The Black–Scholes Theory of Derivative Pricing

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1.1 Market Model

- Riskless asset (bond) with price β_t at time t described by the ordinary differential equation

$$d\beta_t = r\beta_t dt$$

where $t \geq 0$, is the instantaneous interest rate. Setting $\beta_0 = 1$, we have $\beta_t = e^{rt}$ for $t \geq 0$.

- Risky stock or stock index, the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

1.1.1 Brownian Motion

- A Brownian motion is a real-valued stochastic process with continuous trajectories that have independent and stationary increments. The trajectories are denoted by t $\longmapsto W_t$ and for the standard Brownian motion, we have that:
- X is a random process and has stationary increments if for the increment $\mathrm{X}_t - \mathrm{X}_s$ has the same distribution

1.1.1 Brownian Motion

- $W_0 = 0$;
- for any $0 < t_1 < \cdots < t_n$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$ are independent;
- for any $0 \le s < t$, the increment $W_t - W_s$ is a centered (mean-zero) normal random variable with variance $E(W_t - W_s)^2 = t - s$ .In particular, W_t is $\mathcal{N}(0, t)$-distributed.

1.1.1 Brownian Motion

- X_t is $\mathcal{F}_t$-measurable for every t.
- Stochastic process $(\mathrm{X}_t)_{t\geq 0}$ is adapted to the filtration $(\mathrm{F}_t)_{t\geq 0}$

  Another process (Y_t) is such that $\mathrm{X}_t = \mathrm{Y}_t\ \mathbb{P}$ −a.s. for every t

  then (Y_t) is also $(\mathcal{F}_t)$-adapted.

1.1.1 Brownian Motion

- Using conditional characteristic function.
- For $0 \leq s < t$ and $u \in \mathbb{R}$

$$\mathrm{E}\left\{ e^{-iu(W_t - W_s)} \middle| \mathcal{F}_s \right\} = e^{-\frac{u^2(t-s)}{2}}$$

  - Independence of the increments $(W_t - W_s) \xrightarrow{\Pi\text{無限小}} dW_t$
  - Centered normal random variable, variance t-s $\xrightarrow{\Pi\text{無限小}} dt$
  - Continuous process (W_t) is a standard Brownian motion .

1.1.2 Stochastic Integrals

- For T a fixed finite time, $(\mathrm{X}_t)_{0 \le t \le T}$ be a continuous stochastic process adapted to $(\mathcal{F}_t)_{0 \le t \le T}$,

$$E\{\int_0^T X^2 \, dt\} < +\infty$$

- The *stochastic integra*l of (X_t) with respect to the Brownian motion $(\mathrm{W}_t)$is defined as a limit in the mean-square sense($L^2(\Omega)$)

$$\int_0^t X_s \, dW_s = \lim_{n \to \infty} \sum_{i=1}^{n} X_{t_{i-1}} (W_{t_\mathrm{i}} - W_{t_{\mathrm{i}-1}})$$

1.1.2 Stochastic Integrals

- As a function of time t, this stochastic integral defines a continuous square integrable process s.t.

$$\mathrm{E}\left\{(\int_0^t \mathrm{X_s}\, dW_s)^2\right\} = \mathrm{E}\left\{\int_0^t X_s^2\, ds\right\}$$

- And has the martingale property

$$E\left\{\int_0^t \mathrm{X_u}\, d\mathrm{W_u} \middle| \mathcal{F}_\mathrm{s}\right\} = \int_0^s \mathrm{X_u}\, dW_u \quad \mathbb{P} - \text{a.s. for s} \leq \text{t}$$

