TMA4315: Project 1

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Litt usikker på hva slags notasjon vi skal bruke, f. eks. boldface for vektorer eller ikke? Bare si ifra hvis du vil ha noe spesifikit:)

Problem 1

a)

Since the response variables $y_i \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$, the conditional mean is given by $\text{E}y_i = p$, which is connected to the covariates via the following relationship:

$$x_i^T\beta =: \eta_i = \Phi^{-1}(p),$$

which implies that $p = \Phi(\eta_i)$.This results in the likelihood function

$$\begin{aligned} L(\beta) &= \Pi_{i=1}^n p^{y_i}(1-p)^{1-y_i} \\ &= \Pi_{i=1}^n \Phi(\eta_i)^{y_i}(1-\Phi(\eta_i))^{1-y_i}. \end{aligned}$$

Thus, the log-likelyhood becomes

$$l(\beta) := \ln(L(\beta)) = \sum_{i=1}^n y_i \ln(\Phi(\eta_i)) + (1-y_i)\ln(1-\Phi(\eta_i)) = \sum_{i=1}^n l_i(\beta).$$

To find the score function, we calculate

$$\begin{aligned} \frac{\partial l_i(\beta)}{\partial \beta} &= \frac{y_i}{\Phi(\eta_i)}\frac{\partial \Phi(\eta_i)}{\partial \beta} - \frac{1-y_i}{1-\Phi(\eta_i)}\frac{\partial \Phi(\eta_i)}{\partial \beta} \\ &= \frac{y_i}{\Phi(\eta_i)}\phi(\eta_i)x_i - \frac{1-y_i}{1-\Phi(\eta_i)}\phi(\eta_i)x_i \\ &= \frac{y_i(1-\Phi(\eta_i)) - (1-y_i)\Phi(\eta_i)}{\Phi(\eta_i)(1-\Phi(\eta_i))}\phi(\eta_i)x_i \\ &= \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1-\Phi(\eta_i))}\phi(\eta_i)x_i \end{aligned}$$

Consequently, the score function is given by

$$s(\beta) = \sum_{i=1}^n \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1-\Phi(\eta_i))}\phi(\eta_i)x_i.$$

Next, we find the expected Fisher information, $F(\beta)$. We find it by using the result

$$
\begin{aligned}
F(\beta) &= \mathrm{Var}(s(\beta)) = \mathrm{Var}\left(\sum_{i=1}^{n} \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1-\Phi(\eta_i))}\phi(\eta_i)x_i\right) \\
&= \sum_{i=1}^{n} \underbrace{\left[\frac{\phi(\eta_i)}{\Phi(\eta_i(1-\Phi(\eta_i)))}\right]^2}_{=:\xi_i} \mathrm{Var}(y_i x_i) = \sum_{i=1}^{n} \xi_i x_i^T \mathrm{Var}(y_i) x_i \\
&= \sum_{i=1}^{n} \xi_i p(1-p) x_i^T x_i.
\end{aligned}
$$

Where in the third equality we have used that the y_i's are independent.