_l Q$, the lines of one ruling being the directions of some α planes, and of the other those of some β planes. See [WW90] for the details of this structure on Q .

The intrinsic contact structure α on the space of contact elements $L = \mathbb{P}T^*M$ is usually defined in terms of the Liouville 1-form on T^*M ([CdS01] page 61). It can alternatively be defined in terms of the immersion $M \rightarrow P^3$ as follows.

By passing to sufficiently small neighborhoods of M , assume M is embedded in P^3 . The natural isomorphism peculiar to dimension 2, $\mathbb{P}TM \cong \mathbb{P}T^*M$, implies that L can be regarded as the space of tangent lines to M in P^3 . That is, L is a hypersurface in the 4-dimensional manifold Q . The directions belonging to the contact planes in L are given by the tangent directions to the curves of contact elements (tangent directions) along smooth curves in M . The corresponding families of tangent lines trace developable surfaces in P^3 by construction, and so these directions are null when regarded as directions in the larger space Q . That is, the projective 2-plane $\mathbb{P}T_l L$ meets the null quadric surface $N_l \subset \mathbb{P}T_l Q$ along a line of the α ruling which is the projectivization of the usual contact hyperplane of L .

On the other hand, since N_l is the doubly-ruled quadric surface: 2-planes like $\mathbb{P}T_l L$ containing a line belonging to N_l are precisely tangent planes of to N_l at some point. But tangent planes of this quadric are precisely the spans of *two* lines, one from one ruling and one from the other, which meet at the contact point of the tangent plane with the quadric.

The second projective line in $\mathbb{P}T_l L \subset \mathbb{P}T_l Q$ therefore determines a second hyperplane field β in L .

Proposition 20. *Assume that M is non-degenerate in P^3 .*

1. *The involution Π_M exchanges the hyperplane fields α and β .*
2. *β is a contact structure.*

Proof. We'll use the version of Π_M appearing as Definition 18, the composition of line duality and the tangent map of $M \rightarrow M^*$.

(1) The plane fields α and β in $\mathbb{P}T^*M$ are defined by the respective α and β null rulings of the null quadric $N \subset \mathbb{P}T_l Q$ at the points l which label tangent lines of X , points of L .

In precisely the same manner, plane fields α^* and β^* in $\mathbb{P}T^*M^*$ are defined over $X^* \subset P^{3*}$, with the change of notation to record the change of underlying projective space from P^3 to P^{3*} .

Under line duality, α corresponds to β^* and β corresponds to α^* . On the other hand, the tangent map $M \rightarrow M^*$ corresponds α with α^* , and β with β^* , since α and α^* are the intrinsic contact structure of the manifold $M \cong M^*$. Therefore Π_M , which is the composition of these two maps, exchanges the α and β plane fields.

(2) is an immediate consequence of (1). Π is a diffeomorphism. Since α is a contact structure, the pullback $\beta = \Pi_M^*(\alpha)$ is also a contact structure. \square

Definition 21. *β will be called the extrinsic contact structure over the surface $M \subset P^3$.*

Proposition 22. *A differentiable fiber-wise involution of $\mathbb{P}T^*M$ exchanging α and β is uniquely determined.*

Proof. Suppose Π and Π' are two automorphisms of the bundle $\mathbb{P}T^*M$ exchanging α and β . Their composition preserves both. Since this composition preserves the standard contact structure, and acts fiberwise, it fixes pointwise all values of Legendre lifts of curves of M . All points of $\mathbb{P}T^*M$ belong to such a lift. Therefore this composition is the identity, and $\Pi = \Pi'$. \square

For completeness let us record this definition formally.

Definition 23. *The involution Π of $\mathbb{P}TM \cong \mathbb{P}T^*M$ at non-degenerate points of M is the unique involution exchanging contact structures α and β .*

2.7 Smooth convex projective surfaces

In this section suppose that M is a closed non-degenerate submanifold of P^3 , that it belongs to an affine chart, and that it is convex in this chart. Evidently Π_M has Riemannian signature, so that it is a conformal structure on the surface M in the usual sense. By selecting an orientation for M , we can regard Π_M as a 1-dimensional complex structure on M . By the uniformization theorem, M admits a rational parameterization $\mathbb{CP}^1 \cong M$, unique up to precomposition by $PSL(2, \mathbb{C})$.

The image in M of the 3-dimensional collection of circles of \mathbb{CP}^1 is independent of the parameterization, so that the collection forms a projective invariant of M . Let us call them the *projective circles* of M .

