Project 1, TMA4215 - Numerical Mathematics

August 2020

Problem 1

Show that for the complex square matrix A, the following equality holds:

$$||A||_2 = \rho(A). \tag{1}$$

Proof:

$$\begin{aligned} ||A||_2 &= \sqrt{\rho(A^H A)} = \sqrt{\rho((U\Lambda U^H)^H U\Lambda U^H)} \\ &= \sqrt{\rho(U\Lambda U^H U\Lambda U^H)} = \sqrt{\rho((U\Lambda^2 U^H)} \\ &= \sqrt{\rho(A^2)} = \rho(A). \end{aligned} \tag{2}$$

The last equality follows from the fact that the eigenvalues of A^2 are the eigenvalues of A squared:

$$Ax = \lambda x \iff A^2 x = \lambda A x \iff A^2 x = \lambda^2 x \tag{3}$$

Problem 2

(a)

We find the eigenvalues from the equation $\det(\lambda I - A) = 0$, where the right hand side is a lower triangular matrix with $(-\lambda)$ along the diagonal. Then the determinant becomes

$$\det(\lambda I - A) = (-\lambda)^n = 0. \implies \lambda = 0. \tag{4}$$

Gershgorin's circle theorem tells us that every eigenvalue λ of A must lie inside the unit circle in the complex plane:

$$\lambda \in \{z \in \mathbb{C} : |z| \leq 1\}. \tag{5}$$

(b)

Since we are asked to show that $\rho(\hat{A}) = \varepsilon^{1/n}$, it is hence assumed that ε is positive and real. Now that $\hat{A} = A + \varepsilon e_1 e_n^T$ we get the following equation:

$$\det(\hat{A} - \lambda I) = \begin{vmatrix} -\lambda & \cdots & 0 & \varepsilon \\ 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -\lambda \end{vmatrix} = (-1)^{n+1}\varepsilon + (-\lambda)^n = 0. \tag{6}$$

This is equivalent to

$$\lambda = \begin{cases} \pm\varepsilon^{1/n} & \text{when } n \text{ is even} \\ \varepsilon^{1/n} & \text{when } n \text{ is odd} \end{cases} \tag{7}$$

We clearly see that $\rho(\hat{A}) = \varepsilon^{1/n}$. To find an expression for the eigenvectors, we again turn to the eigenequation

$$(\hat{A} - I\lambda)v = 0, \tag{8}$$

where $v = (v_1, \dots, v_n)^T \neq 0$ is an eigenvector. This results in the set of equations:

$$\begin{aligned} v_1 &= \varepsilon\lambda^{-1}v_n \\ v_2 &= \lambda^{-1}v_1 \\ &\vdots \\ v_n &= \lambda^{-1}v_{n-1}. \end{aligned} \tag{9}$$

We recognize the recursive structure, and write

$$v_i = \varepsilon\lambda^{-i}v_n, \qquad \forall i \in \{1, 2, \dots, n\}. \tag{10}$$

Inserting the eigenvalue found above (regardless of whether n is even or odd), we obtain for $i = n$ that

$$v_n = \varepsilon(\varepsilon^{(1/n)})^{-n}v_n = v_n. \tag{11}$$

In other words, we are free to choose the value of the last element in the eigenvector. Let e.g. $v_n = \nu \in \mathbb{C}$. Then the eigenvectors of $\hat{A}$ are given by

$$v = \varepsilon\nu \begin{pmatrix} \lambda^{-1} \\ \vdots \\ \lambda^{-i} \\ \vdots \\ \lambda^{-(n-1)} \\ 1/\varepsilon \end{pmatrix} \tag{12}$$

(c)

In this task, ε is also assumed to be a positive, real number. To find $K_2(\hat{A})$, we first look at $||\hat{A}||_2 = \sqrt{\rho(\hat{A}^T\hat{A})}$. We notice that

$$A^TA = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon^2 \end{pmatrix}, \tag{13}$$

implying that $\sigma(A^TA) = \{1, \varepsilon^2\}$, where $\sigma(\cdot)$ denotes the set of eigenvalues to the matrix. This means that

$$\rho(\hat{A}^T\hat{A}) = \begin{cases} \varepsilon^2, & \varepsilon > 1 \\ 1, & \varepsilon \leq 1 \end{cases}. \tag{14}$$

Now we turn to $||\hat{A}^{-1}||$. From (13) we readily observe that

$$\hat{A}^{-1} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 1 \\ 1/\varepsilon & 0 & \cdots & 0 \end{pmatrix} := B. \tag{15}$$