1.1.3 Risky Asset Price Model

- The corresponding formula for the infinitesimal return is

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

- In integral form,

$$X_t = X_0 + \mu \int_0^t \mu(t, X_t)\, ds + \int_0^t \sigma(t, X_t)\, dW_s$$

1.1.4 Ito's Formula

- We suppose in the following that the function g is twice continuously differentiable, bounded, and has bounded derivatives. ((W_t) is not differentiable)

$$g(W_t) - g(W_0) = \sum_{i=1}^{n} g'\left(W_{t_{\mathrm{i}-1}}\right)\left(W_{t_{\mathrm{i}}} - W_{t_{\mathrm{i}-1}}\right)$$

$$+\frac{1}{2}\sum_{i=1}^{n} g''\left(W_{t_{\mathrm{i}-1}}\right)\left(W_{t_{\mathrm{i}}} - W_{t_{\mathrm{i}-1}}\right)^2 + R$$

$$\Longrightarrow g(W_t) - g(W_0) = \int_0^{\mathrm{t}} \mathrm{g}'(W_s)\, dW_s + \frac{1}{2}\int_0^{\mathrm{t}} \mathrm{g}''(W_s)\, ds$$

1.1.4 Ito's Formula

- General formula for a function g , $dX_t = \mu X_t dt + \sigma X_t dW_t$

$dg(t, X_t)$

$$= \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)d[X]_t$$

$$= (\frac{\partial g}{\partial t} + \mu(t, X_t)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 g}{\partial x^2})dt + \sigma(t, X_t)\frac{\partial g}{\partial x}dW_t$$

1.1.5 Lognormal Risky Asset Price

- Differential of $log X_t$ by applying Ito's formula

$$dlogX_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

- Leads to stock price:

$$X_t = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

- The process (X_t) is also called geometric Brownian motion.

1.1.6 Ornstein–Uhlenbeck Process

- simplest example of a mean-reverting diffusion is the O–U process

$$dY_t = \alpha(m - Y_t)dt + \beta dW_t$$

where α and β are positive constants.

  - $Y_t = m + (Y_0 - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)}\, dW_t$
  - $Y_t \sim \mathcal{N}(m + (Y_0 - m)e^{-\alpha t}, \frac{\beta}{2\alpha}(1 - e^{-2\alpha t}))$

- As $t \to \infty$, $Y_t \sim \mathcal{N}\left(m, \frac{\beta}{2\alpha}\right)$, doesn't depend on starting value.

1.2 Derivative Contracts

- European options,
- American options,
- path-independent options,
- or path-dependent options.

1.2.1 European Call and Put Options

- European Call Options Payoff:

$$h(X_T) = (X_T - K)^+ = \begin{cases} X_T - K, & X_T > K \\ 0 \qquad , & X_T \leq K \end{cases}$$

- European Put Options Payoff:

$$h(X_T) = (K - X_T)^+ = \begin{cases} K - X_T, & X_T < K \\ 0 \qquad , & X_T \geq K \end{cases}$$

1.2.1 European Call and Put Options

- The option price $\mathrm{P}(0, \mathrm{x})$ at time 0 and $X_0 = x$:

$$\mathrm{P}(0, \mathrm{x}) = \mathrm{E}\{e^{-\mathrm{r}t} h(X_T)\} = \mathrm{E}\left\{e^{-\mathrm{r}t} h\left(X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}\right)\right\}$$

  - The expected value of its discounted payoff.

1.2.2 American Options

- American Put Options Payoff:

$$h(X_\tau) = (K - X_\tau)^+$$

- maximize the expected value of the discounted payoff over all the stopping times $\tau \leq T$

$$\mathrm{P}(0, \mathrm{x}) = \sup_{\tau \leq T} E\{e^{-\mathrm{r}\tau} h(X_\tau)\}$$

1.2.3 Other Exotic Options

- Barrier options payoff:

$$h(X_T) = (X_T - K)^+ \mathbb{I}(\inf_{t \le T} X_\mathrm{t} > \mathrm{B})$$

- Lookback Call / Put options payoff:

$$h(X_T) = \left(X_T - \inf_{t \le T} X_\mathrm{t}\right)^+$$

$$h(X_T) = \left(\sup_{t \le T} X_\mathrm{t} - X_T\right)^+$$

1.2.3 Other Exotic Options

- Forward−start or cliquet option

$$h(X_T) = \left(X_T - X_{T_1}\right)^+$$

- Compound options payoff:

$$h(X_T) = \left(\mathrm{C}_{T_1}(K, T) - K_1\right)^+$$

- Asian options payoff:

$$h(X_T) = \left(X_T - \frac{1}{\mathrm{T}}\int_0^T X_s\, ds\right)^+$$

1.3 Replicating Strategies

- At time t
- Stock price X_{t}
- Bond price $e^{r\mathrm{t}}$
- $(a_{\mathrm{t}}, b_{\mathrm{t}})$ the number of units held at time t of the underlying asset and the riskless bond, respectively.
- portfolio value at time t is $a_{\mathrm{t}}X_{\mathrm{t}} + b_{\mathrm{t}}e^{r\mathrm{t}} = h(X_{\mathrm{t}})$