Question 24. *What is a direct description of the family of projective circles of M in terms of projective geometry?*

The answer is well-known in one case:

Fact 25. *If M is an ellipsoid, the projective circles of M are the intersections of hyperplanes of P^3 with M .*

3 Generalized Wilczynski equations

There is a structure theorem in classical smooth projective surface theory, due to Wilczynski, which applies near hyperbolic points.

Tabachnikov and Ovsienko showed that it has a natural formulation in terms of an integrable system [TO04]. The treatment given below is an adaptation of their formulation to the general case, to include elliptic points.

3.1 Jet notation

The details concerning the notation introduced in this section can be found in the preliminary chapter of [Gro86].

Temporarily denote by M an arbitrary smooth manifold. The jet bundles of a smooth fiber bundle F over M always form a tower

$$J^0 \leftarrow J^1 \leftarrow J^2 \dots$$

in which J^s is an affine bundle over J^{s-1} . The structure vector bundle for this affine bundle is $S^s T^* M \otimes \text{Vert}$, where Vert denotes the pullback to J^{s-1} of the vertical bundle over J^0 , the kernel of $TJ^0 \rightarrow TM$.

From now on let J^s denote $J^s(M, \mathbb{R})$, the bundle of s -order jets of real-valued functions on M . Since $M \times \mathbb{R} \rightarrow \mathbb{R}$ is a trivial vector bundle, they naturally form a nested sequence

$$J^0 \subset J^1 \subset J^2 \dots$$

of *vector* bundles over M . The associated graded vector bundle of quotients is naturally isomorphic to the bundle of graded algebras $S^\bullet T^* M \rightarrow M$. The quotient map identifies the grades with the successive kernels of the original projections $J^s \rightarrow J^{s-1}$, so that

$$J^s = \bigoplus_{k=0}^{k=s} S^k T^* M$$

3.2 The differential equations and Π as a characteristic variety

We re-specialize to the case that M is a smooth 2-dimensional manifold immersed in P^3 . V still denotes a 4-dimensional real vector space, and $P^3 := \mathbb{P}V$. Denote $J^s := J^s(M, \mathbb{R})$ and $\bar{J}^s := J^s(M, V) = J^s \otimes V$.

Proposition 26. *Suppose that M in P^3 is non-degenerate. Choose a lift $X : M \rightarrow V \setminus \{0\}$ representing it.*

1. *The $GL(V)$ orbits of X all satisfy a common second-order system of two linear partial differential equations $\bar{E} \subset \bar{J}^2 = J^2 \otimes V$ of the form $\bar{E} = E \otimes V$, where $E \subset J^2$ is a rank 4 vector subbundle. The component functions of these orbits satisfy E .*
2. *The fundamental form $\Pi_M \subset S^2 T^* M$ is the characteristic variety of E , the common kernel of the two-dimensional system in $S^2 TM \cong (S^2 T^* M)^*$ consisting of the symbols of the operators defining E .*

Proof. Since M is non-degenerate, there are only two cases: hyperbolic and elliptic surfaces M .

The proof in the hyperbolic case follows directly from [TO04] page 111-112, or from Wilczynski's original work [Wil07]. We will summarize their

reasoning in the present language and then proceed to the the proof in the elliptic case.

Select a local parameterization for M whose coordinate directions are the asymptotic directions. In other words, a parameterization $X(u, v) \in V$ in which the fundamental form Π_M is the conformal class of $dudv$. It follows from Proposition 8 that

$$X_{uu} \equiv X_{vv} \equiv 0 \pmod{(X_u, X_v, X)}$$

In other words, there are functions a, b, c, p, q, r of (u, v) such that

$$\begin{aligned} X_{uu} + aX + bX_u + cX_v &= 0 \\ X_{vv} + pX + qX_u + rX_v &= 0 \end{aligned}$$

Set \bar{E} equal to the codimension 2 locus in \bar{J}^2 described by these equations. Set E equal to the locus in J^2 satisfied by the same system,

$$\begin{aligned} x_{uu} + ax + bx_u + cx_v &= 0 \\ x_{vv} + px + qx_u + rx_v &= 0 \end{aligned} \tag{3.1}$$

where now $x, x_u, x_v, x_{uu}, x_{vv}, x_{uv}$ denote the fiber coordinates of J^2 . By construction $j^2X \subset \bar{E}$ and $\bar{E} = E \otimes V$. Since \bar{E} is $GL(V)$ invariant, all $GL(V)$ translates of j^2X also lie in \bar{E} .

