# Theory of Locality Sensitive Hashing

CS246: Mining Massive Datasets
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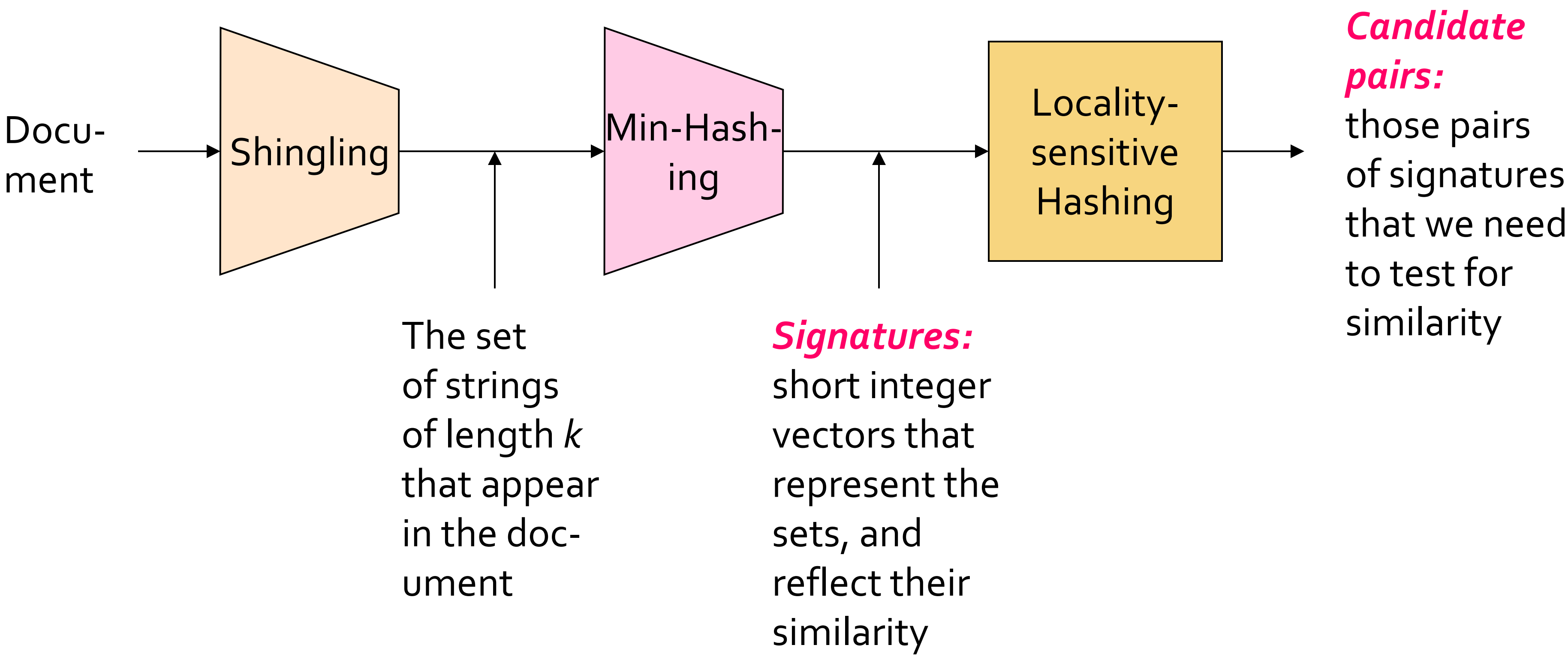
# Recap: Finding similar documents

- **Task:** Given a large number (***N*** in the millions or billions) of documents, find "near duplicates"
- **Applications:**
  - Mirror websites, or approximate mirrors
    - Don't want to show both in a single set of search results
- **Problems:**
  - Many small pieces of one document can appear out of order in another
  - Too many documents to compare all pairs
  - Documents are so large or so many that they cannot fit in main memory

# Recap: 3 Essential Steps

1. ***Shingling***: Convert docs to sets of items
   - Document is a set of k-**shingles**
2. ***Min-Hashing***: Convert large sets into short signatures, while preserving similarity
   - Want hash func. that **Pr**$[h_\pi(C_1) = h_\pi(C_2)] = sim(C_1, C_2)$
     - For the Jaccard similarity Min-Hash has this property!
3. ***Locality-sensitive hashing***: Focus on pairs of signatures likely to be from similar documents
   - Split signatures into bands and hash them
   - Documents with similar signatures get hashed into same buckets: **Candidate pairs**

# Recap: The Big Picture



# Recap: Shingles

- A *k*-shingle (or *k*-gram) is a sequence of *k* tokens that appears in the document
  - Example: **k=2**; $\mathbf{D_1}$ = abcab
    Set of 2-shingles: $\mathbf{C_1}$ = S($D_1$) = {ab, bc, ca}
- Represent a doc by the set of hash values of its *k*-shingles
- A natural document similarity measure is then the **Jaccard similarity:**

$$\boldsymbol{sim(D_1, D_2) = |C_1 \cap C_2| / |C_1 \cup C_2|}$$

  - Similarity of two documents is the Jaccard similarity of their shingles

# Recap: Minhashing

- Prob. $h_\pi(C_1) = h_\pi(C_2)$ is the same as $sim(D_1, D_2)$:

$$\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(D_1, D_2)$$

**Permutation $\pi$**

| | | |
|---|---|---|
| 2 | 4 | 3 |
| 3 | 2 | 4 |
| 7 | 1 | 7 |
| 6 | 3 | 2 |
| 1 | 6 | 6 |
| 5 | 7 | 1 |
| 4 | 5 | 5 |

**Input matrix (Shingles x Documents)**

| | | | |
|---|---|---|---|
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 |

**Signature matrix *M***

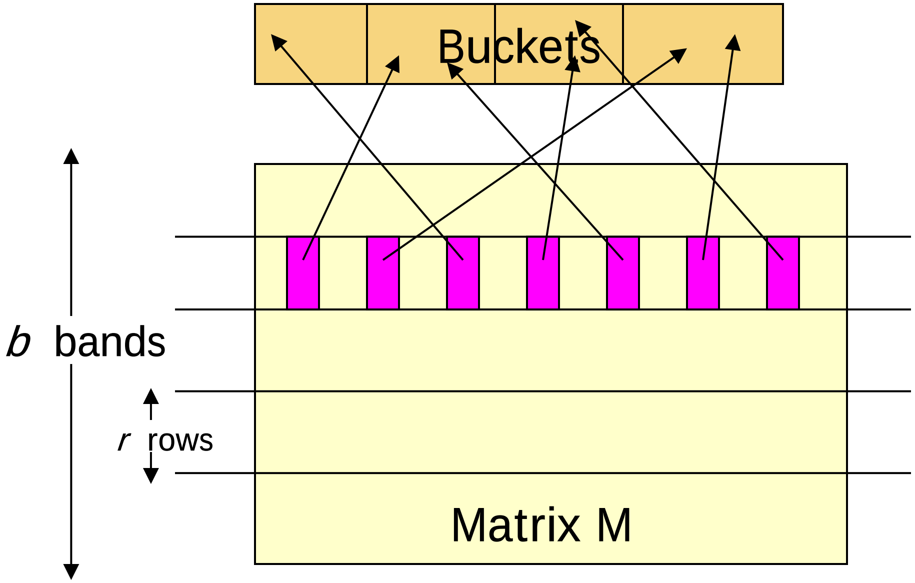
| | | | |
|---|---|---|---|
| 2 | 1 | 2 | 1 |
| 2 | 1 | 4 | 1 |
| 1 | 2 | 1 | 2 |

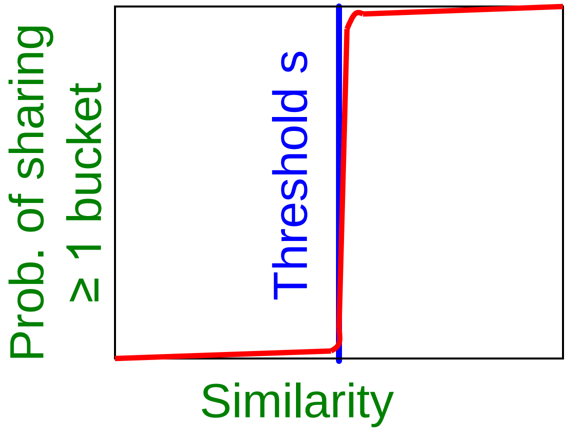
**Similarities of columns and signatures (approx.) match!**

| | 1-3 | 2-4 | 1-2 | 3-4 |
|---|---|---|---|---|
| **Col/Col** | 0.75 | 0.75 | 0 | 0 |
| **Sig/Sig** | 0.67 | 1.00 | 0 | 0 |

