Online Algorithms

Performance-based Advertising

Mining of Massive Datasets
Leskovec, Rajaraman, and Ullman
Stanford University

Online Algorithms

- **Classic model of algorithms**
  - You get to see the entire input, then compute some function of it
  - In this context, "offline algorithm"
- **Online Algorithms**
  - You get to see the input one piece at a time, and need to make irrevocable decisions along the way
  - **Similar to the data stream model**

Online Algorithms

Bipartite Matching

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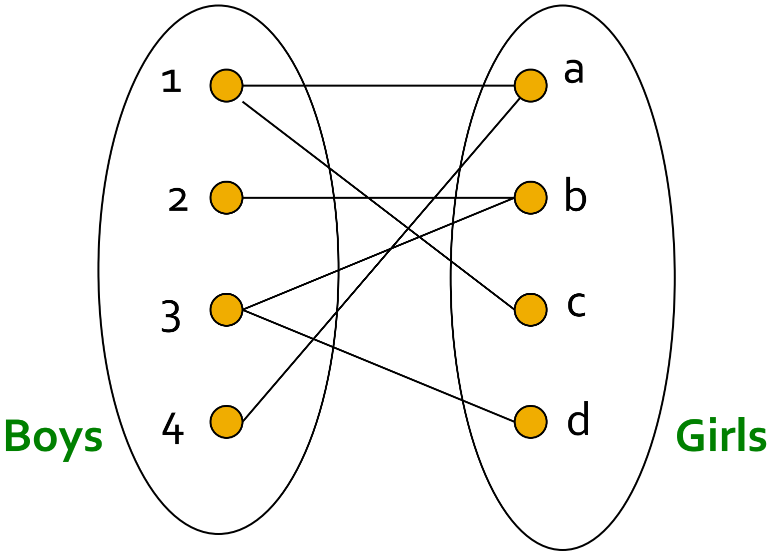
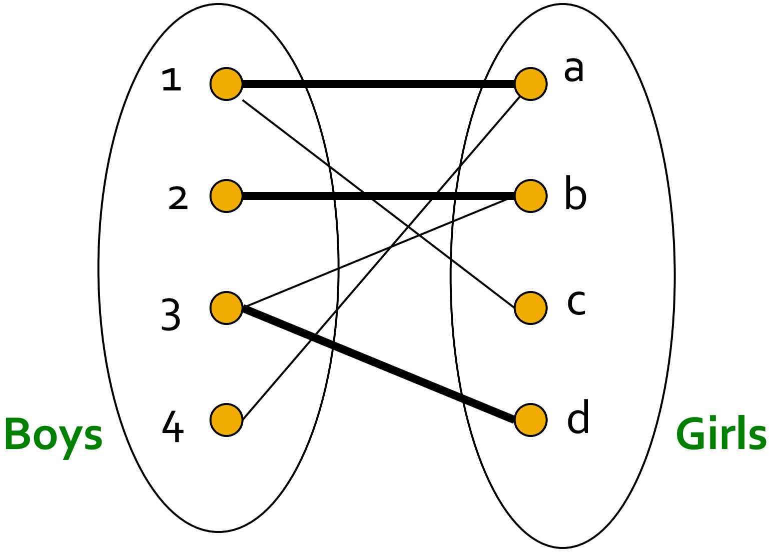
Example: Bipartite Matching



