

## Lecture 16: (Continue) Min-Max Optimization

### 1 Saddle points in min-max optimization

#### The goal of min-max optimization

Consider the following optimization problem:

$$\inf_{x \in X} \sup_{y \in Y} g(x, y)$$

where  $g : X \times Y \rightarrow \mathbb{R}$  is a given function,  $X$  and  $Y$  are sets over which the optimization is performed,  $\inf$  denotes the infimum (or greatest lower bound), and  $\sup$  denotes the supremum (or least upper bound).

#### Definition of saddle points

**Definition 1** (Saddle Points/Nash Equilibrium). *Let  $x \in X$  and  $y \in Y$  and  $g(\cdot, \cdot) : X \times Y \rightarrow \mathbb{R}$ . A pair of points  $(x_*, y_*) \in X \times Y$  is a saddle point of  $g(\cdot, \cdot)$  if*

$$g(x_*, y) \leq g(x_*, y_*) \leq g(x, y_*), \forall x \in X, y \in Y.$$

**Remark.** This condition implies that at the saddle point,  $g(x_*, y_*)$  represents a Nash equilibrium in the sense that no player can unilaterally improve their payoff by changing their strategy from  $x_*$  or  $y_*$ .

**Theorem 1.** *Let  $g : X \times Y \rightarrow \mathbb{R}$ , where  $X$  and  $Y$  are non-empty sets. A point  $(x_*, y_*)$  is a saddle point of  $g$  if and only if the following conditions are satisfied:*

1. *The supremum in  $\sup_{y \in Y} \inf_{x \in X} g(x, y)$  is attained at  $y_*$ .*
2. *The infimum in  $\inf_{x \in X} \sup_{y \in Y} g(x, y)$  is attained at  $x_*$ .*
3. *Moreover,  $\sup_{y \in Y} \inf_{x \in X} g(x, y) = \inf_{x \in X} \sup_{y \in Y} g(x, y)$ .*

#### Remarks.

1. If  $\inf \sup$  and  $\sup \inf$  have different values, then there is no saddle point.

2. If a saddle point exists, then:

- There might be multiple ones, all of them must have the same minimax value, i.e.,  $\sup_{y \in Y} \inf_{x \in X} g(x, y) = \inf_{x \in X} \sup_{y \in Y} g(x, y)$
- The set of saddle points is the Cartesian product  $X_* \times Y_*$  when nonempty.
- The set  $x_*$  is the optimal solution to  $\inf_{x \in X} \sup_{y \in Y} g(x, y)$ .
- The set  $y_*$  is the optimal solution to  $\sup_{y \in Y} \inf_{x \in X} g(x, y)$ .

## Example of No Saddle Points

Consider the function  $g(x, y) = (x - y)^2$  with  $X = [-1, 1]$  and  $Y = [-1, 1]$ . Then, we evaluate the infimum and supremum as follows:

$$\inf_{x \in X} \sup_{y \in Y} (x - y)^2 = \inf_{x \in X} (1 + |x|)^2 = 1,$$

where the infimum is taken over the maximum value the function can achieve for each  $x$ , realizing that the maximum occurs at the endpoints of  $Y$ . Similarly,

$$\sup_{y \in Y} \inf_{x \in X} (x - y)^2 = \sup_{y \in Y} 0 = 0,$$

where the infimum for each  $y$  is achieved when  $x = y$ , leading to a minimum value of 0 for all  $y$ .

This discrepancy between the infimum of the supremum and the supremum of the infimum indicates that there are no saddle points for  $g(x, y) = (x - y)^2$  over the given domain.

## 2 Metric to measure the progress of min-max optimization

In the context of min-max optimization, it is paramount to quantify the progress of optimization from the perspectives of participating entities. For a given function  $g : X \times Y \rightarrow \mathbb{R}$ , where  $X$  and  $Y$  represent the strategy sets for two players within the optimization problem, we define two metrics,  $\ell(x)$  and  $h(y)$ , to assess progress from the viewpoints of the  $x$ -player and  $y$ -player respectively.

