1 Variance

Let X_i for $i = 1, 2, \ldots, n$ be independent and identically-distributed random variables with $\sigma^2 = \mathrm{Var}(X_i)$ and $\mu = E(X_i)$. We denote

$$\hat{\sigma}^2 = \frac{1}{K_1(n)} \sum_{i=1}^{n} (X_i - \mu)^2$$

and

$$S^2 = \frac{1}{K_2(n)} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = \sum_{i=1}^{n} X_i / n$.

1. (a) Find $K_1(n)$ satisfying $E(\hat{\sigma}^2) = \sigma^2$. You can find $K_1(n)$ mathematically (without normality assumption).

   (b) Find $K_2(n)$ satisfying $E(S^2) = \sigma^2$. You can find $K_2(n)$ mathematically (without normality assumption).

2. (a) Using the Monte Carlo simulation (R program), estimate $K_1(n)$ satisfying $E(\hat{\sigma}^2) = \sigma^2$ under the assumption that X_i are from a normal distribution. (Set $n = 5$ for example).

   (b) Using the Monte Carlo simulation (R program), estimate $K_2(n)$ satisfying $E(S^2) = \sigma^2$ under the assumption that X_i are from a normal distribution. (Set $n = 5$ for example).

3. (a) Using the Monte Carlo simulation (R program), estimate $K_1(n)$ satisfying $E(\hat{\sigma}^2) = \sigma^2$ under the assumption that X_i are from a **exponential** distribution ($\mu = 1$). (Set $n = 5$ for example).

(b) Using the Monte Carlo simulation (R program), estimate $K_2(n)$ satisfying $E(S^2) = \sigma^2$ under the assumption that X_i are from a **exponential** distribution ($\mu = 1$). (Set $n = 5$ for example).

2 Standard deviation

Let X_i for $i = 1, 2, \ldots, n$ be independent and identically-distributed random variables with $\sigma = \sqrt{\mathrm{Var}(X_i)}$. We denote

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2},$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$.

1. Find $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ under the assumption that X_i are from a normal distribution. You can find $c_4(n)$ mathematically.

2. Can you find $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ under any other distribution?

3. Using the Monte Carlo simulation (R program), estimate $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ under the assumption that X_i are from a normal distribution.

4. Using the Monte Carlo simulation (R program), estimate $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ under the assumption that X_i are from a **exponential** distribution ($\mu = 1$).