# Recap: LSH

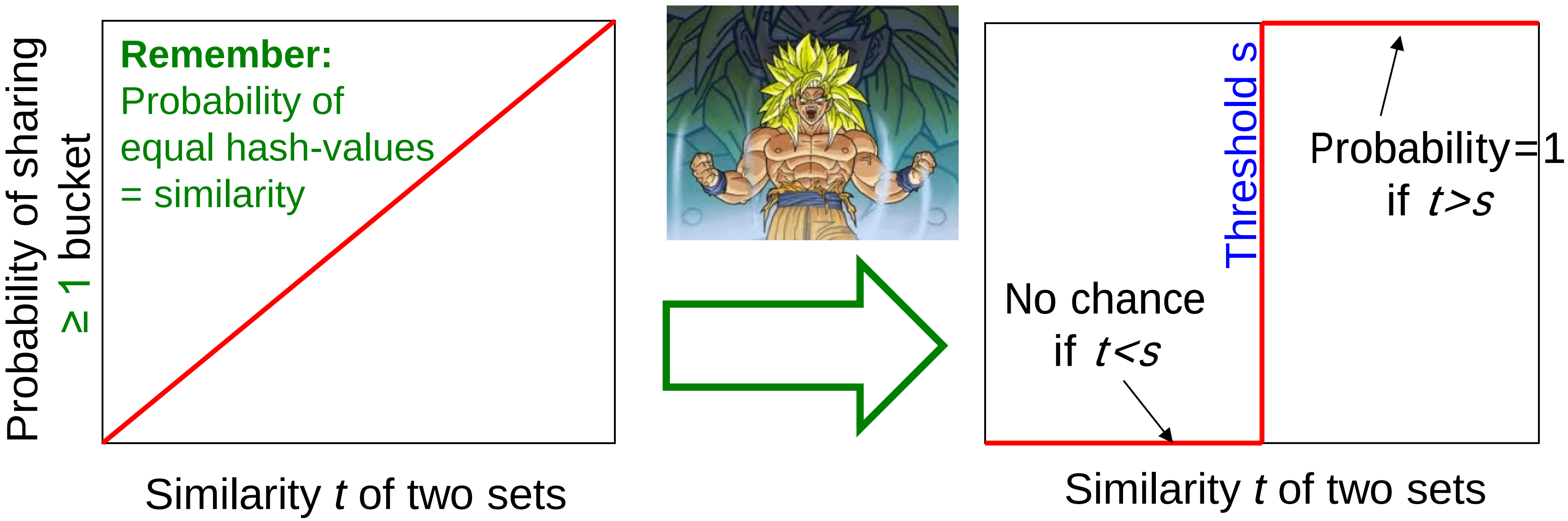
- **Hash columns of the signature matrix *M*:** **Similar columns likely hash to same bucket**
  - Divide matrix *M* into ***b*** bands of ***r*** rows (M=b·r)
  - ***Candidate*** column pairs are those that hash to the same bucket for ≥ **1** band



# Recap: The S-Curve

- **The S-curve is where the "magic" happens**




**This is what 1 hash-code gives you**

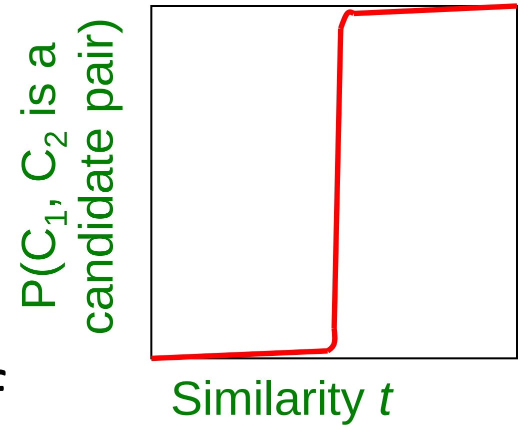
$\Pr[h_\pi(C_1) = h_\pi(C_2)] = sim(D_1, D_2)$

**This is what we want!**

**How to get a step-function?**

**By choosing *r* and *b*!**

# How Do We Make the S-curve?



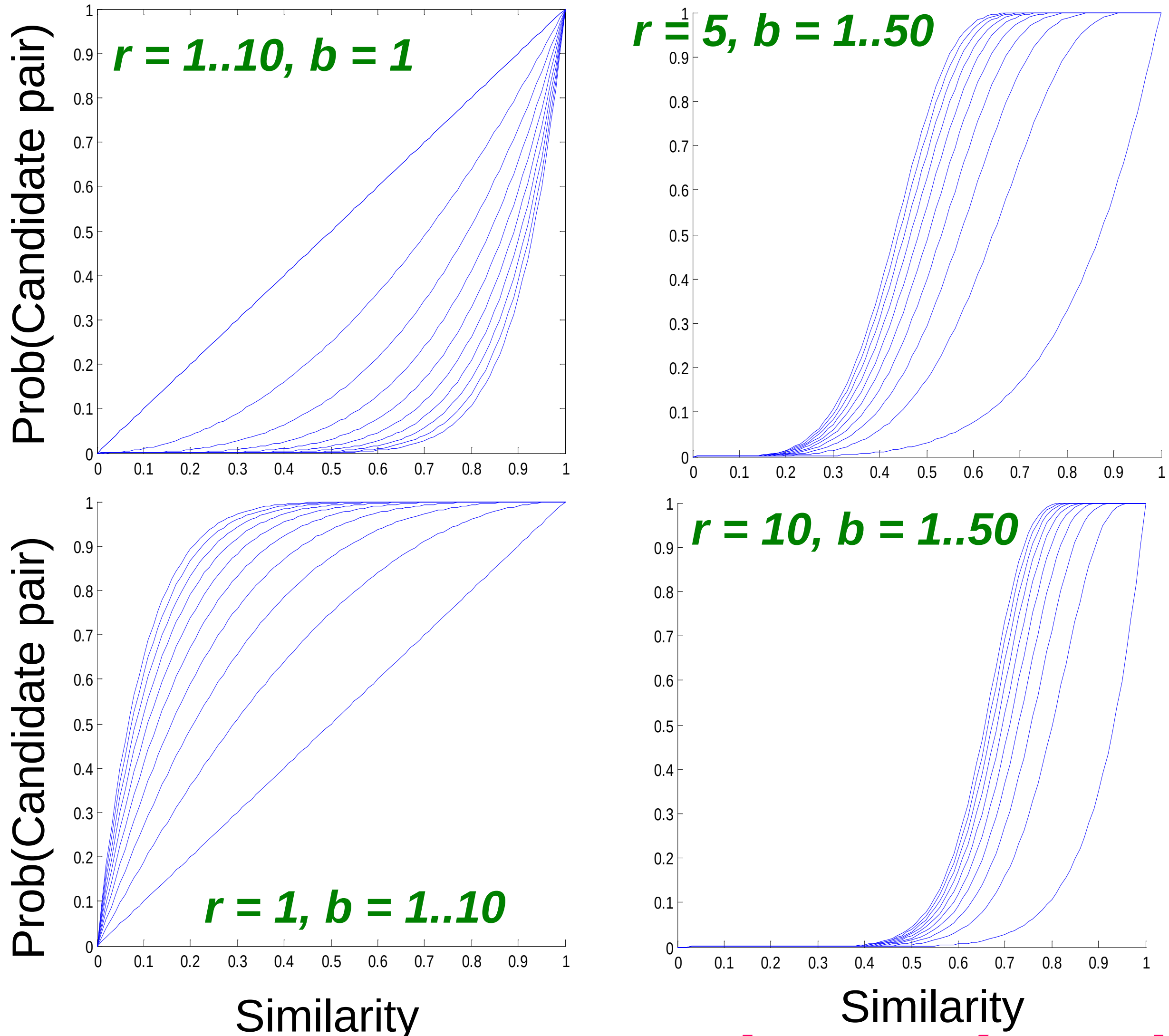

- **Remember:** ***b*** bands, ***r*** rows/band
- Let $sim(C_1, C_2) = t$
- Pick some band (***r*** rows)
  - Prob. that elements in a single row of columns $C_1$ and $C_2$ are equal $= t$
  - Prob. that all rows in a band are equal $= t^r$
  - Prob. that some row in a band is not equal $= 1 - t^r$
- Prob. that all bands are not equal $= (1 - t^r)^b$
- Prob. that at least 1 band is equal $= 1 - (1 - t^r)^b$

