TMA4315: Project 3

jototlan@stud.ntnu.no (10018), martigtu@stud.ntnu.no (10037)

```
long <- read.csv("https://www.math.ntnu.no/emner/TMA4315/2020h/eliteserie.csv", colClasses = c("factor"
head(long)
```

```
##                    attack              defence home goals
## 1                   Molde Sandefjord_Fotball  yes     5
## 2 Sandefjord_Fotball               Molde   no     0
## 3      Stroemsgodset             Stabaek  yes     2
## 4            Stabaek       Stroemsgodset   no     2
## 5                Odd            Haugesund  yes     1
## 6          Haugesund                 Odd   no     2
```

a)

We consider the model

```
library(glmmTMB)
mod <- glmmTMB(goals ~ home + (1|attack) + (1|defence),  poisson, data=long, REML=TRUE)
```

The distributional assumption on the i'th response (number of goals) is $y_i|\gamma_{j(i)}, \gamma_{k(i)} \sim \text{Poisson}(\lambda_i)$, $i = 1, 2, \ldots, n = 480$. The conditional mean is connected to the covariates by the canonical link function:

$$\lambda_i = \exp\left(\beta_0 + \beta_h x_i + \gamma_{j(i)}^a + \gamma_{k(i)}^d\right).$$

Here, β_h is the effect of playing home, $\gamma_{j(i)}^a$ is the effect of team $j(i)$ attacking, $\gamma_{k(i)}^d$ is the effect of team $k(i)$ defending, and ε_i is the error term. The random effects are independent and identically distributed, such that

$$\gamma_{j(i)}^a \sim \mathcal{N}(0, \tau_a) \quad \text{and} \quad \gamma_{k(i)}^d \sim \mathcal{N}(0, \tau_d),$$

where $j(i) = 1, 2, \ldots, m$, $k(i) = 1, 2, \ldots, m = 16$

Distributional Assumption on Response

Assuming that the response follows a Poisson distribution amounts to making the following assumptions:

1. Goals are scored independently, i.e. the number of goals scored within disjoint time intervals is independent.

2. The number of goals scored in a time interval is proportional to the length of the interval.

3. Two (or more) goals cannot be scored at exactly the same instance.

The last two assumptions seem very reasonable; two goals can obviously not occur at the same time, and more time gives more oppurtunities for goal scoring. The first one, however, is more questionable. For example, a team might which has just conceded a goal close to the end of the game might play more aggressively to salvage a draw, hence increasing the likelihood of more goals being scored. Despite this, the Poisson distribution seems like a reasonable choice to model this process. Trenger flere antagelser? Diskutere REML?

b)

```
summary(mod)
```

```
##  Family: poisson  ( log )
## Formula:          goals ~ home + (1 | attack) + (1 | defence)
## Data: long
##
##      AIC      BIC   logLik deviance df.resid
##   1147.2   1163.1   -569.6   1139.2      382
##
## Random effects:
##
## Conditional model:
##  Groups  Name        Variance Std.Dev.
##  attack  (Intercept) 0.007478 0.08647
##  defence (Intercept) 0.016383 0.12800
## Number of obs: 384, groups:  attack, 16; defence, 16
##
## Conditional model:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  0.12421    0.07809   1.591    0.112
## homeyes      0.40716    0.08745   4.656 3.22e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
ranef(mod)
```

```
## $attack
##                    (Intercept)
## BodoeGlimt         -0.036781062
## Brann               0.012026209
## Haugesund           0.011223106
## Kristiansund       -0.011367328
## Lillestroem        -0.049915996
## Molde               0.078390643
## Odd                 0.003654179
## Ranheim_TF          0.023375599
## Rosenborg           0.050622609
## Sandefjord_Fotball -0.058333079
## Sarpsborg08         0.026946364
## Stabaek            -0.026801293
## Start              -0.060500163
## Stroemsgodset       0.024556017
## Tromsoe             0.005756700
## Vaalerenga          0.007147494
##
## $defence
##                    (Intercept)
## BodoeGlimt         -0.042616090
## Brann              -0.123934761
## Haugesund          -0.061931278
## Kristiansund        0.008112432
## Lillestroem         0.030699257
## Molde              -0.036630979
```

```
## Odd                -0.052013600
## Ranheim_TF           0.062209734
## Rosenborg           -0.152631173
## Sandefjord_Fotball   0.133164228
## Sarpsborg08          0.006574064
## Stabaek              0.085376126
## Start                0.081958112
## Stroemsgodset        0.040486666
## Tromsoe             -0.009852817
## Vaalerenga           0.031030079
```