This results in

$$B^T B = \begin{pmatrix} 1/\varepsilon^2 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{16}$$

and since this matrix also only has entries on the diagonal, we see that $\sigma(B^TB) = \{1, 1/\varepsilon^2\}$. This means that

$$\rho(B^TB) = \begin{cases} 1, & \varepsilon \geq 1 \\ 1/\varepsilon^2, & \varepsilon < 1 \end{cases}. \tag{17}$$

From the results above, we conclude that

$$K_2(\hat{A}) = \begin{cases} \varepsilon, & \varepsilon > 1 \\ 1/\varepsilon, & \varepsilon < 1 \\ 1, & \varepsilon = 1 \end{cases}. \tag{18}$$

Problem 3

(a)

Below is the semi-log plot of $K(H_n)$ against n. From figure 1 we can clearly see that there is an exponential relationship between $K(H_n)$ and n.

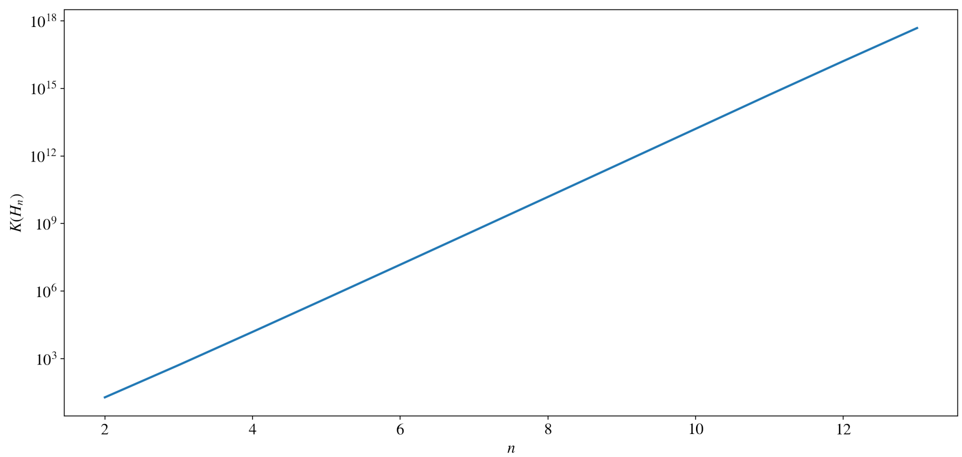

Figure 1: $K(H_n)$ plotted with a logarithmic scale against n.

(b)

In the lectures we have seen that by considering $Ax = b$ with smalle perturbations δA, δb and thus δx, such that

$$(A + \delta A)(x + \delta x) = (b + \delta b), \tag{19}$$

we obtain the following relationship:

$$K(A) = ||A|| \cdot ||A^{-1}|| \geq \frac{||\delta x||/||x||}{||\delta b||/||b|| + ||\delta A||/||A||} := Q. \tag{20}$$

We exemplify this inequality by solving

$$(H_n + \delta H)(x + \delta x) = b + \delta b, \tag{21}$$

where H_n is the Hilbert matrix with dimension n and $b = (1, \ldots, 1)^T$. The perturbations are of order 10^{-16}. In figure 2 you can see an average of the ratio Q plotted together with the condition number. We see that the condition number constitutes an upper bound as we would expect. In figure 3 the norm of the perturbation in the solution is plotted.

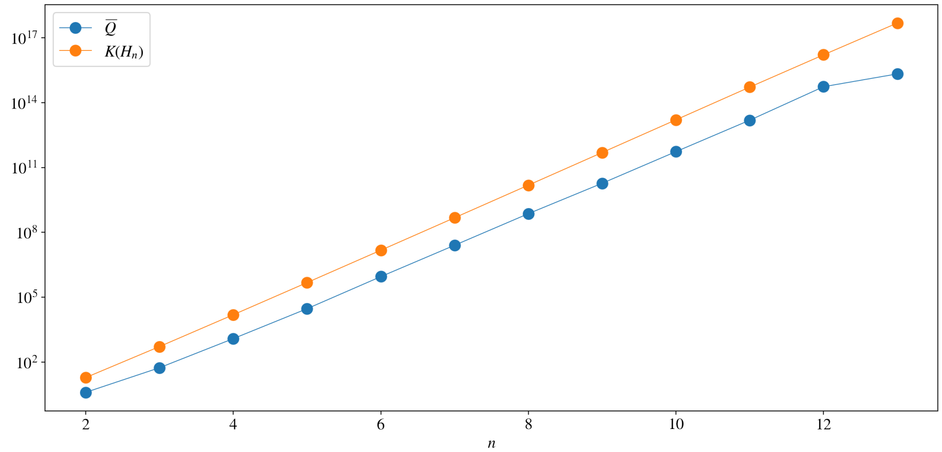

Figure 2: The average of ratio Q from (20) with randomized perturbations and the condition number, both using the Hilbert matrix, plotted against the dimension of the Hilbert matrix: $n. = 1, \dots, 13$.