$$P(C_1, C_2 \text{ is a candidate pair}) = 1 - (1 - t^r)^b$$

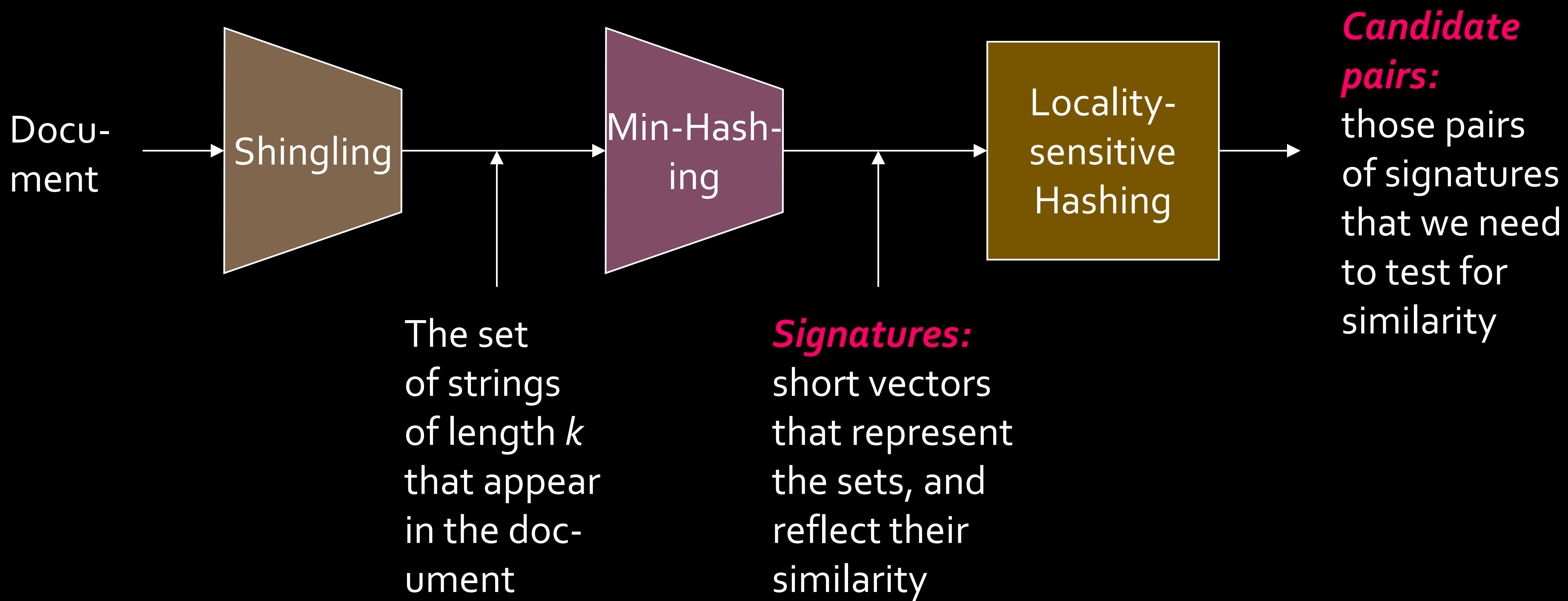
# S-curves as a func. of *b* and *r*

Given a fixed threshold ***s***.

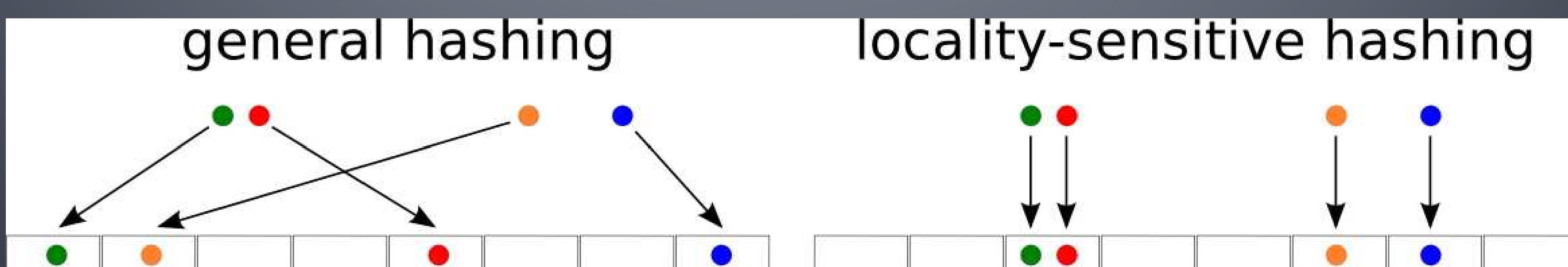
We want choose ***r*** and ***b*** such that the ***P(Candidate pair)*** has a "step" right around ***s***.




***prob = 1 - (1 - t ^r^)^b^***




# Theory of LSH



# Theory of LSH

- **We have used LSH to find similar documents**
  - More generally, we found similar columns in large sparse matrices with high Jaccard similarity
    - For example, customer/item purchase histories
- **Can we use LSH for other distance measures?**
  - e.g., Euclidean distances, Cosine distance
  - **Let’s generalize what we’ve learned!**

# Families of Hash Functions

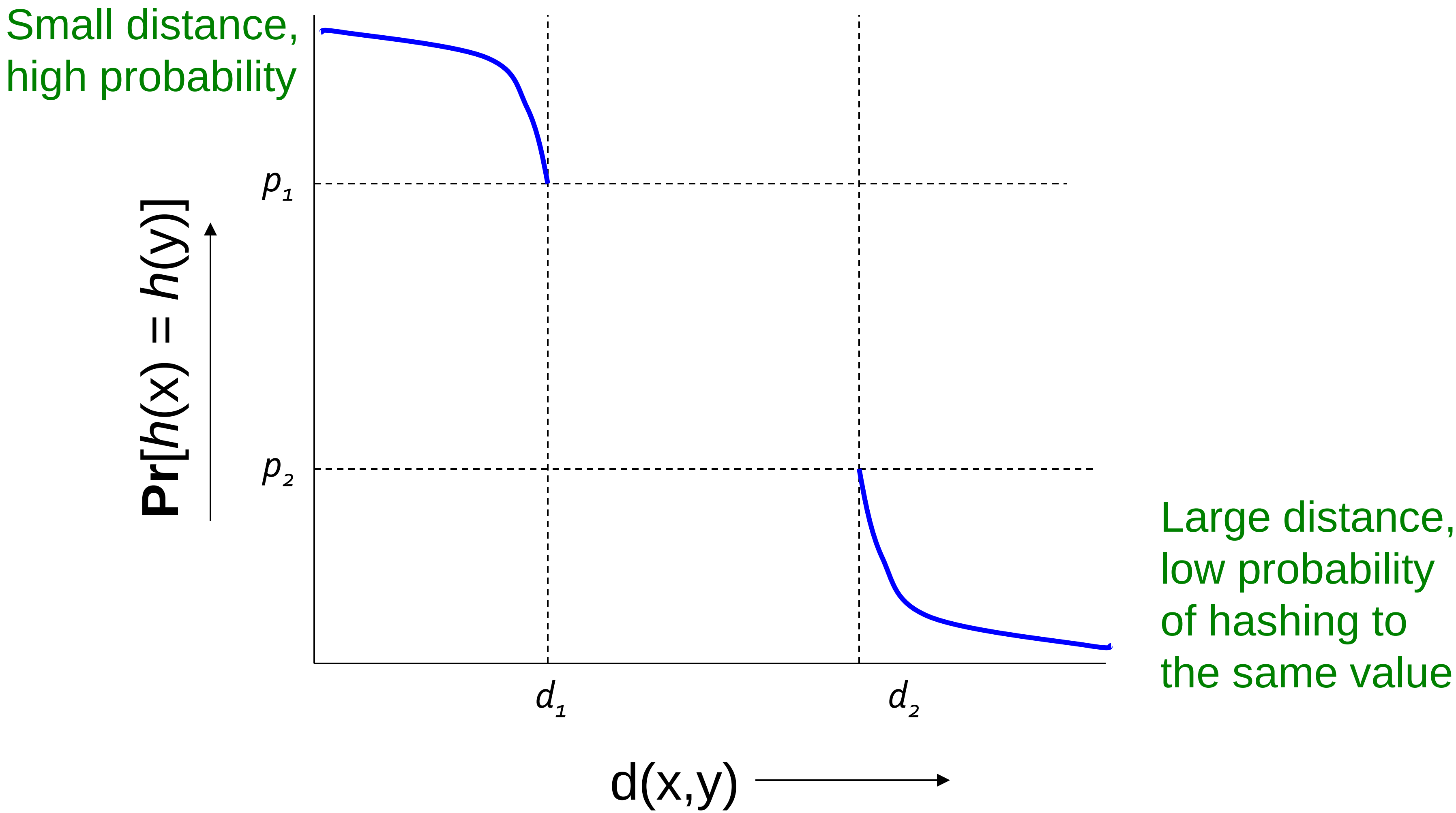
- **For Min-Hashing signatures, we got a Min-Hash function for each permutation of rows**
- A "hash function" is any function that takes ***two*** elements and says whether they are "equal"
  - **Shorthand:** ***h(x) = h(y)*** means "***h says x and y are equal***"
- A ***family*** of hash functions is any set of hash functions from which we can ***pick one at random efficiently***
  - **Example:** The set of Min-Hash functions generated from permutations of rows

# Locality-Sensitive (LS) Families

- **Suppose we have a space *S* of points with a distance measure *d(x,y)***
- A family ***H*** of hash functions is said to be ***($d_1$, $d_2$, $p_1$, $p_2$)-sensitive*** if for any ***x*** and ***y*** in ***S***:
  1. If $d(x, y) \leq d_1$, then the probability over all $h \in H$, that $h(x) = h(y)$ is at least $p_1$
  2. If $d(x, y) \geq d_2$, then the probability over all $h \in H$, that $h(x) = h(y)$ is at most $p_2$

Note: Here x,y are "fixed" and the randomness is over the hash functions

# A $(d_1,d_2,p_1,p_2)$-sensitive function



# Example of LS Family: Min-Hash

- **Let:**
  - $\boldsymbol{S}$ = space of all sets,
  - $\boldsymbol{d}$ = Jaccard distance,
  - $\boldsymbol{H}$ is family of Min-Hash functions for all permutations of rows
- Then for any hash function $\boldsymbol{h} \in \boldsymbol{H}$:

$$Pr[h(x) = h(y)] = 1 - d(x, y)$$

  - Simply restates theorem about Min-Hashing in terms of distances rather than similarities

# Example: LS Family – (2)

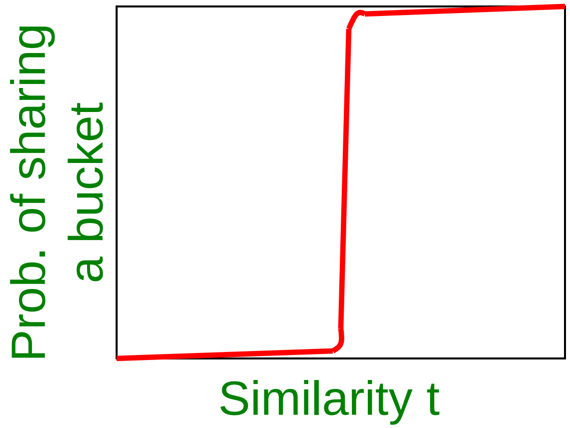
- **Claim: Min-hash *H*** is a ($\boxed{1/3}$, 2/3, $\boxed{2/3}$, 1/3)-sensitive family for ***S*** and ***d***.

