

# TMA4250 Spatial Statistics

## Project 1 - Random Fields and Gaussian Random Fields

Christian Moen, Jim Totland

2/3/2022

### Problem 1

a)

The positive semi-definite (PSD) property of the correlation function can be stated as follows.  $\forall m \in \mathbb{Z}_+$ ,  $\forall a_1, \dots, a_m \in \mathbb{R}$  and  $\forall \mathbf{s}_1, \dots, \mathbf{s}_m \in \mathcal{D}$ , we have

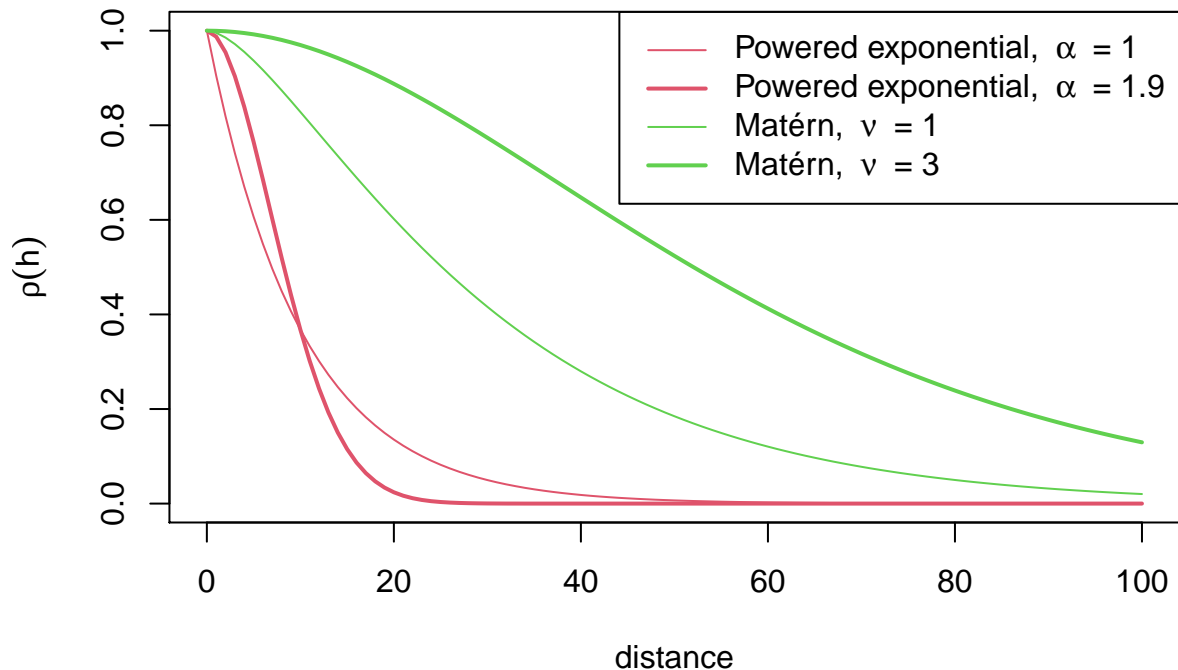
$$\sum_{i=1}^m \sum_{j=1}^m a_i a_j \rho(\mathbf{s}_i, \mathbf{s}_j) \geq 0.$$

To explain why this requirement is necessary, we observe that (in this case)  $\rho(\mathbf{s}_i, \mathbf{s}_j) = \sigma^{-2} c(\mathbf{s}_i, \mathbf{s}_j)$ , where  $c$  is the covariance function. Consequently,

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^m a_i a_j \rho(\mathbf{s}_i, \mathbf{s}_j) &= \sigma^{-2} \sum_{i=1}^m \sum_{j=1}^m a_i a_j c(\mathbf{s}_i, \mathbf{s}_j) \\ &= \sigma^{-2} \text{Var} \left[ \sum_{i=1}^m a_i X(\mathbf{s}_i) \right]. \end{aligned}$$