**For the  $x$ -Player**

Define  $\ell(x)$  as the supremum of  $g(x, y)$  over all  $y \in Y$ :

$$\ell(x) := \sup_{y \in Y} g(x, y).$$

From the  $x$ -player's perspective, the progress is measured as:

$$\ell(x) - \inf_{x \in X} \ell(x).$$

**For the  $y$ -Player**

Define  $h(y)$  as the infimum of  $g(x, y)$  over all  $x \in X$ :

$$h(y) := \inf_{x \in X} g(x, y).$$

For the  $y$ -player's perspective, the progress is captured by:

$$\sup_{y \in Y} h(y) - h(y).$$

Let  $g : X \times Y \rightarrow \mathbb{R}$  be a given function, and  $\hat{x} \in X$ ,  $\hat{y} \in Y$  represent specific selections within their respective domains. By the definition of sup and inf, the following relation holds:

$$\sup_{y \in Y} g(\hat{x}, y) \geq g(\hat{x}, \hat{y}) \geq \inf_{x \in X} g(x, \hat{y}).$$

Combining the optimality gap of each player, we have that

$$\begin{aligned} \text{Gap}(\hat{x}, \hat{y}) &:= \ell(\hat{x}) - \inf_{x \in X} \ell(x) + \sup_{y \in Y} h(y) - h(\hat{y}) \\ &= \sup_{y \in Y} g(\hat{x}, y) - \inf_{x \in X} \sup_{y \in Y} g(x, y) + \sup_{y \in Y} \inf_{x \in X} g(x, y) - \inf_{x \in X} g(x, \hat{y}) \\ &= \sup_{y \in Y} g(\hat{x}, y) - \inf_{x \in X} g(x, \hat{y}), \end{aligned}$$

where the second-to-the-last line is by the assumptoin that assuming the existence of a saddle point.

**Definition 2** (Duality Gap). *The duality gap  $\text{Gap}(\hat{x}, \hat{y})$  is defined as:*

$$\text{Gap}(\hat{x}, \hat{y}) := \sup_{y \in Y} g(\hat{x}, y) - \inf_{x \in X} g(x, \hat{y}),$$

**Remark.** Duality gap is always non-negative even if the saddle point does not exist. By the definition of sup and inf, we have

$$\sup_{y \in Y} g(\hat{x}, y) \geq g(\hat{x}, \hat{y}) \geq \inf_x g(x, \hat{y})$$

Therefore,

$$\begin{aligned} \text{Gap}(\hat{x}, \hat{y}) &:= \sup_{y \in Y} g(\hat{x}, y) - \inf_{x \in X} g(x, \hat{y}) \\ &= \sup_{y \in Y} g(\hat{x}, y) - g(\hat{x}, \hat{y}) + g(\hat{x}, \hat{y}) - \inf_x g(x, \hat{y}) \\ &\geq 0. \end{aligned}$$

### $\epsilon$ -equilibrium / $\epsilon$ -saddle point

Assume a saddle point of  $g(\cdot, \cdot)$  exists. Let us define the value  $v_*$  as follows:

$$v_* = \inf_{x \in X} \sup_{y \in Y} g(x, y) = \sup_{y \in Y} \inf_{x \in X} g(x, y).$$

**Definition 3** ( $\epsilon$ -equilibrium /  $\epsilon$ -saddle point). *A pair  $(\hat{x}, \hat{y}) \in X \times Y$  is an  $\epsilon$ -equilibrium or  $\epsilon$ -saddle point if*

$$v_* - \epsilon \leq \inf_{x \in X} g(x, \hat{y}) \leq v_* \leq \sup_{y \in Y} g(\hat{x}, y) \leq v_* + \epsilon.$$

**Remark.** This definition extends the concept of a saddle point by introducing a margin of  $\epsilon$ , allowing for a near-optimal equilibrium within an  $\epsilon$  range of the optimal value  $v_*$ . Using the following inequality,

$$\sup_{y \in Y} g(\hat{x}, y) \geq g(\hat{x}, \hat{y}) \geq \inf_{x \in X} g(x, \hat{y}),$$

we can derive the following two inequalities

$$\begin{aligned} v_* - \epsilon &\leq \inf_{x \in X} g(x, \hat{y}) \leq v_* \leq g(\hat{x}, \hat{y}) \\ g(\hat{x}, \hat{y}) &\leq \sup_{y \in Y} g(\hat{x}, y) \leq v_* + \epsilon \end{aligned}$$

