EE 517 Optimal Control Homework 3

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Link to code: MATLAB Code

1 Problem 1

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad R = 1$$

Recall that the HJB equation is given as

$$0 = J^*(x(t)) + \mathcal{H}(x(t), u^*(t), J_x^*) \tag{1}$$

$$\mathcal{H}(x(t), u^*(t), J_x^*) = g(x(t), u(t)) + J_x^{*T}[f(x(t), u(t))] \tag{2}$$

$$\mathcal{H}(x(t), u^*(t), J_x^*) = g(x(t), u(t)) + J_x^{*T}[f(x(t), u(t))] \tag{3}$$

$$\mathcal{H} = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2}[u][1][u] + J_x^* \left\{ \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \right\} \tag{4}$$

$$\frac{\partial \mathcal{H}}{\partial u} = u^* + J^* B = 0 \tag{5}$$

$$u^* = -B^T J_x^* \tag{6}$$

Substituting 6 into 4, we obtain:

$$-J_t^* = \frac{1}{2}x^T Q x + J_x^{*T} A x - \frac{1}{2} J_x^{*T} B R^{-1} B^T J_x^* \tag{7}$$

$$-J_t^* = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + J_x^* \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{1}{2} J_x^{*T} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} J_x^* \tag{8}$$

1.1 Scenario 1, S1

For a finite time of $t_f = 5sec$, the system above is simulated and results for P(t), K(t), x(t), and u(t) obtained as shown below in Fig.

For the same case, When $t_f \to \infty$, the values of gain (K) at steady state, input u(t) and states are obtained when $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

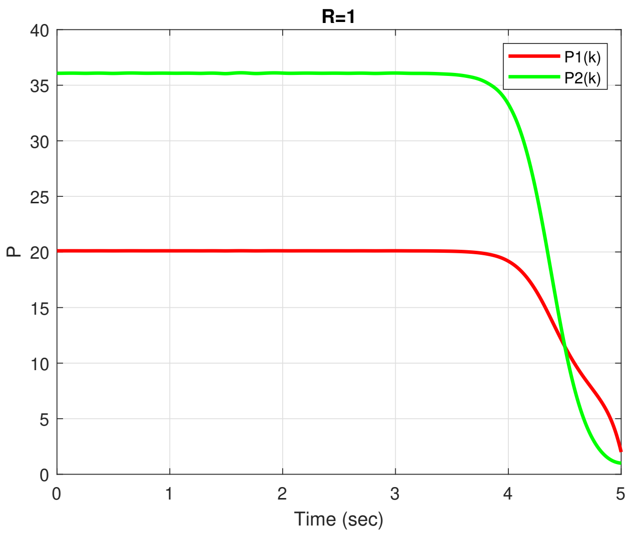

Figure 1: S1, Plot of P

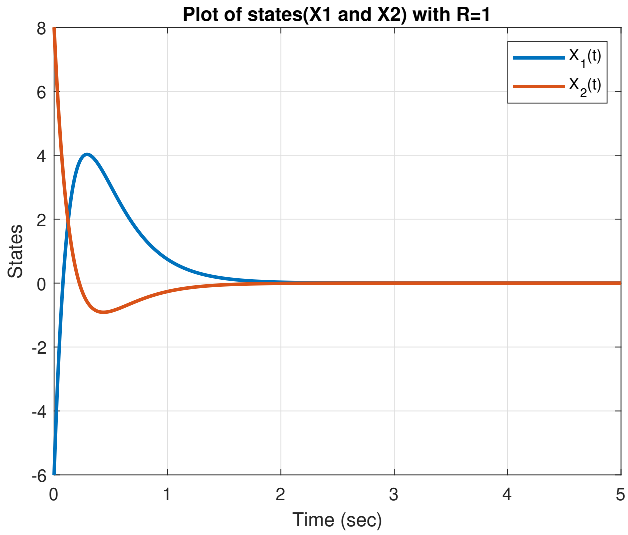

Figure 2: S1, Plot of x(t)

