# Assignment 6

a) $\langle f, g \rangle_w = \int_0^1 \frac{1}{\sqrt{x}} f(x) g(x) \, dx$

Find first 3 orthogonal polynomials wrt. $\langle \cdot , \cdot \rangle_w$ :

- $\underline{q_0 = 1}$, $\quad q_1 = x + b_1$

$\langle q_0, q_1 \rangle_w = \int_0^1 \sqrt{x_1} + b_1 x^{-1/2} dx = \Big|_0^1 \frac{2}{3} x^{\frac{3}{2}} + 2b_1 x^{1/2}$

$= \frac{2}{3} + 2b_1 = 0$

$\Rightarrow b_1 = -\frac{1}{3} \qquad \Rightarrow \underline{q_1 = x - \frac{1}{3}}$

- $q_2 = x^2 + b_2 x + c_2$

$\Rightarrow \langle q_0, q_2 \rangle = \int_0^1 x^{3/2} + b_2 x^{1/2} + c_2 x^{-1/2} dx = \Big|_0^1 \frac{2}{5} x^{\frac{5}{2}} + \frac{2}{3} x^{\frac{3}{2}} + 2c_2 x^{1/2}$

$= \frac{2}{5} + \frac{2}{3} b_2 + 2c_2 = 0 \qquad (\mathrm{I})$

$\langle q_1, q_2 \rangle_w = \int_0^1 x^{-\frac{1}{2}} (x - \frac{1}{3})(x^2 + b_2 x + c_2) \, dx$

$= \int_0^1 x^{\frac{5}{2}} + (b_2 - \frac{1}{3}) x^{\frac{3}{2}} + (c_2 - \frac{1}{3} b_2) x^{1/2} - \frac{c_2}{3} x^{-\frac{1}{2}} dx$

$$= \Big|_0^1 \frac{2}{7}x^{\frac{7}{2}} + \frac{2}{5}\left(b_2 - \frac{1}{3}\right) + \frac{2(c_2 - \frac{1}{3}b_2)}{3} x^{\frac{3}{2}} - \frac{2}{3}c_2 x^{\frac{1}{2}}$$

$$= \frac{2}{7} + \frac{2}{5}\left(b_2 - \frac{1}{3}\right) + \frac{2}{3}\left(c_2 - \frac{1}{3}b_2\right) - \frac{2}{3}c_2 = 0$$

$$\Rightarrow \frac{8}{45}b_2 + \frac{4}{3}c_2 + \frac{16}{105} = 0$$

Fra (I): $b_2 = \frac{45}{8} \cdot \left(-\frac{16}{105}\right) = -\frac{6}{7}$

$$\Rightarrow \quad c_2 = \frac{3}{35} \quad \Rightarrow \quad q_2 = x^2 - \frac{6}{7}x + \frac{3}{35}$$

b)

The roots of $q_2$ are $x_1 = \frac{3}{7} - \frac{2}{35}\sqrt{30}$, $x_2 = \frac{3}{7} + \frac{2}{35}\sqrt{30}$

Gaussian quadrature gives:

$$A_1 = \int_0^1 \frac{1}{\sqrt{x}} l_1(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}} \frac{x - x_2}{x_1 - x_2}\,dx = 1 + \frac{\sqrt{30}}{18}$$

$$A_2 = \int_0^1 \frac{1}{\sqrt{x}} \frac{x - x_1}{x_2 - x_1}\,dx = 1 - \frac{\sqrt{30}}{18}$$

$f(x) = x \sin x$

$$\Rightarrow I = \int_0^1 \sqrt{x} \sin x\,dx \approx Q_w[f] = A_1 f(x_1) + A_2 f(x_2)$$

$$= 0.36584953$$

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a)

We know: $T_{2^k} = \frac{1}{2}\left(T_{2^{k-1}} + U_{2^{k-1}}\right)$

$$T_1 = \frac{h_1}{2} \cdot f(0) + \frac{h_1}{2} f\left(\frac{\pi}{2}\right) = \frac{\pi}{4} \cdot (0+0) = 0$$

$$T_2 = h_2 \cdot f(h_2) = 0.4921753$$

$$T_4 = \frac{1}{2}\left(T_2 + U_2\right) = 0.6365562$$

$$T_8 = \frac{1}{2}\left(T_4 + U_4\right) = 0.6819061$$

$$T_{16} = \frac{1}{2}\left(T_8 + U_8\right) = 0.6966345$$

b)

$$T_n = I + C_{\frac{3}{2}} n^{-\frac{3}{2}} + C_2 n^{-2} + \dots$$

$$= I + \sum_{i=1}^{\infty} C_{1+\frac{i}{2}} \cdot n^{1+\frac{i}{2}}$$

Let $H_8 = \dfrac{T_8 - \frac{1}{2\sqrt{2}} T_4}{\frac{4-\sqrt{2}}{4}} = I + \dfrac{4(1-\sqrt{2})}{64(4-\sqrt{2})} \cdot C_2 + \dots$

$$H_{16} = \frac{T_{16} - \frac{1}{2\sqrt{2}} T_8}{\frac{4-\sqrt{2}}{4}} = I + \frac{1-\sqrt{2}}{64(4-\sqrt{2})} \cdot C_2 + \dots$$

Then we have:

$$G_{16} = \frac{H_{16} - \frac{1}{4} H_8}{3/4} = I + \tilde{C}_{\frac{5}{2}} n^{-\frac{5}{2}} + \dots$$

$n = 16$ here.

$H_8 = 0.7067088$

$H_{16} = 0.7046897$

$\Rightarrow G_{16} = \underline{\underline{0.7040167}}$