NTNU | Norwegian University of Science and Technology

TMA4300 - COMPUTER INTENSIVE STATISTICAL METHODS

Exercise 1: Problem C

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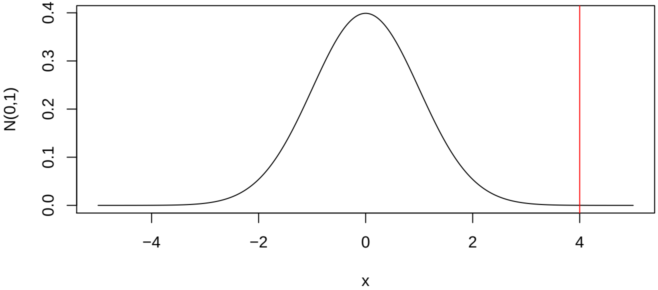
Problem C: Monte Carlo integration and variance reduction

Aim: Learn about variance reduction techniques in Monte Carlo integration.

Problem: Find $\theta = \text{Prob}(X > 4)$ when $X \sim N(0,1)$.



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Use sample mean as estimator:

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^n \underbrace{\mathbb{I}(x_i > 4)}_{h(x_i)}$$

C.1: Confidence Intervals

Estimate variance:

$$\mathrm{Var}[\hat{\theta}] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}_f[h(X_i)] = \frac{1}{n}\mathrm{Var}_f[h(X)] \approx \frac{1}{n}\left(\frac{1}{n-1}\sum_{i=1}^{n}(h(x_i)-\hat{\theta})^2\right) =: \hat{\sigma}^2$$

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Using the central limit theorem: $\hat{\theta} \sim \mathcal{N}(\theta, \mathrm{Var}[\hat{\theta}]) \implies \frac{\hat{\theta}-\theta}{\hat{\sigma}} \sim t_{n-1}$.

A 95% confidence interval is therefore

$$[\hat{\theta} - t_{n-1,\,0.025}\cdot\hat{\sigma},\ \hat{\theta} + t_{n-1,\,0.025}\cdot\hat{\sigma}].$$

C.2: Importance sampling

$$\hat{\theta}_{IS} = \frac{1}{n}\sum_{i=1}^{n}\frac{h(X_i)f(X_i)}{g(X_i)} = \frac{1}{n}\sum_{i=1}^{n} h(X_i)w(X_i).$$

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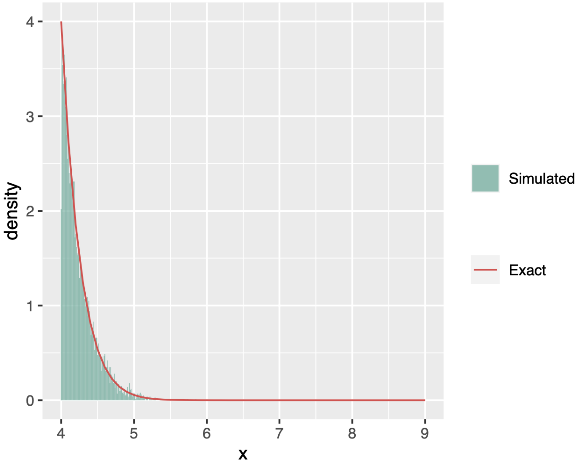
Need to sample from proposal distribution $g(x)$.

C.2: Importance sampling

$$g(x) = \begin{cases} cx \exp -\frac{1}{2}x^2 & , x > 4 \\ 0 & , \text{else} \end{cases}$$



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$$\begin{aligned}\mathrm{Var}[\hat{\theta}_{IS}] &= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}_g[h(X_i)w(X_i)] = \frac{1}{n}\mathrm{Var}_g[h(X)w(X)] \\ &\approx \frac{1}{n}\left(\frac{1}{n-1}\sum_{i=1}^{n}(h(x_i)w(x_i) - \hat{\theta}_{IS})^2\right) =: \hat{\sigma}^2_{IS}.\end{aligned}$$

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95% Confidence interval: $\left[\hat{\theta}_{IS} - t_{n-1,0.025}\cdot\hat{\sigma}_{IS},\ \hat{\theta}_{IS} + t_{n-1,0.025}\cdot\hat{\sigma}_{IS}\right]$

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The $X_j^{(1)}$'s are based on u while the $X_j^{(2)}$'s are based on $u-1$ in the inversion sampling of $g(x)$.

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$\implies$ Only need $n = 50000$ to make $2n$ (antithetic) samples from $g(x)$.

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Y_i's independent $\implies$ can use t-distribution to create C.I.

C.3: Confidence intervals part 2

Use sample variance:

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From which we construct a confidence interval.