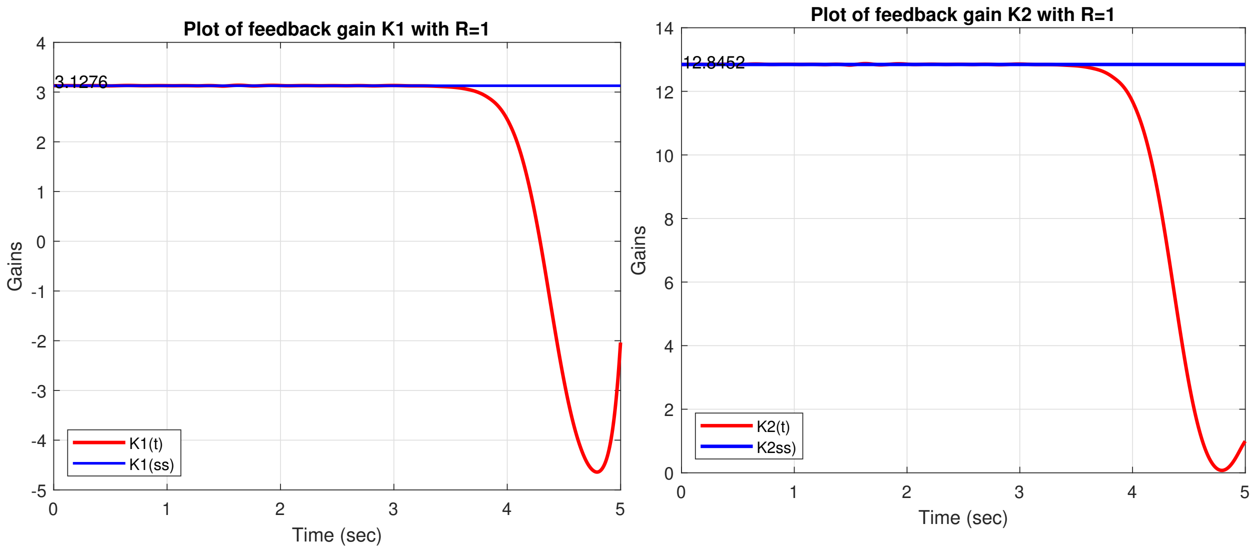

Figure 3: S1, Plot of steady state gain and time varying gain

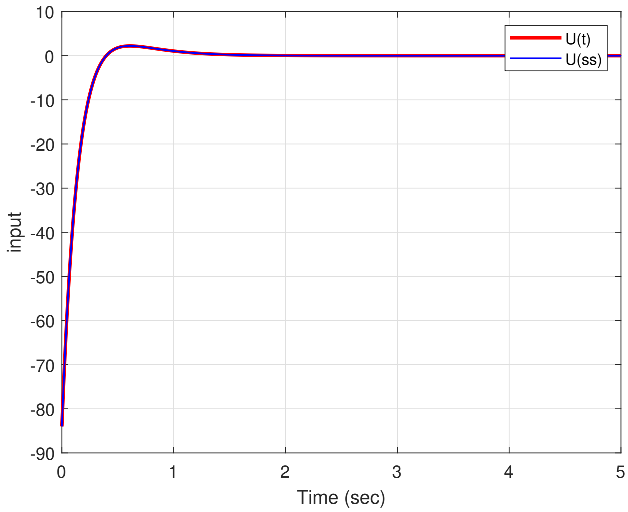

Figure 4: S1, Plot of u(t)

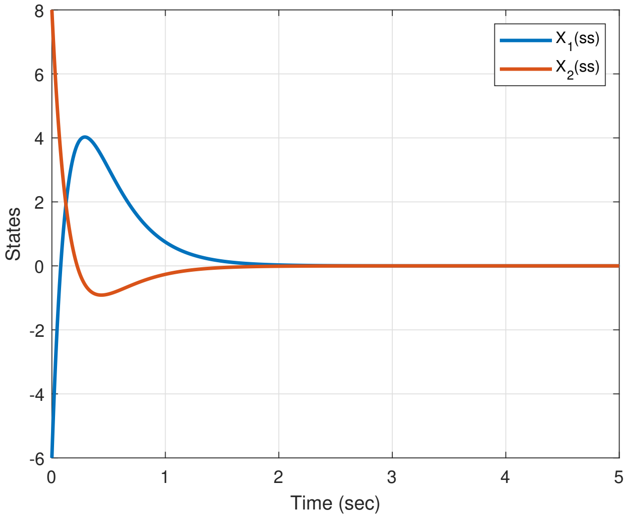

Figure 5: S1,Plot of x(t) with $Q = diag[50, 0.01]$

1.2 Scenario 2,S2

When $t_f = 5sec$ and $t_f \to \infty$, the values of time varying and steady state gain (K) is obtained and the corresponding value of input u(t) and states are obtained when $Q = \begin{bmatrix} 50 & 0 \\ 0 & 0.01 \end{bmatrix}$