Nodes: Boys and Girls; Edges: Compatible Pairs

Goal: Match as many compatible pairs as possible

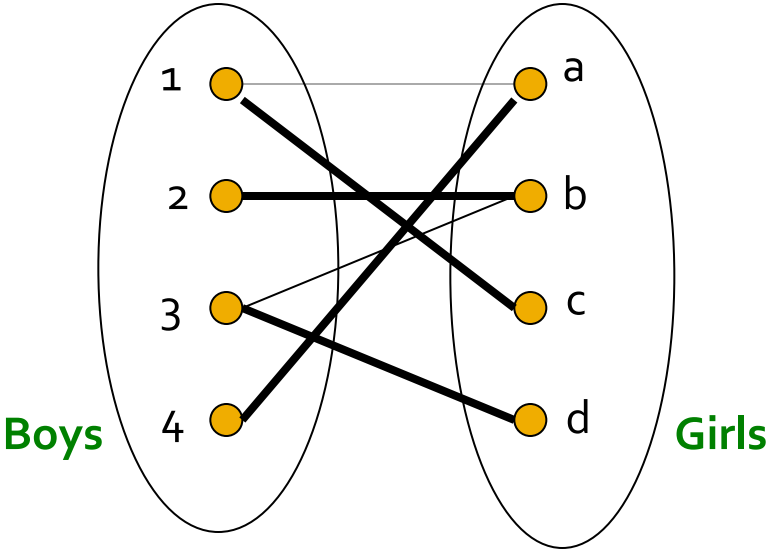
Example: Bipartite Matching



M = {(1,a),(2,b),(3,d)} is a **matching**

Cardinality of matching = |M| = 3

Example: Bipartite Matching



M = {(1,c),(2,b),(3,d),(4,a)} is a **perfect matching**

Perfect matching … all vertices of the graph are matched
Maximum matching … a matching that contains the largest possible number of matches

Matching Algorithm

- **Problem: Find a maximum matching for a given bipartite graph**
    - A perfect one if it exists

- There is a polynomial-time offline algorithm based on augmenting paths (Hopcroft & Karp 1973, see http://en.wikipedia.org/wiki/Hopcroft-Karp_algorithm)

- **But what if we do not know the entire graph upfront?**

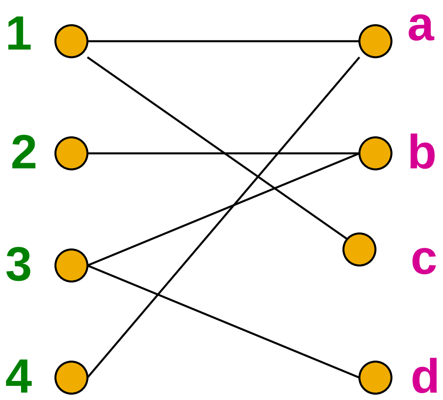
Online Graph Matching Problem

- Initially, we are given the set **boys**
- In each **round**, **one girl's choices are revealed**
  - That is, girl's **edges** are revealed
- **At that time, we have to decide to either:**
  - Pair the **girl** with a **boy**
  - Do not pair the **girl** with any **boy**

- **Example of application:**

  Assigning tasks to servers

Online Graph Matching: Example



(1,a)
(2,b)
(3,d)

Greedy Algorithm

- **Greedy algorithm for the online graph matching problem:**
  - Pair the new girl with **any** eligible boy
    - If there is none, do not pair girl
- **How good is the algorithm?**

Competitive Ratio

- For input ***I***, suppose greedy produces matching M_{greedy} while an optimal matching is M_{opt}

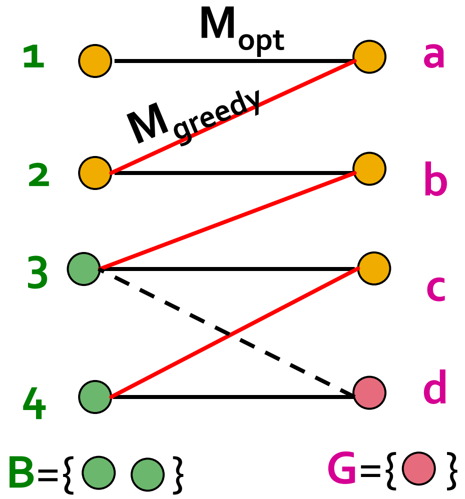
Competitive ratio =

$$\min_{\text{all possible inputs } I} (|M_{greedy}|/|M_{opt}|)$$

(what is greedy's worst performance over all possible inputs *I*)

Analyzing the Greedy Algorithm

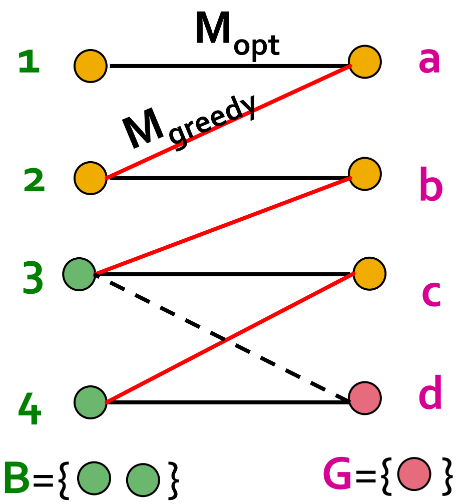
- Suppose $M_{greedy} \neq M_{opt}$
- Consider the set ***G*** of girls matched in M_{opt} but not in M_{greedy}
- (1) $|\mathbf{M}_{\mathbf{opt}}| \leq |\mathbf{M}_{\mathbf{greedy}}| + |\mathbf{G}|$