The kernel of the system of symbols $\langle \partial_u^2, \partial_v^2 \rangle$ of E is the conformal class of $dudv$, which is equal to Π_M .

For the locally-convex case, recall that it is always possible to select coordinates (u, v) on M in which a given positive-definite metric is conformal to the flat metric $du^2 + dv^2$. Such coordinates are called *isothermal*. They are usually selected with respect to the *first* fundamental form of a surface.

Select Π_M -isothermal coordinates (u, v) . To ascertain the consequences of this choice, we will apply the formula for the second fundamental form in an affine chart. With respect to some fixed isomorphism $V \cong \mathbb{R}^4$, select a local affinization of M by $(y, 1)$, where $y : U \rightarrow \mathbb{R}^3$ for a chosen sufficiently small neighborhood U in the coordinate space $\{(u, v)\}$.

According to [dC76] page 154, Π_M is conformal to

$$|y_u, y_v, y_{uu}|du^2 + |y_u, y_v, y_{vv}|dv^2 + |y_u, y_v, y_{uv}|dudv \tag{3.2}$$

where $|\cdot, \cdot|$ denotes the standard volume form of \mathbb{R}^3 .

Then $\Pi_M \equiv du^2 + dv^2$ implies

$$y_{uu} - y_{vv} \equiv y_{uv} \equiv 0 \pmod{(y_u, y_v)}$$

Setting $X = f \cdot (y, 1)$ for an appropriate function f on M , we conclude that, without the need for affine charts, the Π_M -isothermal condition is

$$X_{uu} - X_{vv} \equiv X_{uv} \equiv 0 \pmod{(X, X_u, X_v)}$$

In other words, there are functions a, b, c, p, q, r of (u, v) such that

$$\begin{aligned} X_{uu-vv} + aX + bX_u + cX_v &= 0 \\ X_{uv} + pX + qX_u + rX_v &= 0 \end{aligned} \tag{3.3}$$

As before, they determine a differential system $\bar{E} \subset \bar{J}^2$ satisfied by X and its $GL(V)$ translates, and a differential system $E \subset J^2$ satisfied by their linear coordinate functions. The kernel of the system of symbols $\langle \partial_{uu-vv}, \partial_{uv} \rangle$ is the conformal class of $du^2 + dv^2$, which is Π_M . □

3.3 The flat connections

Proposition 27. *As in Proposition 26, suppose that M is non-degenerate in P^3 with a chosen lift $X : M \rightarrow V \setminus \{0\}$.*

1. *There is an open subset of \bar{E} which has a holonomic foliation, i.e. one whose leaves are integrals of the restriction of the contact system of \bar{J}^2 to \bar{E} . The leaves are the $GL(V)$ orbits of j^2X . The leaves are also flat sections of a $GL(V)$ -invariant flat connection $\bar{\nabla}$ on the vector bundle \bar{E} .*
2. *$\bar{\nabla}$ is the tensor product of a flat connection ∇ on E and the trivial connection on the trivial vector bundle with fiber V .*
3. *The linear form $p : E \rightarrow J^0 \cong \mathbb{R}$ induces an isomorphism from the 4-dimensional vector space of ∇ -flat sections of E over M onto a subspace $\Gamma \subset C^\infty(M, \mathbb{R})$. Γ also consists of the restrictions of the linear coordinate functions of V to $X(M)$.*

The mapping $M \rightarrow \mathbb{P}\Gamma^$ assigning to $m \in M$ the hyperplane*

$$\{\gamma \in \Gamma \mid \gamma(m) = 0\}$$

is equivalent to the original surface $M \rightarrow \mathbb{P}V$, with respect to the evident isomorphism $\Gamma^ \cong V$.*

Proof. (1 and 2) The action of $GL(V)$ is free on the non-degenerate 2-jets belonging to \bar{J}^2 . In particular, it has 16-dimensional $GL(V)$ orbits there. The dimension of \bar{E} is $2 + 16$. Therefore the 16-dimensional $GL(V)$ orbits of the 2-dimensional holonomic section $j^2X \subset \bar{E}$ foliate an open subset of \bar{E} .

First observe that the tangent distribution of this foliation, so far just an integrable Ehresmann connection on an open subbundle of \bar{E} , is actually the horizontal plane field of a linear connection $\bar{\nabla}$ on \bar{E} , by construction the connection induced by a connection ∇ on E . Namely, ∇ is the expression of the contact system of J^2 , the codimension 3 Pfaffian exterior differential system generated by the 1-forms

$$\begin{aligned} dx - x_u du - x_v dv \\ dx_u - x_{uu} du - x_{uv} dv \\ dx_v - x_{uv} du - x_{vv} dv, \end{aligned}$$

upon restriction to E by application of the system of equations (3.1) in the hyperbolic case and (3.3) in the elliptic case.