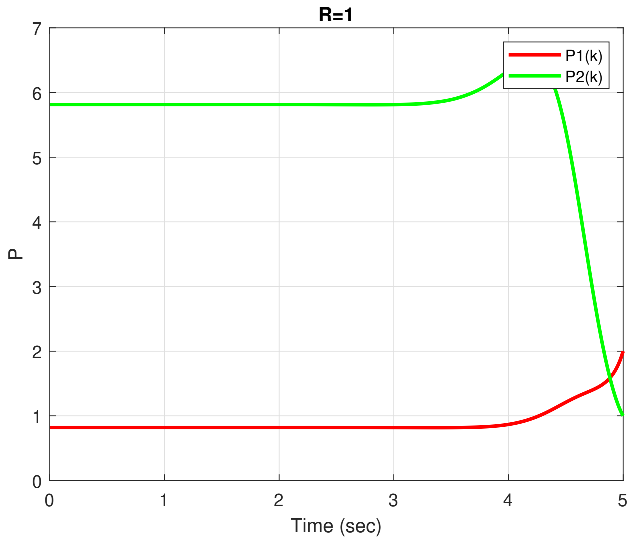

Figure 6: S2,Plot of P with $Q = diag[50, 0.01]$

1.3 Scenario 3(S3)

When $t_f = 5sec$ and $t_f \to \infty$, the values of time varying and steady state gain (K) is obtained and the corresponding value of input u(t) and states are obtained when $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and R = 25