- Every boy ***B*** adjacent to girls in ***G*** is already matched in M_{greedy}
- (2) $|M_{greedy}| \geq |B|$

Analyzing the Greedy Algorithm

- **So far:**
  - G matched in M_{opt} but not in M_{greedy}
  - Boys B adjacent to girls G
  - (1) $|\mathbf{M_{opt}}| \leq |\mathbf{M_{greedy}}| + |\mathbf{G}|$
  - (2) $|M_{greedy}| \geq |B|$

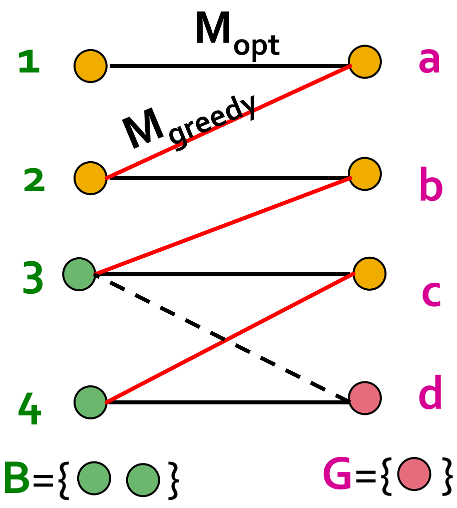


- Optimal matches all the girls in G to boys in B
  - (3) $|G| \leq |B|$

- Combining (2) and (3):
  - (4) $|G| \leq |\mathbf{B}| \leq |M_{greedy}|$

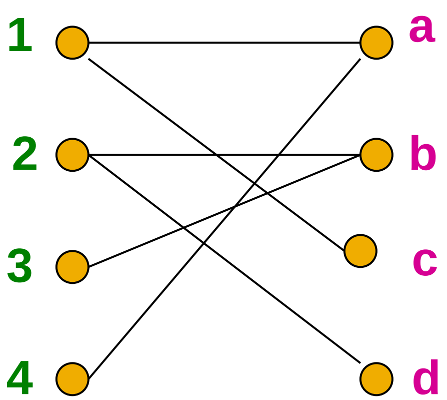
Analyzing the Greedy Algorithm

- So we have:
  - (1) $|M_{opt}| \leq |M_{greedy}| + |G|$
  - (4) $|G| \leq |B| \leq |M_{greedy}|$



- Combining (1) and (4):
  - $|M_{opt}| \leq |M_{greedy}| + |M_{greedy}|$
  - $|M_{opt}| \leq 2|M_{greedy}|$
  - $|M_{greedy}|/|M_{opt}| \geq 1/2$

Worst-case Scenario



(1,a)
(2,b)

Online Algorithms

Performance-based Advertising
The AdWords Problem

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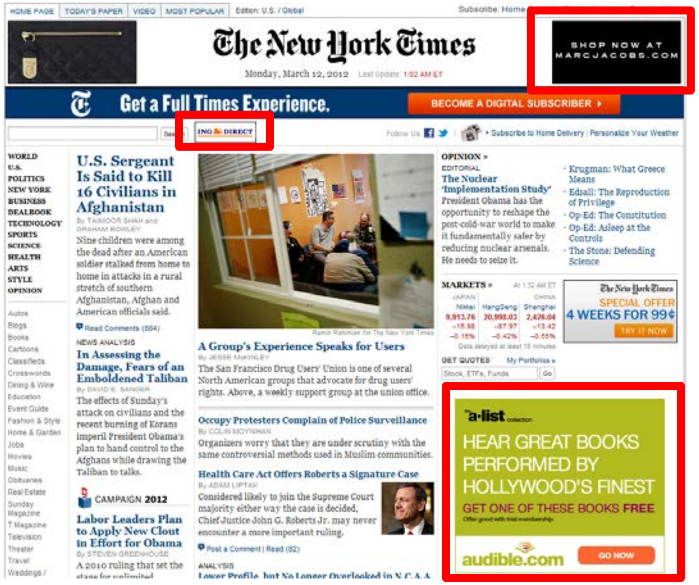
History of Web Advertising