The effect of playing home is positive and statistically significant. According to the output, it almost worth half a goal (0.40716). This seems reasonable from an intuitive perspective. Looking at the estimated random effects, we can e.g. consider $\gamma_{\text{Rosenborg}}^{\text{defence}} \approx -0.153$. This is the lowest value among all teams, which indicates that Rosenborg is the best defending team. To check this, we calculate the average number of goals conceded by every team:

```
no.NA = long[is.na(long$goals) == 0, c("defence", "goals")]
agg = aggregate(no.NA$goals, by = list(no.NA$defence), FUN = mean)
colnames(agg) <- c("Team", "Avg. # of conceded goals")
knitr::kable(agg)
```

Team	Avg. # of conceded goals
BodoeGlimt	1.2500000
Brann	0.9583333
Haugesund	1.1666667
Kristiansund	1.4583333
Lillestroem	1.5416667
Molde	1.2500000
Odd	1.2083333
Ranheim_TF	1.6666667
Rosenborg	0.8333333
Sandefjord_Fotball	1.9583333
Sarpsborg08	1.4166667
Stabaek	1.7916667
Start	1.7500000
Stroemsgodset	1.5833333
Tromsoe	1.3750000
Vaalerenga	1.5416667

As expected, Rosenborg has the lowest average number of conceded goals.

we denote the team of average attack strength by A, and the team of average defense strength by D. Let y_A be the number of goals scored by team A, and similarly y_D be the number of goals scored by team D Then, we want to estimate skal epsilon være med egt?

$$\mathrm{E}\left[y_A\right] = \exp\left(\beta_h + \gamma_A^{\text{attack}} + \gamma_D^{\text{defence}} + \varepsilon_i\right),$$

as well as

$$\mathrm{Var}(y_A)$$

mangler informasjon for yB?

Notation

The linear predictor is given as

$$\eta_{ijk} = \beta_0 + \beta_{hk} + \gamma_{ai} + \gamma_{dj}.$$

The random effects γ_{ai} and γ_{dj} are IID normal with

$$\gamma_{ai} \sim \mathrm{N}(0, \ \tau_a^2), \qquad \gamma_{dj} \sim \mathrm{N}(0, \ \tau_d^2)$$

whenever $i, j \in \{1, \cdots, 16\}$ and $i \neq j$.

Using the log-link, the conditional mean of the response is

$$\mu_{ijk} = \mathrm{E}[y_{ijk} \mid \gamma_{ai}, \gamma_{dj}] = \exp(\eta_{ijk}).$$

c)

The expected value and variance of goals scored by two randomly selected teams can be found by the laws of total expectation and total variance conditioned on the random effect given by the two teams. For this, we use that the distribution Lognormal$(0, \tau^2)$ has mean $\exp(\tau^2/2)$ and variance $(\exp(\tau^2) - 1)\exp(\tau^2)$.

The law of total expectation gives

$$
\begin{aligned}
\mathrm{E}[y_{ijk}] &= \mathrm{E}\left[\mathrm{E}\left[y_{ijk} \mid \gamma_{ai}, \gamma_{dj}\right]\right] \\
&= \mathrm{E}\left[\exp\left(\beta_0 + \beta_h x_{ijk} + \gamma_{ai} + \gamma_{dj}\right)\right] \\
&= \exp\left(\beta_0 + \beta_h x_{ijk}\right) \mathrm{E}\left[\exp\left(\gamma_{ai} + \gamma_{dj}\right)\right] \\
&= \exp\left(\beta_0 + \beta_h x_{ijk}\right) \exp\left(\frac{\tau_a^2 + \tau_d^2}{2}\right).
\end{aligned}
$$

The law of total variance gives

$$
\begin{aligned}
\mathrm{Var}[y_{ijk}] &= \mathrm{E}\left[\mathrm{Var}\left[y_{ijk} \mid \gamma_{ai}, \gamma_{dj}\right]\right] + \mathrm{Var}\left[\mathrm{E}\left[y_{ijk} \mid \gamma_{ai}, \gamma_{dj}\right]\right] \\
&= \mathrm{E}\left[\exp\left(\beta_0 + \beta_h x_{ijk} + \gamma_{ai} + \gamma_{dj}\right)\right] + \mathrm{Var}\left[\exp\left(\beta_0 + \beta_h x_{ijk} + \gamma_{ai} + \gamma_{dj}\right)\right] \\
&= \mathrm{E}[y_{ijk}] + \exp\left(2(\beta_0 + \beta_h x_{ijk})\right)\mathrm{Var}\left[\exp\left(\gamma_{ai} + \gamma_{dj}\right)\right] \\
&= \exp\left(\beta_0 + \beta_h x_{ijk}\right)\exp\left(\frac{\tau_a^2 + \tau_d^2}{2}\right) + \exp\left(2\left(\beta_0 + \beta_h x_{ijk}\right)\right)\left(\exp\left(\tau_a^2 + \tau_d^2\right) - 1\right)\exp\left(\tau_a^2 + \tau_d^2\right).
\end{aligned}
$$