Thus, the above definition implies that

$$v_* - \epsilon \leq g(\hat{x}, \hat{y}) \leq v_* + \epsilon.$$

**Lemma 1.** *Given that the duality gap  $\text{Gap}(\hat{x}, \hat{y}) \leq \epsilon$  and assuming the existence of a saddle point, it follows that the pair  $(\hat{x}, \hat{y}) \in X \times Y$  constitutes an  $\epsilon$ -equilibrium or  $\epsilon$ -saddle point.*

*Proof.* By definition of the duality gap

$$\begin{aligned} \text{Gap}(\hat{x}, \hat{y}) &:= \sup_{y \in Y} g(\hat{x}, y) - \inf_{x \in X} g(x, \hat{y}) \leq \varepsilon \\ &\Leftrightarrow \sup_{y \in Y} g(\hat{x}, y) \leq \inf_{x \in X} g(x, \hat{y}) + \varepsilon. \end{aligned}$$

Given the optimal value

$$v_* = \inf_{x \in X} \sup_{y \in Y} g(x, y),$$

it follows from the definition that

$$v_* \leq \sup_{y \in Y} g(\hat{x}, y).$$

Therefore, we can establish the chain of inequalities

$$v_* = \inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} g(\hat{x}, y) \leq \inf_{x \in X} g(x, \hat{y}) + \varepsilon.$$

This sequence demonstrates the relationship between the optimal value  $v_*$ , the supremum over  $y$  for a fixed  $\hat{x}$ , and the adjusted infimum over  $x$  for a fixed  $\hat{y}$  by an  $\varepsilon$  margin, reflecting the bounds within which  $v_*$  is situated.

The duality gap for a pair  $(\hat{x}, \hat{y})$  is defined as:

$$\text{Gap}(\hat{x}, \hat{y}) := \sup_{y \in Y} g(\hat{x}, y) - \inf_{x \in X} g(x, \hat{y}) \leq \varepsilon$$

This can be equivalently expressed as:

$$\sup_{y \in Y} g(\hat{x}, y) \leq \inf_{x \in X} g(x, \hat{y}) + \varepsilon \leq \sup_{y \in Y} \inf_{x \in X} g(x, y) + \varepsilon = v_* + \varepsilon$$

Using similar arguments, we can prove the left side of the chain of inequalities. Therefore, we have proven that

$$v_* - \varepsilon \leq \inf_{x \in X} g(x, \hat{y}) \leq v_* \leq \sup_{y \in Y} g(\hat{x}, y) \leq v_* + \varepsilon.$$

□

**Definition 4.** Given a pair  $(\hat{x}, \hat{y}) \in X \times Y$ , it is considered to be an  $\varepsilon$ -equilibrium or  $\varepsilon$ -saddle point if the following condition holds:

$$v_* - \varepsilon \leq \inf_{x \in X} g(x, \hat{y}) \leq v_* \leq \sup_{y \in Y} g(\hat{x}, y) \leq v_* + \varepsilon.$$

### 3 The algorithmic aspect of min-max optimization

#### Review of online convex optimization

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**Algorithm 1** Online convex optimization

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1: for  $t = 1, 2, \dots$  do  
2:   Commit a point  $z_t$  with its convex decision space  $Z \subset \mathbb{R}^d$ .  
3:   Receive a loss function  $\ell_t(\cdot) : Z \rightarrow \mathbb{R}$  and incurs a loss  $\ell_t(z_t)$ .  
4: end for
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The goal of online convex optimization is to learn to be competitive with the best-fixed predictor from the convex set  $S$ , which is captured by minimizing the regret. Formally, the regret of the algorithm relative to any fixed benchmark  $z^*$  in  $Z$  when running on a sequence of  $T$  examples is defined as

$$\text{Regret}_T(z_*) = \sum_{t=1}^T \ell_t(z_t) - \sum_{t=1}^T \ell_t(z_*).$$

The regret of the algorithm relative to a convex set  $Z$  is defined as

$$\text{Regret}_T(Z) = \arg\max_{z_* \in Z} \text{Regret}_T(z_*)$$

#### The $x$ -Player Perspective

Consider the  $x$ -player who, at each time step  $t$ , plays a strategy  $x_t \in X$ . Upon choosing this strategy, the  $x$ -player receives a loss function defined as:

$$\ell_t(x) := g(x, y_t),$$

where  $g : X \times Y \rightarrow \mathbb{R}$  is a given function that determines the loss based on the player's choice  $x_t$  and the strategy  $y_t$  chosen by the opponent at time  $t$ .