Since the variance must be non-negative, it is clear that the PSD property above must be satisfied. Below, the different correlation functions are illustrated. **ingen grunn til to plots, fordi ulike varianser gir samme korrelasjon?!**

```
curve(1/5 * cov.spatial(x, cov.mod = "powered.exponential", cov.pars = c(5, 10), kappa = 1),
      from = 0, to = 100, col = 2, xlab = "distance", ylab = expression(rho(h)))
curve(1/5 * cov.spatial(x, cov.mod = "powered.exponential", cov.pars = c(5, 10), kappa = 1.9),
      from = 0, to = 100, col = 2, lwd = 2, add = TRUE)
curve(1/5 * cov.spatial(x, cov.mod = "matern", cov.pars = c(5, 20), kappa = 1),
      from = 0, to = 100, col = 3, add = TRUE)
curve(1/5 * cov.spatial(x, cov.mod = "matern", cov.pars = c(5, 20), kappa = 3),
      from = 0, to = 100, col = 3, lwd = 2, add = TRUE)
legend("topright", c(expression("Powered exponential, " ~alpha~ " = 1"),
                     expression("Powered exponential, " ~alpha~ " = 1.9"),
                     expression("Matérn, " ~nu~ " = 1"), expression("Matérn, " ~nu~ " = 3")),
      col = c(2,2,3,3), lwd = c(1,2,1,2))
```