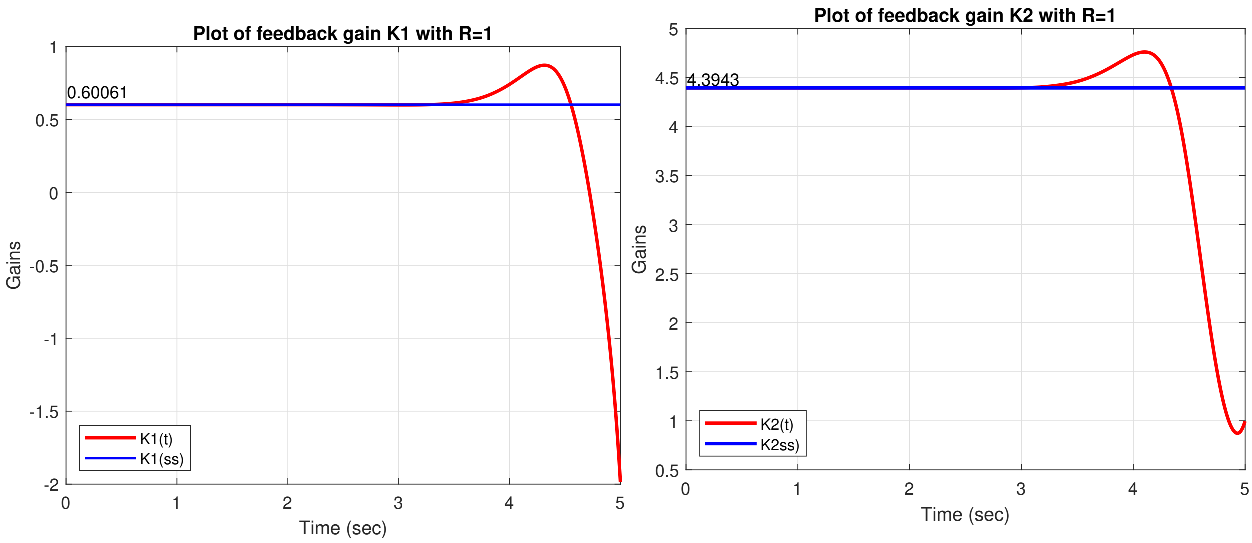

Figure 7: S2, Plot of steady state gain and time varying gain

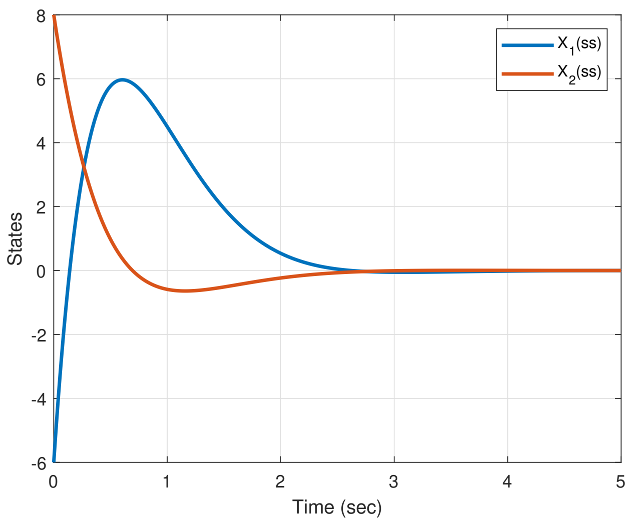

Figure 8: S2, Plot of x(t) with steady state gain

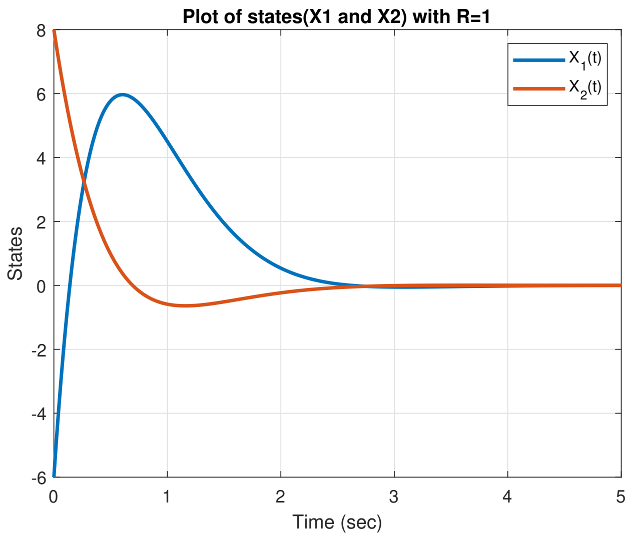

Figure 9: S2, Plot of x(t) with time varying K

1.4 Discussion

We observe that for $Q = \begin{bmatrix} 50 & 0 \\ 0 & 0.01 \end{bmatrix}$ in scenario 2 and $R = 25$, there is an overshoot in value of x_2, as against scenario when $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $R = 1$ where the overshoot is not as high as scenario 2 and 3. The overshoot is highest in scenario 3.

Furthermore, in scenario, when we compare Fig. 2 and Fig. 5, we see there is literally no difference in the response observed. Therefore, using the steady state value or the time changing value of gain produces similar transient response. In similar vain, when we compare Fig. 8 and Fig. 9 showing steady state gain response and time varying gain response, similar transient response is observed as earlier discussed.