- **Banner ads (1995-2001)**
  - Initial form of web advertising
  - Popular websites charged *X*\$ for every 1,000 “impressions” of the ad
    - Called “**CPM**” rate (Cost per thousand impressions)
    - Modeled similar to TV, magazine ads
  - From **untargeted** to **demographically targeted**
  - **Low click-through rates**
    - Low ROI for advertisers



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U.S. Sergeant Is Said to Kill 16 Civilians in Afghanistan

By TAIMOOR SHAH and GRAHAM BOWLEY

Nine children were among the dead after an American soldier stalked from home to home in attacks in a rural stretch of southern Afghanistan, Afghan and American officials said.

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NEWS ANALYSIS

In Assessing the Damage, Fears of an Emboldened Taliban

By DAVID E. SANGER

The effects of Sunday's attack on civilians and the recent burning of Korans imperil President Obama's plan to hand control to the Afghans while drawing the Taliban to talks.

CAMPAIGN 2012

Labor Leaders Plan to Apply New Clout in Effort for Obama

By STEVEN GREENHOUSE

A 2010 ruling that set the stage for unlimited

Ramin Rahman for The New York Times

A Group's Experience Speaks for Users

By JESSE McKINLEY

The San Francisco Drug Users' Union is one of several North American groups that advocate for drug users' rights. Above, a weekly support group at the union office.

Occupy Protesters Complain of Police Surveillance

By COLIN MOYNIHAN

Organizers worry that they are under scrutiny with the same controversial methods used in Muslim communities.

Health Care Act Offers Roberts a Signature Case

By ADAM LIPTAK

Considered likely to join the Supreme Court majority either way the case is decided, Chief Justice John G. Roberts Jr. may never encounter a more important ruling.

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ANALYSIS

Lower Profile, but No Longer Overlooked in N.C.A.A.

OPINION »

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President Obama has the opportunity to reshape the post-cold-war world to make it fundamentally safer by reducing nuclear arsenals. He needs to seize it.

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Performance-based Advertising

- **Introduced by Overture around 2000**
  - Advertisers **bid** on **search keywords**
  - When someone searches for that keyword, the **highest bidder's ad is shown**
  - Advertiser is charged only if the ad is clicked on
- Similar model adopted by Google with some changes around 2002
  - Called **Adwords**

Algorithmic Challenges

- **Performance-based advertising works!**
  - Multi-billion-dollar industry
- **What ads to show for a given query?**
  - (Today’s lecture)
- **If I am an advertiser, which search terms should I bid on and how much should I bid?**
  - (Not focus of today’s lecture)

AdWords Problem

- A stream of queries arrives at the search engine: q_1, q_2, …
- Several advertisers bid on each query
- When query q_i arrives, search engine must pick a subset of advertisers whose ads are shown

- **Goal: Maximize search engine's revenues**

- **Clearly we need an online algorithm!**

Expected Revenue

Advertiser	Bid	CTR	Bid * CTR
A	$1.00	1%	1 cent
B	$0.75	2%	1.5 cents
C	$0.50	2.5%	1.125 cents

Click through rate

Expected revenue

The Adwords Innovation

Instead of sorting advertisers by bid, sort by expected revenue!

Advertiser	Bid	CTR	Bid * CTR
B	$0.75	2%	1.5 cents
C	$0.50	2.5%	1.125 cents
A	$1.00	1%	1 cent

Adwords Problem

- **Given:**
  - A set of bids by advertisers for search queries
  - A click-through rate for each advertiser-query pair
  - A budget for each advertiser (say for 1 day, month...)
  - A limit on the number of ads to be displayed with each search query

- **Respond to each search query with a set of advertisers such that:**
  - The size of the set is no larger than the limit on the number of ads per query
  - Each advertiser has bid on the search query
  - Each advertiser has enough budget left to pay for the ad if it is clicked upon

Limitations of Simple Algorithm

Instead of sorting advertisers by bid, sort by expected revenue!