#### The $y$ -Player Perspective

From the perspective of the  $y$ -player, the game proceeds as follows: at each time step  $t$ , the  $y$ -player selects a strategy  $y_t \in Y$ . Upon making this selection, the  $y$ -player receives a loss function, which is defined as:

$$h_t(y) := -g(x_t, y),$$

where  $g : X \times Y \rightarrow \mathbb{R}$  is the function determining the outcome based on the strategy  $x_t$  chosen by the opponent and the  $y$ -player's own choice  $y$  at time  $t$ .

## Meta-algorithm for solving min-max problems

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### Algorithm 2 Meta-algorithm for Solving Min-Max Problems

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- 1: Initialize  $\text{OAlg}^x$  (OCO Algorithm for  $x$ ) and  $\text{OAlg}^y$  (OCO Algorithm for  $y$ ).
- 2: Define weight sequence  $\alpha_1, \alpha_2, \dots, \alpha_T$ .
- 3: **for**  $t = 1, 2, \dots, T$  **do**
- 4:    $x$  plays  $x_t \leftarrow \text{OAlg}^x(\alpha_1 \ell_1, \alpha_2 \ell_2, \dots, \alpha_{t-1} \ell_{t-1})$
- 5:    $y$  plays  $y_t \leftarrow \text{OAlg}^y(\alpha_1 h_1, \alpha_2 h_2, \dots, \alpha_{t-1} h_{t-1})$
- 6:    $x$  receives  $\alpha_t \ell_t(x) := \alpha_t g(x, y_t)$
- 7:    $y$  receives  $\alpha_t h_t(y) := -\alpha_t g(x_t, y)$
- 8: **end for**
- 9: Output the average strategies  $x_T$  and  $y_T$ , where:

$$x_T := \frac{\sum_{t=1}^T \alpha_t x_t}{A_T}, \quad y_T := \frac{\sum_{t=1}^T \alpha_t y_t}{A_T},$$

$$\text{with } A_T := \sum_{t=1}^T \alpha_t.$$


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### From the $x$ -player perspective:

- Play  $x_t \in X$ .
- Receives the loss function at  $t$ ,  $\alpha_t \ell_t(x) := \alpha_t g(x, y_t)$ .

(Weighted) Regret of the  $x$ -player:

$$\alpha\text{-Regret}^x := \sum_{t=1}^T \alpha_t \ell_t(x_t) - \inf_{x \in X} \sum_{t=1}^T \alpha_t \ell_t(x).$$

(Weighted) Average regret of the  $x$ -player:

$$\overline{\alpha\text{-Regret}^x} := \frac{\alpha\text{-Regret}^x}{A_T},$$

where  $A_T := \sum_{t=1}^T \alpha_t$ .

**From the  $y$ -player perspective:**

- Play  $y_t \in Y$ .
- Receives the loss function at  $t$ ,  $h_t(y) := -\alpha_t g(x_t, y)$ .

(Weighted) Regret of the  $y$ -player:

$$\alpha\text{-Regret}^y := \sum_{t=1}^T \alpha_t h_t(y_t) - \inf_{y \in Y} \sum_{t=1}^T \alpha_t h_t(y).$$

(Weighted) Average regret of the  $y$ -player:

$$\overline{\alpha\text{-Regret}^y} := \frac{\alpha\text{-Regret}^y}{A_T},$$

where  $A_T := \sum_{t=1}^T \alpha_t$ .