If distance $\leq$ 1/3
(so similarity ≥ 2/3)

Then probability
that Min-Hash values
agree is $\geq$ 2/3

- **For Jaccard similarity, Min-Hashing gives a *$(d_1,d_2,(1-d_1),(1-d_2))$*-sensitive family for any *$d_1<d_2$***
- **Theory leaves unknown what happens to pairs that are at distance between *$d_1$* and *$d_2$***
  - **Consequence:** No guarantees about fraction of false positives in that range

# Amplifying a LS-Family




- **Can we reproduce the "S-curve" effect we saw before for any LS family?**
- The "**bands**" technique we learned for signature matrices carries over to this more general setting
  - So we can do LSH with any **$(d_1, d_2, p_1, p_2)$*-sensitive* family**
- **Two constructions:**
  - **AND** construction like "rows in a band"
  - **OR** construction like "many bands"

# Amplifying Hash Functions: AND and OR

# AND of Hash Functions

- Given family $H$, construct family $H'$ consisting of $r$ functions from $H$
- For $h = [h_1,\ldots,h_r]$ in $\mathbf{H'}$, we say $\mathbf{h(x) = h(y)}$ if and only if $\mathbf{h_i(x) = h_i(y)}$ for **all** $i$ $\quad 1 \le i \le r$
  - Note this corresponds to creating a band of size **r**
- **Theorem:** If $H$ is $(d_1, d_2, p_1, p_2)$-sensitive, then $\mathbf{H'}$ is $(d_1, d_2, (p_1)^r, (p_2)^r)$-sensitive
- **Proof:** Use the fact that $h_i$'s are **independent**

# Subtlety Regarding Independence

- **Independence of hash functions (HFs) really means that the prob. of two HFs saying "yes" is the product of each saying "yes"**
  - **But** two hash functions could be highly correlated
    - For example, in Min-Hash if their permutations agree in the first one million entries
  - **However**, the probabilities in definition of a LSH-family are over all possible members of ***H***, ***H'***

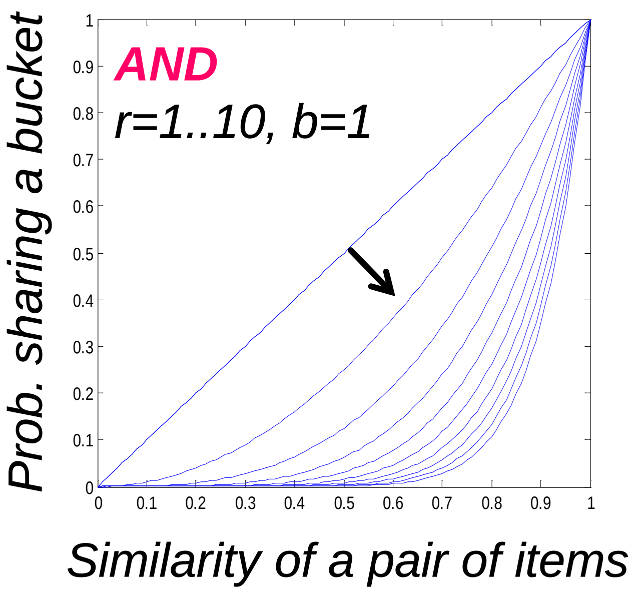
# OR of Hash Functions

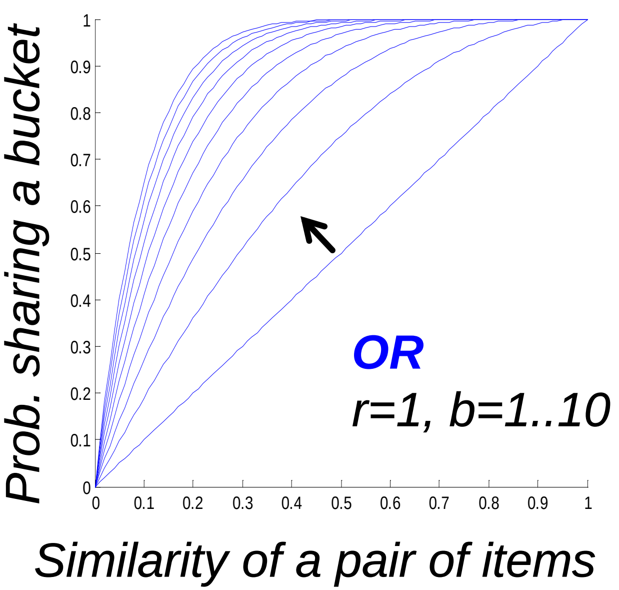
- Given family $\boldsymbol{H}$, construct family $\boldsymbol{H'}$ consisting of $\boldsymbol{b}$ functions from $\boldsymbol{H}$
- For $h = [h_1,\ldots,h_b]$ in $H'$,
  **h(x) = h(y)** if and only if $\mathbf{h_i(x) = h_i(y)}$ for **at least 1** $\boldsymbol{i}$
- **Theorem:** If $H$ is $(d_1, d_2, p_1, p_2)$-sensitive, then $H'$ is $(d_1, d_2, 1-(1-p_1)^b, 1-(1-p_2)^b)$-sensitive
- **Proof:** Use the fact that $h_i$'s are **independent**

# Effect of AND and OR Constructions

- **AND** makes all probs. **shrink**, but by choosing ***r*** correctly, we can make the lower prob. approach 0 while the higher does not

- **OR** makes all probs. **grow**, but by choosing ***b*** correctly, we can make the upper prob. approach 1 while the lower does not



# Composing Constructions

- ***r*-way AND** followed by ***b*-way OR** construction
  - **Exactly what we did with Min-Hashing**
    - If bands match in all ***r*** values hash to same bucket
    - Cols that are hashed into ≥ 1 common bucket → **Candidate**
- Take points ***x*** and ***y*** s.t. $\boldsymbol{Pr[h(x) = h(y)] = p}$
  - ***H*** will make **(x,y)** a candidate pair with prob. **p**
- Construction makes **(x,y)** a candidate pair with probability $\boldsymbol{1-(1-p^r)^b}$ **The S-Curve!**
  - **Example:** Take **H** and construct **H'** by the **AND** construction with $\boldsymbol{r = 4}$. Then, from **H'**, construct **H''** by the **OR** construction with $\boldsymbol{b = 4}$

# Table for Function $1-(1-p^4)^4$

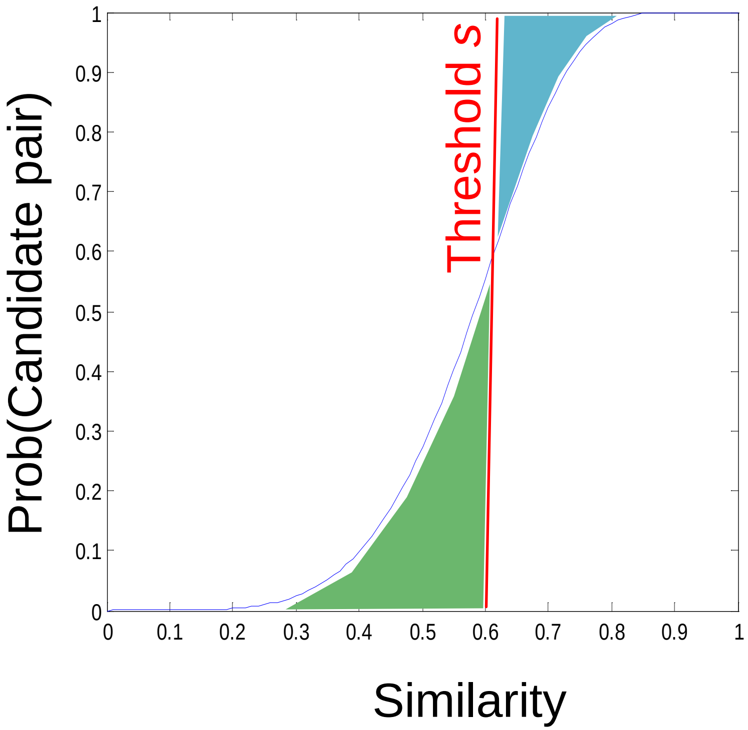
| p | $1-(1-p^4)^4$ |
|---|---|
| .2 | .0064 |
| .3 | .0320 |
| .4 | .0985 |
| .5 | .2275 |
| .6 | .4260 |
| .7 | .6666 |
| .8 | .8785 |
| .9 | .9860 |

***r* = 4, *b* = 4** transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.8785,.0064)-sensitive family.

# How to choose *r* and *b*

# Picking *r* and *b*: The S-curve

- **Picking *r* and *b* to get desired performance**
  - **50 hash-functions (*r* = 5, *b* = 10)**