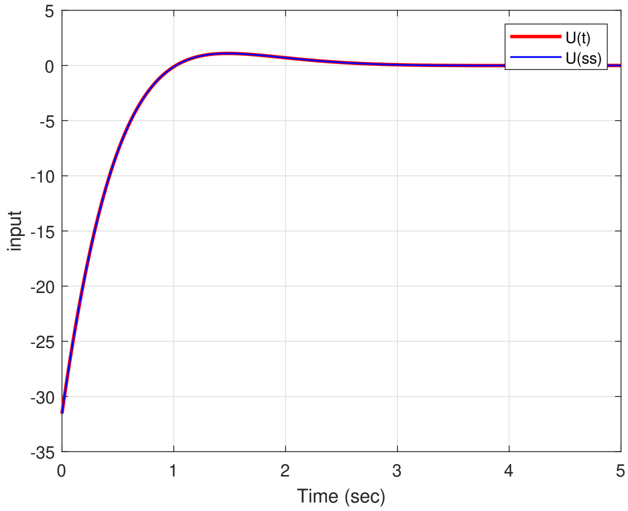

Figure 10: S2,u(t) with K(t) and Kss

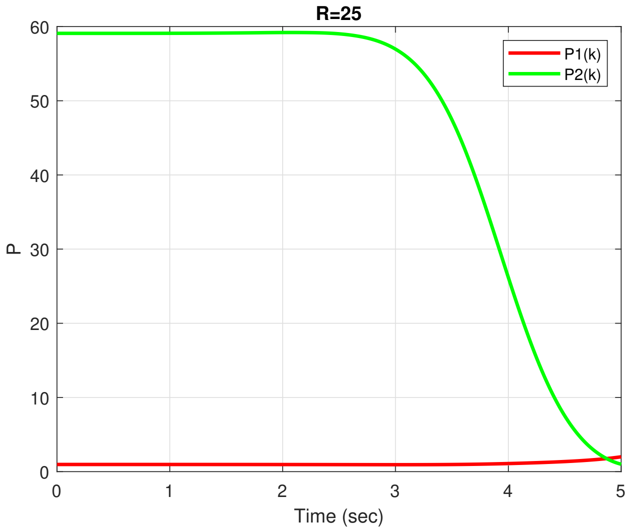

Figure 11: S3,Plot of P(t)

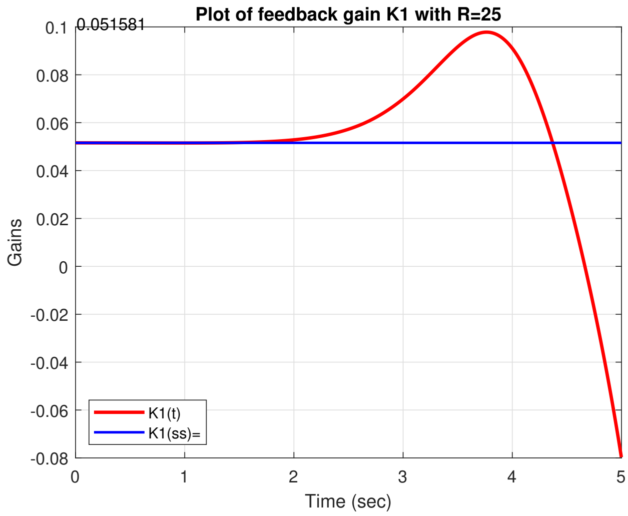

Figure 12: S3, Plot of gain, $K_1(t)$

2 Problem 2

2.1 Question

Consider the following continuous time linear quadratic regulator problem:

