

TMA4315: Project 1

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Litt usikker på hva slags notasjon vi skal bruke, f. eks. boldface for vektorer eller ikke? Bare si ifra hvis du vil ha noe spesifikt:)

Problem 1

a)

Since the response variables $y_i \sim \text{Bernoulli}(p_i)$, where $p_i = \Pr(y_i = 1 \mid x_i) = \Phi(x_i^T \beta)$, the conditional mean is given by $Ey_i = p_i$, which is connected to the covariates via the following relationship:

$$x_i^T \beta =: \eta_i = \Phi^{-1}(p_i),$$

which implies that $p_i = \Phi(\eta_i)$. This results in the likelihood function

$$\begin{aligned} L(\beta) &= \prod_{i=1}^n p_i^{y_i} (1 - p_i)^{1-y_i} \\ &= \prod_{i=1}^n \Phi(\eta_i)^{y_i} (1 - \Phi(\eta_i))^{1-y_i}. \end{aligned}$$

Thus, the log-likelihood becomes

$$l(\beta) := \ln(L(\beta)) = \sum_{i=1}^n y_i \ln(\Phi(\eta_i)) + (1 - y_i) \ln(1 - \Phi(\eta_i)) = \sum_{i=1}^n l_i(\beta).$$

To find the score function, we calculate

$$\begin{aligned} \frac{\partial l_i(\beta)}{\partial \beta} &= \frac{y_i}{\Phi(\eta_i)} \frac{\partial \Phi(\eta_i)}{\partial \beta} - \frac{1 - y_i}{1 - \Phi(\eta_i)} \frac{\partial \Phi(\eta_i)}{\partial \beta} \\ &= \frac{y_i}{\Phi(\eta_i)} \phi(\eta_i) x_i - \frac{1 - y_i}{1 - \Phi(\eta_i)} \phi(\eta_i) x_i \\ &= \frac{y_i(1 - \Phi(\eta_i)) - (1 - y_i)\Phi(\eta_i)}{\Phi(\eta_i)(1 - \Phi(\eta_i))} \phi(\eta_i) x_i \\ &= \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1 - \Phi(\eta_i))} \phi(\eta_i) x_i \end{aligned}$$

Consequently, the score function is given by

$$s(\beta) = \sum_{i=1}^n \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1 - \Phi(\eta_i))} \phi(\eta_i) x_i.$$

Next, we find the expected Fisher information, $F(\beta)$. We find it by using the result

$$\begin{aligned}
F(\beta) &= \text{Var}(s(\beta)) = \text{Var}\left(\sum_{i=1}^n \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1 - \Phi(\eta_i))} \phi(\eta_i) x_i\right) \\
&= \sum_{i=1}^n \underbrace{\left[\frac{\phi(\eta_i)}{\Phi(\eta_i)(1 - \Phi(\eta_i))}\right]^2}_{=:\xi_i} \text{Var}(y_i x_i) = \sum_{i=1}^n \xi_i x_i^T \text{Var}(y_i) x_i \\
&= \sum_{i=1}^n \xi_i p_i (1 - p_i) x_i^T x_i \\
&= \sum_{i=1}^n \frac{\phi(\eta_i)^2}{\Phi(\eta_i)(1 - \Phi(\eta_i))} x_i^T x_i
\end{aligned}$$

Where in the third equality we have used that the y_i 's are independent.