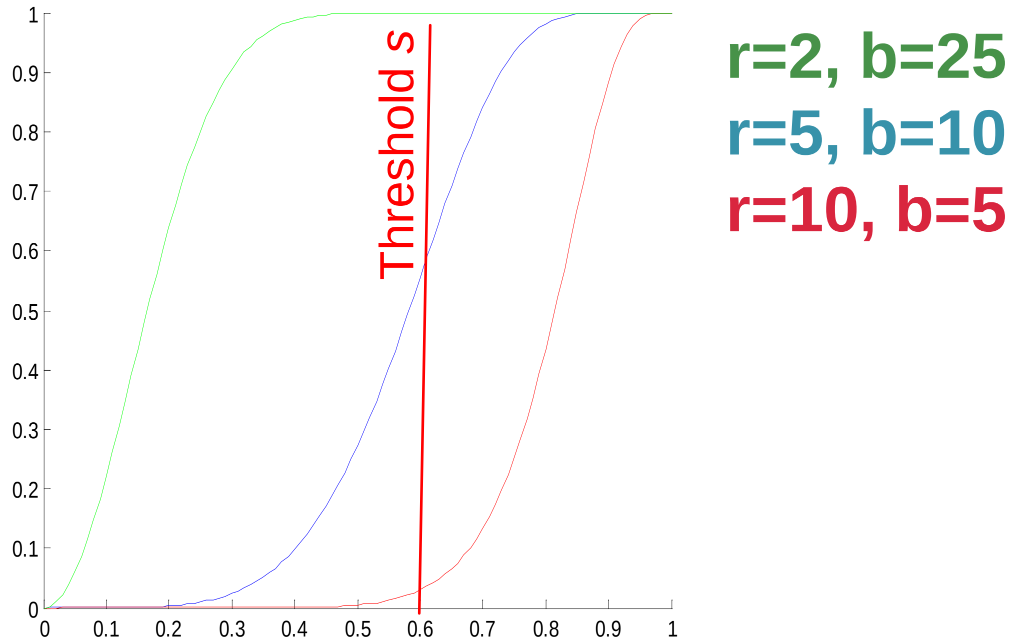



**Blue area *X*: False Negative rate**
These are pairs with ***sim* > *s*** but the ***X*** fraction won't share a band and then will **never become candidates.** This means we will never consider these pairs for (slow/exact) similarity calculation!

**Green area Y: False Positive rate**
These are pairs with ***sim* < *s*** but we will consider them as candidates. This is not too bad, we will consider them for (slow/exact) similarity computation and discard them.

# Picking *r* and *b*: The S-curve

- **Picking *r* and *b* to get desired performance**
  - **50 hash-functions (*r* * *b* = *50*)**

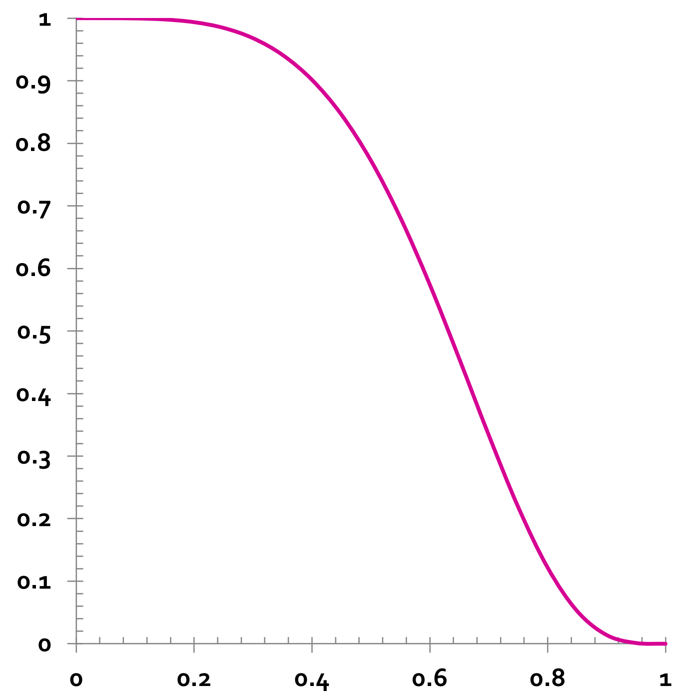


# OR-AND Composition

- Apply a ***b***-way **OR** construction followed by an ***r***-way **AND** construction
- Transforms probability ***p*** into $(1-(1-p)^b)^r$
  - **The same S-curve, mirrored horizontally and vertically**
- **Example:** Take **H** and construct **H'** by the **OR** construction with ***b*** = 4. Then, from **H'**, construct **H''** by the **AND** construction with ***r*** = 4

# Table for Function $(1-(1-p)^4)^4$

| p | $(1-(1-p)^4)^4$ |
|---|---|
| .1 | .0140 |
| .2 | .1215 |
| .3 | .3334 |
| .4 | .5740 |
| .5 | .7725 |
| .6 | .9015 |
| .7 | .9680 |
| .8 | .9936 |




The example transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.9936,.1215)-sensitive family

# Cascading Constructions

- **Example:** Apply the **(4,4) OR-AND** construction followed by the **(4,4) AND-OR** construction
- **Transforms a (.2, .8, .8, .2)-sensitive family into a (.2, .8, .9999996, .0008715)-sensitive family**
  - **Note this family uses 256 (=4*4*4*4) of the original hash functions**

# Summary

- Pick any two distances $d_1 < d_2$
- Start with a $(d_1, d_2, (1- d_1), (1- d_2))$-sensitive family
- Apply constructions to **amplify** $(d_1, d_2, p_1, p_2)$-sensitive family, where $p_1$ is almost 1 and $p_2$ is almost 0
- **The closer to 0 and 1 we get, the more hash functions must be used!**


WANTED
Graders
We need help grading
the problem sets.
Grading is paid 15$/h.
If interested send us email.
$1000 CASH
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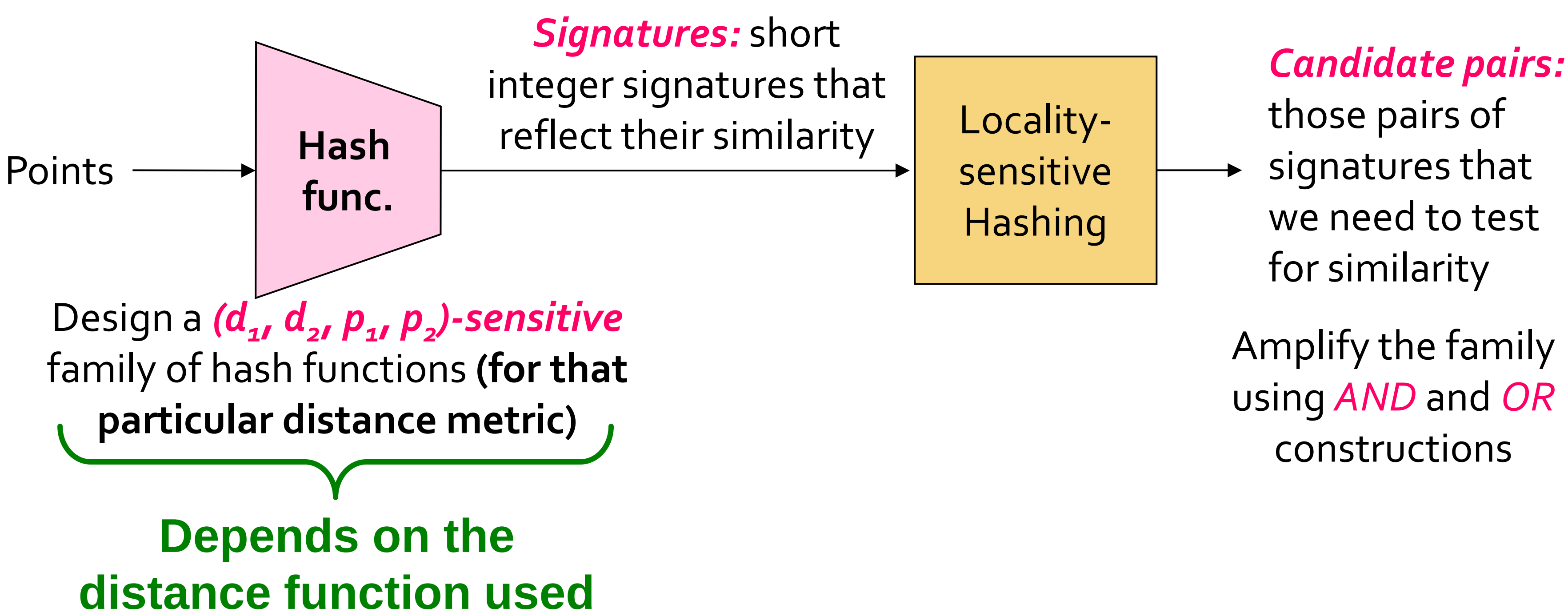
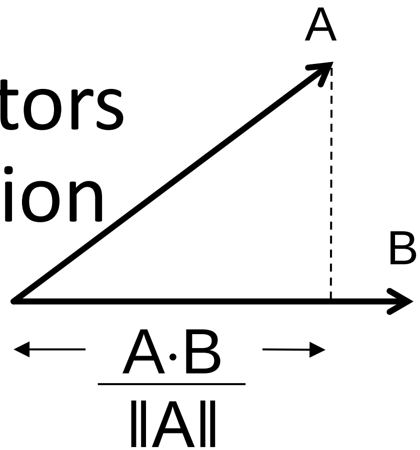
# LHS for other distance metrics