## Guarantees of the meta-algorithm

**Theorem 2.** *Let  $g(x, y)$  be convex w.r.t  $x$  and concave w.r.t.  $y$ . The output  $(\bar{x}_T, \bar{y}_T)$  of the meta-algorithm is an  $\epsilon$ -equilibrium of  $g(\cdot, \cdot)$ , where*

$$\epsilon := \overline{\alpha\text{-Regret}^x} + \overline{\alpha\text{-Regret}^y}.$$

*Also, the duality gap is bounded as*

$$\text{Gap}(\bar{x}_T, \bar{y}_T) := \sup_{y \in Y} g(\bar{x}_T, y) - \inf_{x \in X} g(x, \bar{y}_T) \leq \overline{\alpha\text{-Regret}^x} + \overline{\alpha\text{-Regret}^y}.$$

**x-perspective**  $\ell_t(x) = g(x, y_t)$

$$\frac{1}{A_T} \sum_{t=1}^T \alpha_t g(x_t, y_t) = \frac{1}{A_T} \sum_{t=1}^T \alpha_t \ell_t(x_t)$$

This expression can further be decomposed into the infimum over  $x$  in  $X$  of the weighted outcomes, adjusted by the weighted regret for the  $x$ -player, and be simplified by using the definition of  $\alpha\text{-Regret}^x$  and  $\overline{\alpha\text{-Regret}^x}$ :

$$\begin{aligned} &= \inf_{x \in X} \left( \sum_{t=1}^T \frac{1}{A_T} \alpha_t g(x, y_t) \right) + \frac{\alpha\text{-Regret}^x}{A_T} \\ &= \inf_{x \in X} \left( \sum_{t=1}^T \frac{1}{A_T} \alpha_t g(x, y_t) \right) + \overline{\alpha\text{-Regret}^x} \end{aligned} \tag{1}$$



Using the Jensen's inequality, we have

$$\leq \inf_{x \in X} g \left( x, \sum_{t=1}^T \frac{\alpha_t}{A_t} y_t \right) + \overline{\alpha\text{-Regret}^x} \quad (2)$$

$$\leq \sup_{y \in Y} \inf_{x \in X} g(x, y) + \overline{\alpha\text{-Regret}^x} \quad (3)$$

**y-perspective**  $h_t(y) = -g(x_t, y)$

$$\frac{1}{A_T} \sum_{t=1}^T \alpha_t g(x_t, y_t) = \frac{1}{A_T} - \sum_{t=1}^T \alpha_t h_t(y_t)$$

This expression can further be decomposed into the infimum over  $y$  in  $Y$  of the weighted outcomes, adjusted by the weighted regret for the  $y$ -player, and be simplified by using the definition of  $\alpha\text{-Regret}^y$  and  $\overline{\alpha\text{-Regret}^y}$ :

$$\begin{aligned} &= - \inf_{y \in Y} \left( \sum_{t=1}^T \frac{1}{A_T} \alpha_t g(x_t, y_t) \right) - \frac{\alpha\text{-Regret}^y}{A_T} \\ &= \sup_{y \in Y} \left( \sum_{t=1}^T \frac{1}{A_T} \alpha_t g(x_t, y) \right) - \overline{\alpha\text{-Regret}^y} \end{aligned}$$

Using the Jensen's inequality, we have

$$\geq \sup_{y \in Y} g \left( \sum_{t=1}^T \frac{\alpha_t}{A_t} x_t, y \right) - \overline{\alpha\text{-Regret}^y} \quad (4)$$

$$\geq \inf_{x \in X} \inf_{y \in Y} g(x, y) - \overline{\alpha\text{-Regret}^y} \quad (5)$$

Thus, from (2) and (4), we have

$$\frac{1}{A_T} \sum_{t=1}^T \alpha_t g(x_t, y_t) \leq \inf_{x \in X} g \left( x, \sum_{t=1}^T \frac{\alpha_t}{A_t} y_t \right) + \overline{\alpha\text{-Regret}^x},$$

and

$$\frac{1}{A_T} \sum_{t=1}^T \alpha_t g(x_t, y_t) \geq \sup_{y \in Y} g \left( \sum_{t=1}^T \frac{\alpha_t}{A_t} x_t, y \right) - \overline{\alpha\text{-Regret}^y},$$

which implies that

$$\text{Gap}(\bar{x}_T, \bar{y}_T) := \sup_{y \in Y} g(\bar{x}_T, y) - \inf_{x \in X} g(x, \bar{y}_T) \leq \overline{\alpha - \text{Regret}^x} + \overline{\alpha - \text{Regret}^y}.$$

