TMA4315: Project 1

Jim Totland, Martin Gudahl Tufte

9/8/2021

Problem 1

a)

Since the response variables $y_i \sim \text{Bernoulli}(p)$, the conditional mean is given by $\text{E}y_i = p$, which is connected to the covariates via the following relationship:

$$x_i^T\beta =: \eta_i = \Phi^{-1}(p),$$

which implies that $p = \Phi(\eta_i)$.This results in the likelihood function

$$\begin{aligned}
L(\beta) &= \Pi_{i=1}^n p^{y_i}(1-p)^{1-y_i} \\
&= \Pi_{i=1}^n \Phi(\eta_i)^{y_i}(1-\Phi(\eta_i))^{1-y_i}.
\end{aligned}$$

Thus, the log-likelyhood becomes

$$l(\beta) := \ln(L(\beta)) = \sum_{i=1}^n y_i \ln(\Phi(\eta_i)) + (1-y_i)\ln(1-\Phi(\eta_i)) = \sum_{i=1}^n l_i(\beta).$$

To find the score function, we calculate

$$\begin{aligned}
\frac{\partial l_i(\beta)}{\partial \beta} &= \frac{y_i}{\Phi(\eta_i)}\frac{\partial \Phi(\eta_i)}{\partial \beta} - \frac{1-y_i}{1-\Phi(\eta_i)}\frac{\partial \Phi(\eta_i)}{\partial \beta} \\
&= \frac{y_i}{\Phi(\eta_i)}\phi(\eta_i)x_i^T - \frac{1-y_i}{1-\Phi(\eta_i)}\phi(\eta_i)x_i^T \\
&= \frac{y_i(1-\Phi(\eta_i)) - (1-y_i)\Phi(\eta_i)}{\Phi(\eta_i)(1-\Phi(\eta_i))}\phi(\eta_i)x_i^T \\
&= \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1-\Phi(\eta_i))}\phi(\eta_i)x_i^T
\end{aligned}$$

Consequently, the score function is given by

$$s(\beta) = \sum_{i=1}^n \frac{y_i - \Phi(\eta_i)}{\Phi(\eta_i)(1-\Phi(\eta_i))}\phi(\eta_i)x_i^T.$$

Next, we find the expected Fisher information.