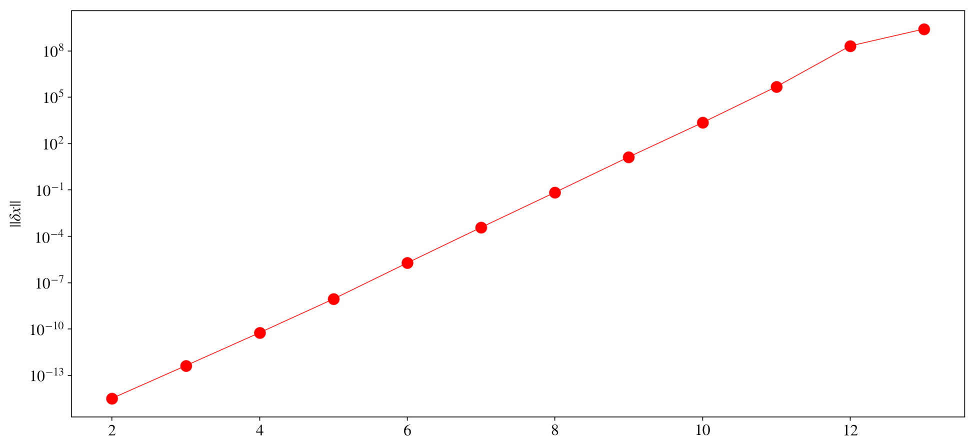

Figure 3: $||\delta x||$ plotted against the dimension n in a semi-log plot.

Problem X

Another way to phrase the problem at hand is that we want to find

$$\min\{||\delta A|| : A + \delta A \quad \text{is singular}\}. \tag{22}$$

Starting with the fact that $A + \delta A$ must be singular, it is apparant that is has an eigenvalue $\lambda = 0$. This gives the equation

$$(A + \delta A)x = \lambda x = 0, \tag{23}$$

where we choose the eigenvector x such that $||x|| = 1$. Multiplying (23) by A^{-1} yields

$$A^{-1}\delta A x = -x. \tag{24}$$

Applying the norm on both sides and using the properties of the induced matrix norm gives

$$||A^{-1}|| \cdot ||\delta A|| \geq 1 \iff ||\delta A|| \geq ||A^{-1}||^{-1}. \tag{25}$$

Now we have to show that there exists such a minimal δA. In order to do this, we let $\delta A = -||A^{-1}||^{-1} x y^T$, where

$$||x|| = 1, \qquad ||y||_* := \max_{z \neq 0} \frac{|y^T z|}{||z||} = 1, \qquad ||A^{-1}|| = y^T A^{-1} x. \tag{26}$$

To show that such an x and y exist, we use the definition of the induced matrix norm on A^{-1}:

$$\begin{aligned} ||A^{-1}|| &= \max_{x \neq 0} \frac{||A^{-1}||}{||x||} = \max_{||x||=1} ||A^{-1}x|| \\ \Longrightarrow \exists x : ||x|| = 1 \quad &\text{and} \quad ||A^{-1}|| = ||A^{-1}x|| > 0. \end{aligned} \tag{27}$$

We simply construct y from x:

$$y = \frac{A^{-1}x}{||A^{-1}x||}. \tag{28}$$

Then, $||y|| = ||A^{-1}x||/||A^{-1}x|| = 1$. The $|| \cdot ||_*$ norm from (26) induces a $|| \cdot ||_{**}$ norm and it can be shown that $||y||_{**} = ||y||$. This guarantees the existence of $||y||_* = 1$ from (26). Now we show that δA has the desired properties. First we compute the norm:

$$||\delta A|| = \max_{z \neq 0} \frac{||x y^T z||}{||A^{-1}|| \cdot ||z||} = \max_{z \neq 0} \frac{|y^T z|}{||z||} \cdot \frac{||x||}{||A^{-1}||} = ||A^{-1}||^{-1}. \tag{29}$$

Next step is to show that $A + \delta A$ is singular. We consider

(A + $\delta A)A^{-1}x = (A - ||A^{-1}||^{-1}xy^T)A^{-1}x$

= x - x $\cdot \frac{y^T A^{-1} x}{||A^{-1}||} = x - x = 0$ (30)

In other words, $A^{-1}x$ is an eigenvector to $A + \delta A$ with eigenvalue equal to zero. Thus, $A + \delta A$ is singular.