## First implication

Recall the Theorem:

**Theorem 3.** *Let  $g(x, y)$  be convex w.r.t  $x$  and concave w.r.t.  $y$ . The output  $(\bar{x}_T, \bar{y}_T)$  of the meta-algorithm is an  $\epsilon$ -equilibrium of  $g(\cdot, \cdot)$ , where*

$$\epsilon := \overline{\alpha\text{-Regret}^x} + \overline{\alpha\text{-Regret}^y}.$$

Also, the duality gap is bounded as

$$\text{Gap}(\bar{x}_T, \bar{y}_T) := \sup_{y \in Y} g(\bar{x}_T, y) - \inf_{x \in X} g(x, \bar{y}_T) \leq \overline{\alpha\text{-Regret}^x} + \overline{\alpha\text{-Regret}^y}.$$

We have the following implication:

Let  $g(x, y)$  be convex w.r.t  $x$  and concave w.r.t.  $y$ . If the decision space  $X$  and  $Y$  are convex and compact and  $g(\cdot, \cdot)$  is Lipschitz continuous, then we know there are sublinear regret algorithms. This implies our second implication.

## Second implication

**Theorem 4.** *Let  $X, Y$  be compact convex subsets of  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively. Let  $g(x, y) : X \times Y \rightarrow \mathbb{R}$  be convex in its first argument and concave in its second, and Lipschitz with respect to both. Then,*

$$\min_{x \in X} \max_{y \in Y} g(x, y) = \max_{y \in Y} \min_{x \in X} g(x, y).$$

*Proof.* From (3) and (5), we have

$$\frac{1}{A_T} \sum_{t=1}^T \alpha_t g(x_t, y_t) \leq \sup_{y \in Y} \inf_{x \in X} g(x, y) + \overline{\alpha\text{-Regret}^x}$$

and

$$\frac{1}{A_T} \sum_{t=1}^T \alpha_t g(x_t, y_t) \geq \inf_{x \in X} \inf_{y \in Y} g(x, y) - \overline{\alpha\text{-Regret}^y}$$

we can derive that

$$\begin{aligned} \sup_y \inf_x g(x, y) + \overline{\alpha\text{-Regret}^x} &\geq \inf_x \sup_y g(x, y) - \overline{\alpha\text{-Regret}^y} \\ \Leftrightarrow \sup_y \inf_x g(x, y) + \overline{\alpha\text{-Regret}^x} + \overline{\alpha\text{-Regret}^y} &\geq \inf_x \sup_y g(x, y) \end{aligned}$$

Recall the following lemma in the last lecture:

**Lemma 2.** Let  $g(\cdot, \cdot) : X \times Y \rightarrow \mathbb{R}$ , where  $X$  and  $Y$  are not empty. Then,

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \geq \sup_{y \in Y} \inf_{x \in X} g(x, y)$$

Therefore, we get

$$\min_{x \in X} \max_{y \in Y} g(x, y) = \max_{y \in Y} \min_{x \in X} g(x, y).$$

□

The above result together with the following theorem that we saw in the last lecture imply that a saddle point exists for when  $g(x, y)$  is convex w.r.t  $x$  and concave w.r.t.  $y$ ,  $g(\cdot, \cdot)$  is Lipschitz continuous, and the decision space  $X$  and  $Y$  are convex and compact.

**Theorem 5.** Let  $g(x, y) : X \times Y \rightarrow \mathbb{R}$ , where  $X$  and  $Y$  are not empty. A point  $(x_*, y_*)$  is a saddle point if and only if

- The supremum in  $\sup_{y \in Y} \inf_{x \in X} g(x, y)$  is attained at  $y_*$  & the infimum in  $\inf_{x \in X} \sup_{y \in Y} g(x, y)$  is attained at  $x_*$ .
- Also,  $\sup_{y \in Y} \inf_{x \in X} g(x, y) = \inf_{x \in X} \sup_{y \in Y} g(x, y)$ .