Advertiser	Bid	CTR	Bid * CTR
B	$0.75	2%	1.5 cents
C	$0.50	2.5%	1.125 cents
A	$1.00	1%	1 cent

- CTR of an ad is unknown
- Advertisers have limited budgets and bid on multiple ads (BALANCE algorithm)

Estimating CTR

- Clickthrough rate (CTR) for a query-ad pair is measured historically
  - Averaged over a time period

- Some complications we won't cover in this lecture
  - CTR is position dependent
    - Ad #1 is clicked more than Ad #2
  - Explore v Exploit: Keep showing ads we already know the CTR of, or show new ads to estimate their CTR?

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The BALANCE Algorithm

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  - Each advertiser has enough budget left to pay for the ad if it is clicked upon

Dealing with Limited Budgets

- **Our setting: Simplified environment**
  - There is **1** ad shown for each query
  - All advertisers have the same budget ***B***
  - All ads are equally likely to be clicked
  - Value of each ad is the same (=**1**)
- **Simplest algorithm is greedy:**
  - For a query pick any advertiser who has bid **1** for that query
  - **Competitive ratio of greedy is 1/2**

Bad Scenario for Greedy

- Two advertisers A and B
  - ***A*** bids on query ***x***, ***B*** bids on ***x*** and ***y***
  - Both have budgets of **$4**
- Query stream: ***x x x x y y y y***
  - Worst case greedy choice: ***B B B B*** **_ _ _ _**
  - Optimal: **A A A A B B B B**
  - **Competitive ratio = ½**
- This is the worst case!
  - **Note:** Greedy algorithm is deterministic – it always resolves draws in the same way

BALANCE Algorithm [MSVV]

- **BALANCE** Algorithm by Mehta, Saberi, Vazirani, and Vazirani
  - **For each query, pick the advertiser with the largest unspent budget**
  - Break ties arbitrarily (**but in a deterministic way**)

Example: BALANCE

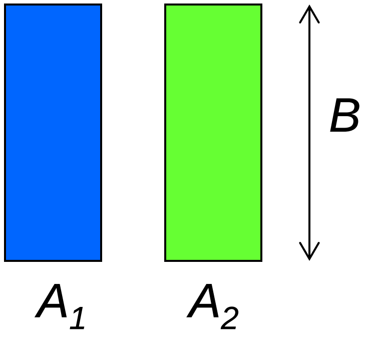
- Two advertisers A and B
  - **A** bids on query ***x***, **B** bids on ***x*** and ***y***
  - Both have budgets of **$4**
- Query stream: ***x x x x y y y y***
- BALANCE choice: **A B A B B B _ _**
  - Optimal: **A A A A B B B B**
- Competitive ratio = ¾
  - **For BALANCE with 2 advertisers**

Analyzing 2-advertiser BALANCE

- **Consider simple case**
  - **2** advertisers, $\mathbf{A_1}$ and $\mathbf{A_2}$, each with budget **B** (≥1)
  - Optimal solution exhausts both advertisers' budgets

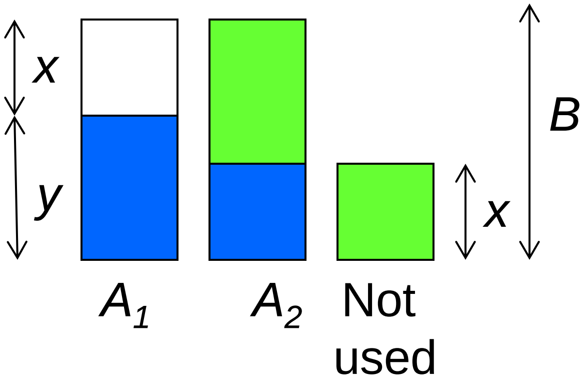
- **BALANCE must exhaust at least one advertiser's budget:**
  - **If not, we can allocate more queries**
  - Assume BALANCE exhausts A_2's budget

Analyzing Balance



🟦 Queries allocated to A_1 in the optimal solution

🟩 Queries allocated to A_2 in the optimal solution

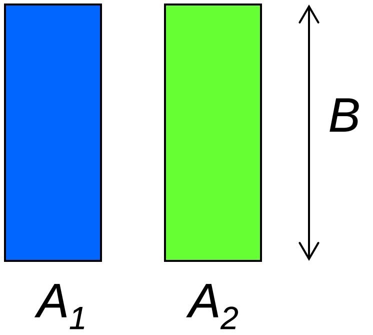


Optimal revenue = **2B**
Balance revenue **= B+y**
Goal: Show that y ¸ B/2

Case 1: BALANCE assigns at least B/2 blue queries to A_1. So y ¸ B/2.