$$x(t) = 1.5x(t) + 0.5u(t);$$

$$x(0) = 10$$

$$J = \frac{1}{2}x^2(t_f) + \frac{1}{2}\int \{x^2(\tau) + 4u^2(\tau)\}d\tau$$

1. Write the Riccati Equation

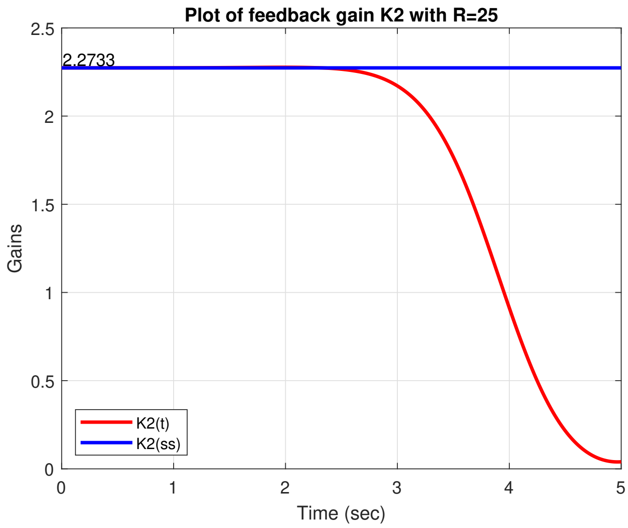

Figure 13: S3, Plot of gain, $K_2(t)$

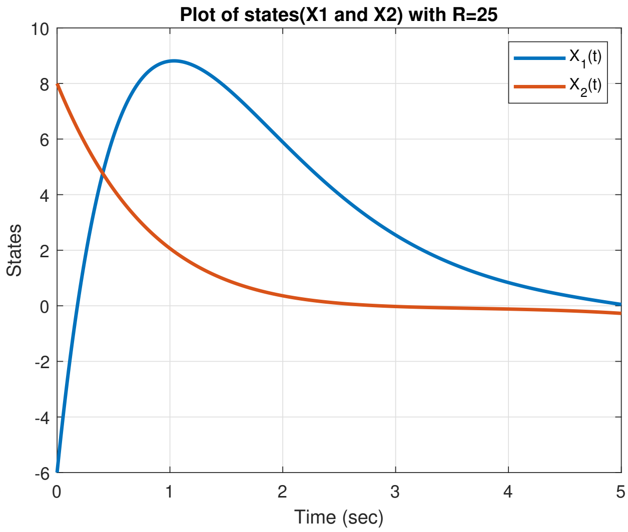

Figure 14: S3,Plot of x(t)

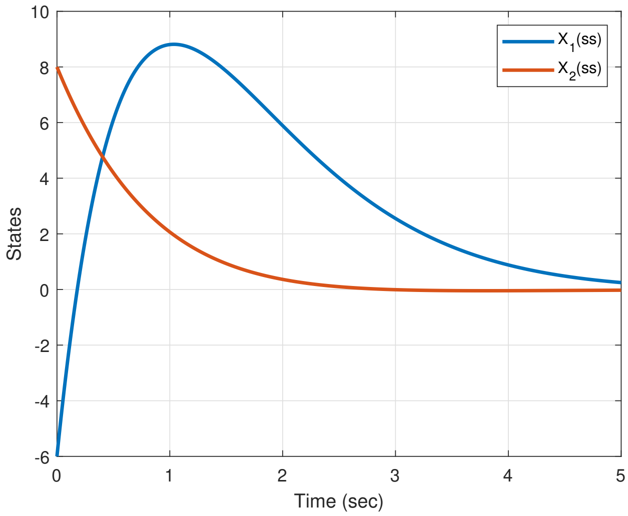

Figure 15: S3, Plot of states: x(t)

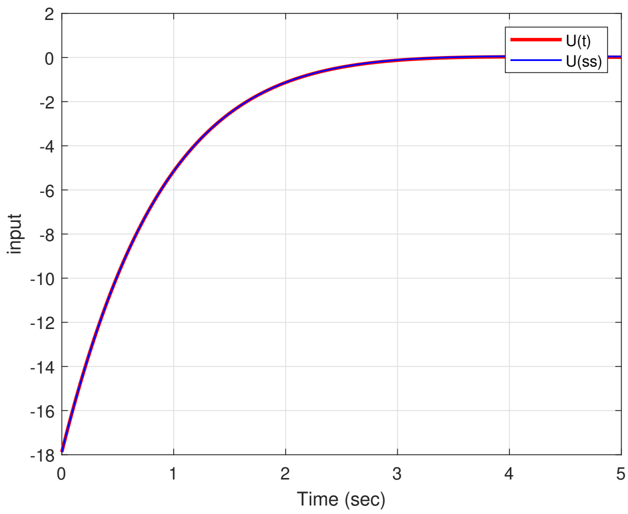

Figure 16: S3, Plot of input, u(t)

2. Solve it using the 2 different approaches discussed in class (numerical solution of the differential equation, and $P = EF^{-1}$)

3. Find and plot K(t)

4. Plot x(t) (closed loop

2.2 Solution 2.1

$$A = 1.5, B = 0.5, H = 0.5, Q = 1, R = 4 \qquad (**)$$

We first derive the Hamiltonian of the system: This is given as:

$$0 = J^*(x(t)) + \mathcal{H}(x(t), u^*(t), J_x^*) \tag{9}$$

$$\mathcal{H}(x(t), u^*(t), J_x^*) = g(x(t), u(t)) + J_x^{*T}[f(x(t), u(t))] \tag{10}$$