## 4 Applications of the min-max theorem

### Boosting as a bilinear game

Denote the training set  $\{z_j \in \mathbb{R}^d, l_j = \{+1, -1\}\}_{j=1}^m$ . Let  $H := \{h_i(\cdot)\}_{i=1}^n$  be a set of prediction functions, i.e.,

$$h_i(\cdot) : \mathbb{R}^d \rightarrow \{+1, -1\}.$$

We can construct the misclassification matrix as

$$A_{i,j} = \begin{cases} 1 & \text{if } h_i(z_j) \neq l_j, \\ 0 & \text{otherwise.} \end{cases}$$

We have that

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} x^\top A y := \min_{x \in \Delta_n} \max_{y \in \Delta_m} \sum_{i=1}^n \sum_{j=1}^m x[i] y[j] \mathbb{I}\{h_i(z_j) \neq l_j\}$$

Assume the existence of a weak learning oracle, i.e.,

$$\sum_{j=1}^m y[j] \mathbb{I}\{h_{i_*}(z_j) \neq l_j\} \leq \frac{1}{2} - \gamma,$$

where  $\gamma > 0$ . Here,  $i_*$  is the index of the predictor that gives a  $y$ -weighted error better than chance. Furthermore, for any  $y \in \Delta_m$ ,

$$\min_{x \in \Delta_n} x^\top Ay \leq e_{i_*}^\top Ay \leq \frac{1}{2} - \gamma.$$

Recall  $v_* = \max_{y \in \Delta_m} \min_{x \in \Delta_n} x^\top Ay$ . These imply that

$$v_* \leq \frac{1}{2} - \gamma < \frac{1}{2}.$$

Thus,

$$\max_{y \in \Delta_m} \min_{x \in \Delta_n} x^\top Ay = v_* \leq \frac{1}{2} - \gamma.$$

As we know the Nash equilibrium/Saddle points  $(x_*, y_*)$  exist,

$$x_*^\top Ay_* = v_* \leq \frac{1}{2} - \gamma.$$

The above implies that there exists  $x_* \in \Delta_n$  such that

$$\forall j \in [m] : \sum_{i=1}^n x_*[i] \mathbb{I}\{h_i(z_j) \neq l_j\} = x_*^\top Ae_j \leq v_* \leq \frac{1}{2} - \gamma < \frac{1}{2}.$$

Less than half of the base predictors misclassify when weighted by  $x_*[i]$  for each sample  $j \in [m]$ . The above implies that

$$\sum_{i=1}^n x_*[i] \mathbb{I}\{h_i(z_j) \neq l_j\} = x_*^\top Ae_j \leq v_* \leq \frac{1}{2} - \gamma < \frac{1}{2}.$$

We can correctly classify all the samples using a weighted majority vote.

## 5 Meta-algorithm for solving min-max problems (Simultaneously Play)

### Instance of the meta-algorithm

$$OAlg^x = FTRL, OMD, OptimisticMD, \dots$$

---

**Algorithm 3** Meta-algorithm for solving min-max problems (Simultaneously Play)

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- 1:  $\text{OAlg}^x$  (OCO Alg. of  $x$ ) and  $\text{OAlg}^y$  (OCO Alg. of  $y$ ).
  - 2: Weight sequence  $\alpha_1, \alpha_2, \dots, \alpha_T$ .
  - 3: **for**  $t = 1, 2, \dots, T$  **do**
  - 4:      $\begin{cases} x \text{ plays } x_t \leftarrow \text{OAlg}^x(\alpha_1 \ell_1, \alpha_2 \ell_2, \dots, \alpha_{t-1} \ell_{t-1}) \\ y \text{ plays } y_t \leftarrow \text{OAlg}^y(\alpha_1 h_1, \alpha_2 h_2, \dots, \alpha_{t-1} h_{t-1}) \end{cases}$
  - 5:      $\begin{cases} x \text{ receives } \alpha_t \ell_t(x) := \alpha_t g(x, y_t) \\ y \text{ receives } \alpha_t h_t(y) := -\alpha_t g(x_t, y) \end{cases}$
  - 6: **end for**
  - 7: Output:  $\left( \bar{x}_T := \frac{\sum_{t=1}^T \alpha_t x_t}{A_T}, \bar{y}_T := \frac{\sum_{t=1}^T \alpha_t y_t}{A_T} \right)$ , where  $A_T := \sum_{t=1}^T \alpha_t$ .
- 