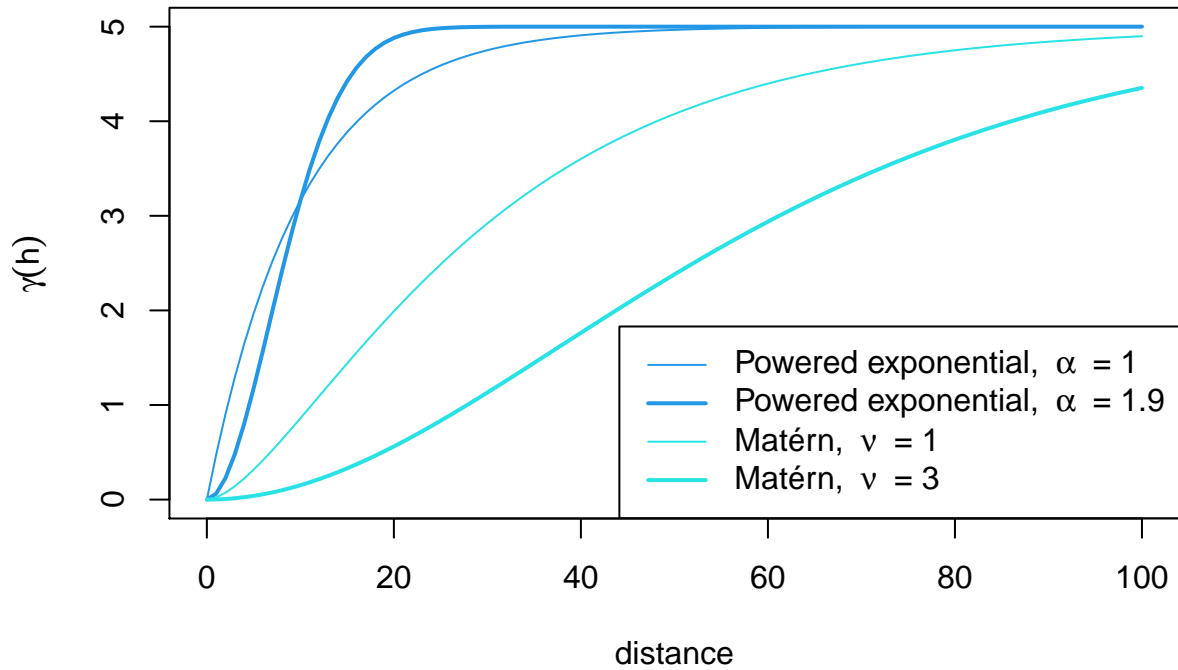


Next,

we plot the semi-variograms. *to figurer her kanskje? litt unødvendig? spør om dette.*

```
semi.variogram <- function(x, ...){
  return(cov.spatial(0, ...) - cov.spatial(x, ...))
}

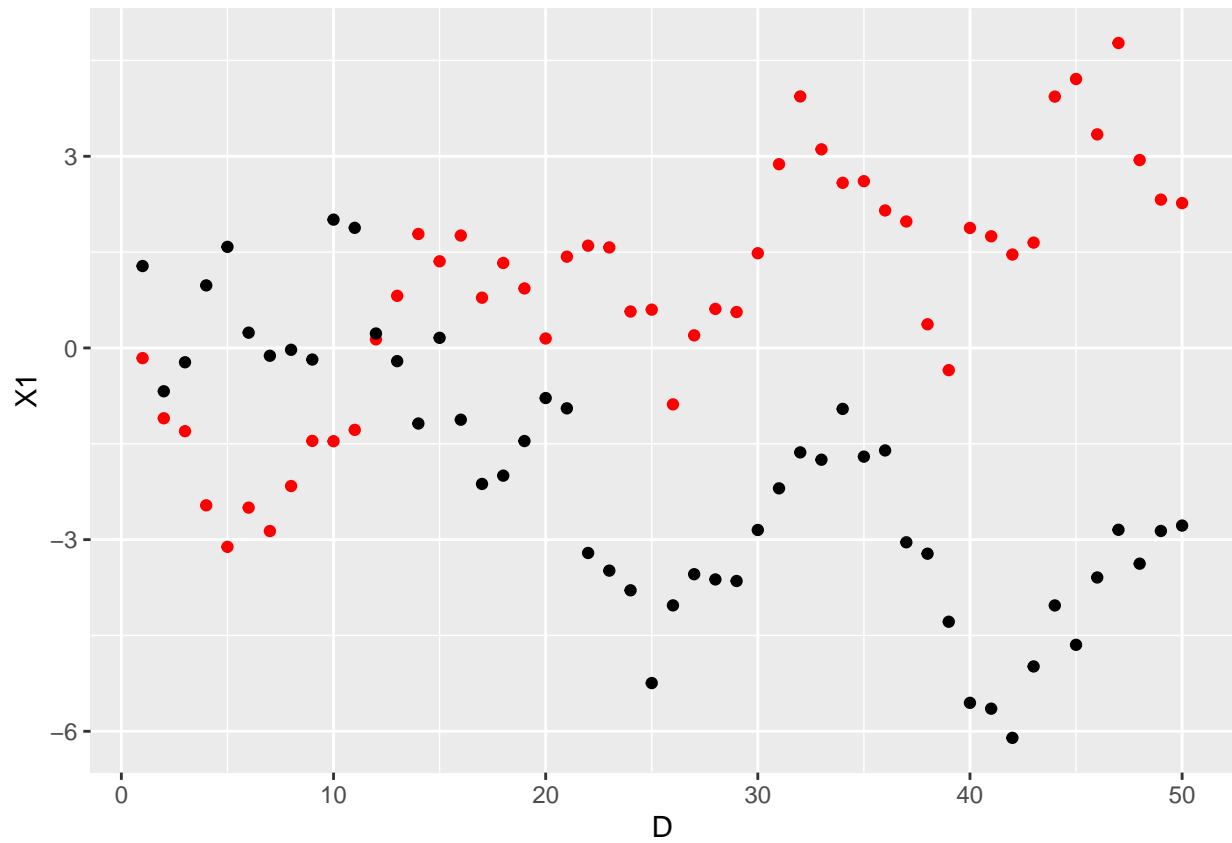
curve(semi.variogram(x, cov.mod = "powered.exponential", cov.pars = c(5, 10), kappa = 1),
      from = 0, to = 100, col = 4, xlab = "distance", ylab = expression(gamma(h)))
curve(semi.variogram(x, cov.mod = "powered.exponential", cov.pars = c(5, 10), kappa = 1.9),
      from = 0, to = 100, col = 4, lwd = 2, add = TRUE)
curve(semi.variogram(x, cov.mod = "matern", cov.pars = c(5, 20), kappa = 1),
      from = 0, to = 100, col = 5, add = TRUE)
curve(semi.variogram(x, cov.mod = "matern", cov.pars = c(5, 20), kappa = 3),
      from = 0, to = 100, col = 5, lwd = 2, add = TRUE)
legend("bottomright", c(expression("Powered exponential, " ~alpha~ " = 1"),
                        expression("Powered exponential, " ~alpha~ " = 1.9"),
                        expression("Matérn, " ~nu~ " = 1"),
                        expression("Matérn, " ~nu~ " = 3")),
      col = c(4,4,5,5), lwd = c(1,2,1,2))
```



b)

By the definition of a GRF,  $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ . The parameters are calculated from the mean- and covariance function of the GRF, such that  $\boldsymbol{\mu} = \mathbf{0}$  and  $\Sigma_{ij} = \sigma^2 \rho(\|i - j\|)$ .

```
n <- 50
D.tilde <- 1:n
mu <- rep(0, n)
Sigma <- cov.spatial(as.matrix(dist(expand.grid(D.tilde))),
                     cov.mod = "powered.exponential", cov.pars = c(5, 10), kappa = 1)
X <- mvrnorm(4, mu, Sigma)
df <- data.frame(t(X), D = D.tilde)
ggplot(df) + geom_point(aes(x = D, y = X1), color = "red") + geom_point(aes(x = D, y = X2))
```



```
plot(D.tilde, X[1,])
```