$$\mathcal{H}(x(t), u^*(t), J_x^*) = \frac{1}{2}x^2 + 2u^2 + J_x^*(1.5x + 0.5u) \tag{11}$$

$$\frac{\partial H}{\partial u} = 4u + J_x^*(0.5) = 0 \tag{12}$$

$$u^* = -\frac{1}{8}J_x^*$$

$$\mathcal{H} = \frac{1}{2}x^2 + 2.(\frac{-1}{8}J_x^*)^2 + J_x^*\{1.5x + 0.5. - \frac{1}{8}J_x^*\} \tag{13}$$

$$\mathcal{H} = -\frac{1}{16}J_x^* + \frac{1}{2}x^2 + \frac{3}{2}J_x^* \tag{14}$$

The Hamiltonian is thus defined as:

$$0 = J_t^* - \frac{1}{16}J_x^{*2} + \frac{1}{2}x^2 + \frac{3}{2}J_x^* \tag{15}$$

where:

$$J^*(t_f) = \frac{1}{4}x(10)^2 \tag{16}$$

We guess a form of the solution of Equation 15

$$J^*(x(t)) = \frac{1}{2}P(t)x(t)^2 \tag{17}$$

We have:

$$J_x^* = Px(t), \;\; J_t^* = \frac{1}{2}\dot{P}x^2$$

Substituting this into equation 15, we have:

$$0 = -\frac{1}{16}P^2x(t))^2 + \frac{1}{2}\dot{P}x^2 + \frac{1}{2}x^2 + Px(\frac{3}{2}x) \tag{18}$$

We then have:

$$\dot{P} = -1 - 3P + \frac{1}{16}P^2 \tag{19}$$

Eqn. 19 is solved numerically using MATLAB solved backward in time. The result obtained is shown in Fig 17.

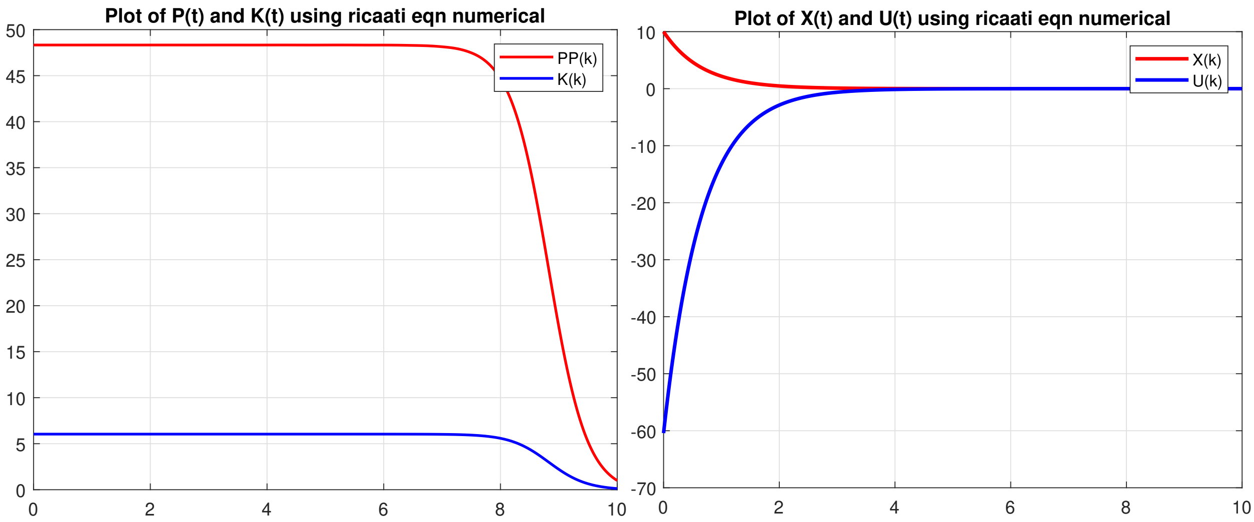

Figure 17: Plot of gain, input(x(t)) and u(t)