$$\text{OAlg}^y = \text{FTRL}, \text{OMD}, \text{OptimisticMD}, \dots$$

Assume that  $\alpha_t = 1$  and  $\bar{x}_T$  and  $\bar{y}_T$  are  $\epsilon$ -equilibrium points

$$\epsilon = \frac{\text{Regret}_T(\text{OMD})}{T} + \frac{\text{Regret}_T(\text{OMD})}{T} = \frac{\mathcal{O}(\sqrt{T})}{T} \rightarrow 0, \text{ as } T \rightarrow \infty$$

**Question:** Can we get a better rate than  $\mathcal{O}(\frac{1}{\sqrt{T}})$ ? Yes!

## Recall Online Mirror Descent

The function  $\ell_t(z)$  is convex but not necessarily differentiable.  $g_t \in \partial \ell_t(z_t)$  is the subgradient of  $\ell_t(\cdot)$  at  $z_t$ .

---

**Algorithm 4** Online Mirror Descent

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- 1: **for**  $t = 1, 2, \dots$  **do**
  - 2:      $z_{t+1} = \arg \min_{z \in C} \langle g_t, z - z_t \rangle + \frac{1}{\eta} D_{z_t}^\phi(z)$ .
  - 3: **end for**
- 

Mirror Descent has

$$\sum_{t=1}^T \ell_t(z_t) - \ell_t(z^*) \leq \frac{1}{\eta} D_{z_1}^\phi(z^*) + \sum_{t=1}^T \frac{\eta}{2} \|g_t\|_*^2,$$

for any benchmark  $z^* \in Z$ .

If the loss  $\ell_t(\cdot)$  is scaled by  $\alpha_t$ ,

$$\alpha\text{-Regret}_z(z^*) \leq \frac{1}{\eta} D_{z_1}^\phi(z^*) + \sum_{t=1}^T \frac{\eta}{2} \|\alpha_t g_t\|_*^2,$$

for any benchmark  $z^* \in Z$ .

Assume there is a good guess  $m_t$  of  $g_t$ .

---

**Algorithm 5** Optimistic Mirror Descent

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```

1: for  $t = 1, 2, \dots$  do
2:    $z_{t-\frac{1}{2}} = \arg \min_{z \in C} \alpha_{t-1} \langle g_{t-1}, z \rangle + \frac{1}{\eta} D_{z_{t-\frac{3}{2}}}^\phi(z)$ .
3:    $z_t = \arg \min_{z \in C} \alpha_t \langle m_t, z \rangle + \frac{1}{\eta} D_{z_{t-\frac{1}{2}}}^\phi(z)$ .
4: end for

```

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We have that

$$\alpha\text{-Regret}^z(z^*) \leq \frac{1}{\eta} D_{z_1}^\phi(z_*) + \sum_{t=1}^T \frac{\eta}{2} \|\alpha_t(g_t - m_t)\|_*^2,$$

for any benchmark  $z^* \in Z$ .

By putting two Optimistic Mirror Descent against each other, we can get  $\mathcal{O}(\frac{1}{T})$  in a min-max problem, see e.g., [3] for details.

## 6 Bibliographic notes

More materials about min-max optimization can be found in [1],[2],[3],[4].

## References

- [1] Francesco Orabona, *A Modern Introduction to Online Learning*, Chapter 11.
- [2] Jun-Kun Wang, Jacob Abernethy, and Kfir Y. Levy, *No-Regret Dynamics in the Fenchel Game: A Unified Framework for Algorithmic Convex Optimization*, Mathematical Programming, 2023.
- [3] Vasilis Syrgkanis, Alekh Agarwal, Haipeng Luo, and Robert E. Schapire, *Fast Convergence of Regularized Learning in Games*, NeurIPS 2015.
- [4] Robert E. Schapire and Yoav Freund, *Boosting: Foundations and Algorithms*, MIT Press, 2012.