Compulsory exercise 1: Group 39

TMA4268 Statistical Learning V2021

Alexander J Ohrt, Jim Totland

16 February, 2021

Problem 1

a)

By extending the given univariate regression problem to a multivariate regression problem that allows for several observations, we have that $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \varepsilon I$, where I is the identity matrix of the correct dimensions. Hence, we have $\mathsf{E}(\mathbf{Y}) = \mathsf{E}(\mathbf{X}\boldsymbol{\beta} + \varepsilon I) = \mathbf{X}\boldsymbol{\beta}$ and $\mathsf{Cov}(\mathbf{Y}) = \sigma I$, without assuming anything else than that each of the ε's are independent of each other. Then

$$\mathsf{E}(\widetilde{\boldsymbol{\beta}}) = \mathsf{E}((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{Y}) = (\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathsf{E}(\mathbf{Y}) = (\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$$

and

$$\begin{aligned}\mathsf{Cov}(\widetilde{\boldsymbol{\beta}}) &= \mathsf{Cov}((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{Y}) = ((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T)\mathsf{Cov}(\mathbf{Y})((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T)^T \\ &= ((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T)\sigma^2 I\mathbf{X}(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-T} = \sigma^2((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-T}) \\ &= \sigma^2((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}),\end{aligned}$$

where we have used that $(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1} = (\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-T}$ in the last equality. In both these equations it is apparent that the moments are equal to those of the OLS estimator when $\lambda = 0$.

b)

The requested moments of $\widetilde{f}(\mathbf{x}_0) = \mathbf{x}_0^T\widetilde{\boldsymbol{\beta}}$ are

$$\mathsf{E}(\widetilde{f}(\mathbf{x}_0)) = \mathsf{E}(\mathbf{x}_0^T\widetilde{\boldsymbol{\beta}}) = \mathbf{x}_0^T\mathsf{E}(\widetilde{\boldsymbol{\beta}}) = \mathbf{x}_0^T(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta}$$

and

$$\begin{aligned}\mathsf{Cov}(\widetilde{f}(\mathbf{x}_0)) &= \mathsf{Cov}(\mathbf{x}_0^T\widetilde{\boldsymbol{\beta}}) = \mathbf{x}_0^T\mathsf{Cov}(\widetilde{\boldsymbol{\beta}})\mathbf{x}_0 \\ &= \sigma^2\mathbf{x}_0^T((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1})\mathbf{x}_0.\end{aligned}$$

c)

The expected MSE at $\mathbf{x}_0$ is

$$\begin{aligned}
\mathsf{E}[(y_0 - \widetilde{f}(\mathbf{x}_0))^2] &= [\mathsf{E}(\widetilde{f}(\mathbf{x}_0) - f(\mathbf{x}_0)]^2 + \mathsf{Var}(\widetilde{f}(\mathbf{x}_0)) + \mathsf{Var}(\varepsilon) \\
&= [\mathsf{E}(\widetilde{f}(\mathbf{x}_0)) - \mathsf{E}(f(\mathbf{x}_0))]^2 + \mathsf{Cov}(\widetilde{f}(\mathbf{x}_0)) + \sigma^2 \\
&= [\mathbf{x}_0^T(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}\boldsymbol{\beta} - \mathbf{x}_0^T\boldsymbol{\beta}]^2 + \sigma^2\mathbf{x}_0^T((\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X} + \lambda\mathbf{I})^{-1})\mathbf{x}_0 + \sigma^2.
\end{aligned}$$

Since there is no obvious way to simplify this, it will be left like this. This is also practical since it is easy to distinguish between the irreducible error, the variance of prediction and the squared bias.

d)

```
id <- "1X_8OKcoYbng1XvYFDirxjEWr7LtpNr1m" # google file ID
values <- dget(sprintf("https://docs.google.com/uc?id=%s&export=download", id))
X = values$X
dim(X)
```

```
#> [1] 100  81
```

```
x0 = values$x0
dim(x0)
```

```
#> [1] 81  1
```

```
beta=values$beta
dim(beta)
```