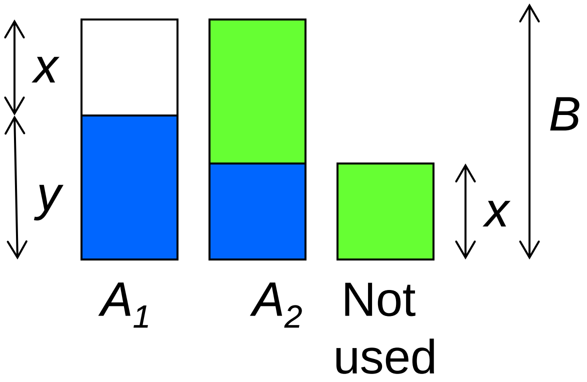
Case 2: BALANCE assigns more than B/2 blue queries to A_2.
Consider the last blue query assigned to A_2.
At that time, A_2's unspent budget must have been at least as big as A_1's.
That means at least as many queries have been assigned to A_1 as to A_2.
At this point, we have already assigned at least B/2 queries to A_2.
So y ¸ B/2.

Analyzing BALANCE



🟦 Queries allocated to $\boldsymbol{A_1}$ in the optimal solution

🟩 Queries allocated to $\boldsymbol{A_2}$ in the optimal solution



Optimal revenue OPT = **2B**
Balance revenue BAL **= B+y**

We have shown that y ¸ B/2
BAL ¸ B+B/2 = 3B/2
BAL/OPT ¸ 3/4

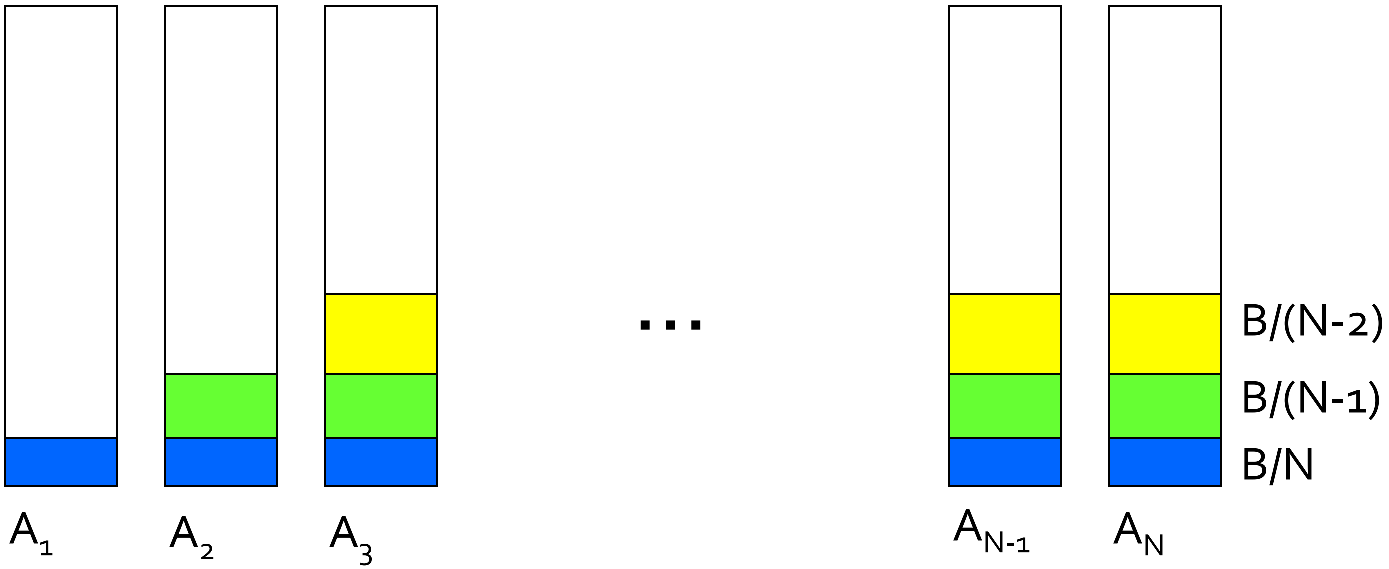
BALANCE: General Result

- **In the general case, worst competitive ratio of BALANCE is 1–1/e = approx. 0.63**
  - Interestingly, no online algorithm has a better competitive ratio!
- **Let’s see the worst case example that gives this ratio**