1.3.1 Replicating Self-Financing Portfolios

$$a_{t_n} X_{t_{n+1}} + b_{t_n} e^{r t_{n+1}}$$
$$= a_{t_{n+1}} X_{t_{n+1}} + b_{t_{n+1}} e^{r t_{n+1}}$$

1.3.2 The Black–Scholes PDE

- $a_{\mathrm{t}}X_{\mathrm{t}} + b_{\mathrm{t}}e^{r\mathrm{t}} = \mathrm{P}(\mathrm{t}, X_t)$ for any $0 \leq \mathrm{t} < \mathrm{T}$

$$\mathrm{r}\left(\mathrm{P} - X_t \frac{\partial P}{\partial x}\right) = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 {X_t}^2 \frac{\partial^2 \mathrm{P}}{\partial x^2}$$

- P(t,x) is the solution of the BS partial differential equation

$$\mathcal{L}_{\mathrm{BS}}(\sigma)\mathrm{P} = 0$$

- where the Black–Scholes operator is defined by

$$\mathcal{L}_{\mathrm{BS}}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot) \cdot$$

1.3.3 Pricing to Hedge

- At time t
- Hold A_t units of the risky asset X_T
- sell $\mathrm{N_t}$ options
- this portfolio is riskless

$$\mathrm{A}_t dX_t - N_t dP_t = \mathrm{r}(\mathrm{A}_t X_t - N_t P_t)\mathrm{dt}$$

- Using Ito's formula we have

$$\mathcal{L}_{\mathrm{BS}}(\sigma)\mathrm{P} = 0$$

1.3.4 The Black–Scholes Formula

- Black–Scholes formula

$$C_{BS}(t, x) = xN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2}}\,dy$$

1.3.5 The Greeks

- Delta of a call option

$$\Delta_{BS} = \frac{\partial \mathrm{C_{BS}}}{\partial x} = N(\mathrm{d}_1)$$

- Gamma

$$\Gamma_{BS} = \frac{\partial^2 \mathrm{C_{BS}}}{\partial \mathrm{x}^2} = \frac{e^{-\frac{\mathrm{d_1}^2}{2}}}{x\sigma\sqrt{2\pi(T-t)}}$$

- Vega

$$\upsilon_{BS} = \frac{\partial \mathrm{C_{BS}}}{\partial \sigma} = \frac{e^{-\frac{\mathrm{d_1}^2}{2}}\sqrt{(T-t)}}{\sqrt{2\pi}}$$

1.3.5 The Greeks

- Differentiating

$$\mathcal{L}_{\mathrm{BS}}(\sigma)\mathrm{P} = 0$$

- with respect to σ leads to the following equation for the Vega:

$$\mathcal{L}_{\mathrm{BS}}(\sigma)\vartheta_{BS} + \sigma \mathrm{x}^2 \frac{\partial^2 \mathrm{P}}{\partial \mathrm{x}^2} = 0$$

#Reference: Commutator formulas

[L,Q]P=L[Q[P]]-Q[L[P]]

1.4 Risk-Neutral Pricing

- discounted price $\widetilde{X_t} = e^{-rt}X_t$ is not a martingale since

$$d\widetilde{X_t} = (\mu - r)\widetilde{X_t}dt + \sigma\widetilde{X_t}dW_t$$

- Construct probability measure $\mathbb{P}^*$ equivalent to P s.t.
  - the discounted price $\widetilde{X_t}$ is a martingale and
  - the expected value under $\mathbb{P}^*$ of the discounted payoff of a derivative gives its no-arbitrage price.
- $\mathbb{P}^*$ describing a risk-neutral world is called an equivalent martingale measure.