# LSH for other Distance Metrics

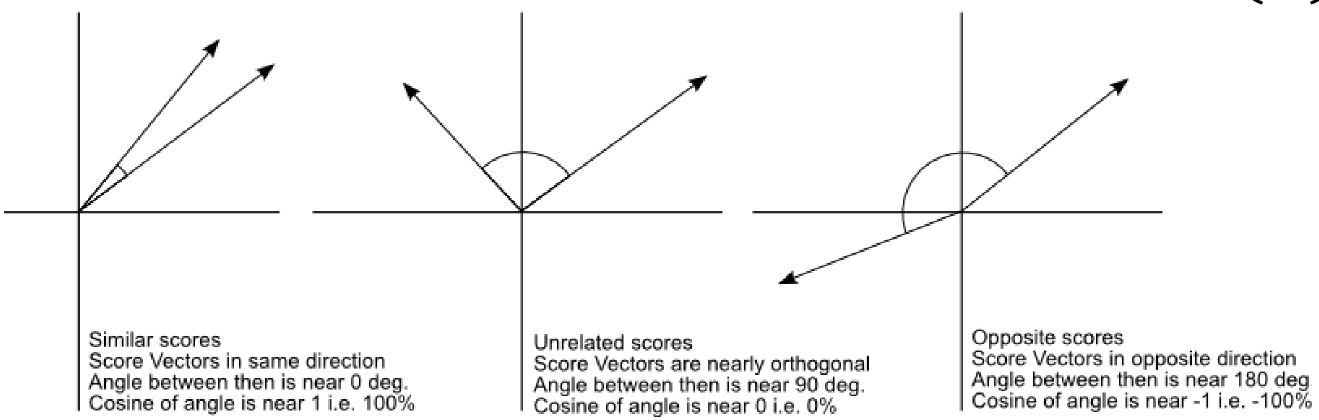
- **LSH methods for other distance metrics:**
  - **Cosine distance: Random hyperplanes**
  - **Euclidean distance: Project on lines**



# Cosine Distance




- ***Cosine distance*** = angle between vectors from the origin to the points in question
  **d(A, B) = θ = arccos(A·B / ‖A‖·‖B‖)**
  - Has range $\mathbf{0} \ldots \boldsymbol{\pi}$ (equivalently 0...180°)
  - Can divide θ by $\boldsymbol{\pi}$ to have distance in range 0…1
- **Cosine similarity = 1-d(A,B)**
  - But often defined as **cosine sim:** $\cos(\theta) = \dfrac{A \cdot B}{\|A\| \|B\|}$




- Has range -1…1 for general vectors
- Range 0..1 for non-negative vectors (angles up to 90°)

# LSH for Cosine Distance

- For **cosine distance**, there is a technique called **Random Hyperplanes**
  - Technique similar to Min-Hashing

- **Random Hyperplanes** method is a ***$(d_1, d_2, (1-d_1/\pi), (1-d_2/\pi))$**-sensitive* family for any ***$d_1$*** and ***$d_2$***

- **Reminder:** *$(d_1, d_2, p_1, p_2)$-sensitive*
  1. If ***$d(x,y) \leq d_1$***, then prob. that ***$h(x) = h(y)$*** is at least ***$p_1$***
  2. If ***$d(x,y) \geq d_2$***, then prob. that ***$h(x) = h(y)$*** is at most **$p_2$**