Worst case for BALANCE

- ***N* advertisers:** $A_1, A_2, \ldots A_N$
  - Each with budget $B > N$
- **Queries:**
  - $N \cdot B$ queries appear in N rounds of B queries each
- **Bidding:**
  - Round **1** queries: bidders $A_1, A_2, \ldots, A_N$
  - Round **2** queries: bidders $A_2, A_3, \ldots, A_N$
  - Round ***i*** queries: bidders $A_i, \ldots, A_N$
- **Optimum allocation:**
  Allocate round ***i*** queries to A_i
  - Optimum revenue $N \cdot B$

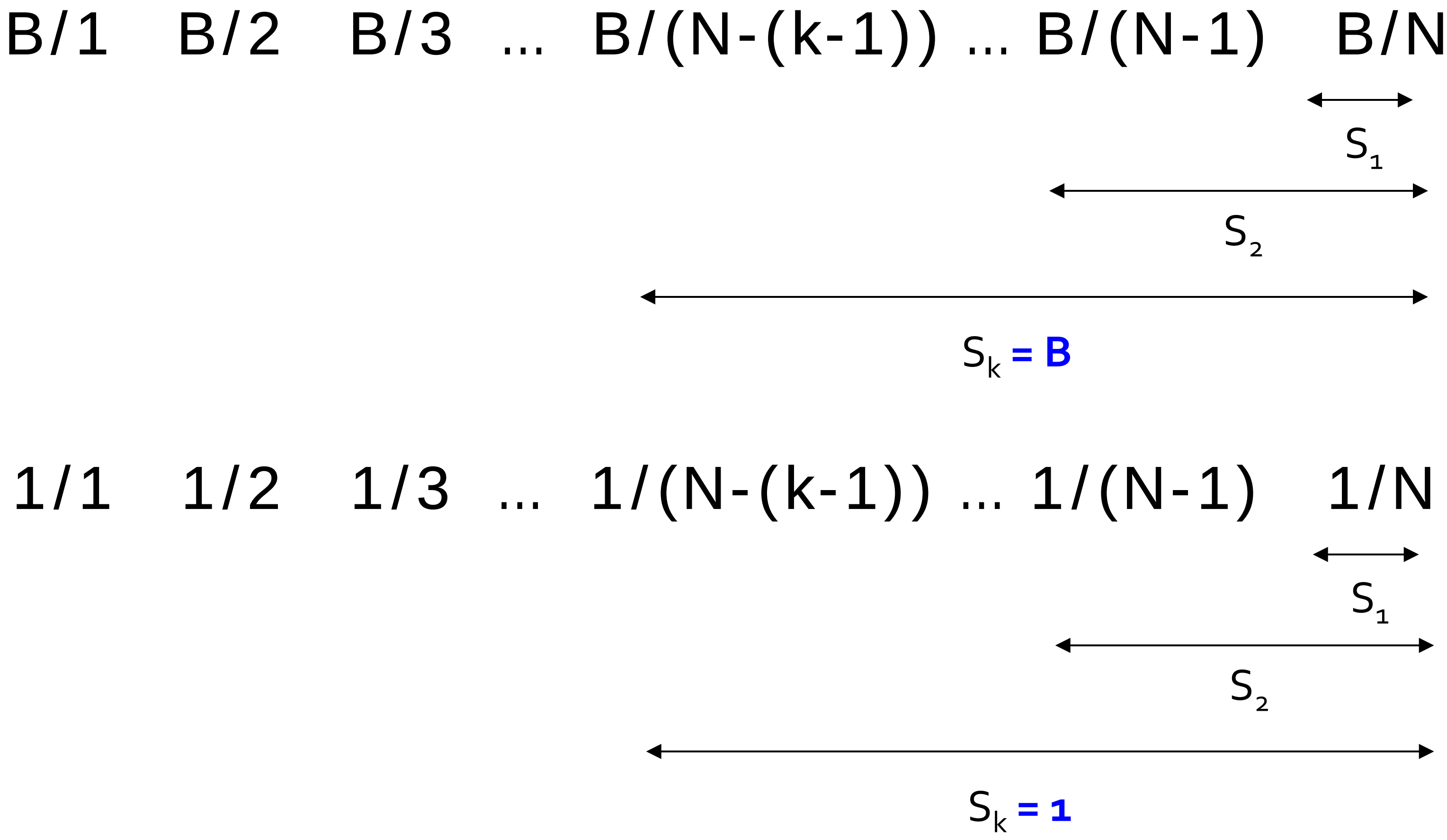
BALANCE Allocation



After k rounds, the allocation to advertiser k is:
$S_k = \sum_{1 \cdot i \cdot k} B/(N-i+1)$