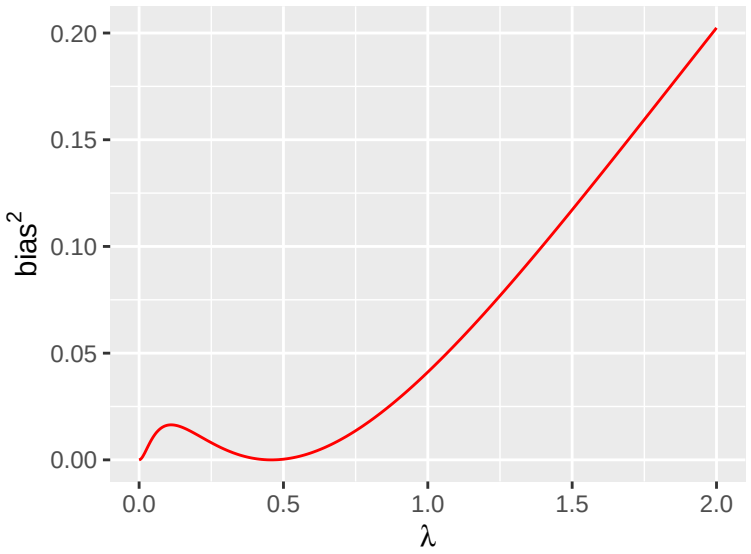
```
#> [1] 81  1
```

```
sigma=values$sigma
sigma
```

```
#> [1] 0.5
```

```
bias = function(lambda,X,x0,beta)
{
  p = ncol(X)
  value <-  (t(x0) %*% solve(t(X) %*% X + lambda * diag(p)) %*% t(X) %*% X %*% beta - t(x0) %*% beta)^2
  return(value)
}
lambdas = seq(0, 2, length.out = 500)
BIAS = rep(NA,length(lambdas))
for (i in 1:length(lambdas)) BIAS[i] = bias(lambdas[i], X, x0, beta)
dfBias = data.frame(lambdas = lambdas, bias = BIAS)
ggplot(dfBias, aes(x = lambdas, y = bias)) +
  geom_line(color = "red")+

  xlab(expression(lambda))+
  ylab(expression(bias^2))
```

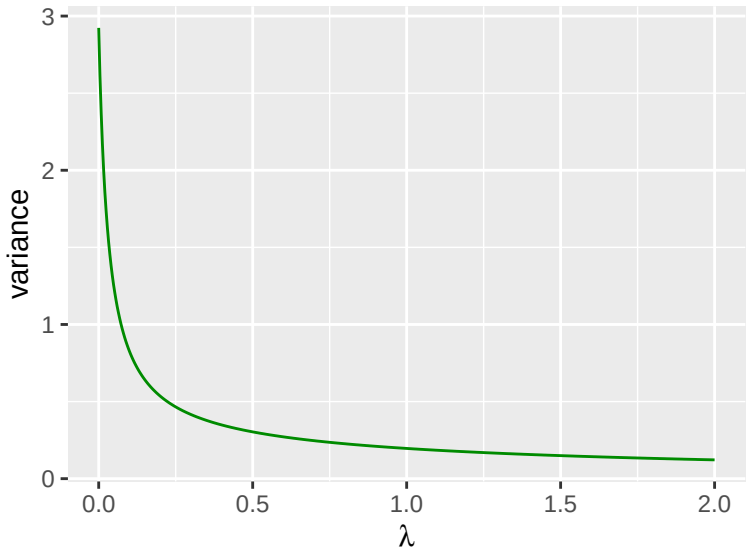


Comments: The graph shows that the bias of the ridge regression estimator increases as λ grows. OLS is unbiased, so, as expected, the bias is zero when $\lambda = 0$. Note that $\lambda \approx 0.5$ appears to be a sweet spot for this estimator when comparing all $\lambda > 0$, since the bias is low there. Perhaps this can be useful later.

e)

```
variance = function(lambda, X, x0, sigma)
{
  p = ncol(X)
  inv = solve(t(X)%*%X+lambda*diag(p))
  value = sigma^2*t(x0) %*% (inv %*% t(X) %*% X %*% inv) %*% x0
  return(value)
}
lambdas = seq(0, 2, length.out = 500)
VAR=rep(NA,length(lambdas))
for (i in 1:length(lambdas)) VAR[i]=variance(lambdas[i], X, x0, sigma)
dfVar = data.frame(lambdas = lambdas, var = VAR)
ggplot(dfVar, aes(x = lambdas, y = var))+
  geom_line(color = "green4")+

  xlab(expression(lambda))+
  ylab("variance")
```



Comments: The variance begins at ≈ 2.923 and decreases with λ. Hence, it is apparent that the ridge regression estimator is advantageous, when looking at solely variance, compared to the OLS estimator, since the variance is decreasing with λ. Despite this, when adding the bias and the variance, the OLS estimator may still have a lower expected MSE than the ridge regression estimator. Finally, note that the changes in the variance are larger for $\lambda \in [0, 2]$ than the changes in the bias (compared to the plot in task d)), which indicates that the variance dominates the change in expected MSE for the ridge regression estimation.