Note that the total marginal variance consists of two terms, given as

$$\mathrm{Var}[y_{ijk}] = \underbrace{\mathrm{E}\left[\mathrm{Var}\left[y_{ijk} \mid \gamma_{ai}, \gamma_{dj}\right]\right]}_{\text{Variance of the game}} + \underbrace{\mathrm{Var}\left[\mathrm{E}\left[y_{ijk} \mid \gamma_{ai}, \gamma_{dj}\right]\right]}_{\text{Varaince of team strengths}}.$$

The first term is the expected variance of goals, which is a measure of the inherent randomness of the football game given the two teams. The second term is the variance of the expected goals, which is attributed to the variance in the strength of the two teams, since the expected goals are constant given two teams, so the randomness comes from the difference in strengths among the teams. We now calculate the proportion of variance explained by the two terms without and with home filed advantage.

Estimating the proportions with no home field advantage:

```r
# Parameters from model
beta <- summary(mod)$coefficients$cond[,1]
beta0 = beta[1]
beta.h = beta[2]
```

```
# Variance of random effects
tau2.attack <- summary(mod)$var$cond$attack[1]
tau2.defence <- summary(mod)$var$cond$defence[1]
tau2 <- tau2.attack + tau2.defence

# Marginal variance
var0.game <- exp(beta0 + tau2/2)
var0.strength <- exp(2*beta0)*(exp(tau2)-1)*exp(tau2)
var0 <- var0.game + var0.strength

# Proportion of variance
prop0.game = var0.game/var0
prop0.strength = var0.strength/var0
```

With no home field advantage for the attacking team, the total marginal variance of the number of goals is 1.1775525, where the inherent randomness of the game accounts for 0.9731 and the variation in team strengths account for 0.0269.

Estimating the proportions with home field advantage:

```
# Marginal variance
var1.game <- exp(beta0 + beta.h + tau2/2)
var1.strength <- exp(2*(beta0 + beta.h))*(exp(tau2)-1)*exp(tau2)
var1 <- var1.game + var1.strength

# Proportions of variance
prop1.game = var1.game / var1
prop1.strength = var1.strength / var1
```

If the attacking team has a home field advantage, the total marginal variance of the number of goals is 1.7932623, where the inherent randomness of the game accounts for 0.9601 and the variation in team strengths account for 0.0399. We note that also here the majority of variance can be explained by the game itself.

An interesting observation is that when the home field advantage is present, the proportion of variance explained by the strengths of the teams is higher. This follows directly from the equation of the total marginal variance when $\beta_h > 0$, as the second term is multiplied by an extra factor of $\exp(\beta_h) > 1$.

d)

We want to test if the two random effects in the model are significant. Since $\gamma_a \sim N(0, \tau_a^2)$ and $\gamma_d \sim N(0, \tau_d^2)$, this is equivalent to testing whether the variance of each random effect is positive. The hypothesis test for determining if the effect of attacking is significant can be formulated as

$$\text{H}_0 : \tau_a^2 = 0 \quad \text{vs} \quad \text{H}_1 : \tau_a^2 > 0,$$

and a similar test can be constructed for the effect of defending.

This can be carried out using a likelihood-ratio test. The test statistic is $\lambda_{LR} := -2(l_0 - l_1)$, where l_0 and l_1 are the log-likelihoods of the two models. We cannot apply Wilks' theorem directly here, as standard asymptotic theory is violated. The reason for this is that there is a 50% chance that under H_1, the maximum likelihood estimator $\hat{\tau}_a^2$ falls on the boundary of the parameter space. This is due to the expected value of the score function $s(\tau_a^2) := \frac{\partial l_1}{\partial \tau_a^2}$ evaluated at zero is zero, $E[s(0)] = 0$, so whenever the slope of $s(0)$ is negative, the maximum likelihood estimator returns zero. This results in a mixture of two distributions

$$\lambda_{RT} \sim \frac{1}{2}\chi_0^2 : \frac{1}{2}\chi_1^2$$

where χ_0^2 is simply the point mass distribution located at zero. The critical value C of the likelihood-ratio test is given as

$$P(\lambda_{RT} \geq C) = \frac{1}{2}P(\chi_0^2 \geq C) + \frac{1}{2}P(\chi_1^2 \geq C) = \alpha.$$

Since $C > 0$ for $\alpha < \frac{1}{2}$ the first term vanishes, so C is the upper 2α quantile of a chi-squared distribution with one degree of freedom. Using $\alpha = 0.05$ we get the critical value

```
alpha <- 0.05
qchisq(1-2*alpha, df=1)
```

```
## [1] 2.705543
```

Testing significance of effect of attack

```
mod.no_attack <- glmmTMB(goals ~ home + (1|defence), poisson, data=long, REML=TRUE)
LRT.no_attack <- -2*as.numeric(logLik(mod.no_attack) - logLik(mod))

p.value <- 0.5 * pchisq(LRT.no_attack, df=0, lower.tail = FALSE) +
           0.5 * pchisq(LRT.no_attack, df=1, lower.tail = FALSE)
p.value
```

```
## [1] 0.2587558
```

The p-value suggest that we can not reject H_0, meaning that the effect of attack is not significant. The test statistic has value 0.4188496, which is lower than the critical value.