2.3 Approach 2 of solving using Hamiltonian Matrix, $P = EF^{-1}$

$$H_M = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{20}$$

Substituting terms as we have in Eqn. **, we have

$$Hm = \begin{bmatrix} 1.5000 & -0.0625 \\ -1.0000 & -1.5000 \end{bmatrix}$$

$$\begin{bmatrix} \dot{F} \\ \dot{E} \end{bmatrix} = \begin{bmatrix} 1.5000 & -0.0625 \\ -1.0000 & -1.5000 \end{bmatrix} \begin{bmatrix} F = 1 \\ E = H \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} \dot{F} \\ \dot{E} \end{bmatrix} = \begin{bmatrix} 1.5000 & -0.0625 \\ -1.0000 & -1.5000 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \tag{22}$$

We know that:

$$e^{Hm} = \mathcal{L}^{-1}[(s\mathcal{I} - Hm)] = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \tag{23}$$

Eqn 23 is computed with MATLAB and the results shown below are obtained for P, K, u and x(t)

$$\begin{bmatrix} F(t_2) \\ E(t_2) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \tag{24}$$

P is obtained by finding $P = EF^{-1}$ on matlab and computing the result. The resulting plots obtained is shown below in Fig. 18

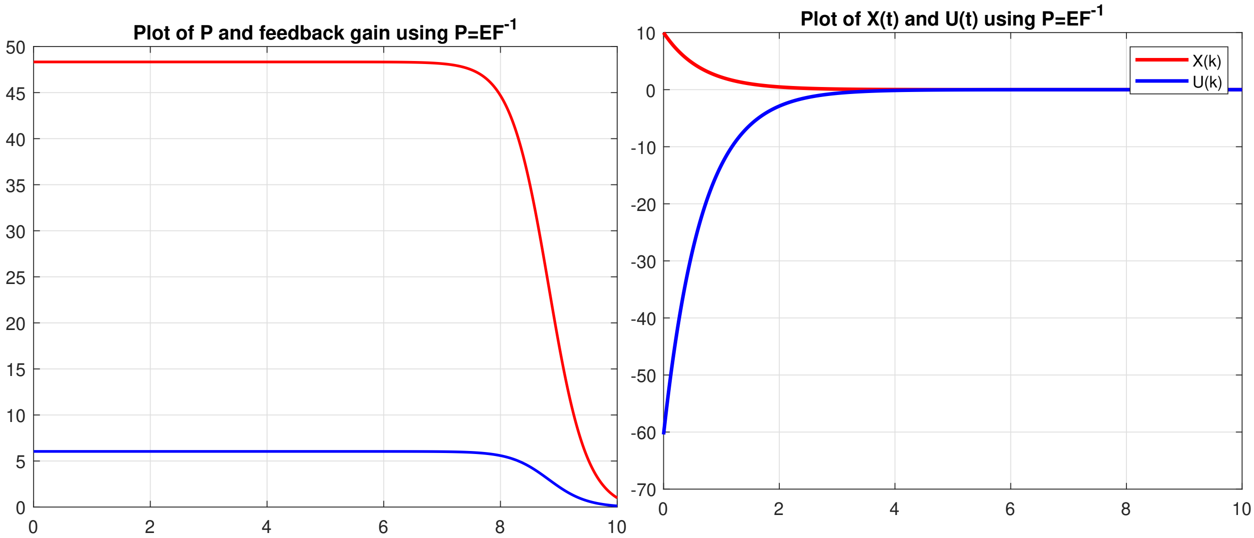

Figure 18: time varying P and K(t) , closed loop x(t) and u(t) result

2.4 Another Faster Approach for Ricatti Equation:

The ricatti equation derived from 23 and 19 could have also been derived directly using:

$$-\dot{P} = Q + PA + A^T P - PBR^{-1}B^T P, \;\; P(t_f) = H \tag{25}$$

$$-\dot{P} = 1 + P.1.5 + 1.5.P - P.0.5.(\frac{1}{4}) \times 0.5 \times P \tag{26}$$

$$-\dot{P} = 1 + 3P + \frac{1}{16}P^2 \tag{27}$$

$$\dot{P} = -1 - 3P + \frac{1}{16}P^2 \tag{28}$$

2.5 Discussion

The problem presented solved with the numerical and analytical approach result is shown in 17 for the numerical solution and Fig. 18 for the analytical solution. We observe there is no difference in the response obtained for the analytical solution or the analytical solution solving backward.