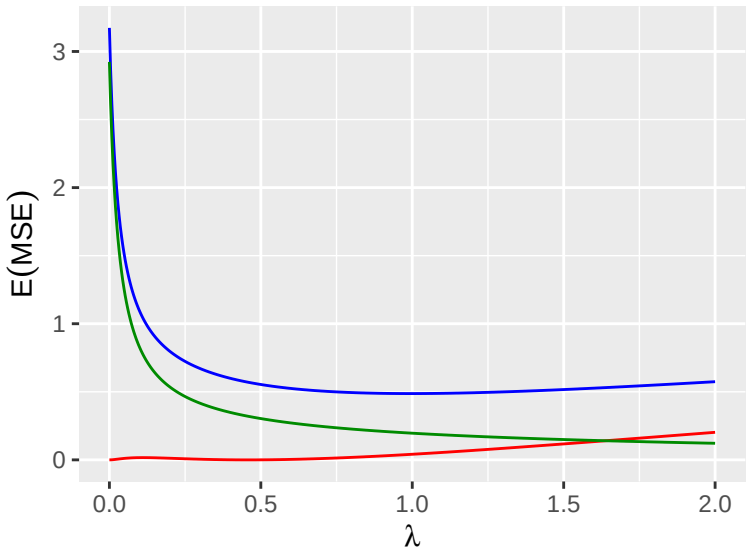
f)

```
exp_mse = BIAS + VAR + sigma^2
lambdas[which.min(exp_mse)]
```

```
#> [1] 0.993988
```

```
dfAll = data.frame(lambda = lambdas, bias = BIAS, var = VAR, exp_mse = exp_mse)
ggplot(dfAll)+
  geom_line(aes(x = lambda, y = exp_mse), color = "blue")+

  geom_line(aes(x = lambda, y = bias), color = "red")+
  geom_line(aes(x = lambda, y = var), color = "green4")+
    xlab(expression(lambda))+
    ylab(expression(E(MSE)))
```



Comments: Now we are able to conclude that the ridge regression estimator has a lower expected MSE compared to the OLS estimator, as gathered from the blue line in the plot. The optimal value of λ, which minimizes MSE, is ≈ 0.994. Hence, despite the fact that the bias is higher for the ridge regression estimator, the total expected MSE is lower, since the decrease in variance counters the increase in bias.

Problem 2

```
# read file
id <- "1yYlEl5gYY3BEtJ4d7KWaFGIOEweJIn__" # google file ID
d.corona <- read.csv(sprintf("https://docs.google.com/uc?id=%s&export=download", id),header=T)
```

a)

The tables are reported below.

```
# knitr::kable(table(d.corona$deceased)) # Hvorfor ta bort de som fungerer?

table1 <- summarise(d.corona, deceased = sum(d.corona$deceased),
                    non_deceased = nrow(d.corona) - deceased)
knitr::kable(table1)
```

deceased	non_deceased
105	1905

```
table2 <- table(d.corona$country, d.corona$sex)
knitr::kable(table2)
```

	female	male
France	60	54
indonesia	30	39
japan	120	174
Korea	879	654

```r
table3 <- table(d.corona$deceased, d.corona$sex)
rownames(table3) = c("non-deceased", "deceased")
knitr::kable(table3)
```

	female	male
non-deceased	1046	859
deceased	43	62

```r
d.france <- filter(d.corona, country == "France")
table4 <- table(d.france$deceased, d.france$sex)
rownames(table4) = c("non-deceased", "deceased")
knitr::kable(table4)
```

	female	male
non-deceased	55	43
deceased	5	11

b)

```r
glm.fit <- glm(deceased ~ ., family = "binomial", data = d.corona)
summary(glm.fit)
```

```
#>
#> Call:
#> glm(formula = deceased ~ ., family = "binomial", data = d.corona)
#>
#> Deviance Residuals:
#>     Min       1Q   Median       3Q      Max
#> -0.9050  -0.3508  -0.2761  -0.2144   3.1165
#>
#> Coefficients:
#>                   Estimate Std. Error z value Pr(>|z|)
#> (Intercept)      -3.993485   0.462190  -8.640  < 2e-16 ***
#> sexmale           0.626068   0.209045   2.995  0.00275 **
#> age               0.027134   0.004736   5.729 1.01e-08 ***
#> countryindonesia -0.411855   0.550051  -0.749  0.45400
#> countryjapan     -1.343383   0.417196  -3.220  0.00128 **
#> countryKorea     -0.773895   0.307980  -2.513  0.01198 *
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> (Dispersion parameter for binomial family taken to be 1)
#>
#>     Null deviance: 824.32  on 2009  degrees of freedom
#> Residual deviance: 766.16  on 2004  degrees of freedom
#> AIC: 778.16
#>
#> Number of Fisher Scoring iterations: 6
```

```r
coef(glm.fit)[3]
```

```
#>        age
#> 0.02713421
```