In the hyperbolic case the explicit formula is

$$\begin{aligned} \nabla = d + \begin{bmatrix} 0 & -1 & 0 & 0 \\ a & b & c & 0 \\ 0 & 0 & 0 & -1 \\ cp - a_v & cq - b_v & cr - a - cv & -b \end{bmatrix} du + \\ \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ p & q & r & 0 \\ qa - p_u & qb - p - q_u & qc - r_u & -r \end{bmatrix} dv \end{aligned} \quad (3.4)$$

where the matrices are the matrices of linear transformations of the fibers of the vector bundle E , written with respect to the trivialization dual to the trivialization by linear forms on E equal to the restrictions of the 4 coordinate functions x, x_u, x_v, x_{uv} on J^2 . The key step in obtaining (3.4) is to use the equations $(x_{uv})_u = (x_{uu})_v$ and $(x_{uv})_v = (x_{vv})_u$, in order to calculate $d(x_{uv})$ as a linear combination of x, x_u, x_v, x_{uv} , with coefficients belonging to $\Omega^1(M)$ (that is, coefficients of the form $f(u, v)du + g(u, v)dv$). In the elliptic case, the reasoning is the same but the calculation is slightly more complicated. One uses the trivialization of E induced by the coordinate

functions x, x_u, x_v, x_{uu+vv} instead. The polynomial identities

$$\begin{aligned}(u^2 + v^2)u &= (u^2 - v^2)u + 2(uv)v \\ (u^2 + v^2)v &= -(u^2 - v^2)v + 2(uv)u\end{aligned}$$

imply

$$d(x_{uu+vv}) = ((x_{uu-vv})_u + 2(x_{uv})_v) du + (-(x_{uu-vv})_v + 2(x_{uv})_u) dv$$

Then substitution of x_{uu-vv} and x_{uv} using equations (3.3) proceeds as in the hyperbolic case. The result is

$$\nabla = d + Adu + Bdv$$

where A and B are given below (transposed for typographical reasons):

$$A^t = \begin{bmatrix} 0 & a/2 & p & (\frac{-b+2r}{2})a + (c+2q)p - a_u - 2p_v \\ -1 & b/2 & q & (\frac{-b+2r}{2})b + (c+2q)q - a - b_u - 2q_v \\ 0 & c/2 & r & (\frac{-b+2r}{2})c + (c+2q)r - c_u - 2p - 2r_v \\ 0 & -1/2 & 0 & \frac{-b-2r}{2} \end{bmatrix}$$

$$B^t = \begin{bmatrix} 0 & p & -a/2 & (\frac{c+2q}{2})a + (-b+2r)p + a_v - 2p_u \\ 0 & q & -b/2 & (\frac{c+2q}{2})b + (-b+2r)q + b_v - 2p - 2q_u \\ -1 & r & -c/2 & (\frac{c+2q}{2})c + (-b+2r)r + a + c_v - 2r_u \\ 0 & 0 & -1/2 & \frac{c-2q}{2} \end{bmatrix}$$

The flatness condition $F_\nabla = 0$ is implied by $F_{\bar{\nabla}} = 0$, which is a consequence of the existence of a basis of sections of \bar{E} which are flat; the section $j^2 X$ and sufficiently many of its $GL(V)$ orbits. Clearly it is no trivial matter to write explicitly the integrability conditions in terms of the coefficient functions a, b, c, p, q, r . Nevertheless in the hyperbolic case they appear in Wilczynski's original formulation.

(3) By construction the ∇ -flat sections are the 2-jets of the linear coordinate functions of V restricted to $X(M)$. p -evaluation maps the 2-jet sections back to these original coordinate functions.

In particular, $V^* \cong \Gamma$. The equivalence of $M \rightarrow \mathbb{P}\Gamma^*$ and $M \rightarrow \mathbb{P}V$ is a matter of notation. \square

We conclude that the structure of a non-degenerate projective surface on a 2-manifold M is entirely encoded by a second order integrable system of two linear partial differential equations for functions on M , namely $E \subset J^2$. The form $\Pi \subset S^2T^*M$ is revealed to be its characteristic variety, the common kernel of the system of symbols belonging to $S^2TM \cong (S^2T^*M)^*$.