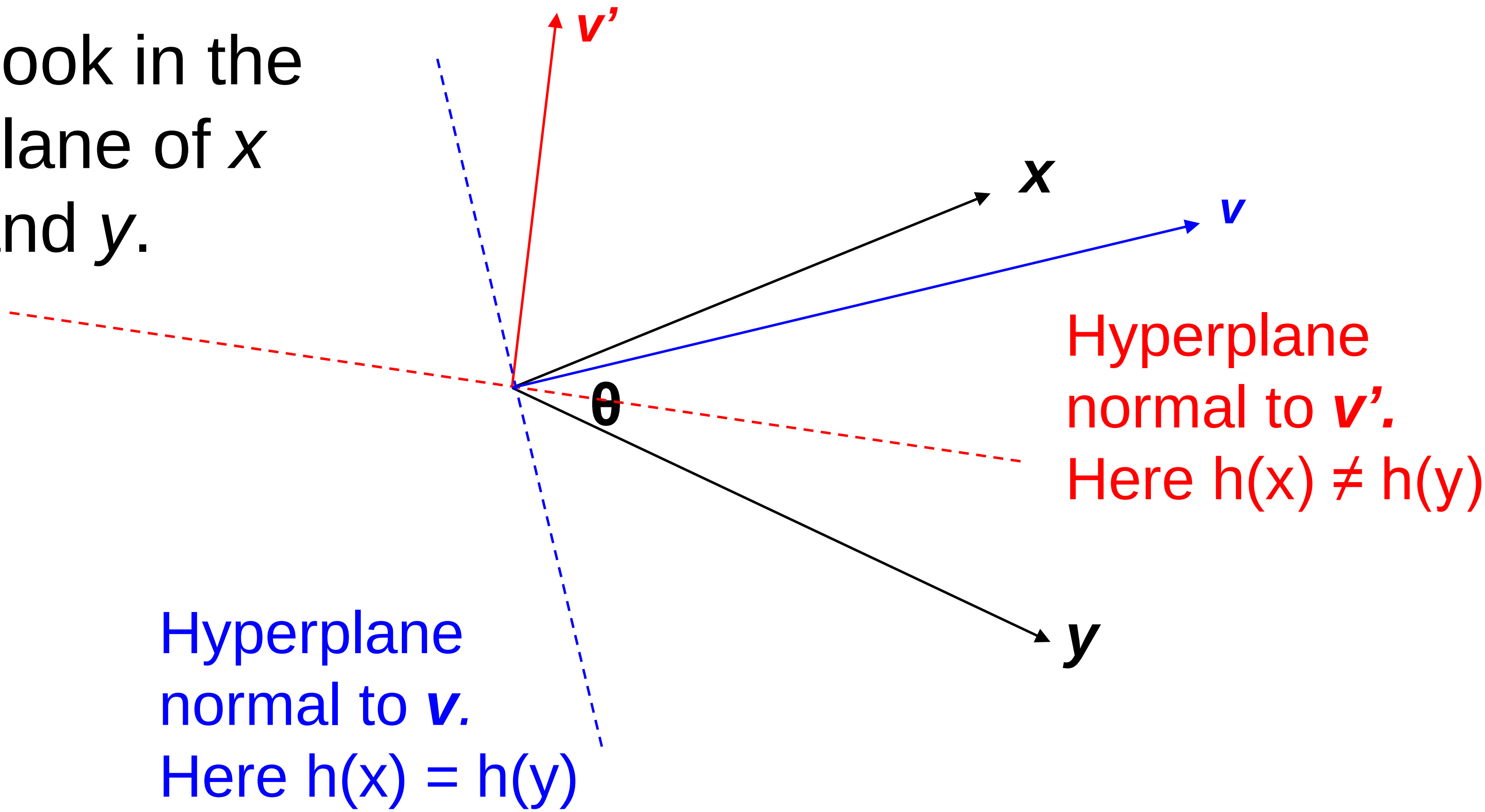
# Random Hyperplanes

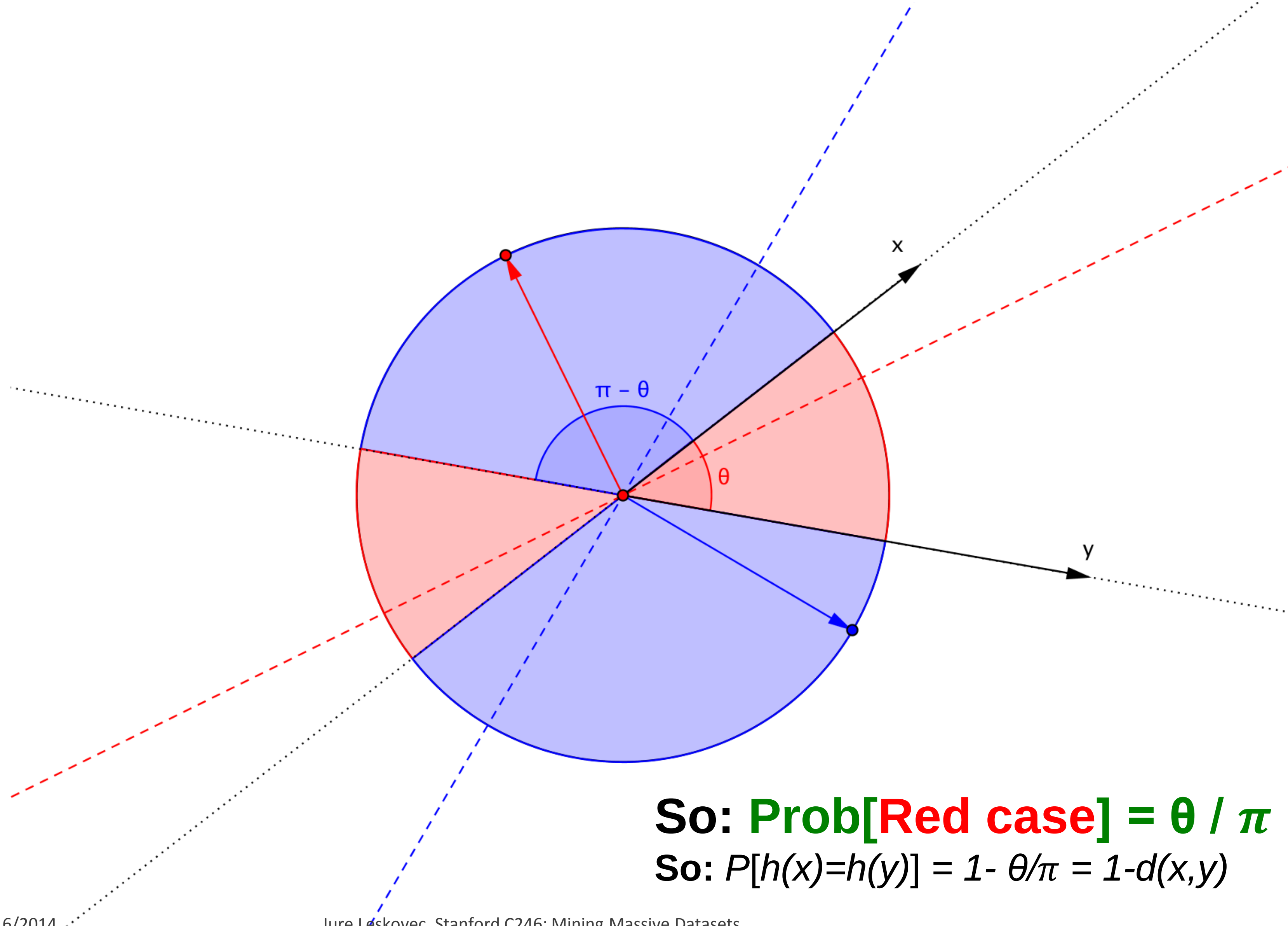
- Pick a random vector ***v***, which determines a hash function $\boldsymbol{h_v}$ with two buckets
- $\mathbf{h_v(x) = +1}$ if $\mathbf{v \cdot x} \geq \mathbf{0}$; $\mathbf{= -1}$ if $\mathbf{v \cdot x < 0}$
- LS-family ***H*** = set of all functions derived from any vector
- **Claim:** For points **x** and **y**,

$$\mathbf{Pr[h(x) = h(y)] \;=\; 1 - d(x,y) / \pi}$$

# Proof of Claim

Look in the plane of *x* and *y*.






**So: Prob[Red case] = θ / $\pi$**

**So:** $P[h(x)=h(y)] = 1 - \theta/\pi = 1 - d(x,y)$

# Signatures for Cosine Distance

- Pick some number of random vectors, and hash your data for each vector
- The result is a signature (*sketch*) of **+1**'s and **–1**'s for each data point
- Can be used for LSH like we used the Min-Hash signatures for Jaccard distance
- Amplify using **AND/OR** constructions

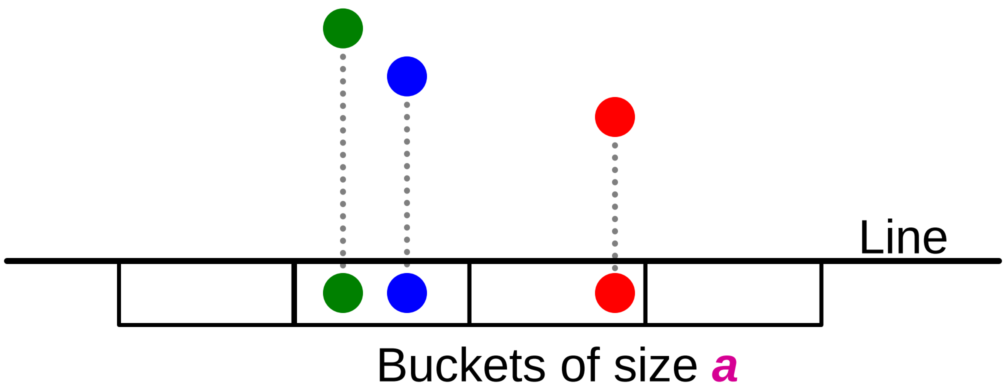
# How to pick random vectors?

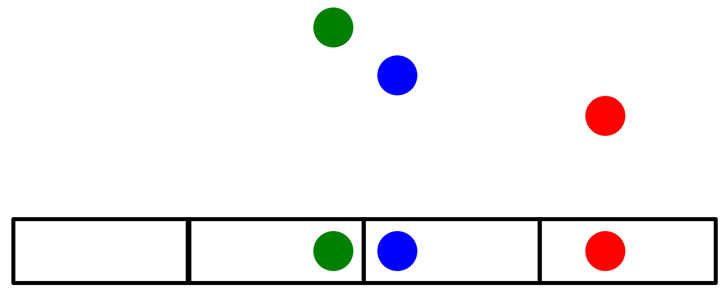
- Expensive to pick a random vector in ***M*** dimensions for large ***M***
  - Would have to generate ***M*** random numbers
- **A more efficient approach**
  - It suffices to consider only vectors ***v*** consisting of +1 and –1 components
  - Why is this more efficient?

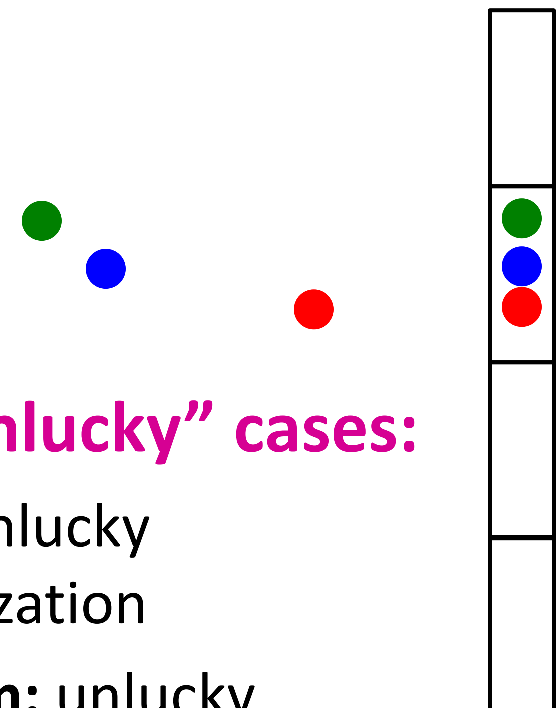
# LSH for Euclidean Distance

- **Simple idea: Hash functions correspond to lines**
- Partition the line into buckets of size ***a***
- **Hash each point to the bucket containing its projection onto the line**
- **Nearby points are always close**;
  distant points are rarely in same bucket