(i) The probability to die of covid for a male age 75 living in Korea can be predicted from the model. The prediction is found by

```
x0 = data.frame(sex = "male", age = 75, country = "Korea")
predict(glm.fit, newdata = x0, type = "response")
```

```
#>         1
#> 0.1084912
```

(ii) The p-value associated with `sexmale` is relatively small, and since the coefficient is positive, this constitutes some evidence that males have a higher probability of dying.

(iii) Yes. Both the `countryjapan` and `countryKorea` coefficients have relatively low p-values and are negative, which could be used as evidence that the probability of decease is lower in Japan and Korea compared to the reference category, France. `countryIndonesia` is not significant (large p-value), so there is no evidence that the probability of decease is any higher in Indonesia than in France.

(iv) Since we have used logistic regression, the odds of decease, given an observation $\boldsymbol{x}$ is

$$\frac{p(\boldsymbol{x})}{1-p(\boldsymbol{x})} = e^{\boldsymbol{x}^T\hat{\boldsymbol{\beta}}}.$$

Thus, the odds of decease increases by a factor of $e^{10\hat{\beta_{\text{age}}}} \approx 1.312$ when `age` increases by 10 and all other covariates are held constant.

c)

```
log.fit1 <- glm(deceased ~. + sex:age, data = d.corona, family="binomial")
summary(log.fit1)
```

```
#>
#> Call:
#> glm(formula = deceased ~ . + sex:age, family = "binomial", data = d.corona)
#>
#> Deviance Residuals:
#>     Min       1Q   Median       3Q      Max
#> -0.9111  -0.3500  -0.2768  -0.2150   3.1078
#>
#> Coefficients:
#>                   Estimate Std. Error z value Pr(>|z|)
#> (Intercept)      -3.953580   0.571712  -6.915 4.67e-12 ***
#> sexmale           0.556745   0.624834   0.891 0.372913
#> age               0.026485   0.007261   3.648 0.000265 ***
#> countryindonesia -0.410197   0.550304  -0.745 0.456030
#> countryjapan     -1.344440   0.417364  -3.221 0.001276 **
#> countryKorea     -0.772596   0.308244  -2.506 0.012195 *
#> sexmale:age       0.001111   0.009443   0.118 0.906350
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> (Dispersion parameter for binomial family taken to be 1)
#>
#>     Null deviance: 824.32  on 2009  degrees of freedom
#> Residual deviance: 766.15  on 2003  degrees of freedom
```

```
#> AIC: 780.15
#>
#> Number of Fisher Scoring iterations: 6
```

(i) As seen above, we have fitted the full logistic regression with an interaction term between `age` and `sex`. As the summary shows, the interaction effect between `sexmale` and `age` is not significant, since the p-value is large. Hence, although the coefficient is slightly positive, we cannot conclude that age is a greater risk factor for men.

```
log.fit2 <- glm(deceased ~ . + age:country, data = d.corona, family="binomial")
summary(log.fit2)
```

```
#>
#> Call:
#> glm(formula = deceased ~ . + age:country, family = "binomial",
#>     data = d.corona)
#>
#> Deviance Residuals:
#>     Min       1Q   Median       3Q      Max
#> -1.2681  -0.3447  -0.2768  -0.2180   3.0170
#>
#> Coefficients:
#>                       Estimate Std. Error z value Pr(>|z|)
#> (Intercept)          -7.04272    1.72789  -4.076 4.58e-05 ***
#> sexmale               0.62777    0.21056   2.981  0.00287 **
#> age                   0.06693    0.02124   3.151  0.00163 **
#> countryindonesia      4.34249    2.16594   2.005  0.04497 *
#> countryjapan          2.13091    2.03299   1.048  0.29456
#> countryKorea          2.37162    1.75357   1.352  0.17623
#> age:countryindonesia -0.07189    0.03310  -2.172  0.02986 *
#> age:countryjapan     -0.04630    0.02668  -1.736  0.08260 .
#> age:countryKorea     -0.04142    0.02189  -1.892  0.05854 .
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> (Dispersion parameter for binomial family taken to be 1)
#>
#>     Null deviance: 824.32  on 2009  degrees of freedom
#> Residual deviance: 759.71  on 2001  degrees of freedom
#> AIC: 777.71
#>
#> Number of Fisher Scoring iterations: 6
```