If we find the smallest *k* such that $S_k \geq B$, then after *k* rounds we cannot allocate any queries to any advertiser

BALANCE: Analysis



B/1 B/2 B/3 … B/(N-(k-1)) … B/(N-1) B/N

S_1

S_2

S_k = **B**

1/1 1/2 1/3 … 1/(N-(k-1)) … 1/(N-1) 1/N

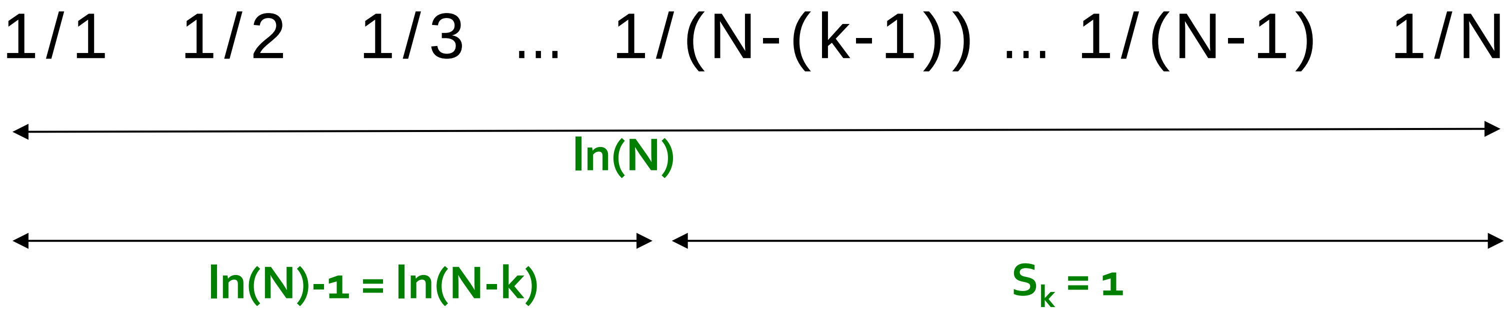
S_1

S_2

S_k = **1**

BALANCE: Analysis

- **Fact:** for large ***n***
  - Result due to Euler



$1/1 \quad 1/2 \quad 1/3 \quad \ldots \quad 1/(N-(k-1)) \quad \ldots \quad 1/(N-1) \quad 1/N$

ln(N)

ln(N)-1 = ln(N-k)

S_k = 1

ln(N-k) = ln(N) – 1
ln(N/(N-k)) = 1
N/(N-k) = e
k = N(1 – 1/e)

BALANCE: Analysis

- So after the first **k=N(1-1/e)** rounds, we cannot allocate a query to any advertiser
- **Revenue = B·N (1-1/e)**
- **Competitive ratio = 1-1/e**

General Version of the Problem

- So far: all bids = 1, all budgets equal (=B)

- **In a general setting BALANCE can be terrible**
  - Consider query **q**, two advertisers $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$
  - $\boldsymbol{A_1}$: ***bid*** = **1**, ***budget*** = **110**
  - $\boldsymbol{A_2}$: ***bid*** = **10**, ***budget*** = **100**
  - Suppose we see **10** instances of **q**
  - BALANCE always selects $\boldsymbol{A_1}$ and earns **10**
  - Optimal earns **100**

Generalized BALANCE

- Consider query ***q***, bidder ***i***
  - Bid = x_i
  - Budget = b_i
  - Amount spent so far = m_i
  - Fraction of budget left over $f_i = 1-m_i/b_i$
  - Define $\psi_i(q) = x_i(1-e^{-f_i})$
- Allocate query ***q*** to bidder ***i*** with largest value of $\psi_i(q)$
- **Same competitive ratio (1-1/e)**