Drill 5

In this exercise, we perform the two-sided hypothesis testing for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. We assume that $X_1, X_2, \ldots, X_n$ are from the normal with mean μ and variance σ^2. This test is well known as z-test (when σ is known) or t-test (when σ is unknown) in the statistics literature. Note that the rejection region of the z-test and t-test are given by

$$Z = \frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2} \quad \text{and} \quad T = \frac{|\bar{X} - \mu_0|}{S/\sqrt{n}} > t_{\alpha/2}.$$

1. (a) When the variance known, obtain the theoretical power function of the z-test

When the variance σ^2 is known, the power function of the z-test is then given by

$$\begin{aligned} K_z(\mu) &= P\Big[\frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2}\Big] \\ &= 1 - P\Big[\frac{|\bar{X} - \mu_0|}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\Big] \\ &= 1 - P\Big[-z_{\alpha/2} \leq \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\Big]. \end{aligned} \tag{1}$$

Notice that X_i are from $N(\mu, \sigma^2)$, not from $N(\mu_0, \sigma^2)$. Thus, $(|\bar{X} - \mu|)/(\sigma/\sqrt{n})$ is distributed as the standard normal distribution, $N(0, 1)$, but $(|\bar{X} - \mu_0|)/(\sigma/\sqrt{n})$ is not.

We rewrite the power function in (1) by

$$\begin{aligned} K_z(\mu) &= 1 - P\Big[-z_{\alpha/2} + \frac{\mu_0}{\sigma/\sqrt{n}} \leq \frac{\bar{X}}{\sigma/\sqrt{n}} \leq z_{\alpha/2} + \frac{\mu_0}{\sigma/\sqrt{n}}\Big] \\ &= 1 - P\Big[-z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2} + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\Big] \\ &= 1 - P\Big[-z_{\alpha/2} - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2} - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\Big] \\ &= 1 - P\Big[-z_{\alpha/2} - \frac{\mu - \mu_0}{\sigma/\sqrt{n}} \leq Z \leq z_{\alpha/2} - \frac{\mu - \mu_0}{\sigma/\sqrt{n}}\Big], \end{aligned}$$

where $Z \sim N(0,1)$. Thus, we have

$$K_z(\mu) = 1 - \left\{\Phi\Big(z_{\alpha/2} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big) - \Phi\Big(-z_{\alpha/2} - \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big)\right\},$$

where $\Phi()$ is the cdf of the standard normal distribution. Using the relation $\Phi(-z) = 1 - \Phi(z)$, we can rewrite the above by

$$K_z(\mu) = 1 - \left\{1 - \Phi\Big(-z_{\alpha/2} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big) - 1 + \Phi\Big(z_{\alpha/2} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big)\right\}.$$

Therefore, the power function of the z-test is given by

$$\begin{aligned} K_z(\mu) &= 1 + \Phi\Big(-z_{\alpha/2} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big) - \Phi\Big(z_{\alpha/2} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big) \\ &= 1 - \Phi\Big(z_{\alpha/2} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big) + \Phi\Big(-z_{\alpha/2} + \frac{\mu-\mu_0}{\sigma/\sqrt{n}}\Big). \end{aligned}$$

(b) When the variance unknown, obtain the theoretical power function of the t-test

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with μ and variance σ^2. Let $\bar{X}$ and S^2 denote the sample mean and variance, respectively. Then the following t-test statistic under the local alternative H_1 : $\mu = \mu_0 + \delta\sigma/\sqrt{n}$*

$$\frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

has a non-central t-distribution with $n-1$ degrees of freedom and non-centrality δ.

Proof. Recall the definition of a non-central t-distribution. Let $Z \sim N(0,1)$ and V has a chi-square distribution with r degrees of freedom Suppose that Z and V are independent. Then the quotient below has a non-central t-distribution with r degrees of freedom and non-centrality δ:

$$\frac{Z+\delta}{\sqrt{V/r}}.$$

Let $V = (n-1)S^2/\sigma^2$ for convenience. Then V has a chi-square distribution with $n-1$ degrees of freedom. See Theorem 5.3.1 of **?**. We have

$$\begin{aligned}\frac{\sqrt{n}(\bar{X}-\mu_0)}{S} &= \frac{\sqrt{n}(\bar{X}-\mu_0)/\sigma}{\sqrt{V/(n-1)}} \\ &= \frac{\sqrt{n}(\bar{X}-\mu)/\sigma+\sqrt{n}(\mu-\mu_0)/\sigma}{\sqrt{V/(n-1)}} \\ &= \frac{Z+\sqrt{n}(\mu-\mu_0)/\sigma}{\sqrt{V/(n-1)}},\end{aligned}$$

where $Z=(\bar{X}-\mu)/(\sigma/\sqrt{n})$ and $Z\sim N(0,1)$. Thus, under the local $H_1:\mu=\mu_0+\delta\sigma/\sqrt{n}$, we have

$$\frac{\bar{X}-\mu_0}{S/\sqrt{n}}=\frac{Z+\delta}{\sqrt{V/(n-1)}}.$$

Since S^2 and $\bar{X}$ are independent, V and Z are also independent. This completes the proof. $\square$

Next, we want to obtain the power function for testing $H_0:\mu=\mu_0$ versus $H_1:\mu\neq\mu_0$. Since $\mu=\mu_0+\{\sqrt{n}(\mu-\mu_0)/\sigma\}\cdot\{\sigma/\sqrt{n}\}$, it is immediate upon using Theorem 1 that $(\bar{X}-\mu_0)/(S/\sqrt{n})$ under H_1 has the non-central t-distribution with $\nu=n-1$ degrees of freedom and non-centrality $\delta=(\mu-\mu_0)/(\sigma/\sqrt{n})$. The critical region for testing $H_0:\mu=\mu_0$ versus $H_1:\mu\neq\mu_0$ is given by

$$\frac{|\bar{X}-\mu_0|}{S/\sqrt{n}}>t_{\alpha/2}.$$

For convenience, we let $T_{n-1}(\delta)=(\bar{X}-\mu_0)/(S/\sqrt{n})$. Then the critical region can be rewritten as $|T_{n-1}(\delta)|>t_{\alpha/2}$. Then the power function is given by

$$\begin{aligned}K_t(\mu) &= P\big(|T_{n-1}(\delta)|>t_{\alpha/2}\big) \\ &= P(T_{n-1}(\delta)>t_{\alpha/2})+P(T_{n-1}(\delta)<-t_{\alpha/2}) \\ &= 1-\Phi_{\nu,\delta}(t_{\alpha/2})+\Phi_{\nu,\delta}(-t_{\alpha/2}),\end{aligned}$$

where $\Phi_{\nu,\delta}(\cdot)$ is the cdf of the non-central t-distribution with $\nu=n-1$ degrees of freedom and non-centrality $\delta=(\mu-\mu_0)/(\sigma/\sqrt{n})$.

2. Obtain the simulated power functions of the z-test and t-test for testing $H_0:\mu=1/2$ versus $H_1:\mu\neq 1/2$ with the significance level $\alpha=0.05$. Generate a sample of size $n=5$ from the normal distribution with mean μ and $\sigma=1$, where μ varies from -1 to 2.

3. Compare the theoretical and simulated power functions of two tests. (The results should be similar to the following plot).