(ii) As seen above, we have fitted the full logistic regression with an interaction term between `age` and `country`. Since `France` is the reference level, and the interaction-term coefficient `age:countryindonesia` has the value -0.072 with a somewhat low p-value, we can infer that age is a greater risk factor for the French population than for the Indonesian population.

d)

TRUE, TRUE, TRUE and FALSE.

EVENTUALLY WE WILL ONLY LEAVE THE LINE ABOVE, BUT THOUGHTS ARE WRITTEN BELOW FIRST.

Jeg tror vi må lage modeller for dette for å klare å besvare de siste 3 spmene! Modeller er lagt til nedenfor. Disse SKAL fjernes før innlevering!

(i) 5.2238806 % is the amount of deaths. Hence: TRUE

(ii) Following the argumentation in the above point, LDA is useless if the misclassification rate is higher than approx 5 %. Need to make a LDA in code to test? In this case, we are probably more interested in the probabilites than the classification, so LDA is not so relevant? spørre om dette på mandan! tja, men LDA kan fortsatt brukes til å finne posterior-sannsynlighetene også

```
lda.fit2 <- lda(deceased~., data = d.corona)
prob <- predict(lda.fit2)$posterior
cl <- predict(lda.fit2)$class
table(predicted = cl, true = d.corona$deceased)
```

```
#>          true
#> predicted    0    1
#>         0 1905  105
#>         1    0    0
```

From the model above it can be seen that the LDA-fit classifies all non-deceased correctly, but it classifies none of the true deceased correctly. This means that the specificity is 1 and the sensitivity is 0. Hence, LDA is not very useful for this dataset, since it always predicts non-deceased (and hence has the null rate for misclassification as discussed above). Hence: TRUE

(iii) TRUE, as seen from the model above.

(iv) As seen from the model below, the sensitivity of QDA is higher, since it classifies some true deceased correctly, opposite to LDA, which classifies none of the true deceased correctly. Hence: FALSE

```
qda.fit2 <- qda(deceased~., data = d.corona)
prob <- predict(qda.fit2)$posterior
cl2 <- predict(qda.fit2)$class
table(predicted = cl2, true = d.corona$deceased)
```

```
#>          true
#> predicted    0    1
#>         0 1751   84
#>         1  154   21
```

Problem 3

```
# Read file.
id <- "1i1cQPeoLLC_FyAH0nnqCnnrSBpn05_hO" # Google file ID.
diab <- dget(sprintf("https://docs.google.com/uc?id=%s&export=download", id))
t = MASS::Pima.tr2
train = diab$ctrain
test = diab$ctest
```

a)

```
logReg = glm(diabetes~., data = train, family = "binomial")
summary(logReg)
```

```
#>
#> Call:
#> glm(formula = diabetes ~ ., family = "binomial", data = train)
#>
#> Deviance Residuals:
```

```
#>     Min       1Q   Median       3Q      Max
#> -2.8155  -0.6367  -0.3211   0.6147   2.2408
#>
#> Coefficients:
#>               Estimate Std. Error z value Pr(>|z|)
#> (Intercept) -10.583538   1.428276  -7.410 1.26e-13 ***
#> npreg         0.105109   0.062721   1.676 0.093775 .
#> glu           0.035586   0.005892   6.039 1.55e-09 ***
#> bp           -0.014654   0.013982  -1.048 0.294615
#> skin          0.020379   0.020575   0.990 0.321962
#> bmi           0.094683   0.031265   3.028 0.002458 **
#> ped           1.931666   0.529573   3.648 0.000265 ***
#> age           0.038291   0.020247   1.891 0.058594 .
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> (Dispersion parameter for binomial family taken to be 1)
#>
#>     Null deviance: 381.91  on 299  degrees of freedom
#> Residual deviance: 253.84  on 292  degrees of freedom
#> AIC: 269.84
#>
#> Number of Fisher Scoring iterations: 5
```