1.4.1 Equivalent Martingale Measure

$$d\widetilde{X_t} = (\mu - r)\widetilde{X_t}dt + \sigma\widetilde{X_t}dW_t = \sigma\widetilde{X_t}\left[dW_t + \frac{\mu - r}{\sigma}dt\right]$$

- setting $\theta = \frac{\mu - r}{\sigma}$

$$d\widetilde{X_t} = \sigma\widetilde{X_t}dW_t^*,$$

- By Girsanov's theorem,we find

$$z_T^\theta = e^{-\theta W_T - \frac{1}{2}\theta^2 T}$$

$$z_T^\theta = \frac{d\mathbb{P}^*}{d\mathbb{P}}$$

1.4.1 Equivalent Martingale Measure

- z_T^θ is a martingale,

$$E\{z_T^\theta|\mathcal{F}_t\} = e^{-\theta W_t - \frac{1}{2}\theta^2 t} = z_t^\theta \text{, for } 0 \le \mathrm{t} < \mathrm{T}$$

- For any Integrable random variable Y we have

$$E^*\{Y\} = E\{z_T^\theta Y\}$$

- The process (z_t^θ) is called the Radon–Nikodym process.

$$E^*\{Y_t|\mathcal{F}_s\} = \frac{1}{z_s^\theta} E\{z_t^\theta Y_t|\mathcal{F}_s\} \text{, for } 0 \le \mathrm{s} \le \mathrm{t} < \mathrm{T}$$

1.4.2 Self-Financing Portfolios

- The discounted value of the portfolio is a martingale under the risk-neutral probability $\mathbb{P}^*$.
- $d\widetilde{\mathrm{V}}_t = de^{-\mathrm{rt}}\mathrm{V}_t = -re^{-\mathrm{rt}}\mathrm{V}_t dt + e^{-\mathrm{rt}}\mathrm{V}_t\mathrm{dV}_t$

$$(\because d\mathrm{V}_t = a_{\mathrm{t}}dX_{\mathrm{t}} + \mathrm{r}b_{\mathrm{t}}e^{r\mathrm{t}}dt)$$

$$= -re^{-\mathrm{rt}}a_{\mathrm{t}}X_{\mathrm{t}}dt + e^{-r\mathrm{t}}a_{\mathrm{t}}dX_{\mathrm{t}} = a_{\mathrm{t}}\mathrm{d}(e^{-\mathrm{rt}}X_{\mathrm{t}})$$

$$= a_{\mathrm{t}}\mathrm{d}\widetilde{X_t} \text{ (self-financing)}$$

$$= \sigma a_{\mathrm{t}}\widetilde{X_t}dW_t^* \text{ (martingale under } \mathbb{P}^* \text{)}$$

$$
\begin{aligned}
\mathbb{E}^{\star}\left\{e^{iu(W_t^{\star}-W_s^{\star})} \mid \mathscr{F}_s\right\} &= \frac{1}{\xi_s^{\theta}}\mathbb{E}\left\{\xi_t^{\theta} e^{iu(W_t^{\star}-W_s^{\star})} \mid \mathscr{F}_s\right\} \\
&= e^{\theta W_s+\frac{1}{2}\theta^2 s}\mathbb{E}\left\{e^{-\theta W_t-\frac{1}{2}\theta^2 t} e^{iu(W_t-W_s+\theta(t-s))} \mid \mathscr{F}_s\right\} \\
&= e^{\left(-\frac{1}{2}\theta^2+iu\theta\right)(t-s)}\mathbb{E}\left\{e^{i(u+i\theta)(W_t-W_s)} \mid \mathscr{F}_s\right\} \\
&= e^{\left(-\frac{1}{2}\theta^2+iu\theta\right)(t-s)} e^{-\frac{(u+i\theta)^2(t-s)}{2}} \\
&= e^{-\frac{u^2(t-s)}{2}}.
\end{aligned}
$$

- W_t*-W_s* ~N(0 , t-s)
- $e^{iu(Wt^*-Ws^*)}$ is a martingale