Mathematical Problem

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July 27, 2022

1 Calculating eigenvalues

1.1 Problem Statement

Given the matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

where i is the imaginary number, find the eigenvalues of the matrix

$$H = \frac{\Omega_x}{2}\sigma_x \quad + \quad \frac{\Omega_y}{2}\sigma_y + \quad \frac{\omega}{2}\sigma_z$$

where $\Omega_x\Omega_y$ are real scalars

1.2 Solution

a) Substitution into formula

$$H = \frac{\Omega_x}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\Omega_y}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

$$H = \begin{pmatrix} 0 & \frac{\Omega_x}{2} \\ \frac{\Omega_x}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\frac{\Omega_y}{2} \\ i\frac{\Omega_y}{2} & 0 \end{pmatrix} + \begin{pmatrix} \frac{\omega}{2} & 0 \\ 0 & -\frac{\omega}{2} \end{pmatrix} \tag{5}$$

$$H = \begin{pmatrix} \frac{\omega}{2} & \frac{\Omega_x - i\Omega_y}{2} \\ \frac{\Omega_x + i\Omega_y}{2} & -\frac{\omega}{2} \end{pmatrix} \tag{6}$$

b) Applying the definition of eigenvalues

$\left|H-\lambda I\right| = 0$

$$\left| \begin{pmatrix} \frac{\omega}{2} & \frac{\Omega_x - i\Omega_y}{2} \\ \frac{\Omega_x + i\Omega_y}{2} & -\frac{\omega}{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right| \tag{7}$$

$$\begin{vmatrix} \frac{\omega}{2} - \lambda & \frac{\Omega_x - i\Omega_y}{2} \\ \frac{\Omega_x + i\Omega_y}{2} & -\frac{\omega}{2} - \lambda \end{vmatrix} = 0 \tag{8}$$

$$(\frac{\omega}{2} - \lambda)(-\frac{\omega}{2} - \lambda) - \frac{(\Omega_x + i\Omega_y)(\Omega_x - i\Omega_y)}{4} = 0 \tag{9}$$

$$-(\frac{\omega^2}{4} - \lambda^2) - \frac{\Omega_x^2 + \Omega_y^2}{4} = 0 \tag{10}$$

$$\lambda^2 - (\frac{\Omega_x^2 + \Omega_y^2 + \omega^2}{4}) = 0 \tag{11}$$

c) Getting the roots of lambda

$$\lambda_1 = \sqrt{\frac{\Omega_x^2 + \Omega_y^2 + \omega^2}{2}}$$

$$\lambda_2 = -\sqrt{\frac{\Omega_x^2 + \Omega_y^2 + \omega^2}{2}}$$