(i) We have that

$$p_i = \frac{e^{\beta_0+\beta_1 x_{i1}+\beta_2 x_{i2}+\cdots+\beta_7 x_{i7}}}{1+e^{\beta_0+\beta_1 x_{i1}+\beta_2 x_{i2}+\cdots+\beta_7 x_{i7}}} := \frac{e^{\eta_i(x)}}{1+e^{\eta_i(x)}} \implies \frac{p_i}{1-p_i} = \frac{\frac{e^{\eta_i(x)}}{1+e^{\eta_i(x)}}}{1-\frac{e^{\eta_i(x)}}{1+e^{\eta_i(x)}}} = e^{\eta_i(x)}$$

$$\implies \text{logit}(p_i) = \log\left(\frac{p_i}{1-p_i}\right) = \eta_i(x) = \beta_0+\beta_1 x_{i1}+\beta_2 x_{i2}+\cdots+\beta_7 x_{i7}$$

(ii) The classification is done below.

```
glm.probs <- predict(logReg, newdata = test, type = "response")
glm.preds <- ifelse(glm.probs > 0.5, 1, 0)
conf.table.glm <- table(predicted = glm.preds, true = test$diabetes)
conf.table.glm
```

```
#>          true
#> predicted   0   1
#>         0 137  29
#>         1  18  48
```

The sensitivity is $48/(48+29) \approx 0.623$ and the specificity is $137/(137+18) \approx 0.884$.

b)

(i) Skjønner ikke helt hvordan/hva de ønsker at vi skal forklare her. Det står jo explain in this case, men disse variablene har strengt tatt samme betydning uansett hvilken data vi ser på.(are we supposed to give estimates in this case?: "Explain what they are in the diabetes classification problem") π_k is the prior probability, given by $\pi_k = P(y = k)$. $\boldsymbol{\mu}_k$ is the mean vector of class k, when we have assumed that each $f_k(\mathbf{x})$ is normal. In this case, the class 1, i.e. presence of diabetes, has the mean vector $\boldsymbol{\mu}_1$ and the class 0, i.e. non-presence of diabetes, has the mean vector $\boldsymbol{\mu}_0$. $\boldsymbol{\Sigma}$ is the covariance matrix of each class, when assumed that the distribution of each class are normal. In LDA $\boldsymbol{\Sigma}_k = \boldsymbol{\Sigma} \quad \forall k$, whereas

in QDA each $\mathbf{\Sigma}_k$ are allowed to be different. $f_k(x)$ is the density function of X for an observation that comes from class k. These are assumed to be normal in LDA and QDA.

(ii) The fits are seen below

```
lda.diabetes <- lda(diabetes~., data = train)
qda.diabetes <- qda(diabetes~., data = train)

# Only need the prob if diabetes is present.
lda.diabetes.probs <- predict(lda.diabetes, newdata = test)$posterior[, 2]
lda.preds <- ifelse(lda.diabetes.probs > 0.5, 1, 0)
conf.table.lda.diabetes <- table(predicted = lda.preds, true = test$diabetes)
conf.table.lda.diabetes
```

```
#>          true
#> predicted   0   1
#>         0 138  30
#>         1  17  47
```

```
# Only need the prob if diabetes is present.
qda.diabetes.probs <- predict(qda.diabetes, newdata = test)$posterior[, 2]
qda.preds <- ifelse(qda.diabetes.probs > 0.5, 1, 0)
conf.table.qda.diabetes <- table(predicted = qda.preds, true = test$diabetes)
conf.table.qda.diabetes
```

```
#>          true
#> predicted   0   1
#>         0 131  32
#>         1  24  45
```

The sensitivity and specificity for LDA are thus $47/(47+30) \approx 0.61$ and $138/(138+17) \approx 0.89$, respectively. The sensitivity and specificity of QDA are $45/(45+32) \approx 0.584$ and $131/(131+24) \approx 0.845$, respectively.

The difference between the methods is that the covariances are equal across all classes (in this case: both classes) in LDA, which gives linear discriminant functions and a linear decision boundary. On the contrary, the covariances in each class when using QDA are allowed to be different, which gives a quadratic discriminant function in each class and a quadratic decision boundary.

c)

(i) In the KNN approach, a new observation is classified by using the k nearest points (in Euclidean norm) to the observation in question, to estimate the probability of the new point belonging to each of the different classes in the response. The new point is classified to the class with the highest estimated probability.

(ii) We would choose the tuning parameter k based on a K-fold cross validation, with $K = 5$ or $K = 10$. virket som om dette var nok! (elaborate, saw it in the lecture on linear regression, but not quite sure how it is done in practice).