# Projection of Points

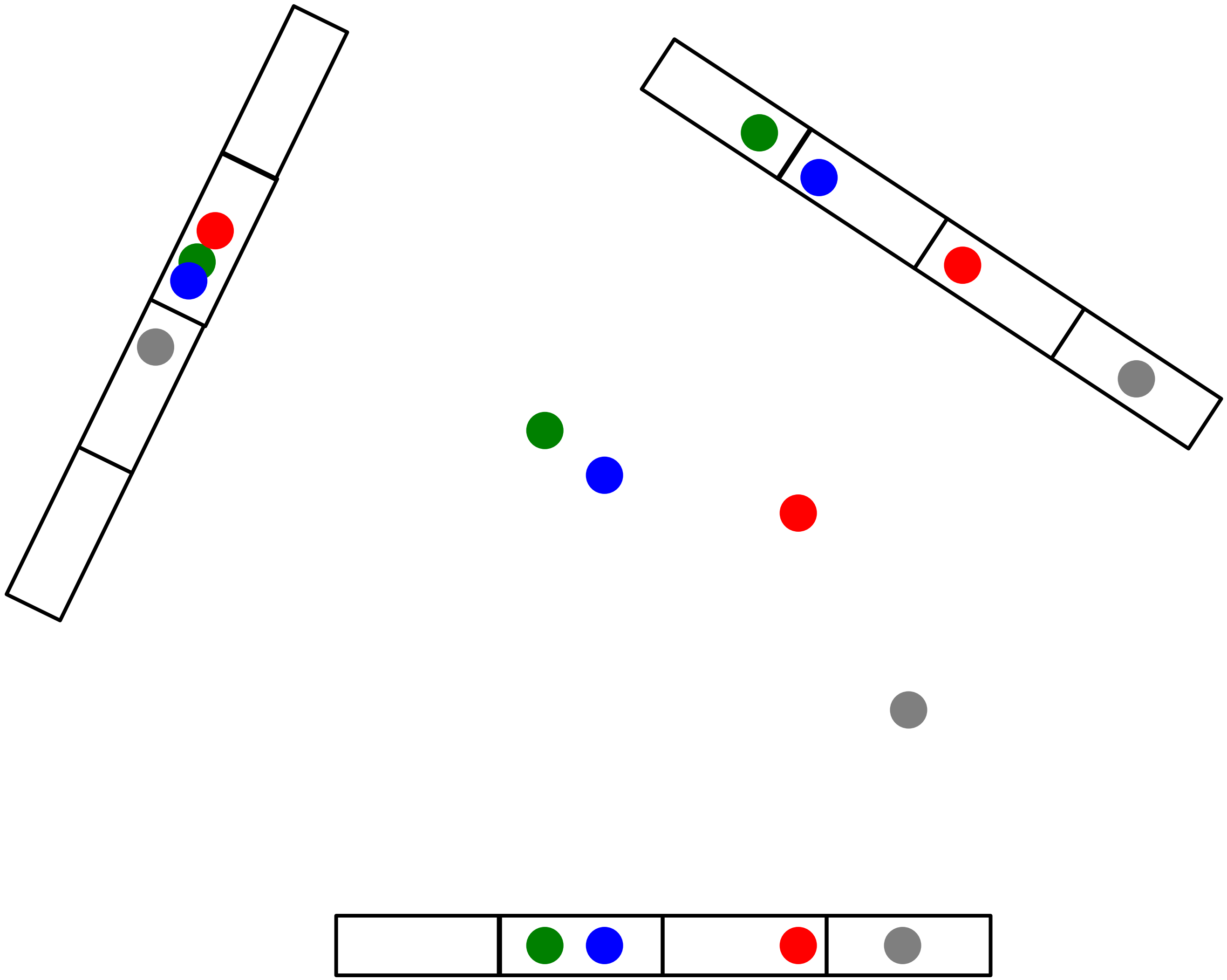




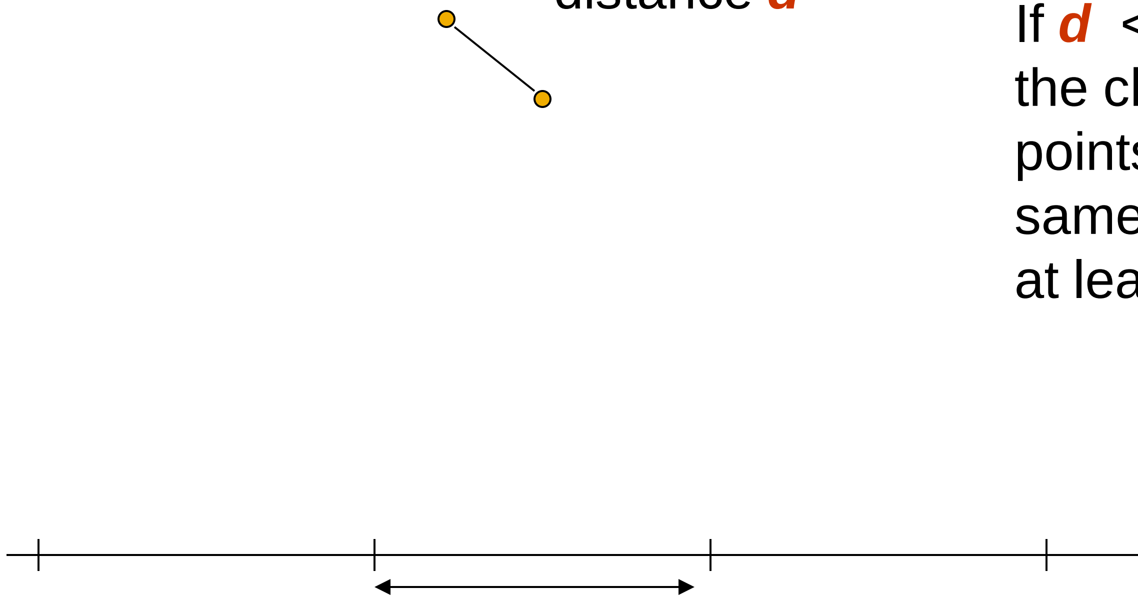
- **"Lucky" case:**
  - Points that are close hash in the same bucket
  - Distant points end up in different buckets
- **Two "unlucky" cases:**
  - **Top:** unlucky quantization
  - **Bottom:** unlucky projection

# Multiple Projections



# Projection of Points

Points at distance ***d***



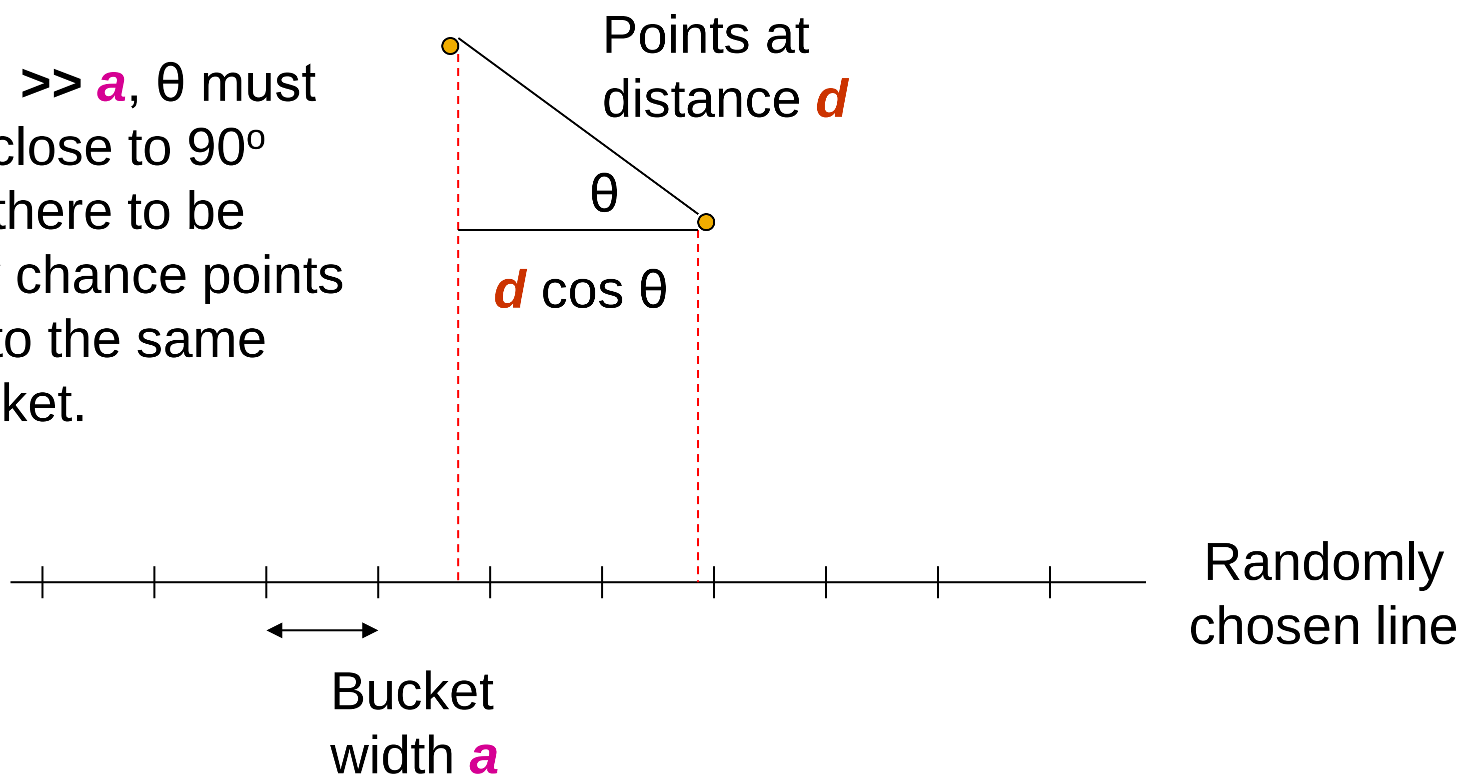
If ***d*** << ***a***, then the chance the points are in the same bucket is at least **1 –** ***d*****/*****a***.

Randomly chosen line

Bucket width ***a***

# Projection of Points

If ***d*** **>>** ***a***, θ must be close to 90$^o$ for there to be any chance points go to the same bucket.



# An LS-Family for Euclidean Distance

- If points are distance $d \leq a/2$, prob. they are in same bucket $\geq 1 - d/a = ½$
- If points are distance $d \geq 2a$ apart, then they can be in the same bucket only if $d \cos \theta \leq a$
  - $\cos \theta \leq ½$
  - $60 \leq \theta \leq 90$, i.e., at most 1/3 probability
- Yields a ***(a/2, 2a, 1/2, 1/3)**-sensitive* family of hash functions for any ***a***
- **Amplify using AND-OR cascades**

# Fixup: Euclidean Distance

- Projection method yields a ***(a/2, 2a, 1/2, 1/3)**-sensitive* family of hash functions
- For previous distance measures, we could start with an ***($d_1$, $d_2$, $p_1$, $p_2$)***-sensitive family for any $d_1 < d_2$, and drive $p_1$ and $p_2$ to 1 and 0 by **AND/OR** constructions
- Note: Here, we seem to need $d_1 \leq 4\, d_2$
  - In the calculation on the previous slide we only considered cases $d \leq a/2$ and $d \geq 2a$

# Fixup – (2)

- But as long as $d_1 < d_2$, the probability of points at distance $d_1$ falling in the same bucket is greater than the probability of points at distance $d_2$ doing so

- Thus, the hash family formed by projecting onto lines is an $(d_1, d_2, p_1, p_2)$-sensitive family for **some** $p_1 > p_2$
  - **Then, amplify by AND/OR constructions**