Testing significance of effect of defence

```
mod.no_defence <- glmmTMB(goals ~ home + (1|attack), poisson, data=long, REML=TRUE)
LRT.no_defence <- -2*as.numeric(logLik(mod.no_defence) - logLik(mod))

p.value <- 0.5 * pchisq(LRT.no_defence, df=0, lower.tail = FALSE) +
           0.5 * pchisq(LRT.no_defence, df=1, lower.tail = FALSE)
p.value
```

```
## [1] 0.09843786
```

The p-value suggest that the effect of defence is not significant. The test statistic has value has value 1.6654006, which is again lower than the critical value.

Testing the effect of home field advantage

Our hypothesis can be formulated as

$$H_0 : \beta_h = 0 \quad \text{vs} \quad H_1 : \beta_h \neq 0,$$

where under H_1, the home field advantage could be either negative or positive.

The restricted maximum likelihood (REML) is often preferred when fitting generalized mixed models as the calculations are cheaper than using normal maximum likelihood estimation. However, this is not possible here, as REML uses a likelihood function on a transformed set of data, so that nuisance parameters have no effect. Under H_0, this is simply the intercept β_0. But under H_1, we also have the parameter β_h for the home field advantage. This means that the two models lead to different means using REML, making the likelihood non-comparable. Setting REML = FALSE avoids this issue and fits the models using normal maximum likelihoods.

```
mod.ML <- glmmTMB(goals ~ home + (1|attack) + (1|defence), poisson, data=long, REML=FALSE)
mod.no_home <- glmmTMB(goals ~ (1|attack) + (1|defence), poisson, data=long, REML=FALSE)

LRT.no_home <- -2*as.numeric(logLik(mod.no_home) - logLik(mod.ML))

p.value <- pchisq(LRT.no_home, df=1, lower.tail = FALSE)
p.value
```

`## [1] 2.507328e-06`

The low p-value is smaller than the significance level, indicating that the effect of home field is significant.

e)

```
frankv <- data.table::frankv

ranking <- function(df){
  n.teams <- length(unique(df$attack))
  info <- data.frame(row.names = unique(df$attack), points = rep(0, n.teams),
                     goal.diff = rep(0, n.teams), goals.scored = rep(0, n.teams))

  for(n in seq(1, dim(df)[1], 2)){
    team1 <- as.character(df$attack[n])
    team2 <- as.character(df$defence[n])
    goals1 = df$goals[n]
    goals2 = df$goals[n+1]

    # skip missing values
    if (is.na(goals1) | is.na(goals2)){
      next
    }
    else if(goals1 > goals2){
      info[team1,"points"] = info[team1,"points"] + 3
    }
    else if(goals1 > goals2){
      info[team2,"points"] = info[team1,"points"] + 3
    }
    else {
      info[team1,"points"] = info[team1,"points"] + 1
      info[team2,"points"] = info[team1,"points"] + 1
    }
    info[team1,"goal.diff"] = info[team1,"goal.diff"] + goals1 - goals2
    info[team2,"goal.diff"] = info[team2,"goal.diff"] + goals2 - goals1
    info[team1,"goals.scored"] = info[team1,"goals.scored"] + goals1
    info[team2,"goals.scored"] = info[team2,"goals.scored"] + goals2
  }
  info$rank <- frankv(info, cols=c("points","goal.diff","goals.scored"),
                      order=-1, ties.method="random")
  info[order(info$rank),]
}
knitr::kable(ranking(long))
```

	points	goal.diff	goals.scored	rank
Start	42	-18	24	1
BodoeGlimt	41	-2	28	2
Rosenborg	31	23	43	3
Kristiansund	31	-3	32	4
Lillestroem	31	-11	26	5
Stabaek	31	-14	29	6
Molde	30	18	48	7
Tromsoe	30	2	35	8
Vaalerenga	30	-2	35	9
Stroemsgodset	29	0	38	10
Odd	28	6	35	11
Sarpsborg08	28	5	39	12
Ranheim_TF	27	-2	38	13
Sandefjord_Fotball	27	-23	24	14
Haugesund	26	8	36	15
Brann	25	13	36	16