(iii) The KNN classification fit can be seen below

```
set.seed(123) # For reproducibility, e.g. in case of ties.
knn.diabetes <- knn(train = train, test = test, cl = train$diabetes, k=25, prob=T)
conf.table.knn <- table(predicted = knn.diabetes, true = test$diabetes)
conf.table.knn
```

```
#>          true
#> predicted   0   1
#>         0 144  36
```

```
#>          1  11  41
```

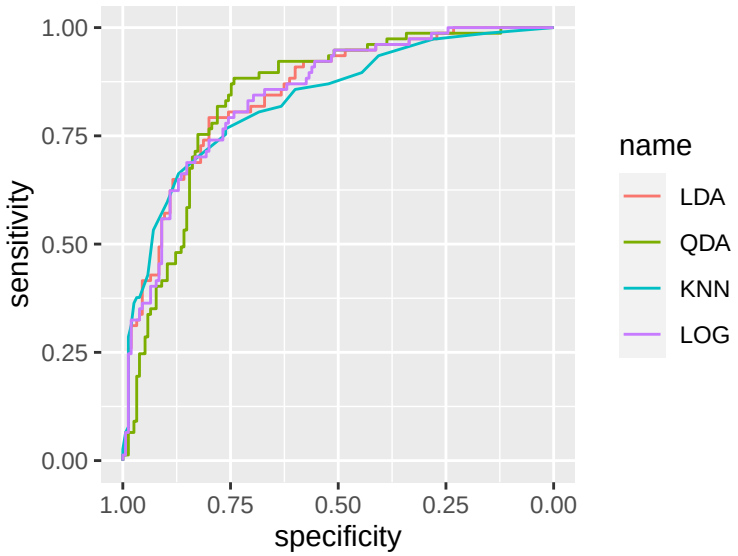
The sensitivity is $41/(41 + 36) \approx 0.532$ and the specificity is $144/(144 + 11) \approx 0.929$.

d) ROC curves

```
knn.probs <- ifelse(knn.diabetes == 0, 1 - attributes(knn.diabetes)$prob, attributes(knn.diabetes)$prob)
glm.log.probs <- predict(logReg, newdata = test, type="response")

ldaroc = roc(response = test$diabetes, predictor = lda.diabetes.probs, direction = "<")
qdaroc = roc(response = test$diabetes, predictor = qda.diabetes.probs, direction = "<")
knnroc = roc(response = test$diabetes, predictor = knn.probs, direction = "<")
glm.logroc = roc(response = test$diabetes, predictor = glm.log.probs, direction = "<")


ggroc(list(LDA  = ldaroc, QDA = qdaroc, KNN = knnroc, LOG = glm.logroc))
```



The area under the ROC curve is 0.849 for LDA, 0.841 for QDA, 0.833 for KNN and 0.845 for logistic regression (LOG). respectively. Based on this, LDA preforms the best. If the task is to create an interpretable model, we would choose the logistic regression, since the AUC is almost as high as for LDA, but the model is more interpretable.

Problem 4

a)

$$
\begin{aligned}
\hat{y}_{(-i)} &= \mathbf{x}_i^T \hat{\boldsymbol{\beta}}_{(-i)} \\
&= \mathbf{x}_i^T (X_{(-i)}^T X_{(-i)})^{-1} X_{(-i)}^T \mathbf{y}_{(-i)} \\
&= \mathbf{x}_i^T (X^T X - \mathbf{x}_i \mathbf{x}_i^T)^{-1} (X^T \mathbf{y} - \mathbf{x}_i y_i) \\
&= \mathbf{x}_i^T \left[(X^T X)^{-1} + \frac{(X^T X)^{-1} \mathbf{x}_i \mathbf{x}_i^T (X^T X)^{-1}}{1 - \mathbf{x}_i^T (X^T X)^{-1} \mathbf{x}_i} \right] (X^T \mathbf{y} - \mathbf{x}_i y_i), \quad \text{Sherman-Morrison} \\
&= \mathbf{x}_i^T \left[(X^T X)^{-1} + \frac{(X^T X)^{-1} \mathbf{x}_i \mathbf{x}_i^T (X^T X)^{-1}}{1 - h_i} \right] (X^T \mathbf{y} - \mathbf{x}_i y_i) \\
&= \mathbf{x}_i^T (X^T X)^{-1} X^T \mathbf{y} - \mathbf{x}_i^T (X^T X)^{-1} \mathbf{x}_i y_i + \mathbf{x}_i^T \frac{(X^T X)^{-1} \mathbf{x}_i \mathbf{x}_i^T (X^T X)^{-1}}{1 - h_i} X^T \mathbf{y} - \mathbf{x}_i^T \frac{(X^T X)^{-1} \mathbf{x}_i \mathbf{x}_i^T (X^T X)^{-1}}{1 - h_i} \mathbf{x}_i y_i \\
&= \mathbf{x}_i^T (X^T X)^{-1} X^T \mathbf{y} - \mathbf{x}_i^T \left[\frac{(X^T X)^{-1} \mathbf{x}_i}{1 - h_i} \right] (y_i (1 - h_i) - \mathbf{x}_i^T (X^T X)^{-1} X^T \mathbf{y} + h_i y_i) \\
&= \mathbf{x}_i^T \hat{\boldsymbol{\beta}} - \mathbf{x}_i^T \left[\frac{(X^T X)^{-1} \mathbf{x}_i}{1 - h_i} \right] (y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}).
\end{aligned}
$$

