1 Variance

Let X_i for $i = 1, 2, \ldots, n$ be independent and identically-distributed random variables with $\sigma^2 = \mathrm{Var}(X_i)$ and $\mu = E(X_i)$. We denote

$$\hat{\sigma}^2 = \frac{1}{K_1(n)} \sum_{i=1}^{n} (X_i - \mu)^2$$

and

$$S^2 = \frac{1}{K_2(n)} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $\bar{X} = \sum_{i=1}^{n} X_i / n$.

1. (a) Find $K_1(n)$ satisfying $E(\hat{\sigma}^2) = \sigma^2$. You can find $K_1(n)$ mathematically (without normality assumption).

Solution:

Since

$$\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} X_i^2 - 2\mu \sum_{i=1}^{n} X_i + n\mu^2$$

and

$$E(X_i^2) = \mathrm{Var}(X_i) + \mu^2 = \sigma^2 + \mu^2,$$

we have

$$E\left[\sum_{i=1}^{n} (X_i - \mu)^2\right] = n(\sigma^2 + \mu^2) - 2\mu(n\mu) + n\mu^2 = n\sigma^2.$$

Thus, $K_1(n) = n$. NOTE: we did not use any distribution assumption.

(b) Find $K_2(n)$ satisfying $E(S^2) = \sigma^2$. You can find $K_2(n)$ mathematically (without normality assumption).

Solution:

Since

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\bar{X})^2 &= \sum_{i=1}^{n}X_i^2 - 2\bar{X}\sum_{i=1}^{n}X_i + n\bar{X}^2\\ &= \sum_{i=1}^{n}X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\\ &= \sum_{i=1}^{n}X_i^2 - n\bar{X}^2,\\ E(X_i^2) &= \mathrm{Var}(X_i) + \mu^2 = \sigma^2 + \mu^2,\end{aligned}$$

and

$$E(\bar{X}^2) = \mathrm{Var}(\bar{X}) + \left\{E(\bar{X})\right\}^2 = \sigma^2/n + \mu^2$$

we have

$$E\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = n(\sigma^2+\mu^2) - n(\sigma^2/n+\mu^2) = (n-1)\sigma^2.$$

Thus, $K_2(n) = n-1$. NOTE: we did not use any distribution assumption.

2. (a) Estimate $K_1(n)$ satisfying $E(\hat{\sigma}^2) = \sigma^2$ using R program under the assumption that X_i are from a normal distribution. (Set $n = 5$ for example).

   (b) Estimate $K_2(n)$ satisfying $E(S^2) = \sigma^2$ using R program under the assumption that X_i are from a normal distribution. (Set $n = 5$ for example).

3. (a) Estimate $K_1(n)$ satisfying $E(\hat{\sigma}^2) = \sigma^2$ using R program under the assumption that X_i are from a **exponential** distribution ($\mu = 1$). (Set $n = 5$ for example).

   (b) Estimate $K_2(n)$ satisfying $E(S^2) = \sigma^2$ using R program under the assumption that X_i are from a **exponential** distribution ($\mu = 1$). (Set $n = 5$ for example).

2 Standard deviation

Let X_i for $i = 1, 2, \ldots, n$ be independent and identically-distributed random variables with $\sigma = \sqrt{\mathrm{Var}(X_i)}$. We denote

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2},$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$.

1. Find $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ under the assumption that X_i are from a normal distribution. You can find $c_4(n)$ mathematically.

2. Can you find $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ under any distribution?

3. Estimate $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ using R program under the assumption that X_i are from a normal distribution.

4. Estimate $c_4(n)$ satisfying $E(S/c_4(n)) = \sigma$ using R program under the assumption that X_i are from a **exponential** distribution ($\mu = 1$).