Note 28. (Uniqueness) This encoding is not quite unique. Technically we have represented a projective surface in the 4-dimensional centro-affine geometry V , by means of a choice of section of the tautological line bundle over the surface; a fully projective conclusion would require consideration of the effect of this choice.

Note 29. (Algebraic case) The reconstruction of a projective surface M in terms of Γ appearing as Proposition 27(3) is obviously reminiscent of an embedding of an *algebraic variety* into a projective space by the means of the vector space Γ of global sections of the pullback of the line bundle $\mathcal{O}(1)$ from the target. This formulation suggests that flat connections and integrable differential systems are behind the scenes of projective embeddings even in the algebraic setting. The solutions are the variety's coordinate functions in various lifts to the linear model of the projective space.

Note 30. (Higher dimensions). The idea seems to generalize immediately to d -dimensional submanifolds of a projective space of dimension n :

1. Find the first s for which s -jets of d -dimensional submanifolds of \mathbb{P}^n are free under $GL(n+1)$.
2. Determine the minimal vector subbundle $\bar{E} \subset \bar{J}^s$ containing the $GL(n+1)$ orbits of the s -jet of a given submanifold.
3. Extract if possible an s -order integrable differential system $E \subset J^s(M, \mathbb{R})$.

3.4 Projective normals

Recall that the differential systems \bar{E} and E associated to a non-degenerate projective surface M depend on the choice of lift $M \rightarrow V$. Despite this the characteristic variety of E , the fundamental form Π_M , does not. On the other hand the symbol system of $\bar{E} = E \otimes V$ *does* depend on the lift. It specifies a “normal” to M as follows.

Since Π_M is a non-degenerate conformal metric on M , it can be used as an isomorphism $TM \cong T^*M$, well-defined up to scale. In particular, it can be used to express itself as a symmetric bivector field Π_M^* in S^2TM . This is also known as the *dual* of Π_M . (Note that it is not an ordinary linear dual).

Definition 31. Let $M \subset P^3$ be a non-degenerate surface. Choose a lift $\lambda : M \rightarrow V$, and regard it as a section of the tautological line bundle over M . In this context, let λ be called an internal density. Consider the class of $[j^2 X]$ in \bar{J}^2/\bar{J}^1 as a linear map $S^2 TM \rightarrow V$. Define the normal surface of the internal density λ to be the surface in $\mathbb{P}V$ specified by the evaluation $\nu = [j^2 X](\Pi_M^*) : M \rightarrow V$.

Note that the values of ν up to scale only depend on the conformal class of Π_M^* . On the other hand, the values of ν are more significantly altered, not just up to scale, by changes of the density λ .

Assuming that ν does not map points of M to themselves, by joining the points of M to their ν images we obtain a *line congruence* (2-dimensional family of lines), transverse to M , which depends on the internal density λ . Since λ depends on only one functional parameter of M , and a line congruence depends on two, we surmise that the line congruences arising this way must be strongly constrained.

Question 32. What are the properties of the line congruences which occur as “projective normal line fields” to a given surface? What are the properties of their focal surfaces, or their focal nets upon M ?

3.5 Potential application to surface reconstruction

Suppose that you wish to determine a surface in Euclidean space up to isometry. One way is to ascertain its local Euclidean invariants, and attempt a direct solution of the structural differential equations of Gauss and Codazzi. Another way is to factor the direct determination of the surface up to isometry into the steps:

1. Determine the surface up to projective transformation
2. Determine the projective transformation modulo isometries achieving the final surface

This is a natural idea, for example, in computational visual geometry. There the principal measurements involve visual rays, which belong to projective geometry. Step 1 is highly likely to be simpler than a direct determination, because projective geometry is simpler than Euclidean geometry.

If step 2 turns out to be difficult, at least it will be difficult in a very different way from the original problem. It is a finite-dimensional problem which could perhaps be resolved by consideration of a finite number of distances or Gaussian curvatures.

This factorization has practical numerical significance. Discrete methods in general struggle to handle derivatives. It is therefore desirable to confine the parts of an algorithm requiring derivatives to steps which are as simple as possible, as in the factorization suggested.

The results of this paper suggest a way to perform step 1:

- Choose coordinates on the surface and a section of the tautological line bundle adapted to the problem at hand.
- Devise a procedure to measure numerically the coefficients of the connection ∇ on the bundle E .
- Enforce the integrability condition $F_\nabla = 0$ by some perturbative integration scheme.
- Solve the equation $\nabla s = 0$, e.g. by Fourier analysis in case of periodic coordinates, or some other numerical methods.

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