This gives

$$
\begin{aligned}
y_i - \hat{y}_{(-i)} &= y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}} + \mathbf{x}_i^T \left[\frac{(X^T X)^{-1} \mathbf{x}_i}{1 - h_i} \right] (y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}) \\
&= (y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}}) \left(1 + \frac{h_i}{1 - h_i} \right) \\
&= \frac{y_i - \hat{y}_i}{1 - h_i},
\end{aligned}
$$

which gives

$$
CV = \frac{1}{N} \sum_{i=1}^{N} (y_i - \hat{y}_{(-i)})^2 = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{y_i - \hat{y}_i}{1 - h_i} \right)^2,
$$

which completes the proof.

b)

FALSE, TRUE, FALSE and FALSE.

Problem 5

```
id <- "19auu8YlUJJJUsZY8JZfsCTWzDm6doE7C" # google file ID
d.bodyfat <- read.csv(sprintf("https://docs.google.com/uc?id=%s&export=download", id),header=T)
```

a)

```
lm.fit5 <- lm(bodyfat ~ age + weight + bmi, data = d.bodyfat)
```

The R^2 is ≈ 0.5803.

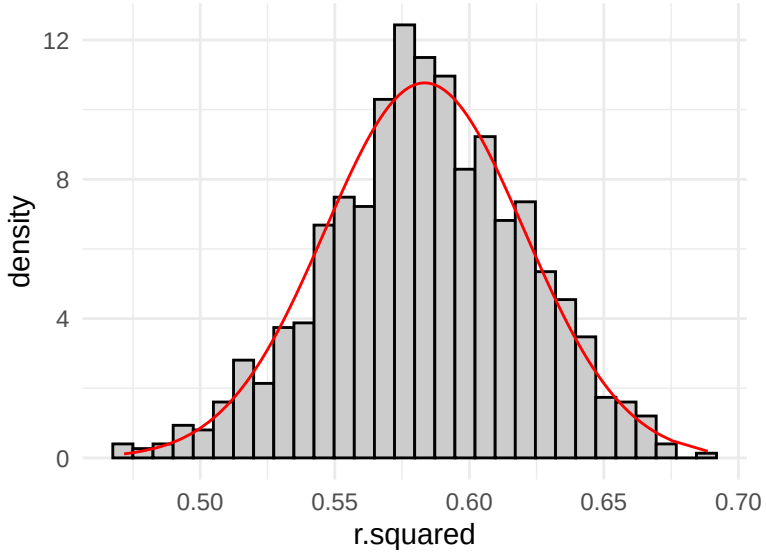
b)

```
set.seed(4268)

boot.fn <-  function(data, index) {
  return(summary(lm(bodyfat ~ age + weight + bmi, data = data, subset = index))$r.squared)
}

B <- 1000
r.squared <- rep(NA, B)
for (b in 1:B){
  r.squared[b] <- boot.fn(d.bodyfat, sample(nrow(d.bodyfat), nrow(d.bodyfat), replace = T))
}

df = data.frame(r.squared = r.squared, norm_den = dnorm(r.squared, mean(r.squared),
                                                        sd(r.squared)))
ggplot(df) + geom_histogram(aes(x = r.squared, y = ..density..), fill = "grey80",color = "black") +
   geom_line(aes(x = r.squared, y = norm_den), color = "red") +
   theme_minimal()
```




```
sd(r.squared)
```

```
#> [1] 0.03705002
```

```
quantile(r.squared, c(0.025, 0.975))
```

```
#>      2.5%     97.5%
#> 0.5090717 0.6534133
```

The standard deviation is 6.385 % of R^2. Also, R^2 is contained in the confidence interval.